CORVINUS ECONOMICS WORKING PAPERS

CORVINUS UNIVERSITY of BUDAPEST

Faculty of Economics

CEWP 2/2014

# A new epistemic model

by

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April 18, 2014

**Abstract**

Meier (2012) gave a "mathematical logic foundation" of the purely measurable universal type space (Heifetz and Samet, 1998). The mathematical logic foundation, however, discloses an inconsistency in the type space literature: a finitary language is used for the belief hierarchies and an infinitary language is used for the beliefs.

In this paper we propose an epistemic model to fix the inconsistency above. We show that in this new model the universal knowledge-belief space exists, is complete and encompasses all belief hierarchies.

Moreover, by examples we demonstrate that in this model the players can agree to disagree – Aumann (1976)'s result does not hold –, and Aumann and Brandenburger (1995)'s conditions are not sufficient for Nash equilibrium. However, we show that if we substitute self-evidence (Osborne and Rubinstein, 1994) for common knowledge, then we get at that both Aumann (1976)'s and Aumann and Brandenburger (1995)'s results hold.

**Keywords**: Incomplete information game, Agreeing to disagree, Nash equilibrium, Epistemic game theory, Knowledge-belief space, Belief hierarchy, Common knowledge, Self-evidence, Nash equilibrium

**JEL Codes**: C70; C72; D80; D82; D83

## 1 Introduction

Looking at the ratings by the three big credit rating companies (Moody's, Standard & Poor's, Fitch Ratings) we can see strange things. On the eve of

*Financial support by the Hungarian Scientific Research Fund (OTKA) and the János Bolyai Research Scholarship of the Hungarian Academy of Sciences is gratefully acknowledged.

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2012 Moody's rated Poland at A2, Slovakia and Slovenia at A3, Standard & Poor's rated Poland at A-, Slovakia at A, and Slovenia at A+, while Fitch Ratings rated Poland at A-, Slovakia at A+, and Slovenia at A. These ratings contradict each other, since Moody's put Poland higher than Slovakia, while Standard & Poor's and Fitch Ratings put Slovakia higher than Poland, furthermore, Standard & Poor's put Slovenia higher than Slovakia, while Fitch Ratings put Slovakia higher than Slovenia.

It seems theory says something else, Aumann (1976)'s seminal result (roughly) says that if the the players' opinions about something are commonly known under a common prior, then those are the same, so the players cannot agree to disagree. However, we see something different in the example above. We can suppose that the three major credit rating companies use the same statistical and economic data, the same scientific and business methods (common prior), they form opinion about the very same thing, their ratings are public, so their ratings are commonly known. However the three big credit rating companies agree to disagree. How can it happen?

In order to make the models of incomplete information situations more amenable to analysis, Harsányi (1967-68) proposed to replace the hierarchies of beliefs by types. Later Mertens and Zamir (1985) introduced the notion of type space, and quite recently Meier (2008) incorporated the knowledge into type spaces, and introduced the concept of knowledge-belief space.

However, there is an inconsistency in the type space literature: while a finitary language is used for the belief hierarchies (see Definition 2 in Meier (2012)) an infinitary language is used for the beliefs (see Definition 3 in Meier (2012)). Namely, the notions of $\sigma$-field and $\sigma$-additive measure assume the players can reason about any countable sequence of events and of probabilities. On the other hand, the belief hierarchies are only about any finite reasoning level (I believe that you believe that I believe and so on). Can we fix this inconsistency, if yes, what kind of model do we get at?

In this paper we fix the above mentioned inconsistency, and introduce a family of classes of knowledge-belief spaces. Each member of the family uses one specific language characterized by an infinite cardinal. E.g. in the case of the smallest infinite cardinal a finitary language is applied, we mean the information structures are fields, the beliefs are additive probability set functions, and the belief hierarchies and common knowledge are as usual, this case is partially covered by Meier (2006). In general, each class uses a $\kappa$-language, where $\kappa$ is an infinite cardinal number, we mean the information structures are $\kappa$-fields, the beliefs are $\kappa$-additive probability measures, and the belief hierarchies and common knowledge are defined as the levels can be any ordinal number smaller than $\kappa$. Therefore we consider the problem in full generality.

Our proposed model of knowledge-belief spaces has remarkable properties. We show that the universal knowledge space exists (from Pintér (2010) we know there is no universal topological type space, topological type spaces are used in e.g. Böge and Eisele (1979), Mertens and Zamir (1985), Brandenburger and Dekel (1993), Heifetz (1993), Mertens et al (1994), Pintér (2005) among others); the universal knowledge-belief space is complete (Pintér (2014) demonstrated that Meier (2008)'s universal knowledge-belief space is not complete); and the universal knowledge-belief space encompasses all belief hierarchies (Heifetz and Samet (1999) showed that the purely measurable universal type space (Heifetz and Samet, 1998), which is complete (Meier, 2012), does not contain all belief hierarchies). To sum up, our model outperforms the previous ones.

The proposed model, moreover, has some further peculiar properties. Neither Aumann (1976)'s nor Aumann and Brandenburger (1995)'s results do not hold in our model. The reason why these seminal results are not valid in our knowledge-belief spaces is that, in contrast with the epistemic models in the literature where the notions of common knowledge and self-evidence are equivalent (see e.g. Osborne and Rubinstein (1994) Proposition 74.2, pp. 74-75), in our model self-evidence is a stronger notion than common knowledge. This is because in our approach, in general, for a player and an event $A$ there is not maximal event among the events from the knowledge structure of the player which is contained by $A$, that is, our knowledge operator is a set valued mapping, it assigns a set of events to event $A$, while an ordinary knowledge operator (Aumann, 1999a; Meier, 2008) assigns only one set, the maximally contained set to event $A$.

A knowledge operator being a set valued mapping in our model is important for the completeness of the universal knowledge-belief space too. By restricting the domain of the knowledge operators Meier (2008) was successful in avoiding the traps around the problem of the existence of universal knowledge-belief space (see e.g. Brandenburger and Keisler (2006)). However, Meier (2008)'s universal knowledge-belief space is not complete (Pintér, 2014). Our notion of knowledge operator is more restricted than Meier (2008)'s, but in our opinion, is still expressive enough, and not least, makes possible that our universal knowledge-belief space is complete.

As we have already mentioned neither Aumann (1976)'s nor Aumann and Brandenburger (1995)'s results are true in our model. We show, however, that if we substitute self-evidence for common knowledge, then both results hold again. In other words, by distinguishing the notions of common knowledge and self-evidence, it turns out that both Aumann (1976)'s and Aumann and Brandenburger (1995)'s results depend rather on self-evidence than on common knowledge.

Back to the three big credit rating companies, even if the ratings of these companies are common knowledge, those are not self-evident. Nobody knows how exactly the credit rating companies calculate their rates, e.g. if Hungary's balance of payments goes up from 6.3 % of GDP (2013 Q3) to 6.5 % of GDP (that is, the surplus improves further), then will Moody's upgrade Hungary from Ba1 negative to Ba1 stable? Nobody knows, the ratings are not self-evident. Therefore, our model does not contradict this real life example. On the other hand, the intuitions behind Aumann (1976)'s and Aumann and Brandenburger (1995)'s results are expressed in our model too, if the rating methods were public, that is, the ratings were self-evident, then those could not be different.

The setup of the paper is as follows. In the next section we discuss counterexamples related to Aumann (1976)'s and Aumann and Brandenburger (1995)'s results. Section 3 is about the notions of knowledge-beliefs space, type morphism, universal knowledge-belief space and complete knowledge-beliefs space. In Section 4 we discuss knowledge and belief hierarchies and put our main result proved in Section 5. Section 6 revisits Aumann (1976)'s and Aumann and Brandenburger (1995)'s results, and the last section briefly concludes. An appendix about inverse systems and limits is enclosed.

# 2 Examples

In this section by examples we show two important consequences of applying our proposed model.

## 2.1 Agreeing to disagree

Our first example is about that in the proposed model Aumann (1976)'s result does not hold, that is, the players can agree to disagree.

