

# Optimal partisan districting on planar geographies

**Attila Tasnádi**

MTA-BCE “Lendület” Strategic Interactions Research Group, Department of Mathematics, Corvinus University of Budapest, H – 1093 Budapest, Fővám tér 8, Hungary, attila.tasnadi@uni-corvinus.hu

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**Abstract.** We show that optimal partisan districting in the plane with geographical constraints is an NP-complete problem.

**Keywords:** districting, gerrymandering, NP-complete problems.

# 1 Introduction

In electoral systems with single-member districts (or even with at least two multi-member districts) redistricting has to be carried out to resolve geographic malapportionment caused by migration and different district population growth rates. An inherent difficulty associated with redistricting is that it may favor a party. The problem becomes even worse if redistricting is manipulated for an electoral advantage, which is referred to as *gerrymandering*.

In the middle of the previous century it was hoped that the problem of gerrymandering could be overcome by computer programs using only data on voters geographic distribution without any statistical information on voters preferences (e.g. Vickrey, 1961) and thus determining an ‘unbiased’ districting. The first algorithm finding all districtings with (i) equally sized, (ii) connected and (iii) compact districts was given by Garfinkel and Nemhauser (1970).<sup>1</sup> The computational difficulty of the problem was clear from the very beginning. Nagel (1972) documented in an early survey the computational limitations of automated redistricting by considering the available programs of his time. Altman (1997) showed that the problems of achieving any of the three mentioned criteria are NP-hard. Moreover, he also demonstrated that maximizing the number of competitive districts is also NP-hard. Because of the computational difficulty of the problem there is a growing literature on new approaches to finding unbiased districtings (see, for instance, Mehrotra et al. (1998), Bozkaya et al. (2003), Bação et al. (2005), Chou and Li (2006), Ricca and Simeone (2008) or Ricca et al. (2008)). For recent surveys we refer to Ricca et al. (2011) and Tasndi (2011).

Though finding an equally sized districting is already computationally hard, from another point of view it is feared by the public that the continuously increasing computational power makes the problem of carrying out an optimal partisan gerrymandering possible. However, the underlying difficulty that hinders us in finding an unbiased districting, does not allow us to determine an optimal partisan redistricting. Indeed, Altman and McDonald (2010) provide recent evidence that current computer programs are far away from finding an optimal gerrymandering.

A formal proof establishing that a simplified version of the optimal gerrymandering problem is NP-complete was given by Puppe and Tasnádi (2009). Though they take geographical constraints into account, planarity is not prescribed explicitly. The current paper overcomes this shortcoming by locating voters in the plane.

## 2 The Framework

We assume that parties  $A$  and  $B$  compete in an electoral system consisting only of single member districts. In addition, voters with known party preferences are located in the plane and have to be divided into a given number of almost equally sized districts. The districting problem is defined by the following structure:

**Definition 1.** A *districting problem* is given by  $\Pi = (X, N, (x_i)_{i \in N}, v, K, \mathcal{D})$ , where

- $X$  is a bounded and strictly connected<sup>2</sup> subset of  $\mathbb{R}^2$ ,

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<sup>1</sup>Earlier Hess et al. (1965) provided an algorithm striving for similar goals; however, their algorithm did not always obtain optimal solutions.

<sup>2</sup>We call a bounded subset  $A$  of  $\mathbb{R}^2$  *strictly connected* if its boundary  $\partial A$  is a closed Jordan curve.

- the finite set of voters is denoted by  $N = \{1, \dots, n\}$ ,
- the distinct locations of voters are given by  $x_1, \dots, x_n \in \text{int}(X)$ ,
- the voters' party preferences are given  $v : N \rightarrow \{A, B\}$ ,
- the set of district labels is denoted by  $K = \{1, \dots, k\}$ , where  $\lfloor n/k \rfloor \geq 3$ , and
- $\mathcal{D}$  denotes the finite set of admissible districts consisting of bounded and strictly connected subsets of  $X$  and each of them containing the location of  $\lfloor n/k \rfloor$  or  $\lceil n/k \rceil$  voters,<sup>3</sup> and furthermore,
- we shall assume that based on their locations the  $n$  voters can be partitioned into  $k$  districts  $\{D_1, \dots, D_k\} \subseteq \mathcal{D}$ .

Observe that in defining the districting problem, we assumed that obtaining an almost equally sized districting is possible, which can be justified by the fact that finding an admissible districting for real-life problems is possible, while finding a districting satisfying additional requirements such as partisan optimality is difficult. In particular, the staff hired to produce a districting map could always construct a districting map consisting of almost equally sized districts although other properties as partisan optimality are difficult to prove or confute. Producing a districting with almost equally sized districts, is a tractable problem if there are not too many geographical restrictions since we can obtain a result by drawing districts from left to right and from top to bottom on a map of a state by keeping the average district size in mind.

