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by

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Abstract

Measuring and allocating risk properly are crucial for performance evaluation and internal capital allocation of portfolios held by banks, insurance companies, investment funds and other entities subject to financial risk. We show that by using coherent measures of risk it is impossible to allocate risk satisfying simultaneously the natural requirements of Core Compatibility, Equal Treatment Property and Strong Monotonicity. To obtain the result we characterize the Shapley value on the class of totally balanced games and also on the class of exact games.

Keywords: Coherent Measures of Risk, Risk Allocation Games, Totally Balanced Games, Exact Games, Shapley value, Core

JEL Classification: C71, G10

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1 Introduction

If a financial enterprise (bank, insurance company, investment fund, portfolio, etc.) consists of divisions (individuals, products, subportfolios, risk factors etc.), not only is it important to *measure* properly the risk of the main entity, but also to *allocate* the risk of the firm to the divisions. Risk allocation can be used for at least two purposes: for performance evaluation or for allocating internal capital requirements. In the first case risk is used to calculate risk adjusted returns for performance evaluation, which crucially depends on the allocated risk. In the second case the measure of risk is used to determine capital requirements, which serve as a cushion against default. Holding capital is costly, hence capital allocation is also important in this case.

A measure of risk assigns a real number to the profit and loss realization vector of a division. When using a *coherent measure of risk* (Artzner et al, 1999) the risk of the portfolio of the firm is at most as much as the sum of the risks of the portfolios of the divisions. Thus there is always a diversification benefit, which should be allocated somehow. *Risk (capital) allocation games* (Denault, 2001) are transferable utility (TU) cooperative games defined to model such risk allocation. In a TU game using the values (the opposite of the risk) of the coalitions (subsets) of the players (divisions) a solution concept (a risk allocation rule) determines how to share the value of the grand coalition (the firm). An allocation is in the core if the total value of the grand coalition is allocated (*Efficiency*) in such a way that no coalition of the players could get a higher value by blocking it and getting its stand-alone value. A totally balanced game has a non-empty core in each of its subgames, where a subgame is obtained by considering only a subset of the players. Csóka et al (2009) show that the class of risk allocation games (using coherent measures of risk) coincides with the class of totally balanced games, that is for any risk allocation game there is a core allocation (a stable way to allocate risk using an allocation rule satisfying *Core Compatibility*) and any totally balanced game can be generated by a properly chosen risk allocation game. Csóka et al (2009) also prove that the class of risk allocation games with no aggregate uncertainty equals the class of *exact games* (Schmeidler, 1972), where for each coalition there is a core allocation allocating the stand-alone value of the coalition.

On top of Core Compatibility in this paper we consider two further fairness properties of risk capital allocation rules: *Equal Treatment Property* and *Strong Monotonicity*. The motivation comes from Young's axiomatization (Young, 1985) of the Shapley value (Shapley, 1953), where he shows that on the class of all games the Shapley value is the only solution concept satisfying Efficiency, Equal Treatment Property and Strong Monotonicity. It is well-known that the Shapley value does not satisfy Core Compatibility in general, hence the properties of Core Compatibility, Equal Treatment Property and Strong Monotonicity can not simultaneously be satisfied on the classes of all games. However, the validity of an axiomatization of a solution concept can vary from subclass to subclass, e.g. Shapley's axiomatization of the Shapley value is valid on the class of monotone games but not valid on the class of strictly monotone games. In the case of risk allocation games (thus totally balanced games or exact games) we generalize Young's result, and show that his axiomatization remain valid on the classes of totally balanced and exact games; and we conclude: on the class of risk allocation games there is no risk allocation rule satisfying Core Compatibility, Equal Treatment Property and Strong Monotonicity at the same time.

