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An asymptotic test for the Conditional Value-at-Risk

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Abstract

Conditional Value-at-Risk (equivalent to the Expected Shortfall, Tail Value-at-Risk and Tail Conditional Expectation in the case of continuous probability distributions) is an increasingly popular risk measure in the fields of actuarial science, banking and finance, and arguably a more suitable alternative to the currently widespread Value-at-Risk. In my paper, I present a brief literature survey, and propose a statistical test of the location of the CVaR, which may be applied by practising actuaries to test whether CVaR-based capital levels are in line with observed data. Finally, I conclude with numerical experiments and some questions for future research.

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Keywords: risk measures, Conditional Value-at-Risk, hypothesis testing, actuarial science

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1 Introduction

In mathematical terms, a risk measure is a mapping between the set of probability distributions and the set of real numbers. In actuarial science and finance, the aim of applying risk measures is to quantify the underlying uncertainty of losses. One of the earliest measures of risk in the literature is the standard deviation, which was used by Markowitz in his famous portfolio optimization model ((17)). Even though it is well-suited for optimization, it is not a particularly good measure of risk as it is not monotone and it makes no distinction between downside and upside risk. Actuarial premium principles (see e.g. (12)) may also be conceived of as special risk measures. Nowadays, the most popular risk measure applied by the actuarial and financial professions is the Value-at-Risk (VaR or V@R), which corresponds to a quantile of the probability distribution of losses on a specific time horizon. Despite its simplicity of interpretation, it has been shown (see e.g. (4) or (23)) that VaR has several undesirable properties: it fails to take the size of any possible losses beyond the VaR into account, thus it is not robust enough and easily manipulated, and in a counter-intuitive way, it is not sub-additive in general, and it is not particularly well-suited for optimization problems due to the possible lack of convexity.

More recently, a popular alternative to VaR, the so-called Conditional Value-at-Risk (CVaR or CV@R) has emerged and gained significant and increasing popularity in academic circles as well as among practitioners of actuarial science and finance. In the important case of

Actually, the standard deviation and several actuarial premium principles do not qualify as risk measures according to a more rigorous definition (see e.g. (4)) used by some authors.

continuous probability distributions, CVaR is equivalent to the Tail Conditional Expectation (TCE), Expected Shortfall (ES) and Tail Value-at-Risk (TVaR or TV@R), which are all equal to the conditional expectation of the loss variable given that the loss exceeds a particular quantile of the distribution. By taking the entire tail of the distribution into account, CVaR is markedly more robust than VaR. Coherence was defined by Arztner et al. (see (4)) as a a set of intuitively desirable properties of a risk measure, and it has been shown that CVaR – as opposed to VaR – is a coherent risk measure (see e.g. (2), (4) or (9)). Additionally, CVaR is more suitable for optimization problems due to its convexity (see e.g. (19) or (3)).

Possible actuarial applications of CVaR include, among others, premium and capital requirement calculations for insurance companies and pension funds as well as asset portfolio optimization for the investment of reserves and surpluses in financial markets. The use of CVaR has been recommended by the American Academy of Actuaries and the Canadian Institute of Actuaries for statutory balance sheet valuation based on the stochastic modeling of liabilities (see e.g. (1), (7) and (8)). Even though the Solvency II legislation specifies solvency capital requirements of European insurers in terms of VaR, the possibility of introducing CVaR has been examined by the European Insurance and Occupational Pensions Authority (see (11)), and the current trend in the literature implies that a shift from VaR to CVaR in practical applications is not at all unlikely to take place in the future.

2 Theoretical and sample VaR and CVaR for continuous distributions

Let X be a continuously distributed random variable that represents a loss and has the cumulative distribution function $F(x) \doteq P(X < x)$ $(x \in \mathbb{R})$. Then the VaR and CVaR of X associated with the confidence level $0 < p < 1$ are given by the following formulas:

$$
\begin{aligned}
VaR_p(X) &= F^{-1}(p), \\
CVaR_p(X) &= E(X|X > F^{-1}(p)).
\end{aligned}
$$