Let $\Omega = [0, 1]$ be set of the states of the world, $N = \{1, 2\}$ be the players set. Moreover, let player 1's knowledge structure be given by field $\mathcal{A}_1$ induced by $\{[0, a) \colon a \in \{1/2^n \colon n \in \mathbb{N}\}\}$, similarly, let player 2's knowledge structure be given by field $\mathcal{A}_2$ induced by $\{[0, a] \colon a \in \{1/2^n \colon n \in \mathbb{N}\}\}$. Then $A \in \mathcal{A}_i$ means for every event $B$ such that $A \subseteq B$, player $i$ knows event $B$ at all states of the world $\omega \in A$.

Let $P'(w) = \begin{cases} \frac{1}{2^{n+2}}, & \text{if there exists } n \in \mathbb{N} : \ w = \frac{1}{2^n} \\ 0 & \text{otherwise} \end{cases}$, and $l$ be the Lebesgue measure, where both measures are defined on $B([0, 1])$, on the Borel $\sigma$-field of $[0, 1]$. Then let the common prior $P = \frac{l}{2} + P'$.

Furthermore, suppose that the players use a finitary language (this is already indicated by that we use fields). Then an event $A \in \mathcal{A}$ is commonly known at state of the world $w \in \Omega$, if there are sequences of events $(A_n^i) \subseteq \mathcal{A}_i$, $i = 1, 2$ such that $A_{n+1}^i \subseteq A_n^1 \cap A_n^2$, for all $n$, $A_1^i \subseteq A$, and $w \in \cap_n A_n^i$, $i = 1, 2$. In other words, at state of word $w$ both players know event $A$ ($A_1^1, A_1^2 \subseteq A$), and both players know that both players know event $A$ ($A_2^1, A_2^2 \subseteq A_1^1 \cap A_1^2$), and so on for any (finite) $n$.

Let $A = \{1/2^n \colon n \in \mathbb{N} \setminus \{0\}\}$ and $B = [0, 1)$. Then we get the following claim:

**Claim 1.** For each $w \in B$, $P_1(A, w) = \frac{1}{3}$ and $P_2(A, w) = \frac{1}{2}$, where $P_i(A, w)$ is player $i$'s belief about event $A$ at state of the world $w$.

*Proof.* We consider two cases.

**Case 1** There exists $n^* \in \mathbb{N}$ such that $w = \frac{1}{2^{n^*}}$: In this case player 1 is in her part $\left[\frac{1}{2^{n^*}}, \frac{1}{2^{n^*-1}}\right)$, and player 2 is in her part $\left(\frac{1}{2^{n^*+1}}, \frac{1}{2^{n^*}}\right]$. Then

$$P_1(A, w) = \frac{P\left(A \cap \left[\frac{1}{2^{n^*}}, \frac{1}{2^{n^*-1}}\right)\right)}{P\left(\left[\frac{1}{2^{n^*}}, \frac{1}{2^{n^*-1}}\right)\right)} = \frac{P\left(\left\{\frac{1}{2^{n^*}}\right\}\right)}{P\left(\left[\frac{1}{2^{n^*}}, \frac{1}{2^{n^*-1}}\right)\right)}$$
$$= \frac{\frac{1}{2^{n^*+2}}}{\frac{1}{2^{n^*+2}} + \frac{1}{2^{n^*+1}}} = \frac{1}{3} ,$$

and

$$P_2(A, w) = \frac{P\left(A \cap \left(\frac{1}{2^{n^*+1}}, \frac{1}{2^{n^*}}\right]\right)}{P\left(\left(\frac{1}{2^{n^*+1}}, \frac{1}{2^{n^*}}\right]\right)} = \frac{P\left(\left\{\frac{1}{2^{n^*}}\right\}\right)}{P\left(\left(\frac{1}{2^{n^*+1}}, \frac{1}{2^{n^*}}\right]\right)}$$
$$= \frac{\frac{1}{2^{n^*+2}}}{\frac{1}{2^{n^*+2}} + \frac{1}{2^{n^*+2}}} = \frac{1}{2} .$$

**Case 2** There exists $n^* \in \mathbb{N}$ such that $w \in \left(\frac{1}{2^{n^*+1}}, \frac{1}{2^{n^*}}\right)$: In this case player 1 is in her part $\left[\frac{1}{2^{n^*+1}}, \frac{1}{2^{n^*}}\right)$, and player 2 is in her part $\left(\frac{1}{2^{n^*+1}}, \frac{1}{2^{n^*}}\right]$. Then

$$P_1(A, w) = \frac{P\left(A \cap \left[\frac{1}{2^{n^*+1}}, \frac{1}{2^{n^*}}\right)\right)}{P\left(\left[\frac{1}{2^{n^*+1}}, \frac{1}{2^{n^*}}\right)\right)} = \frac{P\left(\left\{\frac{1}{2^{n^*+1}}\right\}\right)}{P\left(\left[\frac{1}{2^{n^*+1}}, \frac{1}{2^{n^*}}\right)\right)}$$
$$= \frac{\frac{1}{2^{n^*+3}}}{\frac{1}{2^{n^*+3}} + \frac{1}{2^{n^*+2}}} = \frac{1}{3} ,$$

and

$$P_2(A, w) = \frac{P\left(A \cap \left(\frac{1}{2^{n^*+1}}, \frac{1}{2^{n^*}}\right]\right)}{P\left(\left(\frac{1}{2^{n^*+1}}, \frac{1}{2^{n^*}}\right]\right)} = \frac{P\left(\left\{\frac{1}{2^{n^*}}\right\}\right)}{P\left(\left(\frac{1}{2^{n^*+1}}, \frac{1}{2^{n^*}}\right]\right)}$$

$$= \frac{\frac{1}{2^{n^*+2}}}{\frac{1}{2^{n^*+2}} + \frac{1}{2^{n^*+2}}} = \frac{1}{2} \, .$$

□

Notice that at state of the world 1, $P_1(A, 1) = 1$ and $P_2(A, 1) = \frac{1}{2}$, therefore the event of $P_1(A, w) = \frac{1}{3}$ and $P_2(A, w) = \frac{1}{2}$ is $B$. Then at state of the world 0, $B$ is commonly known, that is, it is common knowledge that $P_1(A, w) = \frac{1}{3} \neq \frac{1}{2} = P_2(A, w)$, so Aumann (1976)'s theorem does not hold here, the players agree to disagree.

It is also worth noticing that since $P(B) = 1$, in this example the players $P$ almost surely agree to disagree; and event $B$ is not self-evident.

## 2.2 Epistemic condition for Nash equilibrium

Our second example is about that in the proposed model Aumann and Brandenburger (1995)'s result (THEOREM B, p. 1168) does not hold, that is, the imposed conditions do not imply that the players play Nash equilibrium.

Let $\Omega = [0, 1]$ be the set of the states of the world, $N = \{1, 2, 3\}$ be the players set. Moreover, let the knowledge structures of players 1 and 2 be given as in the previous example (Section 2.1), and player 3's knowledge structure be given by $B([0, 1])$, by the Borel $\sigma$-field. Furthermore, let the common prior $P$ be also from the previous example (Section 2.1).

Consider the following game in strategic form:

| $T$ | $L$ | $R$ |
|---|---|---|
| $U$ | $(2, 1, 1)$ | $(1, 0, \cdot)$ |
| $D$ | $(1, \cdot, \cdot)$ | $(0, \cdot, \cdot)$ |

| $B$ | $L$ | $R$ |
|---|---|---|
| $U$ | $(1, 1, 0)$ | $(\cdot, 3, \cdot)$ |
| $D$ | $(2, \cdot, \cdot)$ | $(\cdot, \cdot, \cdot)$ |

that is, the actions sets are $A_1 = \{U, D\}$, $A_2 = \{L, R\}$ and $A_3 = \{T, B\}$ respectively, and the $\cdot$s denote not specified payoffs. Suppose that at each state of the world $w \in \Omega$ the players play the above game.

Let the players' conjectures be as follows. Player 1: at each state of the world she believes that player 2 plays action $L$ and that player 3 plays action $T$ if $w \in A$, where $A$ is from the previous example (Section 2.1). In other words, player 1 believes player 3 plays action $T$ with probability $P_1(A, w)$ (see Section 2.1). Similarly, at each state of the world $w \in \Omega$ player 2 believes that player 1 plays action $U$ and that player 3 plays action $T$ with probability $P_2(A, w)$ (see Section 2.1). Finally, at each state of the world player 3 believes that player 1 plays action $U$ and that player 2 plays action $L$.

