Incomplete pairwise comparison matrices and weighting methods

by

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Abstract

A special class of preferences, given by a directed acyclic graph, is considered. They are represented by incomplete pairwise comparison matrices as only partial information is available: for some pairs no comparison is given in the graph. A weighting method satisfies the property linear order preservation if it always results in a ranking such that an alternative directly preferred to another does not have a lower rank. We study whether two procedures, the Eigenvector Method and the Logarithmic Least Squares Method meet this axiom. Both weighting methods break linear order preservation, moreover, the ranking according to the Eigenvector Method depends on the incomplete pairwise comparison representation chosen.

Keywords: Directed acyclic graph, incomplete pairwise comparison matrix, Eigenvector Method, Logarithmic Least Squares Method

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1 Introduction

Pairwise comparisons are widely used in multi-attribute decision making since Saaty published the AHP method (Saaty, 1980). It is assumed that decision makers give a numerical answer to the question ‘How many times is the $i$th alternative more important/better/favorable than the $j$th?’, which are incorporated into a matrix with an appropriate size.

Let $\mathbb{R}_+^n$ denote the positive orthant of the $n$-dimensional Euclidean space and $\mathbb{R}_+^{n \times n}$ denote the set of positive matrices of size $n \times n$.

Definition 1. **Pairwise comparison matrix**: Matrix $A = [a_{ij}] \in \mathbb{R}_+^{n \times n}$ is a pairwise comparison matrix if $a_{ji} = 1/a_{ij}$ for all $i, j = 1, 2, \ldots, n$. 

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The final aim of the use of pairwise comparisons is to determine a weight vector \( w = [w_i] \in \mathbb{R}_+^n \) for the alternatives such that \( w_i/w_j \) somehow approximates \( a_{ij} \).

**Definition 2.** Consistency: Pairwise comparison matrix \( A = [a_{ij}] \) is consistent if \( a_{ik} = a_{ij}a_{jk} \) for all \( i, j, k = 1, 2, \ldots, n \).

Every consistent pairwise comparison matrix can be associated to a weight vector \( w \) where \( a_{ij} = w_i/w_j \) for all \( i, j = 1, 2, \ldots, n \). \( w \) is unique up to multiplication by positive scalars.

Pairwise comparison matrices provided by decision makers are usually do not meet the consistency condition, they are inconsistent. Then the real weight vector \( w \) can only be estimated on the basis of the inconsistent pairwise comparison matrix. A number of weighting methods is proposed for this purpose.

Saaty (1980) has used the Perron theorem (Perron, 1907): a positive matrix has a dominant eigenvalue with multiplicity one and an associated strictly positive (right) eigenvector.

**Definition 3.** Eigenvector Method (EMU) (Saaty, 1980): EMU gives the weight vector \( w_{EMU}(A) \in \mathbb{R}_+^n \) for any pairwise comparison matrix \( A \) such that

\[
Aw_{EMU}(A) = \lambda_{\text{max}}w_{EMU},
\]

where \( \lambda_{\text{max}} \) denotes the maximal eigenvalue, also known as Perron eigenvalue, of matrix \( A \).

Distance-minimization techniques minimize the function \( \sum_i \sum_j d(a_{ij}, w_i/w_j) \) where \( d(a_{ij}, w_i/w_j) \) is some sort of a distance of \( a_{ij} \) from its approximation \( w_i/w_j \). the following is an important example with \( d(a_{ij}, w_i/w_j) = [\log a_{ij} - \log (w_i/w_j)]^2 \).

**Definition 4.** Logarithmic Least Squares Method (LLSM) (Crawford and Williams, 1980, 1985; De Graan, 1980): LLSM gives the weight vector \( w_{LLSM}(A) \in \mathbb{R}_+^n \) for any pairwise comparison matrix \( A \) as the optimal solution of the problem:

\[
\min_{w \in \mathbb{R}_+^n} \sum_{i=1}^n w_i \left( \sum_{j=1}^n \sum_{i=1}^n \left[ \log a_{ij} - \log \left( \frac{w_i}{w_j} \right) \right]^2 \right).
\]

It may also happen that a subset of pairwise comparisons is unknown due to the lack of available data, uncertain evaluations, or other problems. Incomplete pairwise comparison matrices were introduced in Harker (1987).