Let $\{X_i\}_{i=1}^n$ denote an independent and identically distributed (IID) sample from the continuous probability distribution having cumulative distribution function $F(x)$ and probability density function $f(x) \doteq \frac{d}{dx}F(x)$ $(x \in \mathbb{R})$. Furthermore, let $\{Y_i\}_{i=1}^n$ denote the same sample sorted in an ascending order, i.e., Y_j $(j = 1, 2, \ldots, n)$ is the value of the j-th smallest observation in the sample $\{X_i\}_{i=1}^n$. I shall assume for the sake of simplicity that the probability $0 < p < 1$ is chosen so that np is an integer. In this case, the two most commonly used estimators for VaR and CVaR are the sample quantile and the tail sample mean given by

$$
\begin{aligned}
\widehat{VaR}_p(X) &\doteq Y_{np}, \\
\widehat{CVaR}_p(X) &\doteq \frac{1}{n(1-p)} \sum_{i=np+1}^{n} Y_i.
\end{aligned}
$$

This formulation using the right tail of the distribution is only appropriate for loss distributions, whereas the left tail needs to be used for returns. $CVaR_p(X)$ may not exist for some heavy-tailed distributions, however, it always exists if X has a finite second moment.

This simplification is reasonable as my aim is to derive asymptotic results in large samples, where the effect of rounding is negligible.

The distribution of sample quantiles may easily be derived analytically by noting that $Y_j < x$ $(1 \leq j \leq n, x \in \mathbb{R})$ holds if and only if at least j values in the sample $\{Y_i\}_{i=1}^n$ are less than x (see e.g. (20)):

$$\begin{aligned} F_{Y_j}(x) &\triangleq P(Y_j < x) = \sum_{k=j}^{n} \binom{n}{k} F^k(x)(1 - F(x))^{n-k}, \\ f_{Y_j}(x) &\triangleq \frac{d}{dx} F_{Y_j}(x) = \frac{n!}{(j-1)!(n-j)!} F^{j-1}(x) f(x)(1 - F(x))^{n-j}. \end{aligned}$$

Nevertheless, these formulas quickly become computationally intractable for large values of n, which is the case of particular interest to us. As a reasonable approximation for large samples, it has been shown in the literature that both the sample VaR and CVaR have normal asymptotic distributions (see (13) and (16)). More precisely, the following asymptotic relationships hold as $n \to \infty$:

$$\begin{aligned} \sqrt{n}(\widehat{VaR}_p(X) - VaR_p(X)) &\xrightarrow{D} N\left(0, \frac{p(1-p)}{f^2(VaR_p(X))}\right), \qquad (1) \\ \sqrt{n}(\widehat{CVaR}_p(X) - CVaR_p(X)) &\xrightarrow{D} N\Big(0, \frac{Var(X \,|X > VaR_p(X)) + p(CVaR_p(X) - VaR_p(X))^2}{1-p}\Big). \end{aligned}$$

These results imply that the sample quantile $\widehat{VaR}_p(X)$ and the tail sample mean $\widehat{CVaR}_p(X)$ are consistent estimators of the VaR and CVaR.

Additionally, sample VaRs of different confidence levels p and q $(0 < p < q < 1)$ have a bivariate normal asymptotic joint distribution (see (13)) with limiting correlation coefficient

$$\lim_{n \to \infty} Corr(\widehat{VaR}_p(X), \widehat{VaR}_q(X)) = \sqrt{\frac{p(1-q)}{q(1-p)}},$$

Here $\xrightarrow{D}$ denotes convergence in distribution and $N(\mu, \sigma^2)$ denotes the normal distribution with mean μ and variance σ^2 (later on, the symbol $N(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ will be used to denote the multivariate normal distribution with mean vector $\boldsymbol{\mu}$ and covariance matrix $\boldsymbol{\Sigma}$). The asymptotic variance of the tail sample mean is guaranteed to exist if X has a finite third moment.

and the sample VaR and CVaR for the same confidence level $0 < p < 1$ also follow a bivariate normal asymptotic joint distribution (see (16)) with limiting covariance

$$\lim_{n\to\infty} nCov(\widehat{VaR}_p(X), \widehat{CVaR}_p(X)) = p\frac{CVaR_p(X) - VaR_p(X)}{f(VaR_p(X))}.$$

Besides the tail sample mean, other nonparametric estimators for the CVaR have been proposed in the literature. Based on the observation that kernel smoothing increases the precision of VaR estimates, a kernel smoothed version of the tail sample mean was proposed analogously by Scaillet in (22). Nevertheless, Chen argues in (6) that this estimator does not increase the precision of CVaR estimates, and recommends the use of the simple tail sample mean instead. Another, more intricate kernel estimator obtained by inverting the weighted double kernel local linear estimate of the conditional distribution function was proposed by Cai and Wang in (5), who showed their estimator to be consistent and asymptotically normally distributed, similarly to the tail sample mean. Lan et al. presented a two-stage simulation approach based on simulated financial scenarios in (15) to compute scenario-based confidence intervals for the CVaR. The estimation of the CVaR based on Efron's bootstrap method ((10)) was explored in (14).