Moreover, suppose that at each state of the world player 1 plays action $D$, player 2 plays action $R$ and player 3 plays action $T$.

Then it is clear that for each state of the world $w \in [0, 1)$ player 1's conjecture is $\left(L, \frac{1}{3}T - \frac{2}{3}B\right)$, player 2's conjecture is $\left(U, \frac{1}{2}T - \frac{1}{2}B\right)$, and player 3's conjecture is $(U, L)$. Moreover, at each state of the world $w \in [0, 1)$, each player maximizes her own expected payoffs, that is, all players are rational.

Summing up, at state of the world 0, the event of the players' conjectures are $\left(L, \frac{1}{3}T - \frac{2}{3}B\right)$, $\left(U, \frac{1}{2}T - \frac{1}{2}B\right)$, $(U, L)$ respectively, and all players are rational, and the game above is played is commonly known, however, the players play action profile $(D, R, T)$ which is not a Nash equilibrium.

Finally, it is worth noticing that in this example we impose stronger condition than Aumann and Brandenburger (1995)'s, and stronger than Polak (1999)'s, moreover, the event of the players' conjectures are $\left(L, \frac{1}{3}T - \frac{2}{3}B\right)$, $\left(U, \frac{1}{2}T - \frac{1}{2}B\right)$, $(U, L)$ respectively, and all players are rational, and the game above is played is not self-evident, but happens $P$ almost surely.

# 3 The knowledge-belief space

*Notation*: Throughout the paper $\kappa$ is an infinite cardinal. Let $N$ be the set of the players, w.l.o.g. we can assume that $0 \notin N$, and let $N_0 = N \cup \{0\}$, where 0 is for the nature as a player.

Let $\#A$ be the cardinality of set $A$, and $\mathcal{P}(A)$ is the power set of $A$. A set system $\mathcal{A} \subseteq \mathcal{P}(X)$ is a $\kappa$-field, if $(A_i)_{i \in I} \subseteq \mathcal{A}$ such that $\#I < \kappa$ implies $\bigcup_{i \in I} A_i \in \mathcal{A}$. Notice that if $\kappa$ is the smallest infinite cardinal then $\kappa$-field means field, if $\kappa$ is the smallest uncountable infinite cardinal, then $\kappa$-field means $\sigma$-field. Let $\mathcal{A} \subseteq \mathcal{P}(X)$ be a set system, then $\kappa(\mathcal{A})$ denotes the coarsest $\kappa$-field that contains $\mathcal{A}$. Furthermore, $(X, \mathcal{A})$ is a $\kappa$-measurable space if $\mathcal{A}$ is a $\kappa$-field on $X$.

Let $(X, \mathcal{M})$ and $(Y, \mathcal{N})$ be $\kappa$-measurable spaces, then $(X \times Y, \mathcal{M} \otimes \mathcal{N})$ or briefly $X \otimes Y$ is the $\kappa$-measurable space on the set $X \times Y$ equipped with the $\kappa$-field $\kappa(\{A \times B : A \in \mathcal{M},\ B \in \mathcal{N}\})$.

The $\kappa$-measurable spaces $(X, \mathcal{M})$ and $(Y, \mathcal{N})$ are measurable isomorphic if there is a bijection $f$ between them such that both $f$ and $f^{-1}$ are measurable.

Let $\mathcal{A}$ be a field and $\mu$ be an additive set function on $\mathcal{A}$. Then $\mu$ is $\kappa$-additive, if for each generalized sequence (net) $(A_i)_{i \in I}$ from $\mathcal{A}$, such that $\#I < \kappa$, $i \geq j$ implies $A_i \subseteq A_j$, and $\bigcap_{i \in I} A_i = \emptyset$: $\lim_{i \in I} \mu(A_i) = 0$. Notice that if $\kappa$ is the smallest infinite cardinal than $\kappa$-additivity means additivity, if $\kappa$ is the smallest uncountable infinite cardinal, then $\kappa$-additivity means $\sigma$-additivity.

The triplet $(X, \mathcal{A}, \mu)$ is a $\kappa$-measure space if $(X, \mathcal{A})$ is a $\kappa$-measurable space and $\mu$ is a $\kappa$-additive set function on $\mathcal{A}$. If $\mu(X) = 1$ then $\mu$ is a probability $\kappa$-measure, and $(X, \mathcal{A}, \mu)$ is a probability $\kappa$-measure space.

For an ordinal number $\omega$ we say $\omega < \kappa$, if the cardinality of $\omega$ is less than $\kappa$.

The fixed infinite cardinal $\kappa$ refers to the language of the model, if $\kappa$ is the smallest infinite cardinal, then the language is finite, otherwise the language is a $\kappa$-language, so less than $\kappa$-many operations (unions, etc.) can be applied. In other words, somehow $\kappa$ refers to the cognitive power of the players.

In the following, we use terminologies which are similar to Heifetz and Samet (1998)'s and Meier (2008)'s.

**Definition 2.** *Let $(X, \mathcal{M})$ be a $\kappa$-measurable space and denote $\Delta(X, \mathcal{M})$ the set of probability $\kappa$-measures on it. Then the $\kappa$-field $\mathcal{A}^*$ on $\Delta(X, \mathcal{M})$ is defined as follows:*

$$\mathcal{A}^* = \kappa(\{\{\mu \in \Delta(X, \mathcal{M}) : \mu(A) \geq p\},\ A \in \mathcal{M},\ p \in [0, 1]\}) \ .$$

*In other words, $\mathcal{A}^*$ is the smallest $\kappa$-field among the $\kappa$-fields that contain the sets $\{\mu \in \Delta(X, \mathcal{M}) : \mu(A) \geq p\}$, where $A \in \mathcal{M}$ and $p \in [0, 1]$ are arbitrarily chosen.*

In incomplete information situations it is recommended to consider events like a player believes with probability at least $p$ that a certain event occurs (beliefs operator see e.g. Aumann (1999b)). For this reason, for any $A \in \mathcal{M}$ and $p \in [0, 1]$, $\{\mu \in \Delta(X, \mathcal{M}) : \mu(A) \geq p\}$ must be an event (a measurable set). To keep the class of events as small (coarse) as possible, we use $\kappa$-field $\mathcal{A}^*$.

Notice that $\mathcal{A}^*$ is not a fixed $\kappa$-field, it depends on the measurable space on which the probability $\kappa$-measures are defined. Therefore $\mathcal{A}^*$ is similar to the *weak*$^*$ topology, which depends on the topology of the base (primal) space.

**Assumption 3.** *Let the parameter space $(S, \mathcal{A})$ be a $\kappa$-measurable space.*

Henceforth we assume that $(S, \mathcal{A})$ is the fixed parameter space which consists of the states of the nature.

**Definition 4.** *Let $\Omega$ be the space of the states of the world, and for each player $i \in N_0$, let $\mathcal{M}_i$ be a $\kappa$-field on $\Omega$. The $\kappa$-field $\mathcal{M}_i$ is player $i$'s knowledge structure, that is, at each state of the world $w \in A \in \mathcal{M}_i$, player $i$ knows event $A$. The $\kappa$-field $\mathcal{M}_0$ is the nature's knowledge structure, that is, this is the representation of $\mathcal{A}$ (the $\kappa$-field of the parameter space $S$). Let $\mathcal{M} = \kappa(\bigcup_{i \in N_0} \mathcal{M}_i)$, the smallest $\kappa$-field that contains all $\kappa$-fields $\mathcal{M}_i$.*

Each point in $\Omega$ provides a complete description of the actual state of the world. It includes both the state of nature and the players' states of the mind. The different $\kappa$-fields are for modeling the informedness of the players, these have the same role as e.g. the partitions in Aumann (1999a)'s paper have. Therefore, if $w, w' \in \Omega$ are not distinguishable [1] in the $\kappa$-field $\mathcal{M}_i$, then player $i$ is not able to discern the difference between them, that is, she knows, believes the same things and behaves in the same way at the two states $w$ and $w'$. $\mathcal{M}$ represents all information available in the model, it is the $\kappa$-field got by pooling the information of the players and the nature.