We also interpret the axioms in the finance setting as follows. Core Compatibility is satisfied if all the risk of the main unit is allocated in such a way that no group of the divisions can improve by allocating only the risk of the group, the risk allocation can be seen as stable. Core Compatibility can also be viewed as the allocated risk to each coalition should be at least as much as the risk increment the coalition causes by joining the complementary coalition. Equal Treatment Property makes sure that if two divisions have the same stand-alone risk and also they contribute the same risk to all the subsets of divisions not containing them, then they are treated equally, that is the same risk capital is allocated to them. In other words, Equal Treatment Property states that if two divisions are not distinguishable from a riskiness point of view, then they must be evaluated equally. Strong Monotonicity requires that if a division weakly reduces its stand-alone risk and also its risk contribution to all the subsets of the other divisions (hence weakly increases its relative performance), then its allocated capital should not increase. Therefore Strong Monotonicity is a sort of incentive compatibility notion.

The other axiomatic approaches in the literature related to our work are as follows. Denault (2001) considers the original axiomatization of the Shapley value by Shapley (1953), and concludes that there is no allocation rule which satisfies Core Compatibility, Symmetry (a slightly stronger axiom than Equal Treatment Property), Dummy Player Property (a riskless division should be allocated its stand-alone risk) and Additivity (after adding two risk allocation games the allocations obtained by the allocation rule can also be added). However, he does not prove that the original axiomatization is true on the class of totally balanced games. On top of that Additivity is not so natural to require, since by adding the portfolios of two risk allocation games the resulting game will usually not be the addition of the games we started with, there is a diversification effect here as well. Thus Additivity only has a bite if the portfolios for each coalition in the games to be added are comonotonic, that is they are a positive function of the same random variable. Drehmann and Tarashev (2013) also considers the original axiomatization of the Shapley value for systemic risk allocation games but has the same shortcomings and on top of that they interpret Additivity as Efficiency.

Valdez and Chernih (2003) show that for elliptically contoured distributions the covariance (or beta) method satisfies Stability, Symmetry and Consistency (requiring that the allocation method should be independent of the hierarchical structure of the main unit). However, Kim and Hardy (2009) show that it is not even true that the covariance method satisfies Stability in this setting. On top of that profit and loss distributions of financial assets are not elliptically contoured, but heavy tailed (see for instance Cont (2001)), hence our approach is clearly more relevant by not restricting the probability distributions.

Kalkbrener (2005) shows that Linear Aggregation, Diversification and Continuity characterizes the gradient principle (or Euler method, where risk is allocated as a result of slightly increasing the weights of the divisions) to be the only allocation which satisfies those requirements. Although those requirements are also natural, they are not related to the properties of Equal Treatment Property and Strong Monotonicity. In fact Kalkbrener (2005) explicitly assumes that the risk allocated to a division does not depend on the decomposition of the other divisions, only on the main unit, which is a strange and strong assumption.

Finally, there is a related impossibility result by Buch and Dorfleitner (2008). They show that if one uses the gradient principle to allocate risk and Symmetry is satisfied, then the measure of risk must be linear, not allowing for any diversification benefits.

Summing up, in this paper we apply the cooperative game theory toolkit to consider risk allocations. Our main message is that there is no such risk allocation rule satisfying Core Compatibility, Equal Treatment Property and Strong Monotonicity at the same time, so it is necessary to give up at least one of these properties.

The setup of the paper is as follows. In the following section we introduce the notations and notions for transferable utility cooperative games in general and risk allocations games in particular. In Section 3 we present our impossibility result, and the last section concludes.

2 Risk allocation games

We will use the following notations and notions. $|N|$ is for the cardinality of finite set N, 2^N is the power set of N; moreover $A \subset B$ means $A \subseteq B$ and $A \neq B$.

Let N denote the finite *set of players*. *A cooperative game with transferable utility* (*game*, for short) is a function $v : 2^N \to \mathbb{R}$ such that $v(\emptyset) = 0$. The class of games with player set N is denoted by $\mathcal{G}^N$. For a game $v \in \mathcal{G}^N$ and a coalition $C \in 2^N$, a *subgame* v_C is obtained by restricting v to the subsets of C.