We shall mention that in reality the basic units of a districting problem from which districts have to be created are census blocks or counties rather than voters. In this case voter preferences  $v : N \rightarrow \{A, B\}$  have to be replaced by a function of type  $v' : N' \rightarrow [0, 1]$ , where  $N'$  stands for the finite set of counties, expressing the fraction of party  $A$  voters. However, our results obtained in this paper can be extended to this more general setting, by allowing the case of almost equally sized counties, for which district outcomes are determined by the number of winning counties for party  $A$ , which happens to be the case, for instance, if  $v'(N') = \{\alpha, 1 - \alpha\}$  for a given  $\alpha \in [0, 1/2)$ .

Turning back to our districting problem defined on the level of voters, we have to assign each voter to a district.

**Definition 2.** An  $f : N \rightarrow \mathcal{D}$  is a *districting* for problem  $\Pi$  if there exists a set of districts  $D_1, \dots, D_k \in \mathcal{D}$  such that

- $f(N) = \{D_1, \dots, D_k\}$ ,
- $D_i \cap D_j = \emptyset$  if  $i \neq j$  and  $i, j \in K$ ,
- $\{x_i \mid i \in f^{-1}(D_j)\} \subset \text{int}(D_j)$  for any  $j \in K$ .

Observe that without loss of generality we do not explicitly require that a districting covers the entire country but just the inhabited areas.

**Definition 3.** Two districtings  $f : N \rightarrow \mathcal{D}$  and  $g : N \rightarrow \mathcal{D}$  with districts  $D_1, \dots, D_k$  and  $D'_1, \dots, D'_k$ , respectively, are *equivalent* if there exists a bijection between the series of sets  $\{x_i \mid i \in f^{-1}(D_1)\}, \dots, \{x_i \mid i \in f^{-1}(D_k)\}$  and the series of sets  $\{x_i \mid i \in g^{-1}(D'_1)\}, \dots, \{x_i \mid i \in g^{-1}(D'_k)\}$  such that the respective sets are identical.

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<sup>3</sup> $\lfloor x \rfloor$  stands for the largest integer not greater than  $x \in \mathbb{R}$  and  $\lceil x \rceil$  stands for the smallest integer not less than  $x \in \mathbb{R}$ .

Clearly, by defining equivalent districtings we have defined an equivalence relation above the set of districtings for problem  $\Pi$ .

A districting  $f$  and voters' preferences  $v$  determine the number of districts won by parties  $A$  and  $B$ , which we denote by  $F(f, v, A)$  and  $F(f, v, B)$ , respectively. If the two parties should receive the same number of votes in a district, its winner is determined by a predefined tie-breaking rule  $\tau : \mathcal{D} \rightarrow \{A, B\}$ .

**Definition 4.** For a given problem  $\Pi$  and tie-breaking rule  $\tau$  a districting  $f : N \rightarrow \mathcal{D}$  is *optimal* for party  $I \in \{A, B\}$  if  $F(f, v, I) \geq F(g, v, I)$  for any districting  $g : N \rightarrow \mathcal{D}$ .

Note that due to the above defined equivalence relation the set of districtings has finitely many equivalence classes, there exists at least one optimal districting for each party.

### 3 Determining an optimal districting is NP-complete

We establish that even the decision problem associated with the optimization problem of determining an optimal partisan districting, i.e. deciding for a given districting problem  $\Pi$  whether there exists a districting with at least  $m$  winning districts for a party, say party  $A$ , is an NP-complete problem; we call this problem WINNING DISTRICTS. In order to prove this, we shall reduce the INDEPENDENT SET problem on planar cubic<sup>4</sup> graphs, a proven NP-complete problem (see Garey and Johnson; 1979, pp. 195), to WINNING DISTRICTS. The INDEPENDENT SET problem asks whether a given graph  $G$  has a set of non-neighborings vertices of cardinality not less than  $m$ .

**Theorem 1.** *WINNING DISTRICTS is NP-complete.*

*Proof.* Whether a districting possesses at least  $m$  winning districts for party  $A$  can be verified easily in polynomial time, and therefore WINNING DISTRICTS is in NP.

We establish that INDEPENDENT SET on planar cubic graphs reduces to WINNING DISTRICTS. We define the mapping that assigns to an arbitrary planar cubic graph a districting problem in two steps.

*Step 1:* We start with constructing party  $A$  winning districts from a planar cubic graph  $G = (V, E)$ . Let each vertex be a party  $A$  voter and replace the 'midpoint' of each edge with a party  $A$  voter. In addition, we associate with each vertex  $v \in V$  a four member district containing the party  $A$  voter assigned to vertex  $v$  and the three party  $A$  voters replacing the three edges adjacent to vertex  $v$ . Hence, so far we have  $|V| + |E| = 5|V|/2$  party  $A$  voters and  $|V|$  districts, where each of them is consisting of four party  $A$  voters. Step 1 is illustrated in Figure 1.