For the sake of brevity, henceforth – if it does not make confusion – we do not indicate the $\kappa$-fields. E.g. instead of $(S, \mathcal{A})$ we write $S$, or $\Delta(S)$ instead of $(\Delta(S, \mathcal{A}), \mathcal{A}^*)$. However, in some cases we refer to the non-written $\kappa$-field: e.g. $A \in \Delta(X, \mathcal{M})$ is a set from $\mathcal{A}^*$, that is, this is a measurable set in the $\kappa$-measurable space $(\Delta(X, \mathcal{M}), \mathcal{A}^*)$; but $A \subseteq \Delta(X, \mathcal{M})$ keeps its original meaning: $A$ is a subset of $\Delta(X, \mathcal{M})$.

Before we introduce our notion of knowledge-belief space we discuss the notion of knowledge operator in details. In Meier (2008) player $i$'s knowledge operator is a mapping from $\mathcal{M}$ to $\mathcal{M}_i$. The intuition is clear, for every event from $\mathcal{M}$ the knowledge operator gives the set of all states of the world where player $i$ knows the event. Formally, to event $A \in \mathcal{M}$, player $i$'s knowledge operator assigns set $\{w \in \Omega\colon$ there exists $A' \in \mathcal{M}_i,\ w \in A'$ and $A' \subseteq A\} = \bigcup_{w \in A' \subseteq A,\ A' \in \mathcal{M}_i} A'$. Since in Meier (2008) the players' knowledge structures are $\sigma$-fields the range of the knowledge operators are $\mathcal{M}_i$s.

In our model, however, e.g. if we use a finite language, then the players knowledge structures are only fields, so e.g. for player $i$, set $\bigcup_{w \in A' \subseteq A,\ A' \in \mathcal{M}_i} A'$ is not necessarily in $\mathcal{M}_i$. In other words, in our model the set of states of the world at which a player knows an event is not necessarily an event. Therefore,

[1] Let $(X, \mathcal{T})$ be a $\kappa$-measurable space and $x, y \in X$ be two points. Points $x$ and $y$ are measurably indistinguishable if for all $A \in \mathcal{T}$: $(x \in A) \Leftrightarrow (y \in A)$.

we must use a more general notion than ordinary mapping to capture the intuition of the knowledge operator.

In this paper we use set valued mappings as knowledge operators, e.g. for player $i$, events $A \in \mathcal{M}$ and $A' \in \mathcal{M}_i$, event $A'$ is in the image of event $A$ by the player's knowledge operator, if $A' \subseteq A$, that is, if at each state of the world $w \in A'$, player $i$ knows event $A$. Therefore, we must formalize mutual knowledge and common knowledge differently from Aumann (1999a) or Meier (2008); for the details see Sections 4 and 6.

**Definition 5.** *Let $\{(\Omega, \mathcal{M}_i)\}_{i \in N_0}$ be the space of the states of the world. Then the tuple $(S, \{(\Omega, \mathcal{M}_i)\}_{i \in N_0}, g, \{k_i\}_{i \in N}, \{f_i\}_{i \in N})$ is a knowledge-belief space based on the parameter space $S$, where*

1. *function $g \colon \Omega \to S$ is $\mathcal{M}_0$-measurable,*
2. *for each player $i \in N$: the set valued mapping $k_i \colon \mathcal{M} \to \mathcal{P}(\mathcal{M}_i)$ is player $i$'s knowledge operator defined as: $A \in \mathcal{M}$, $k_i(A) = \{A' \in \mathcal{M}_i \colon A' \subseteq A\}$.*
3. *for each player $i \in N$: $f_i \colon \Omega \to \Delta(\Omega, \mathcal{M}_{-i})$ is player $i$'s type function, such that*
   - *a) $f_i$ is $\mathcal{M}_i$-measurable,*
   - *b) for each $w \in \Omega$, $A \in \mathcal{M}_{-i}$ such that there exists $A' \in \mathcal{M}_i$, $w \in A'$ and $A' \subseteq A$: $f_i(w)(A) = 1$,*

*where $\mathcal{M}_{-i} = \kappa(\bigcup_{j \in N_0 \setminus \{i\}} \mathcal{M}_j)$.*

In other words Definition 5 says that $S$ is the parameter space, it consists of the "types" of the nature. $\mathcal{M}_i$ represents the information available for player $i$, hence it corresponds to the concept of types (Harsányi, 1967-68). Set valued mapping $k_i$ is player $i$'s knowledge operator, and mapping $f_i$ is the type function of player $i$, it assigns player $i$'s (subjective) beliefs to her types.

It is worth elaborating on the knowledge operator a bit further. Our definition says that for any player there are some events which are self-evident for her, and these events – because of the language / cognitive power of the player – form a $\kappa$-field. This $\kappa$-field is the player's knowledge structure. Then the knowledge operator at event $A \in \mathcal{M}$ ($\mathcal{M}$ is induced by the events self-evident for a player or for the nature) is the set of the player's self-evident events implying event $A$. Furthermore, this means, that a player knows event $A$ is not (necessarily) an event; but we can express that a player knows that another player knows an event and the like in a different way, see Section 4. Furthermore, it is easy to see that for any player $i$ and event $A$, $k_i(A)$ is closed under $\kappa$-unions and $\kappa$-intersections.

In its spirit the above notion of knowledge-belief space is similar to Meier (2008)'s notion, but differs from it in two main points: (1) we use $\kappa$-models ($\kappa$-field, probability $\kappa$-measures), (2) our knowledge operators are set valued mappings. On the other hand, even if our knowledge operator is not the same as Aumann (1999a)'s or Meier (2008)'s, the intuitions behind all three notions – in our opinion – are the same, which is demonstrated by the following (obvious) lemma.

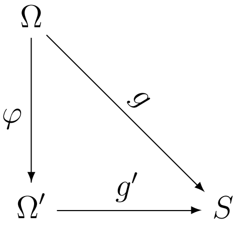
**Lemma 6.** *Consider the knowledge-belief space* $(S, \{(\Omega, \mathcal{M}_i)\}_{i \in N_0}, g, \{k_i\}_{i \in N}, \{f_i\}_{i \in N})$. *Then for each player* $i$, *the knowledge operator* $k_i$ *meets the following points, for all events* $A, B \in \mathcal{M}$ *and* $(A_j)_{j \in J} \subseteq \mathcal{M}_i$ *such that* $\#J < \kappa$*:*

1. $A' \in k_i(A)$ *implies* $A' \subseteq A$,
2. $\bigcap_{j \in J} k_i(A_j) = k_i\left(\bigcap_{j \in J} A_j\right)$,
3. *(Monotonicity)* $A \subseteq B$ *implies* $k_i(A) \subseteq k_i(B)$,
4. *(Positive introspection)* $k_i(A) \subseteq \bigcup_{A' \in k_i(A)} k_i(A')$,
5. *(Negative introspection)* $\complement k_i(A) \subseteq \bigcup_{A' \in \complement k_i(A)} k_i(A')$,
6. $A \in k_i(A)$ *in case of* $A \in \mathcal{M}_i$.

Next we define the notion of type morphism.

**Definition 7.** *A mapping* $\varphi \colon \Omega \to \Omega'$ *is a type morphism between knowledge-belief spaces* $(S, \{(\Omega, \mathcal{M}_i)\}_{i \in N_0}, g, \{k_i\}_{i \in I}, \{f_i\}_{i \in N})$ *and* $(S, \{(\Omega', \mathcal{M}'_i)\}_{i \in N_0}, g', \{k'_i\}_{i \in N}, \{f'_i\}_{i \in N})$ *if*

1. $\varphi$ *is an* $\mathcal{M}$*-measurable mapping,*
2. *Diagram* (1) *is commutative, that is, for each state of the world* $w \in \Omega$*:* $g' \circ \varphi(w) = g(w)$,




(1)

3. *for each player $i \in N$, Diagram* (2) *is commutative, that is, for every event $A \in \mathcal{M}'$: $k_i \circ \varphi^{-1}(A) = \varphi^{-1} \circ k'_i(A)$,*

$$\begin{CD} \mathcal{M}' @>{k'_i}>> \mathcal{P}(\mathcal{M}'_i) \\ @V{\varphi^{-1}}VV @VV{\varphi^{-1}}V \\ \mathcal{M} @>{k_i}>> \mathcal{P}(\mathcal{M}_i) \end{CD} \tag{2}$$