**Definition 5.** Incomplete pairwise comparison matrix: Matrix \( A = [a_{ij}] \) of size \( n \times n \) is an incomplete pairwise comparison matrix if \( a_{ii} = 1 \) for all \( i = 1, 2, \ldots, n \), and for all \( i \neq j \), \( a_{ji} = 1/a_{ij} \in \mathbb{R}_+ \) or both \( a_{ij} \) and \( a_{ji} \) are missing.

**Notation 1.** Missing elements of pairwise comparison matrices are denoted by \(*\).

**Example 1.** An incomplete pairwise comparison matrix of size \( 4 \times 4 \) is the following:

\[
A = \begin{pmatrix}
1 & * & a_{13} & a_{14} \\
* & 1 & a_{23} & * \\
1/a_{13} & 1/a_{23} & 1 & a_{34} \\
1/a_{14} & * & 1/a_{34} & 1
\end{pmatrix}.
\]
Generalization of $EM$ to incomplete pairwise comparison matrices requires some comment on measuring inconsistency. Saaty (Saaty, 1980) defined the index $CR$ as

$$CR(A) = \frac{(\lambda_{\text{max}}(A) - n)}{(n - 1)} = \frac{\lambda_{\text{max}}(A) - n}{\lambda_{\text{max}}^n - n},$$

where $\lambda_{\text{max}}^n$ denotes the average value of the maximal eigenvalue of randomly generated pairwise comparison matrices of size $n \times n$ such that each element $a_{ij}$, $i < j$ is chosen from the set $\{1/9; 1/8; \ldots; 1/2; 1; 2; \ldots; 8; 9\}$ with equal probability. $CR(A)$ is a positive linear transformation of $\lambda_{\text{max}}(A)$. $CR(A) \geq 0$ and $CR(A) = 0$ if and only if $A$ is consistent. Saaty recommended the rule of acceptability $CR < 0.1$.

The idea that larger $\lambda_{\text{max}}$ indicates higher ($CR$) inconsistency led (Shiraishi et al., 1998; Shiraishi and Obata, 2002) to introduce variables for missing elements, arranged in vector $x$ and consider the eigenvalue optimization problem

$$\min_{x > 0} \lambda_{\text{max}}(A(x))$$

in order to find a completion that minimizes the maximal eigenvalue, or, equivalently, $CR$.

Extension of distance-based weighting methods to the incomplete case seems to be straightforward: in calculating the optimal weights, only the known terms are considered in the objective function (Kwiesielewicz, 1996; Bozóki et al., 2010).

Bozóki et al. (2010) discuss the question of uniqueness of the optimal solution for $EM$ and $LLSM$ in the incomplete case, solve the $LLSM$ problem\footnote{See also Kaiser and Serlin (1978).} and propose an algorithm for finding the best completion of an incomplete pairwise comparison matrix according to $EM$. We will use their results extensively.

This paper investigates a special class of preferences described by incomplete pairwise comparison matrices (Section 2), for which some natural rankings of the alternatives exists. Section 3 reveals that $LLSM$ does not result in one of these orders. Section 4 presents that $EM$ do not meet the required condition either, moreover, the ranking depends on the representation chosen. They are the main results of our paper. Finally, in Section 5 we pose some related questions.

## 2 Linear order preservation

Sometimes the decision maker is unable to give a numerical answer to the question 'How many times is the $i$th alternative more important/better/favorable than the $j$th?'. However, he/she can provide an ordinal information such as the $i$th alternative is preferred to the $j$th. In this model, incomplete pairs (missing comparisons) are allowed but draws are excluded: when the $i$th and the $j$th alternatives have been compared, the $i$th or the $j$th is preferred to the other.

**Definition 6.** *Ordinal pairwise comparison matrix:* Incomplete pairwise comparison matrix $A = [a_{ij}]$ of size $n \times n$ is an *ordinal pairwise comparison matrix* if $a_{ii} = 1$ for all $i = 1, 2, \ldots, n$, and for all $i \neq j$, $a_{ij} \in \{b; 1/b\}$ or both $a_{ij}$ and $a_{ji}$ are missing. $b > 1$ is an arbitrarily fixed real number.

Note that the value $b > 1$ corresponds to the (strict) preference relation between the alternatives.
Ordinal pairwise comparison matrices can be represented by directed graphs. Let \( A \) be an ordinal pairwise comparison matrix of size \( n \times n \). Then \( G := (V, E) \) where \( V = \{1, 2, \ldots, n\} \), the vertices correspond to the alternatives, and \( E = \{e(i, j) : a_{ij} = b, i \neq j\} \), there is a directed edge from vertex \( i \) to vertex \( j \) if and only if the \( i \)th alternative is preferred to the \( j \)th. The directed graph associated to an ordinal pairwise comparison matrix \( A \) is independent of the value \( b > 1 \).