An *allocation* is a vector $x \in \mathbb{R}^N$, where x_i is the payoff of player $i \in N$. An allocation x yields payoff $x(C) = \sum_{i \in C} x_i$ to a coalition $C \in 2^N$. An allocation $x \in \mathbb{R}^N$ is called *Efficient*, if $x(N) = v(N)$ and *Coalitionally Rational* if $x(C) \geq v(C)$ for all $C \in 2^N$. The *core* (Gillies, 1959) is the set of Efficient and Coalitionally Rational allocations. The core of game v is denoted by core (v).

Let $v \in \mathcal{G}^N$ and $i \in N$ be a game and a player, and for all $C \subseteq N$ let $v'_i(C) = v(C \cup \{i\}) - v(C)$ denote player i's marginal contribution to coalition C in game v. Then v'_i is called player i's *marginal contribution function* in game v. Moreover, players i and j are *equivalent* in game v, $i \sim^v j$, if for any $C \subseteq N$ such that $i, j \notin C$ we have that $v'_i(C) = v'_j(C)$.

A game is *totally balanced*, if each of its subgame has a non-empty core. Let $\mathcal{G}^N_{\text{tb}}$ denote the class of totally balanced games with player set N.

An interesting subclass of totally balanced games is the class of *exact games* (Schmeidler, 1972). A game v is *exact*, if for each coalition C there exists an allocation $x \in$ core (v) such that $x(C) = v(C)$. Let $\mathcal{G}^N_{\text{e}}$ denote the class of exact games with player set N.

Throughout the paper we consider single-valued solutions. The function $\psi : A \to \mathbb{R}^N$, defined on $A \subseteq \mathcal{G}^N$, is called *solution* on the class of games A. In the context of risk allocation, we refer to solutions as *risk allocation rule*s.

For any game $v \in \mathcal{G}^N$ the *Shapley solution* ϕ is given by

$$\phi_i(v) = \sum_{C \subseteq N \setminus \{i\}} v'_i(C) \frac{|C|!(|N \setminus C| - 1)!}{|N|!} \quad i \in N,$$

where $\phi_i(v)$ is also called the *Shapley value* (Shapley, 1953) of player i in game v.

To define risk allocation games we use the setup of Csóka et al (2009). Let S denote the finite number of states of nature and consider the set $\mathbb{R}^S$ of realization vectors. State of nature s occurs with probability $p_s > 0$, where $\sum_{s=1}^S p_s = 1$. The vector $X \in \mathbb{R}^S$ represents a division's possible profit and loss realizations at a given time in the future. The amount X_s is the division's payoff in state of nature s. Negative values of X_s correspond to losses.

A *measure of risk* is a function $\rho : \mathbb{R}^S \to \mathbb{R}$ measuring the risk of a division from the present perspective. The measure of risk ρ is a *coherent measure of risk* (Artzner et al, 1999) if it satisfies the axioms of *Monotonicity*, *Subadditivity*, *Positive homogeneity* and *Translation invariance*. For the definition and an interpretation of the axioms see Acerbi and Scandolo (2008), who justify them to incorporate liquidity risk as well.

Resuming to risk allocation, let the matrix of realization vectors corresponding to the divisions be given by $X \in \mathbb{R}^{S \times N}$, and let $X_{\cdot i}$ denote column i of X, the realization vector of division i. For a coalition of divisions $C \in 2^N$, let $X_C = \sum_{i \in C} X_{\cdot i}$.

A *risk environment* is a tuple (N, S, p, X, ρ), where N is the set of divisions, S indicates the number of states of nature, $p = (p_1, \ldots, p_S)$ is the vector of realization probabilities of the various states, X is the matrix of realization vectors, and ρ is a coherent measure of risk.

A risk allocation game assigns to each coalition of divisions the negative of the risk involved in the aggregate portfolio of the coalition.