4. *for each player $i \in N$, Diagram* (3) *is commutative, that is, for each state of the world $w \in \Omega$: $f'_i \circ \varphi(w) = \hat{\varphi}_i \circ f_i(w)$,*

$$\begin{CD} \Omega @>{f_i}>> \Delta(\Omega, \mathcal{M}_{-i}) \\ @V{\varphi}VV @VV{\hat{\varphi}_i}V \\ \Omega' @>{f'_i}>> \Delta(\Omega', \mathcal{M}'_{-i}) \end{CD} \tag{3}$$

*where $\hat{\varphi}_i : \Delta(\Omega, \mathcal{M}_{-i}) \to \Delta(\Omega', \mathcal{M}'_{-i})$ is defined as follows: for all $\mu \in \Delta(\Omega, \mathcal{M}_{-i})$, $A \in \mathcal{M}'_{-i}$: $\hat{\varphi}_i(\mu)(A) = \mu(\varphi^{-1}(A))$. It is an easy calculation to show that $\hat{\varphi}_i$ is a measurable mapping.*

*A type morphism $\varphi$ is a type isomorphism, if $\varphi$ is a bijection and $\varphi^{-1}$ is also a type morphism.*

A type morphism assigns type profiles from a knowledge-belief space to type profiles in a(nother) knowledge-belief space in the way the corresponded types induce equivalent knowledge and beliefs for all players. In other words, the type morphism preserves the players' knowledge and beliefs.

The following result is a direct corollary of Definitions 5 and 7.

**Corollary 8.** *The knowledge-belief spaces based on the parameter space $S$ as objects and the type morphisms form a category. Let $\mathcal{C}^S$ denote this category of knowledge-belief spaces.*

Next we introduce our notion of universal knowledge-belief space.

**Definition 9.** *A knowledge-belief space $(S, \{(\Omega, \mathcal{M}_i)\}_{i \in N_0}, g, \{k_i\}_{i \in N}, \{f_i\}_{i \in N}$ ) is a universal knowledge-belief space, if for every knowledge-belief space $(S,$ $\{(\Omega', \mathcal{M}'_i)\}_{i \in N_0}, g', \{k'_i\}_{i \in N}, \{f'_i\}_{i \in N})$ there exists a unique type morphism $\varphi$: $\Omega' \to \Omega$.*

In other words, a universal knowledge-belief space is the most general, the biggest knowledge-belief space among the knowledge-belief spaces. A universal knowledge-belief contains all types of all knowledge-belief spaces of the given category.

In the language of category theory Definition 9 means the following:

**Corollary 10.** *A universal knowledge-belief space is a terminal (final) object in category* $\mathcal{C}^S$.

Since every terminal object is unique up to isomorphism, from the viewpoint of category theory the uniqueness of universal knowledge-belief space is a straightforward statement.

**Corollary 11.** *The universal knowledge-belief space is unique up to type isomorphism.*

Next, we turn our attention to another property of knowledge-belief spaces, to the completeness.

**Definition 12.** *A knowledge-belief space* $(S, \{(\Omega, \mathcal{M}_i)\}_{i \in N_0}, g, \{k_i\}_{i \in N}, \{f_i\}$ $_{i \in N})$ *is complete, if for each player* $i \in N$*, type function* $f_i$ *is surjective (onto).*

Brandenburger (2003) introduced the concept of complete type space, and Pintér (2014) adapted the notion of completeness to knowledge-belief spaces. The completeness recommends that for any player $i$, any probability $\kappa$-measure on $(\Omega,\ \mathcal{M}_{-i})$ be in the range of the player's type function. In other words, for any player $i$, any $\kappa$-measure on $(\Omega, \mathcal{M}_{-i})$ must be assigned (by the type function $f_i$) to a type of player $i$.

# 4 Knowledge and belief hierarchies

In this section we consider the knowledge operator, by which the notion of mutual knowledge is defined (Aumann, 1999a), and formalize the intuition of hierarchies of beliefs, as Harsányi (1967-68) named the "infinite regress in reciprocal expectations".

First we consider the knowledge hierarchies. Take knowledge-belief space $(S, \{(\Omega, \mathcal{M}_i)\}_{i \in N_0}, g, \{k_i\}_{i \in N}, \{f_i\}_{i \in N})$, and state of the world $w \in \Omega$. Then player $i$ knows event $A$ at state of the world $w$, if there exists $A_i \in k_i(A)$ such that $w \in A_i$. Furthermore, the players mutually know event $A$ at state of the world $w$, if there exist $A_i \in k_i(A)$, $i \in N$, such that $w \in \bigcap_{i \in N} A_i$. For the sake of clear exposition we introduce the following notation, let $k^1(A) = \{X \subseteq \Omega\colon$ there exist $A_i \in k_i(A),\ i \in N,$ such that $X = \bigcap_{i \in N} A_i\}$. Notice that $k^1$ is somehow similar to the notion of first order mutual knowledge operator in Aumann (1999a).

Similarly, the players second order mutually know event $A$ at state of the world $w$, if there exist $A_i \in \mathcal{M}_i$, $i \in N$, $X \in k^1(A)$ such that $A_i \subseteq X$, $i \in N$, and $w \in \bigcap_{i \in N} A_i$. A further notation, let $k^2(A) = \{X \subseteq \Omega\colon$ there exist $A_i \in \mathcal{M}_i,\ i \in N,\ Y \in k^1(A)$ such that $A_i \subseteq Y,\ i \in N,\ X = \bigcap_{i \in N} A_i\}$.

In general, for any ordinal number $\omega$ such that $\omega < \kappa$, the players $\omega + 1$th order mutually know event $A$ at state of the world $w$, if there exist $A_i \in \mathcal{M}_i,\ i \in N,\ X \in k^\omega(A)$ such that $A_i \subseteq X,\ i \in N,$ and $w \in \bigcap_{i \in N} A_i$. The auxiliary notation, let $k^{\omega+1}(A) = \{X \subseteq \Omega\colon$ there exist $A_i \in \mathcal{M}_i,\ i \in N,\ Y \in k^\omega(A)$ such that $A_i \subseteq Y,\ i \in N,\ X = \bigcap_{i \in N} A_i\}$.

Notice that our notion of knowledge operator – which differs from the one by Aumann (1999a), as we have already discussed in the previous section – reflects the very same ideas as Aumann (1999a)'s notion does. Again, the reason why we need more complex notions and notation in handling the higher order knowledge issues is that the knowledge operator is a set valued mapping in our model.

Next we consider the belief hierarchies. The following definition is a reformulation of Mertens et al (1994)'s concept.

**Definition 13.** *Let $i \in N$ be a player, and consider Diagram* (4)

$$
\begin{array}{ccc}
\Theta^i & & \Delta(S \otimes \Theta^{N\setminus\{i\}}) \\
p^i_{\omega+1} \Big\downarrow & & \Big\downarrow \mathrm{id}_S \quad \Big\downarrow p^{N\setminus\{i\}}_\omega \\
\Theta^i_{\omega+1} & = & \Delta(S \otimes \Theta^{N\setminus\{i\}}_\omega) \\
q^i_{1\omega+1} \Big\downarrow & & \Big\downarrow \mathrm{id}_S \quad \Big\downarrow q^{N\setminus\{i\}}_{0\omega} \\
\Theta^i_1 & = & \Delta(S \otimes \Theta^{N\setminus\{i\}}_0)
\end{array}
\tag{4}
$$

*where*

- *$\omega$ is an ordinal number such that $\omega < \kappa$,*
- *$\Theta^i_0$ is a singleton set, $\Theta^{N\setminus\{i\}}_\omega = \bigotimes_{j \in N\setminus\{i\}} \Theta^j_\omega$,*

- *for each* $\mu \in \Theta_{\omega+2}$*:*

$$q^i_{\omega+1\omega+2}(\mu) = \mu|_{S \otimes \Theta^{N\setminus\{i\}}_\omega} \ ,$$

*therefore* $q^i_{\omega+1\omega+2}$ *is a measurable mapping.*

- $\Theta^i = \varprojlim(\Theta^i_\omega, K_\kappa, q^i_{\omega\omega+1})$*, where* $K_\kappa = \{\omega'$ *is an ordinal number*: $\omega' < \kappa\}$*,*

- $p^i_\omega \colon \Theta^i \to \Theta^i_\omega$ *is the canonical projection,*

- $q^{N\setminus\{i\}}_{\omega\omega+1}$ *is the product of the mappings* $q^j_{\omega\omega+1}$*,* $j \in N \setminus \{i\}$*, and so is* $p^{N\setminus\{i\}}_\omega$ *of* $p^j_\omega$*,* $j \in N \setminus \{i\}$*, therefore both mappings are measurable,*

- $\Theta^{N\setminus\{i\}} = \bigotimes_{j\in N\setminus\{i\}} \Theta^j$.