Observe that the given planar cubic graph has an independent set of size  $m$  if and only if we can select  $m$  disjoint districts from the districts drawn in Step 1. However, as the right-hand side of Figure 1 shows, based on the districts drawn so far, we cannot partition the set of voters even if we extend the boundaries of the districts<sup>5</sup> in a way that the set of contained voters remains the same. Hence, we cannot obtain a districting. This is what Step 2 takes care about.

*Step 2:* Now we associate with each district full of party  $A$  voters twelve new party  $B$  voters such that the new voters have to be placed on the 'same side' of the district as illustrated in Figure 2.

<sup>4</sup>A graph is cubic if the degree of each vertex equals 3.

<sup>5</sup>By extending districts appropriately we can assign any uninhabited area of a map to one district.

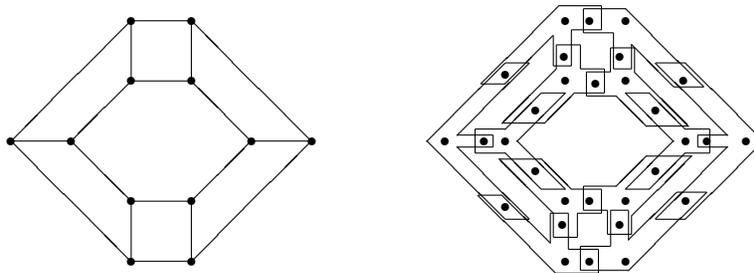


Figure 1: Associating party  $A$  winning districts with a planar cubic graph

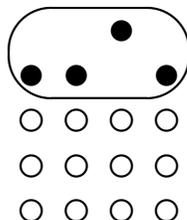


Figure 2: Party  $B$  voters

In addition, we have to form new districts distinguishing between the two cases whether a district full of party  $A$  voters will be included in our districting. First, if a district full of party  $A$  voters is not selected by a districting, then the respective party  $B$  voters should be grouped into party  $B$  winning districts as shown in Figure 3. We will refer to these ‘vertical districts’ as type 1 party  $B$  winning districts. However, we have

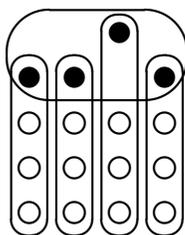


Figure 3: Type 1 party  $B$  winning districts

to be more careful since each of the three party  $A$  voters corresponding to an edge of our initial planar cubic graph  $G = (V, E)$  is even a member of another party  $A$  winning district full of party  $A$  voters. Therefore, if none of these two districts containing the same party  $A$  voter is contained in a districting, then this specific party  $A$  voter can be only included in one of these two type 1 districts. Hence, to make a districting possible we also include three type 2 ‘vertical districts’ associated with each party  $A$  voter corresponding to an edge. Type 2 party  $B$  winning districts are illustrated in

Figure 4 in which the second party  $A$  voter from the left is a voter corresponding to a vertex of our initial planar cubic graph.

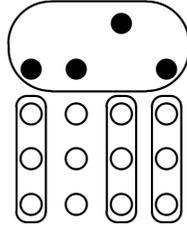


Figure 4: Type 2 party  $B$  winning districts

Second, if a district full of party  $A$  voters is selected by our districting, then the ‘horizontal districts’ illustrated in Figure 5 make a districting possible.

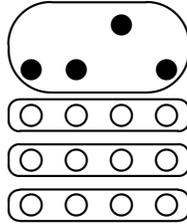


Figure 5: Type 3 party  $B$  winning districts

Observe that the districts introduced in Step 2 make a districting possible and that the given planar cubic graph has at least an independent set of size  $m$  if and only if the associated districting problem has at least  $m$  party  $A$  winning districts.

It remains to be shown that given a planar cubic graph  $G = (V, E)$  its associated districting problem outlined in Steps 1 and 2 can be determined in polynomial time. Constructing a straight line planar drawing of  $G$  with edges of at most five different slopes (see Keszeg et al. 2008) or a planar embedding in the grid of  $G$  (as shown by Liu et al. 1994), which can be obtained in polynomial time, we can easily locate the voters described in Steps 1 and 2 in the plane such that the respective districts can be drawn in polynomial time.  $\square$

*Remark 1.* Considering our reduction, we can observe that the approximability of WINNING DISTRICTS cannot be better than that of INDEPENDENT SET on planar cubic graphs for which Burns (1989) showed that the approximation ratio of the polynomial time algorithm given by Choukhmane and Franco (1986) equals  $7/8$ .

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