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by

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Abstract

In this note we present a cardinally convex game (Sharkey, 1981) with empty core. Sharkey assumes that $V(N)$ is convex, we do not do so, hence we do not contradict Sharkey's result.

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A cooperative game with non-transferable utility V (*game* for short) on a non-empty, finite player set N is a family of sets $V = \{V(S)\}_{S \in 2^N}$ satisfying the following assumptions:

$$V(\emptyset) = \emptyset,$$
$$V(S) = V(S)_S \times \mathbb{R}^{N \setminus S}, \text{ for all } S \subseteq N,$$
$$0^N \in V(S) \text{ for all } S \subseteq N,\ S \neq \emptyset,$$
$$V(S) \text{ is closed for all } S \subseteq N,$$
$$\text{comprehensiveness: if } x \in V(S),\ y \in \mathbb{R}^N,\ y_S \leq x_S,\ \text{then } y \in V(S),$$
$$\text{the sets } V(S)_S \cap (x^S + \mathbb{R}^S_+) \text{ are bounded for all } S \subseteq N \text{ and } x^S \in \mathbb{R}^S,$$

where $\text{Set}_S \subseteq \mathbb{R}^S$ is the coordinate projection of set Set by the coordinates of S. Notice that we do not assume that $V(N)$ is convex, so we are more general than Sharkey (1981).

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The *core* of a game $V \in \mathcal{G}^N$ consists of those elements $x \in V(N)$ for which it holds that there exist no $S \subseteq N$ and no $y \in V(S)$ such that $x_S \ll y_S$.

For a game $V \in \mathcal{G}^N$ and a coalition $S \subseteq N$, $S \neq \emptyset$, let $V^\circ(S) = \{x \in V(S) \colon x_i = 0 \text{ for all } i \in N \setminus S\}$, and let $V^\circ(\emptyset) = \{0^N\}$. A game $V \in \mathcal{G}^N$ is *cardinally convex* (Sharkey, 1981) if for all $S, T \subseteq N$ we have

$$V^\circ(S) + V^\circ(T) \subseteq V^\circ(S \cup T) + V^\circ(S \cap T) \ .$$

The following example is our main result.

Example 1. Let $N = \{1,2,3,4,5,6\}$, $\mathcal{K} = \{\{1,2\},\{3,5\},\{4,6\}\}$, and

$$V(\{i\}) = \{x \in \mathbb{R}^6 \colon x_i \leq 0\},\ i \in N$$

$$V(\{i,j\}) = \begin{cases} \{x \in \mathbb{R}^6 \colon \exists y \in [-10,10],\ (x_i,x_j) \leq (y,-y)\}, & \text{if } \{i,j\} \in \mathcal{K} \\ \{x \in \mathbb{R}^6 \colon x_i, x_j \leq 0\} & \text{otherwise} \end{cases}$$

$$V(\{i,j,k\}) = \begin{cases} \{x \in \mathbb{R}^6 \colon x \in V(\{i,j\}) \text{ and } x_k \leq 0\}, & \text{if } \{i,j\} \in \mathcal{K} \\ \{x \in \mathbb{R}^6 \colon x_i, x_j, x_k \leq 0\} & \text{otherwise} \end{cases}$$

$$V(\{1,2,3,4\}) = \{x \in \mathbb{R}^6 \colon x \in V(\{1,2\}) \cap V(\{3,4\}) \text{ or } x_{\{1,2,3,4\}} \leq (1,1,2,2)\}$$

$$V(\{1,2,5,6\}) = \{x \in \mathbb{R}^6 \colon x \in V(\{1,2\}) \cap V(\{5,6\}) \text{ or } x_{\{1,2,5,6\}} \leq (2,2,1,1)\}$$

$$V(\{3,4,5,6\}) = \{x \in \mathbb{R}^6 \colon x \in V(\{3,4\}) \cap V(\{5,6\}) \text{ or } x_{\{3,4,5,6\}} \leq (1,1,2,2)\}$$

$$V(\{i,j,k,l\}) = \{x \in \mathbb{R}^6 \colon x \in V(\{i,j\}) \cap V(\{k,l\})\},\ \{i,j\} \in \mathcal{K},\ \{k,l\} \notin \mathcal{K}$$

$$V(\{i,j,k,l,m\}) = \{x \in \mathbb{R}^6 \colon x \in V(\{i,j,k,l\}) \text{ and } x_m \leq 0\},\ \{i,j\},\{k,l\} \in \mathcal{K}$$

$$V(\{i,j,k,l,m,n\}) = \{x \in \mathbb{R}^6 \colon x \in V(\{i,j,k,l\}) \cap V(\{m,n\}) \text{ or } \exists \{g,h\} \in \mathcal{K}, \exists y \in \mathbb{R} \text{ such that } (x_g,x_h) \leq (y-1, -y100^{-\operatorname{sgn} y} - 1) \text{ and } x_{N \setminus \{g,h\}} \leq 100\}$$