*Then* $T = S \otimes \Theta^N$ *is called purely measurable beliefs space.*

The interpretation of the purely measurable beliefs space is the following. For any $\theta^i \in \Theta^i$: $\theta^i = (\mu^i_1, \mu^i_2, \ldots)$, where $\mu^i_\omega \in \Theta^i_\omega$ is player $i$'s $\omega$th order belief. Therefore each point of $\Theta^i$ defines an inverse system of probability $\kappa$-measure spaces

$$((S \otimes \Theta^{N\setminus\{i\}}_\omega, p^i_{\omega+1}(\theta^i)), K_\kappa, (\mathrm{id}_S, q^{N\setminus\{i\}}_{\omega\omega+1})) \ , \tag{5}$$

where $(\mathrm{id}_S, q^{N\setminus\{i\}}_{\omega\omega+1})$ is the product of mappings $\mathrm{id}_S$ and $q^{N\setminus\{i\}}_{\omega\omega+1}$. We call the inverse systems of probability $\kappa$-measure spaces like (5) player $i$'s hierarchies of beliefs.

To sum up, $T$ consists of all states of the world: all states of nature: the points of $S$, and all players' all states of the mind: the points of set $\Theta^N$, therefore $T$ contains all players' all hierarchies of beliefs.

Our main result:

**Theorem 14.** *The universal knowledge-belief space exists, is complete, and encompasses all players' all hierarchies of beliefs.*

We present the proof of Theorem 14 in the next section.

# 5 The proof of Theorem 14

The strategy of the proof is to show that the purely measurable beliefs space (see Definition 13) "generates" the universal knowledge-belief space (in category $\mathcal{C}^S$). It is worth mentioning that this proof for the existence of universal knowledge-belief space goes as Heifetz and Samet (1998)'s, Meier (2008)'s proofs, the construction of canonical model in modal logic goes, that is, the same machinery lays behind all these results. We do not go into the details of the common behind these results, only mention that the theory of coalgebras and final coalgebras is the common umbrella for these and other results, see Moss and Viglizzo (2004, 2006); Cirstea et al (2011); Moss (2011) among others.

Mathematically, the key point of the proof is to demonstrate the following lemma:

**Lemma 15.** *For each player $i \in N$, type $\theta^i \in \Theta^i$, the inverse system of probability $\kappa$-measure spaces*

$$((S \otimes \Theta_\omega^{N \setminus \{i\}}, p_{\omega+1}^i(\theta^i)), K_\kappa, (\mathrm{id}_S, q_{\omega\omega+1}^{N \setminus \{i\}})) \tag{6}$$

*admits a unique inverse limit.*

*Proof.* (1) By the Axiom of Choice $\varprojlim((S \times \Theta_\omega^{N \setminus \{i\}}), K_\kappa, (\mathrm{id}_S, q_{\omega\omega+1}^{N \setminus \{i\}})) \neq \emptyset$.

(2) $\bigcup_{\omega \in K_\kappa} (\mathrm{id}_S, p_\omega^N)^{-1}(S \otimes \Theta_\omega^N)$, that is, the union of the inverse images of the $\kappa$-fields on $S \times \Theta_\omega^N$, $\omega \in K_\kappa$ is a $\kappa$-field: First, it is easy to see that $\emptyset \in \bigcup_{\omega \in K_\kappa} (\mathrm{id}_S, p_\omega^N)^{-1}(S \otimes \Theta_\omega^N)$, and if $A \in \bigcup_{\omega \in K_\kappa} (\mathrm{id}_S, p_\omega^N)^{-1}(S \otimes \Theta_\omega^N)$, then $\complement A \in \bigcup_{\omega \in K_\kappa} (\mathrm{id}_S, p_\omega^N)^{-1}(S \otimes \Theta_\omega^N)$.

Let $A_i \in \bigcup_{\omega \in K_\kappa} (\mathrm{id}_S, p_\omega^N)^{-1}(S \otimes \Theta_\omega^N)$, $i \in I$, $\#I < \kappa$. Then by definition for each $A_i$ there exist $\omega(i) \in K_\kappa$ and $B_i \in S \otimes \Theta_{\omega(i)}^N$ such that $A_i = (\mathrm{id}_S, p_{\omega(i)}^N)^{-1}(B_i)$.

Then there exists ordinal number $\omega^* \in K_\kappa$ such that $\omega(i) \leq \omega^*$, $i \in I$ (see e.g. Folland (1999) Section 0.4). Since $(S \otimes \Theta_{\omega^*}^N)$ is a $\kappa$-field and $A_i = ((\mathrm{id}_S, q_{\omega(i)\omega^*}^N) \circ (\mathrm{id}_S, p_{\omega^*}^N))^{-1}(B_i)$, $i \in I$, $\bigcup_i A_i \in \bigcup_{\omega \in K_\kappa} (\mathrm{id}_S, p_\omega^N)^{-1}(S \otimes \Theta_\omega^N)$.

(3) $\mu$ defined by $\mu \circ p_\omega^{-1} = p_\omega^i(\theta^i)$ is $\kappa$-additive. It is clear that $\mu$ is well-defined and additive, then we can take any monotone decreasing $\kappa$-sequence of events with empty limit and apply the reasoning of point (2) to get $\mu$ is $\kappa$-additive. □

Next we show that the beliefs space of Definition 13 induces a knowledge-belief space.

**Lemma 16.** *The purely measurable beliefs space $T$ induces a knowledge-belief space in category $\mathcal{C}^S$.*

*Proof.* For each player $i \in N$, let $pr_i \colon T \to \Theta^i$, $pr_0 \colon T \to S$ be the coordinate projections, and for each player $i \in N \cup \{0\}$, let the $\kappa$-field $\mathcal{M}_i^*$ be induced by $pr_i$. From Lemma 15 for each player $i \in N$:

$$\Theta^i = \Delta(S \otimes \Theta^{N\setminus\{i\}}) , \tag{7}$$

by that we mean, the left hand side and the right hand side are measurable isomorphic.

Furthermore, let $g^* = pr_0$ and for each player $i \in N$, let $f_i^* = pr_i$. Moreover, for each player $i \in N$ and event $A \in \mathcal{M}^*$, let $k_i^*(A) = \{A' \in \mathcal{M}_i^* \colon A' \subseteq A\}$. Then

$$(S, \{(T, \mathcal{M}_i^*)\}_{i\in N}, g^*, \{k_i\}_{i\in N}, \{f_i^*\}_{i\in N})$$

is a knowledge-belief space in category $\mathcal{C}^S$. □

The following proposition is a direct corollary of Equation (7).

**Proposition 17.** *The knowledge-belief space* $(S, \{(T, \mathcal{M}_i^*)\}_{i\in N}, g^*, \{k_i^*\}_{i\in N}, \{f_i^*\}_{i\in N})$ *is complete.*

Next we show that the knowledge-belief space induced by the purely measurable beliefs space is the universal knowledge-belief space.

**Proposition 18.** *The knowledge-belief space* $(S, \{(T, \mathcal{M}_i^*)\}_{i\in N}, g^*, \{k_i^*\}_{i\in N}, \{f_i^*\}_{i\in N})$ *is a universal knowledge-belief space in category* $\mathcal{C}^S$*.*

*Proof.* Let $(S, \{(\Omega, \mathcal{M}_i)\}_{i\in N}, g, \{k_i\}_{i\in N}, \{f_i\}_{i\in N})$ be a knowledge-belief space, and take player $i \in N$ and state of the world $w \in \Omega$.

Player $i$'s first order belief at state of the world $w$, $v_1^i(w)$ is the probability $\kappa$-measure defined as follows, for each $A \in S$:

$$v_1^i(w)(A) = f_i(w)(g^{-1}(A)) .$$

$f_i$ is $\mathcal{M}_i$-measurable, hence $v_1^i$ is also $\mathcal{M}_i$-measurable.