Definition 2.1. Given the risk environment (N, S, p, X, ρ), a *risk allocation game* is a game $v \in \mathcal{G}^N$, where

$$v(C) = -\rho(X_C) \text{ for all } C \in 2^N \setminus \{\emptyset\} . \tag{1}$$

Let $\mathcal{G}_{\mathrm{r}}^N$ denote the family of risk allocation games with player set N. In such games, according to Equation (1), the larger the risk of any subset of divisions, the lower its value. We illustrate the definition of the risk allocation game by the following example.

Example 2.2. Consider the following risk environment (N, S, p, X, ρ). We have 3 divisions, 7 states of nature with equal probability of occurrence. Risk is calculated by using the matrix of realization vectors in the first three columns of Table 1 and the maximum loss, which is a coherent measure of risk (Acerbi and Tasche, 2002).

$S \backslash X_C$	$X_{\{1\}}$	$X_{\{2\}}$	$X_{\{3\}}$	$X_{\{1,2\}}$	$X_{\{1,3\}}$	$X_{\{2,3\}}$	$X_{\{1,2,3\}}$
1	0	1	1	1	1	2	2
2	1	0	1	1	2	1	2
3	1	1	0	2	1	1	2
4	0.5	0.5	1	1	1.5	1.5	2
5	0	1	0	1	0	1	1
6	1	0.5	0.5	1.5	1.5	1	2
7	0	1	0	1	0	1	1
$\rho(C)$	0	0	0	-1	0	-1	-1
$v(C)$	0	0	0	1	0	1	1

Table 1: The matrix of realization vectors of a risk environment and the resulting totally balanced risk allocation game v using the maximum loss.

Note that for all $C \in 2^N \setminus \{\emptyset\}$ the value function is given by $v(C) = -\rho(X_C) = \min_{s \in S} X_{C,s}$.

If the rows of a matrix of realization vectors sum up to the same number, then there is *no aggregate uncertainty*. Formally: a matrix of realization vectors $X \in \mathbb{R}^{S \times N}$ has *no aggregate uncertainty*, if there exists a number $\alpha \in \mathbb{R}$ such that $X_N = \alpha 1^S$. Let $\mathcal{G}_{\mathrm{rnau}}^N$ denote the family of risk allocation games with no aggregate uncertainty and with player set N.

Theorem 2.3. *(Csóka et al, 2009) The class of risk allocation games coincides with the class of totally balanced games, that is $\mathcal{G}_{\mathrm{r}}^N = \mathcal{G}_{\mathrm{tb}}^N$. Moreover, the class of risk allocation games with no aggregate uncertainty equals the class of exact games, that is $\mathcal{G}_{\mathrm{rnau}}^N = \mathcal{G}_{\mathrm{e}}^N$.*

Note that by Theorem 2.3 all risk allocation games are totally balanced (also the one in Example 2.2), and if there is no aggregate uncertainty, then all of them are exact. Moreover, for any totally balanced (exact) game there is a risk environment (with no aggregate uncertainty) – with a coherent risk measure – that generates the game.

A risk allocation rule shows how to share the risk of the main unit among the divisions. Since the situation can be converted into a cooperative game, a risk allocation rule can also be a solution in cooperative game theory, like the Shapley value. We illustrate the marginal contribution function and the fact that the Shapley value is not always in the core by continuing Example 2.2.

Example 2.4. Consider the risk allocation game in Example 2.2.

C	$\emptyset$	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	$\{1,2,3\}$
$v(C)$	0	0	0	0	1	0	1	1
$v_1'(C)$	0	0	1	0	0	0	0	
$v_2'(C)$	0	1	0	1	0	1	0	
$v_3'(C)$	0	0	1	0	0	0	0	

Table 2: The value function of a risk allocation game and the marginal contribution functions of players 1, 2 and 3.

Note that player 2 has a higher marginal contribution than the others, which is also expressed by the Shapley value, since it is $\phi(v) = (1/6, 2/3, 1/6)$. However, coalition $\{1,2\}$ blocks the Shapley allocation, since $v(\{1,2\}) = 1 > 5/6 = 1/6+2/3$, that is $\phi(v) \notin \text{core } (v)$.