The game V is cardinally convex: Take coalitions S and T such that neither $S \subseteq T$ nor $T \subseteq S$, otherwise the proof is obvious. We discuss two cases: First, there does not exist $K \in \mathcal{K}$ such that $K \subseteq S \cap T$. Then for each $i \in S \cap T$ either $V(S)_i \subseteq \mathbb{R}_-$ or $V(T)_i \subseteq \mathbb{R}_-$. Furthermore, if $V(S)_i \subseteq \mathbb{R}_-$, then we can substitute $V^\circ(S \setminus \{i\})$ for $V^\circ(S)$, and similarly if $V(T)_i \subseteq \mathbb{R}_-$, then we can substitute $V^\circ(T \setminus \{i\})$ for $V^\circ(T)$. Therefore, after substituting as above we get two disjoint coalitions $S^* \subseteq S$ and $T^* \subseteq T$, where S^* and T^* are the substitutes for S and T respectively. Then we have $V^\circ(S) + V^\circ(T) = V^\circ(S^*) + V^\circ(T^*) \subseteq V^\circ(S^* \cup T^*) = V^\circ(S \cup T)$.

Otherwise, let $K \in \mathcal{K}$ be a coalition that $K \subseteq S \cap T$. If $S \cup T \neq N$, then $|S \cap T| \leq 3$, so there is only one $K \in \mathcal{K}$ such that $K \subseteq S \cap T$.

If $S \cap T = K$, then either $|S| = 3$ or $|T| = 3$. W.l.o.g. we can assume that $|S| = 3$. Then for $j \in S \setminus T$, $V(S)_j \subseteq \mathbb{R}_-$, hence $V^\circ(T) + V^\circ(S) \subseteq V^\circ(T \cup \{j\}) + V^\circ(S \setminus \{j\}) = V^\circ(S \cup T) + V^\circ(S \cap T)$.

If $|S \cap T| = 3$, then for $i \in (S \cap T) \setminus K$ either $V(S)_i \subseteq \mathbb{R}_-$ or $V(T)_i \subseteq \mathbb{R}_-$. W.l.o.g. we can assume that $V(S)_i \subseteq \mathbb{R}_-$, then for $j \in S \setminus T$, $j \neq i$ (actually there is at most one such player), $V(S)_j \subseteq \mathbb{R}_-$ either. Then $V^\circ(T) + V^\circ(S) \subseteq V^\circ(T \cup \{j\}) + V^\circ(S \setminus \{j\}) = V^\circ(S \cup T) + V^\circ(S \cap T)$.

If $S \cup T = N$, then for each $x \in V(S) + V(T)$, $x_K \leq (4,4)$ or $x_K \notin \mathbb{R}^2_+$, and $x_{N \setminus K} \leq 20^{N \setminus K}$. Moreover, $(6,-6),(-6,6) \in V(K)_K$ and there exist $y, z \in V(N)$ such that $y_K = (-2, 99)$, $z_K = (99, -2)$ and $y_{N \setminus K} = z_{N \setminus K} = 100^{N \setminus K}$, therefore $V^\circ(S) + V^\circ(T) \subseteq V(N) + V^\circ(S \cap T)$.

The game V has empty core: If $x \in V(\{i,j,k,l\}) \cap V(\{m,n\})$, then either there exists $g \in N$ such that $x_g < 0$ or $x_{\{m.n\}} = 0^{\{m,n\}}$. In the first case x is blocked via coalition $\{g\}$, in the second case x is blocked via either coalition $\{1,2,3,4\}$ ($\{i,j,k,l\} = \{3,4,5,6\}$) or coalition $\{1,2,5,6\}$ ($\{i,j,k,l\} = \{1,2,3,4\}$) or coalition $\{3,4,5,6\}$ ($\{i,j,k,l\} = \{1,2,5,6\}$).

If there exist $y \in \mathbb{R}$, $\{g,h\} \in \mathcal{K}$ such that $(x_g, x_h) \leq (y-1, -y100^{-\operatorname{sgn} y} - 1)$, then either $x_g < 0$ or $x_h < 0$, so x is blocked either via coalition $\{g\}$ or coalition $\{h\}$.

References

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