Player $i$'s second order belief at state of the world $w$, $v_2^i(w)$ is the probability $\kappa$-measure defined as follows, for each $A \in S \otimes \Theta_1^{N\setminus\{i\}}$:

$$v_2^i(w)(A) = f_i(w)((g, v_1^{N\setminus\{i\}})^{-1}(A)) ,$$

where for each $w'$: $(g, v_1^{N\setminus\{i\}})(w') = (g(w'), \{v_1^j(w')\}_{j\in N\setminus\{i\}})$, hence $(g, v_1^{N\setminus\{i\}})$ is $\mathcal{M}_{-i}$-measurable. Since $f_i$ is $\mathcal{M}_i$-measurable $v_2^i$ is also $\mathcal{M}_i$-measurable.

For any ordinal number $\omega \in K_\kappa$ player $i$'s $\omega + 1$th order belief at state of the world $w$, $v_{\omega+1}^i(w)$ is the probability $\kappa$-measure defined as follows, for each $A \in S \otimes \Theta_\omega^{N\setminus\{i\}}$:

$$v_{\omega+1}^i(w)(A) = f_i(w)((g, v_\omega^{N\setminus\{i\}})^{-1}(A)) .$$

Since $f_i$ is $\mathcal{M}_i$-measurable $v^i_{\omega+1}$ is also $\mathcal{M}_i$-measurable.

Then, we have got the mapping $\phi\colon \Omega \to T$ defined as follows, for each $w \in \Omega$:

$$\phi(w) = (g(\omega), (v^i_1(w), v^i_2(w), \ldots)_{i\in N}) \ . \tag{8}$$

Then it is easy to verify the following:

(1) $\phi$ is $\mathcal{M}$-measurable,

(2) for each $i \in N$, $w \in \Omega$:

$$g^* \circ \phi(w) = g(w) \ ,$$

and for each event $A \in \mathcal{M}^*$ and player $i \in N$:

$$k_i \circ \phi^{-1}(A) = \phi^{-1} \circ k^*_i(A) \ ,$$

and

$$f^*_i \circ \phi(w) = \hat{\phi}_i \circ f_i(w) \ ,$$

that is, $\phi$ is a type morphism,

(3) Since $\Theta^i$ consists of different inverse systems of probability $\kappa$-measure spaces (hierarchies of beliefs), $\phi$ is the unique type morphism from the knowledge-belief space $(S, \{(\Omega, \mathcal{M}_i)\}_{i\in N}, g, \{k_i\}_{i\in N}, \{f_i\}_{i\in N})$ to the knowledge-belief space $(S, \{(T, \mathcal{M}^*_i)\}_{i\in N}, g^*, \{k^*_i\}_{i\in N}, \{f^*_i\}_{i\in N})$. □

In the above proof we show that each point in a knowledge-belief space induces a hierarchy of beliefs for each player, that is, each point in a knowledge-belief space completely describes the players' hierarchies of beliefs at the states of the world.

It is also worth noticing that in the above proof $\phi$ is not necessarily injective (one-to-one). The $\phi$-image of redundant types, that is, types that generate the same knowledge and hierarchy of beliefs, see e.g. Ely and Peski (2006), is one point in the universal knowledge-belief space. Therefore, there are no redundant types in the universal knowledge-belief space.

*The proof of Theorem 14.* From Corollary 11 and Proposition 18

$$(S, \{(T, \mathcal{M}^*_i)\}_{i\in N}, g^*, \{k^*_i\}_{i\in N}, \{f^*_i\}_{i\in N}) \tag{9}$$

is the universal knowledge-belief space.

From Proposition 17: (9) is complete.

Finally, from Definition 13: (9) encompasses all players' all hierarchies of beliefs. □

# 6 Aumann (1976) and Aumann and Brandenburger (1995) are revisited

In Section 2 we demonstrated by two examples that in our model Aumann (1976)'s and Aumann and Brandenburger (1995)'s results do not hold. As we pointed out in the introduction, the reason for this is that in our model that an event is common knowledge is not necessarily an event, so it can happen – and in the two examples in Section 2 this happens indeed – a player does not know that an event is common knowledge. In other words, the common knowledge might be out of the class of events the player can perceive.

In this section we introduce formally the notions of common knowledge and self-evidence into our model. Thereafter, we show that if we substitute self-evidence for common knowledge in Aumann (1976)'s (and Polak (1999)'s) and Aumann and Brandenburger (1995)'s papers, then these results hold in our model too.

First we introduce the notion of common knowledge.

**Definition 19.** *Consider knowledge-belief space* $(S, \{(T, \mathcal{M}_i)\}_{i \in N}, g, \{k_i\}_{i \in N}, \{f_i\}_{i \in N})$. *Then event* $A \in \mathcal{M}$ *is commonly known at state of the world* $w \in \Omega$, *if for all ordinal numbers* $\omega$ *such that* $\omega < \kappa$, *there exist* $A_i \in \mathcal{M}_i$, $i \in N$, $X \in k^\omega(A)$, *such that* $A_i \subseteq X$, $i \in N$, *and* $w \in \bigcap_{i \in N} A_i$. *In other words, event* $A$ *is commonly known at state of the world* $w$, *if event* $A$ *is* $\omega$*th order mutually known at state of the world* $w$, *for all* $\omega < \kappa$.

The above definition of common knowledge reflects the very same intuition that Aumann (1999a)'s does, that is, an event is commonly known at a state of the world, if it is mutually known on any order the model allows, at the state of the world. In other words, an event is commonly known, if every player knows the event, every player knows that every player knows the event, and so on for all level less than $\kappa$.

In Aumann (1999a)'s model the above definition of common knowledge is equivalent with the following: an event is commonly known, if it is a fixpoint of each player's knowledge operator (in sense of Aumann (1999a)), see e.g. Osborne and Rubinstein (1994) Proposition 74.2, pp. 74-75. Here, however, this definition is not equivalent with the one above, see the examples in Section 2.

**Definition 20.** *Consider knowledge-belief space* $(S, \{(T, \mathcal{M}_i)\}_{i \in N}, g, \{k_i\}_{i \in N}, \{f_i\}_{i \in N})$. *Then event* $A \in \mathcal{M}$ *is self-evident at state of the world* $w \in \Omega$, *if for each player* $i$, $A \in \mathcal{M}_i$, *that is, if* $A \in \wedge_{i \in N} \mathcal{M}_i$, *and* $w \in A$, *where* $\wedge_{i \in N} \mathcal{M}_i$ *is the finest* $\kappa$*-field which is included by all* $\kappa$*-fields* $\mathcal{M}_i$.

Obviously, a self-evident event is a fixpoint of the players' knowledge operators. The following lemma is also apparent:

**Lemma 21.** *If event $A$ is self-evident at a state of the world, then it is commonly known at the state of the world.*

It is clear that the contrary of the above statement does not hold, in the examples of Section 2 the event $[0, 1)$ is commonly known, but not self-evident.

The difference between the two notions is that if at a state of the world an event is self-evident, then this is not only commonly known, but it is an event that the event is commonly known, and it is commonly known that the event is commonly known and so on at any level independently from $\kappa$.

In the following we revisit Aumann (1976)'s and Aumann and Brandenburger (1995)'s results.

**Theorem 22.** *Consider knowledge-belief space $(S, \{(T, \mathcal{M}_i)\}_{i \in N}, g, \{k_i\}_{i \in N}, \{f_i\}_{i \in N})$ with common prior $P$, and event $A \in \mathcal{M}$. Then, if at state of the world $w \in \Omega$, there exist $p_i \in [0, 1]$ and event $B \in \mathcal{M}$ such that $B \subseteq \bigcap_{i \in N} \{w \in \Omega \colon f_i(w)(A) = p_i\}$, $B$ is self-evident, and $P(B) > 0$, then $p_i = p_j$, $i, j \in N$.*

*Proof.* From the definition of self-evidence (Definition 20), $B \in \wedge_{i \in N} \mathcal{M}_i$. Therefore, for each player $i \in N$:

$$P(A \cap B) = \int_B f_i(\cdot)(A) \, \mathrm{d}P = p_i P(B) \ .$$

Since $P(B) \neq 0$, $p_i = p_j$, $i, j \in N$. □

The proof above is the same as Aumann (1976)'s. Therefore, the distinction between common knowledge and self-evidence is relevant.