3 The impossibility result

Next, we introduce four basic properties (axioms) one might think a risk allocation rule (solution) should meet.

Definition 3.1. The solution ψ on class $A \subseteq \mathcal{G}^N$ satisfies

- *Core Compatibility* if for each $v \in A$: $\psi(v) \in \text{core } (v)$,
- *Efficiency* if for each $v \in A$: $\sum_{i \in N} \psi_i(v) = v(N)$,

- *Equal Treatment Property* if for each $v \in A$, $i, j \in N$: $i \sim^v j$ implies $\psi_i(v) = \psi_j(v)$,
- *Strong Monotonicity* if for any $v, w \in A$, $i \in N$: $v'_i \leq w'_i$ implies $\psi_i(v) \leq \psi_i(w)$.

The financial interpretations of the axioms are as follows.

Core Compatibility is satisfied if the risk allocation rule results in a core allocation, that is all the risk of the main unit is allocated in such a way that no group of the divisions can improve by allocating only the risk of the group, the risk allocation can be seen as stable. Notice that for a Coalitionally Rational allocation x we have that $x(N \setminus C) \geq v(N \setminus C)$ for all $C \in 2^N$, which, together with Efficiency imply that $x(C) \leq v(N) - v(N \setminus C)$ for all $C \in 2^N$. That is in a core allocation the diversification gain allocated to each coalition can be at most as much as its contribution to the complementary coalition. Or to put it differently, the allocated risk to each coalition should be at least as much as the risk increment the coalition causes by joining the complementary coalition.

Efficiency is implied by Core Compatibility, since it requires that all the risk of the main unit should be allocated to the divisions.

Equal Treatment Property makes sure that if two divisions have the same stand-alone risk and also they contribute the same risk to all the subsets of divisions not containing them, then they are treated equally, that is the same risk capital is allocated to them. In other words, if two divisions are the same from a riskiness point of view, then by the Equal Treatment Property the same risk is assigned to both.

Strong Monotonicity requires that if a division weakly reduces its stand-alone risk and also its risk contribution to all the subsets of the other divisions (hence weakly increases its relative performance), then its allocated capital should not increase. Thus as a kind of incentive compatibility notion, weakly better relative performance is weakly rewarded. Note that it follows from Strong Monotonicity that for any $v, w \in A$, $i \in N$: $v'_i = w'_i$ implies $\psi_i(v) = \psi_i(w)$ (called Marginality).

Theorem 3.2. *(Young, 1985) Let ψ be a solution on the class of all games. Then solution ψ satisfies Efficiency, Equal Treatment Property and Strong Monotonicity (Marginality) if and only if $\psi = \phi$, that is if and only if it is the Shapley solution.*

Note that risk allocation games form a proper subset of all games, since they are always totally balanced. We prove a Theorem 3.2 type result on the class of totally balanced (exact) games in Theorem 3.3. Formally:

Theorem 3.3. *Let ψ be a solution on the class of totally balanced (exact) games. Then solution ψ satisfies Efficiency, Equal Treatment Property and Strong Monotonicity (Marginality) if and only if $\psi = \phi$, that is if and only if it is the Shapley solution.*

Proof. If: See e.g. Young (1985).

Only if: Let u_T denote the unanimity game on coalition T, that is for all $C \subseteq N$: $u_T(C) = \begin{cases} 1, & \text{if } C \supseteq T \\ 0 & \text{otherwise} \end{cases}$.

Next, we generalize Young (1985) (see p. 71) for such classes of games where if a game v is in the class, then for all $\alpha > 0$ and coalition T we have that $v + \alpha u_T$ is also in the class. Notice that the class of totally balanced (exact) games is such class of games, since we get the required core allocations by distributing α among the members of coalition T in a smart way.