Note that different choice of the parameter \( b > 1 \) is equivalent to take a corresponding positive power of \( A \) by elements. In other words, the associated directed graph is the same for every \( A^{(h)} = [a_{ij}^h] \), \( h > 0 \).

**Definition 7.** Existence of a linear order of the alternatives: Let \( A = [a_{ij}] \) be an ordinal pairwise comparison matrix of size \( n \times n \). There exists a linear order of the alternatives if there is a permutation \( \sigma : \{1; 2; \ldots; n\} \rightarrow \{1; 2; \ldots; n\} \) on the set of alternatives such that \( C = [c_{ij}] \) is the permuted ordinal pairwise comparison matrix given by \( c_{ij} = a_{\sigma(i)\sigma(j)} \) for all \( i, j = 1, 2, \ldots, n \) and \( c_{ij} = b \) if \( i < j \) and \( c_{ij} \) is known.

Existence of a linear order of the alternatives means that the ordinal pairwise comparison matrix can be permuted such that every known value above the diagonal is \( b > 1 \). Regarding the directed graph representation, it is equivalent to acyclicity.

The following condition concerns the weighting methods for ordinal pairwise comparison matrices. A similar requirement has been introduced by [Bana e Costa and Vansnick (2008)] called **Condition of Order Preservation** (COP), however, it is defined on complete pairwise comparison matrices and takes into account the intensity of preferences.

**Definition 8.** Linear order preservation (LOP): Let \( A = [a_{ij}] \) be an ordinal pairwise comparison matrix of size \( n \times n \) such that there exists a linear order of the alternatives. It can be assumed without loss of generality that \( a_{ij} = b \) if \( i < j \) and \( a_{ij} \) is known. A weighting method associating a vector \( w(A) \in \mathbb{R}_+^n \) to \( A \) satisfies linear order preservation if \( w_i(A) \geq w_j(A) \) for all \( i < j \) such that \( a_{ij} \) is known (\( a_{ij} = b \)).

In an ordinal pairwise comparison matrix exhibiting a linear order of the alternatives, there exist some 'natural rankings'. Linear order preservation requires that the ranking according to the weighting method examined always corresponds to one of them.

Note that a weighting method associating the same weight for each alternative meets the property LOP.

### 3 Linear order preservation and the Logarithmic Least Squares Method

In this section it will be scrutinized whether LLSM satisfies the property LOP.

**Notation 2.** \( y(A) \in \mathbb{R}^n \) is given by \( y_i(A) = \log w_i^{LLSM}(A) \) for all \( i = 1, 2, \ldots, n \).

**Proposition 1.** Let \( A = [a_{ij}] \) be an ordinal pairwise comparison matrix. \( w^{LLSM}(A) \) is unique if and only if the directed graph associated to \( A \) is weakly connected: for all \( k, \ell = 1, 2, \ldots, n \), there exists a sequence of alternatives \( k = n_0, m_1, \ldots, m_{t-1}, m_t = \ell \) such that \( a_{n_s-1\ell_s} \) is known for all \( s = 1, 2, \ldots, t \).

Then the ranking of alternatives is independent of the value of \( b > 1 \), that is, \( w_i^{LLSM}(A) \geq w_j^{LLSM}(A) \iff w_i^{LLSM}(A^{(h)}) \geq w_j^{LLSM}(A^{(h)}) \) for all \( i, j = 1, 2, \ldots, n \) and \( h > 0 \) where \( A^{(h)} = [a_{ij}^h] \).
Proof. The necessary and sufficient condition for uniqueness is given by Bozóki et al. (2010, Theorem 4).

\[ y(A) = D(A)r(A) \]

where \( r(A) = \left[ \sum_{j:i,j \text{is known}} \log a_{ij} \right] \in \mathbb{R}^n \), so \( r(A(h)) = hr(A) \) and \( D(A) \) depends only on the positions of known comparisons but is not affected by their values (Bozóki et al., 2010, Remark 3). Therefore \( y(A(h)) = hy(A) \), which proves Proposition 1.

Since linear order preservation is based on the directed acyclic graph representation of an ordinal pairwise comparison matrix, Proposition 1 provides that it is meaningful to question whether LLSM satisfies LOP.