Next we consider Aumann and Brandenburger (1995)'s result (THEOREM B, p. 1168).

**Theorem 23.** *Consider a game in strategic form $\Gamma = (N, \{S_i\}_{i \in N}, \{u_i\}_{i \in N})$ and knowledge-belief space $(S, \{(T, \mathcal{M}_i)\}_{i \in N}, g, \{k_i\}_{i \in N}, \{f_i\}_{i \in N})$ with common prior $P$. Moreover, let $B, G, R \in \mathcal{M}$ be such that $B \subseteq \bigcap_{i \in N} \{w \in \omega \colon \psi_i(w) = \bar{\psi}_i\}$, where $\psi_i \colon \Omega \to \Delta(S_{-i})$ is $\mathcal{M}_i$-measurable, player $i$'s conjuncture on the other players' strategies, $\bar{\psi}_i \in \Delta(S_{-i})$, $G$ is the event of that the players play game $\Gamma$, and $R \subseteq \bigcap_{i \in N} R_i$, where $R_i$ is the event of that player $i$ is rational, $i \in N$.*

*Furthermore, let event $A \in \mathcal{M}$ be such that*

1. *at each state of the world $w \in A$, each player knows event $G$,*

2. *at each state of the world $w \in A$, event $B$ is self-evident,*

3. *at each state of the world $w \in A$, each player knows event $R$,*

4. *$P(A) > 0$.*

*Then, at each state of the world $w \in A$, for all players $i, j, k \in N$, $(\psi_i(w))_k = (\psi_j(w))_k$, where $(\psi_i(w))_k$ is player $i$'s conjecture about player $k$'s strategy, and let $\sigma_k = (\psi_i(w))_k$. Furthermore, $\times_{i\in N}\sigma_i \in \Delta(S)$ is a Nash equilibrium in game $\Gamma$.*

In other words, if the played game and the players' rationality are mutually known and the players' conjectures are self-evident, then the players play a Nash equilibrium in the game.

*Proof.* Notice that by the definition of self-evidence (Definition 20), $B \in \wedge_{i\in N}\mathcal{M}_i$, and for all states of the world $w, w' \in B$ and player $i \in N$, $\psi_i(\omega') = \psi_i(\omega'') = \bar{\psi}_i$. Moreover, from Theorem 22 at each state of the world $w \in B$, for all players $i, j, k \in N$: $(\psi_i(w))_k = (\psi_j(w))_k = \sigma_k$.

Then for all states of the world $w \in A$ and players $i, j \in N$, let $s_j \in S_j$ be such that $P_i(\psi_i^{-1}(\{s_j\} \times S_{-i,j}), w) > 0$, where $P_i(\cdot, w)$ is player $i$'s belief at state of the world $w$. Since $P_i(B, w) = P_i(G, w) = P_i(R, w) = 1$ and $P_i(\psi_i^{-1}(\{s_j\} \times S_{-i,j}), w) > 0$:

$$P_i\left(B \bigcap G \bigcap R \bigcap \psi_i^{-1}(\{s_j\} \times S_{-i,j}), w\right) > 0 \ ,$$

hence

$$B \bigcap G \bigcap R \bigcap \psi_i^{-1}(\{s_j\} \times S_{-i,j}) \neq \emptyset \ .$$

Therefore, there exists state of the world $w \in B \bigcap G \bigcap R$ such that $\psi_i(w)(\{s_j\} \times S_{-i,j}) > 0$, that is, $s_j$ is a best response to conjecture $\bar{\psi}_j$.

Finally, since for each player all actions with positive probability in a conjecture is best response to conjecture $\bar{\psi}_i$, $\times_{i\in N}\sigma_i$ is a Nash equilibrium in strategic form game $\Gamma$. □

As in the case of Theorem 22, the proof of the theorem goes as Aumann and Brandenburger (1995)'s proof goes. Therefore again the notion of self-evidence is the one the above results call for.

# 7 Conclusions

In this paper we have uncovered a logical inconsistency in the epistemic models used in the literature. It is common that while a finitary language is used for describing the belief hierarchies and common knowledge, an infinitary one is used for giving the players' beliefs ($\sigma$-additive measures) and information structures ($\sigma$-fields). We have fixed this inconsistency and introduced a model, where the universal knowledge-belief space exists, is complete, and encompasses all belief hierarchies.

Moreover, we have presented examples demonstrating that in the fixed (our) model two famous results of epistemic game theory, Aumann (1976) and Aumann and Brandenburger (1995) do not hold. However, we showed that by substituting self-evidence for common knowledge Aumann (1976)'s and Aumann and Brandenburger (1995)'s results become true again, so these results call for self-evidence rather than for common knowledge.

# A Inverse systems, inverse limits

In this appendix we introduce the basic notions of inverse system and inverse limit.

**Definition 24.** *Let $(I, \leq)$ be a preordered set, $(X_i)_{i \in I}$ be a family of nonvoid sets, and for all $i, j \in I$ such that $i \leq j$, $f_{ij} : X_j \to X_i$. Then $(X_i, (I, \leq), f_{ij})$ is an inverse system if it meets the following:*

- $f_{ii} = id_{X_i}$,
- $f_{ik} = f_{ij} \circ f_{jk}$,

*$i, j, k \in I$ such that $i \leq j$ and $j \leq k$.*

The inverse system, also called projective system, is a family of sets connected by functions.

**Definition 25.** *Let $((X_i, \mathcal{A}_i, \mu_i), (I, \leq), f_{ij})$ be an inverse system such that for all $i \in I$, $(X_i, \mathcal{A}_i, \mu_i)$ is a $\kappa$-measure space. The inverse system $((X_i, \mathcal{A}_i, \mu_i), (I, \leq), f_{ij})$ is an inverse system of $\kappa$-measure spaces if it meets the following:*

- *$f_{ij}$ is an $\mathcal{A}_j$-measurable function,*
- $\mu_i = \mu_j \circ f_{ij}^{-1}$,

*$i, j \in I$ such that $i \leq j$.*

Next we introduce the notion of inverse limit.

**Definition 26.** *Let $(X_i, (I, \leq), f_{ij})$ be an inverse system, $X = \times_{i \in I} X_i$ and $P = \{x \in X :$ for all $i, j$ such that $i \leq j$, $pr_i(x) = f_{ij} \circ pr_j(x)\}$, where for all $i \in I$, $pr_i$ is the coordinate projection from $X$ to $X_i$. Then $P$ is called the inverse limit of the inverse system $(X_i, (I, \leq), f_{ij})$, and it is denoted by $\varprojlim(X_i, (I, \leq), f_{ij})$.*

*Moreover, let $p_i = pr_i|_P$, so for all $i, j \in I$ such that $i \leq j$, $p_i = f_{ij} \circ p_j$. Projection $p_i$ is called canonical projection, $i \in I$.*

In other words, the inverse limit is a generalization of the Cartesian product. If $(I, \leq)$ is such that every element of $I$ is related only to itself, that is, for all $i, j \in I$, $(i \leq j) \Rightarrow (i = j)$, then the inverse limit is the Cartesian product.

**Definition 27.** *Let $((X_i, \mathcal{A}_i, \mu_i), (I, \leq), f_{ij})$ be an inverse system of $\kappa$-measure spaces and $P = \varprojlim(X_i, (I, \leq), f_{ij})$. Then the $\kappa$-measure space $(P, \mathcal{A}, \mu)$ is the inverse limit of the inverse system of $\kappa$-measure spaces $((X_i, \mathcal{A}_i, \mu_i), (I, \leq), f_{ij})$ denoted by $(P, \mathcal{A}, \mu) = \varprojlim((X_i, \mathcal{A}_i, \mu_i), (I, \leq), f_{ij})$, if it meets the following:*

1. *$\mathcal{A}$ is the coarsest $\kappa$-field for which the canonical projections $p_i$ are $\mathcal{A}$-measurable, $i \in I$,*
2. *$\mu$ is a $\kappa$-measure such that $\mu \circ p_i^{-1} = \mu_i$, $i \in I$.*

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