Let v be a totally balanced (exact) game and let us decompose it to the unique sum of unanimity games such as $v = \sum_{T \subseteq N} \alpha_T u_T$. Moreover, let $\alpha^m = \max_{T \subseteq N} \alpha_T$, $v^* = \alpha^m \sum_{T \subseteq N} u_T$, and $v_d = v^* - v$.

Notice that v^* is a totally balanced (exact) game and in $v_d = \sum_{T \subseteq N} \beta_T u_T$ we have that $\beta_T \geq 0$ for all $T \subseteq N$. For each game w define the index $I(w)$ such that $I(w) = |\{\gamma_T \neq 0 : w = \sum_{T \subseteq N} \gamma_T u_T\}|$.

The proof goes by induction on $I(v_d)$.

If $I(v_d) = 0$, then $v = v^*$, all players are equivalent in this game, so Efficiency and Equal Treatment Property imply solution ψ is well-defined (unique) at game v.

Let k be an integer such that $0 < k < 2^{|N|} - 1$. Assume that for each totally balanced (exact) game w such that $I(w_d) \leq k$, $\psi(w)$ is well-defined. Then, let v be a totally balanced (exact) game such that $I(v_d) = k + 1$. Consider the decomposition $v_d = \sum_{T \subseteq N} \beta_T u_T$, that is $v = v^* - \sum_{T \subseteq N} \beta_T u_T$, where $\beta_T \geq 0$ for all $T \subseteq N$.

First, if it exists, take any player i for which there exists $T \subseteq N$, $\beta_T > 0$ such that $i \notin T$. Let $v^k = v + \beta_T u_T$. Notice that since $\beta_T > 0$ we have that v^k is totally balanced (exact). Moreover, $I(v^k_d) = k$ and by induction $\psi(v^k)$ is well-defined. Since $v'_i = (v + \beta_T u_T)'_i$, Strong Monotonicity (Marginality) implies that $\psi_i(v) = \psi_i(v^k)$.

Second, if they exist, take the remaining players, that is take all i such that $i \in T \subseteq N$ for all $T \subseteq N$ where $\beta_T > 0$. These players are equivalent in v (since they are equivalent in all games $\beta_T u_T$, $\beta_T > 0$), so by Equal Treatment Property they get the same value.

Finally, by Efficiency solution ψ is well-defined at game v.

Since the Shapley solution meets Efficiency, Equal Treatment Property and Strong Monotonicity (Marginality) and ψ is uniquely defined using those properties, we have that $\psi = \phi$. □

Our main result is the following.

Theorem 3.4. *There is no risk allocation rule meeting the properties of Core Compatibility, Equal Treatment Property and Strong Monotonicity (Marginality) at the same time.*

Proof. Theorem 3.3 is characterizing the Shapley value as being the only risk allocation rule satisfying Efficiency, Equal Treatment Property and Strong Monotonicity (Marginality) on the class of risk allocation games (with no aggregate uncertainty), which by Theorem 2.3 coincides with the class of totally balanced (exact) games. By Example 2.4 the Shapley value is not always in the core for totally balanced games. Moreover, Rabie (1981) shows by an example that for at least five players the Shapley value is not in the core for exact games either. □

4 Conclusion

We have shown that by using coherent measures of risk it is impossible to allocate risk satisfying simultaneously the natural requirements of Core Compatibility, Equal Treatment Property and Strong Monotonicity. To obtain the result we have characterized the Shapley value on the class of totally balanced games and also on the class of exact games. Both classes are a strict subset of all the TU games, hence it is not obvious that the original characterization by Young (1985) holds on them. We have also interpreted the axioms in the finance application setting.

Recently Balog et al (2014) show that our analytical result is not only a theoretical possibility. For randomly generated risk allocation games on average the Shapley value is not in the core about 40-60% of the cases. Therefore our result raises the practical problem: one has to give up one of Core Compatibility, Equal Treatment Property or Strong Monotonicity (Marginality) to allocate risk in practice too. Depending on the application at hand the literature on cooperative games can help which property to give up.

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