**Corollary 1.** It does not depend on the choice of \( b > 1 \) whether LLSM satisfies LOP or not. In other words, LLSM gives the same ranking for every ordinal pairwise comparison matrix associated to a given directed acyclic graph.

Intuition appears to suggest that LLSM satisfies LOP. The first of our main results contradicts this expectation.

**Theorem 1.** LLSM violates LOP.

**Proof.** It is provided by Example 2.

---

**Example 2.** Consider the directed acyclic graph on Figure 1. The associated ordinal pairwise comparison matrix \( A \) is as follows:

\[
A = \begin{pmatrix}
1 & b & * & * & * & b & b \\
1/b & 1 & b & b & * & * & * \\
* & 1/b & 1 & b & b & * & * \\
* & 1/b & 1/b & 1 & b & b & * \\
* & * & 1/b & 1/b & 1 & b & b \\
1/b & * & * & 1/b & 1/b & 1 & * \\
1/b & * & * & * & 1/b & * & 1 \\
\end{pmatrix}
\]
where $b > 1$.

In Example 2, property $LOP$ is satisfied if $w_1(A) \geq w_2(A), w_1(A) \geq w_6(A), w_1(A) \geq w_7(A)$ as well as $w_i(A) \geq w_{i+1}(A)$ and $w_i(A) \geq w_{i+2}(A)$ for all $i = 2, 3, 4, 5$.

However, $LLSM$ results in

$$y(A) = \begin{bmatrix} 34 & 36 & 24 & 1 & -14 & -42 & -39 \end{bmatrix}^T \log b/49,$$

namely, $w_1^{LLSM}(A) < w_2^{LLSM}(A)$, in contradiction with preservation of linear order.

**Remark 1.** Example 2 is minimal regarding the number of alternatives (7) and among them, with respect to the number of known comparisons (11). However, there exist more than ten examples with 7 alternatives, and some of them contains only 11 known comparisons.

**Remark 2.** There exist some examples to Theorem 1 with 8 alternatives and 10 known comparisons. Two of them are presented in Example 3.

**Figure 2:** The directed acyclic graphs of Example 3

(a) Graph of matrix $A$

(b) Graph of matrix $A'$

**Example 3.** Consider the directed acyclic graphs on Figure 2 and the associated ordinal pairwise comparison matrices $A$ and $A'$. $LLSM$ gives

$$y(A) = \begin{bmatrix} 95 & 103 & 43 & 43 & -17 & -65 & -113 & -89 \end{bmatrix}^T \log b/128,$$

$$y(A') = \begin{bmatrix} 71 & 95 & 47 & -1 & 7 & -53 & -53 & -113 \end{bmatrix}^T \log b/128,$$

where $w_1^{LLSM}(A) < w_2^{LLSM}(A)$ and $w_4^{LLSM}(A) < w_5^{LLSM}(A)$, in contradiction with preservation of linear order.

The violation of $LOP$ can be arbitrarily ‘strong’ regarding the difference of the weights of the alternatives involved.

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$^2$ It can be verified by brute force, examining all ordinal pairwise comparison matrices up to size $6 \times 6$. It is possible because of Corollary 1 which implies that comparisons above the diagonal may have essentially two ‘values’, known or missing. There exist $2^{15} = 32768$ acyclic directed graphs of size $6 \times 6$. 

6
Theorem 2. For every $K \in \mathbb{R}_+$ there exists an ordinal pairwise comparison matrix $A = [a_{ij}]$ exhibiting a linear order of the alternatives with a given $b > 1$ such that $w_i^{LLSM}(A) - w_j^{LLSM}(A) \leq -K$ for some $i < j$.

Proof. It is provided by Example 3 for any $k \geq 2$.

Figure 3: The directed acyclic graph of Example 3

Notation 3. In a directed acyclic graph, $S \rightarrow T$ if and only if there exists an edge from every vertex $i \in S$ to every vertex $j \in T$.

Example 4. Consider the family of directed acyclic graphs with $n = km + 2$ vertices where $C_i = \{(i-1)k+3, (i-1)k+4, \ldots, ik+2\}$ for all $i = 1, 2, \ldots m$ (so $C_i$ has $k$ elements) such that the edges are given by $\{1\} \rightarrow \{2\}, \{1\} \rightarrow C_m, \{2\} \rightarrow C_1$ and $C_i \rightarrow C_{i+1}$ for all $i = 1, 2, \ldots m - 1$. Figure 3 shows a member of this family if $k = 2$ and $m = 3$, that is, $C_1 = \{3, 4\}, C_1 = \{5, 6\}$ and $C_3 = \{7, 8\}$.

The directed graphs of Example 4 are weakly connected for any pair of $k$ and $m$. LLSM weights can be obtained as the solution of the following system of linear equations (note that alternatives of $C_i$ have the same weight $y_{C_i}$ since the LLSM weight vector is unique and alternatives of $C_i$ are symmetric):

\begin{align*}
(k+1)y_1 - y_2 - ky_{C_m} &= k+1; \quad \text{(for the 1st alternative)} \tag{1} \\
(k+1)y_2 - y_1 - ky_{C_1} &= k-1; \quad \text{(for the 2nd alternative)} \tag{2} \\
(k+1)y_{C_1} - y_2 - ky_{C_2} &= k-1; \quad \text{(for alternatives in $C_1$)} \tag{3} \\
2ky_{C_i} - ky_{C_{i-1}} - ky_{C_{i+1}} &= 0; \quad \text{(for alternatives in $C_i$, $i = 2, 3, \ldots, m - 1$)} \tag{4} \\
(k+1)y_{C_m} - y_1 - ky_{C_{m-1}} &= -(k+1). \quad \text{(for alternatives in $C_m$)} \tag{5}
\end{align*}

For instance, the 2nd alternative has $k + 1$ known comparisons: with the first, and with all $k$ alternatives of $C_1$, the 2nd being favorable in the latter $k$ of these. The derivation can be found in Bozóki et al. (2010).

Subtract 2 from 1 in order to get

\[(k+2)(y_1 - y_2) - k(y_{C_m} - y_{C_1}) = 2. \tag{6}\]
The difference of equations (3) and (5) gives,
\[(y_C - y_m) - (y_2 - y_1) - k (y_2 - y_C) - k (y_m - y_{m-1}) = 2k.\] (7)

It follows from equations (4) that
\[y_2 - y_1 = y_m - y_{m-1} = \frac{y_m - y_1}{m - 1}.\] (8)

Equations (7) and (8) lead to
\[(y_C - y_m) - (y_2 - y_1) + \frac{2k}{m-1} (y_C - y_m) = 2k,
which results in
\[y_C - y_m = \frac{m-1}{2k + m-1} [2k + (y_2 - y_1)].\] (9)

Substituting (9) into (6) gives
\[(k + 2)(y_1 - y_2) = 2 - k (y_C - y_m) = 2 - \frac{k(m-1)}{2k + m-1} [2k + (y_2 - y_1)].
After some calculation we infer
\[\frac{(k + 2)(2k + m - 1)(y_1 - y_2) = 4k + 2m - 2 - 2k^2(m - 1) - k(m - 1)(y_2 - y_1);}{(2k^2 + 4k + 2m - 2)(y_1 - y_2) = -2k^2(m - 1) + 4k + 2m - 2.}
It means that
\[y_1 - y_2 = \frac{-k^2(m - 1) + 2k + m - 1}{k^2 + 2k + m - 1},\]
so \[\lim_{m \to \infty} (y_1 - y_2) = -\infty\] for any \(k \geq 2\). Hence \(w_2^{LSTM} - w_1^{LSTM}\) can be arbitrarily large independent of \(b^3\).

Remark 3. In Example 4 \(w_1^{LSTM} - w_2^{LSTM} > 0\) if \(k = 2\) and \(m = 2\) (so there are 6 alternatives), but \(w_1^{LSTM} - w_2^{LSTM} < 0\) if \(k = 3\) and \(m = 2\) or \(k = 2\) and \(m = 3\) (so there are 8 alternatives with 13 and 16 known comparisons, respectively). This family of directed acyclic graphs does not give an example with 7 alternatives as Example 2 does.

4 Linear order preservation and the Eigenvector Method

In this section we examine whether \(EM\) satisfies the property \(LOP\) or not.

Proposition 2. Let \(A = [a_{ij}]\) be an ordinal pairwise comparison matrix. \(w^{EM}(A)\) is unique if and only if the directed graph associated to \(A\) is weakly connected: for all \(k, \ell = 1, 2, \ldots, n\) there exists a sequence of alternatives \(k = m_0, m_1, \ldots, m_{t-1}, m_t = \ell\) such that \(a_{m_{s-1}m_s}\) is known for all \(s = 1, 2, \ldots, t\).

Proof. See Bozóki et al. (2010) Theorem 2. \(\square\)

\(^3\) It is trivial if \(b\) can vary as \(\log b\) may also be arbitrarily large.
**Theorem 3.** *EM may violate LOP.*

**Proof.** Consider the directed acyclic graph on Figure 2. An associated ordinal pairwise comparison matrix $A$ is as follows:

$$
A = \begin{pmatrix}
1 & 3 & * & * & * & * & 3 & 3 \\
1/3 & 1 & 3 & 3 & * & * & * & * \\
* & 1/3 & 1 & * & 3 & 3 & * & * \\
* & 1/3 & * & 1 & 3 & 3 & * & * \\
* & * & 1/3 & 1/3 & 1 & * & 3 & 3 \\
* & * & 1/3 & 1/3 & * & 1 & 3 & 3 \\
1/3 & * & * & * & 1/3 & 1/3 & 1 & * \\
1/3 & * & * & * & 1/3 & 1/3 & * & 1
\end{pmatrix}.
$$

Property $LOP$ is satisfied if $w_1(A) \geq w_2(A)$, $w_1(A) \geq w_6(A)$, $w_2(A) \geq w_3(A)$, $w_2(A) \geq w_4(A)$ as well as $w_i(A) \geq w_j(A)$ for all $i = 3, 4$ and $j = 5, 6$; $i = 5, 6$ and $j = 7, 8$.

However, $EM$ results in $w^{EM}(A) = [0.2404 0.2442 0.1481 0.1481 0.0729 0.0729 0.0367 0.0367]^T$, that is, $w_1^{EM}(A) < w_2^{EM}(A)$, in contradiction with preservation of linear order. $\square$

A parallel of Corollary 1 is not true in the case of $EM$, it may give a different ranking for a certain ordinal pairwise comparison matrix corresponding to the same directed acyclic graph.

**Proposition 3.** Let $A = [a_{ij}]$ be an ordinal pairwise comparison matrix representing a directed acyclic graph. The ranking of the alternatives according to $EM$ depends on the value of $b > 1$.

Note that it does not mean the violation of linear order preservation if the $a_{ij}$ is missing. For instance, the relative ranking of alternatives 6 and 7 can be arbitrary in Example 2.

**Proof.** Consider the directed acyclic graph on Figure 2. Besides $A$, another representation by ordinal pairwise comparison matrix $A'$ is as follows:

$$
A' = \begin{pmatrix}
1 & 4 & * & * & * & * & 4 & 4 \\
1/4 & 1 & 4 & 4 & * & * & * & * \\
* & 1/4 & 1 & * & 4 & 4 & * & * \\
* & 1/4 & * & 1 & 4 & 4 & * & * \\
* & * & 1/4 & 1/4 & 1 & * & 4 & 4 \\
* & * & 1/4 & 1/4 & * & 1 & 4 & 4 \\
1/4 & * & * & * & 1/4 & 1/4 & 1 & * \\
1/4 & * & * & * & 1/4 & 1/4 & * & 1
\end{pmatrix}.
$$

$EM$ gives

$$
w^{EM}(A') = [0.2828 0.2656 0.1404 0.1404 0.0594 0.0594 0.0260 0.0260]^T,
$$

thus $w_1^{EM}(A) < w_2^{EM}(A)$ but $w_1^{EM}(A') > w_2^{EM}(A')$, which verifies Proposition 3. $\square$

**Remark 4.** There exists an example with 6 alternatives and 5 known comparisons (which is minimal provided weak connectedness) demonstrating Proposition 3. It is the smallest in the number of alternatives.
5 Conclusion

Logarithmic Least Squares Method seems to give a counter-intuitive ranking of the alternatives for some incomplete pairwise comparison matrices representing preferences described by a directed acyclic graph. The ranking according to the Eigenvector Method may also contradict to the natural ranking order, while it depends on the correspondence chosen for these preferences, too.

Our results open at least three topics for future research:

1. How can one characterize the set of ordinal pairwise comparison matrices with a linear order of the alternatives for which $LLSM$ obeys $LOP$? \footnote{One may realize that all counterexamples have a certain structure, visible on the family of directed acyclic graphs according to Example 0 the first alternative is preferred to the second and some with a low rank, while the second is preferred to some others with a high rank.}

2. What is the appropriate representation of preferences given by a directed acyclic graph such that the ranking of the alternatives according to $EM$ is well-defined?

3. Which weighting methods perform well with respect to the condition $LOP$?

References