Most authors agree that the tail sample mean is a reliable estimator of the CVaR in sufficiently large samples, and it has the advantage of having a closed-form formula for its variance. In the next section, I shall present a large-sample statistical test of the location of the CVaR based on the tail sample mean.

3 An asymptotic test for the location of the CVaR

In this section, I present a large-sample statistical test to decide upon the hypothesis

$$H_0 : CVaR_p(X) = c \quad (c \in \mathbb{R})$$

given a confidence level $0 < p < 1$, a significance level $0 < \alpha < 1$ and an ordered sample $\{Y_i\}_{i=1}^n$ of an IID sample $\{X_i\}_{i=1}^n$, where the sample size n is sufficiently large, from a continuous probability distribution having a finite third moment.

This problem is conceptually similar to testing the null hypothesis

$$H_0 : VaR_p(X) = c \quad (c \in \mathbb{R})$$

of the location of the VaR. In the latter case, the null hypothesis is equivalent to

$$H_0 : P(X < c) = p,$$

which can be tested using the proportions test statistic

$$U \circeq \frac{\#\{Y_i : 1 \leq i \leq n, Y_i < c\} - np}{\sqrt{np(1-p)}}$$

from elementary statistics textbooks. The asymptotic distribution of the U statistic is standard normal under the null hypothesis due to the De Moivre-Laplace theorem.

Before introducing an appropriate test statistic for the CVaR, I include a lemma on the delta method (see (18) for a presentation and a proof), which is a powerful tool to compute the limiting distribution of a function of an asymptotically multivariate normal random vector, in order to facilitate the proof of the standard normal asymptotic distribution of the new test statistic under the null hypothesis:

The alternative hypothesis may be both one-sided or two-sided.

Lemma. (*Delta method*)

If $\boldsymbol{\theta}$ *is an* r*-dimensional random vector which satisfies for some* $\boldsymbol{b} \in \mathbb{R}^r$ *and* $\boldsymbol{V} \in \mathbb{R}^{r \times r}$ *that*

$$\sqrt{n}(\boldsymbol{\theta} - \boldsymbol{b}) \xrightarrow{D} N(\boldsymbol{0}, \boldsymbol{V}) \quad \text{as} \quad n \to \infty$$

and the function $g : \mathbb{R}^r \to \mathbb{R}$ *is continuously differentiable at* $\boldsymbol{b}$ *then*

$$\sqrt{n}(g(\boldsymbol{\theta}) - g(\boldsymbol{b})) \xrightarrow{D} N(\boldsymbol{0}, \nabla g(\boldsymbol{b})^T \boldsymbol{V} \nabla g(\boldsymbol{b})) \quad \text{as} \quad n \to \infty. \tag{2}$$

Now I introduce the desired test statistic and its asymptotic distribution under the null hypothesis in the following theorem:

Theorem. (*Test statistic for the CVaR*)

Assuming that the random variable X *has a finite third moment and the null hypothesis* $H_0 : CVaR_p(X) = c$ $(c \in \mathbb{R})$ *is true, it holds for the test statistic*

$$Z \mathrel{\hat{=}} \sqrt{n(1-p)} \frac{\widehat{CVaR}_p(X) - c}{\sqrt{\frac{\sum_{i=np+1}^{n} (Y_i - \widehat{CVaR}_p(X))^2}{n(1-p)} + p(\widehat{CVaR}_p(X) - \widehat{VaR}_p(X))^2}}$$

that $Z \xrightarrow{D} N(0, 1)$ *as* $n \to \infty$.

Proof.

First of all, I assume that the dimension parameter r, the vectors $\boldsymbol{\theta}$ and $\mathbf{b}$, the variable vector $\mathbf{x} \in \mathbb{R}^r$ and the function g($\mathbf{x}$) in the delta method are defined as follows:

$$\begin{aligned}
r &\circeq n(1-p)+2, \\
\boldsymbol{\theta} &\circeq (Y_{np}, Y_{np+1}, \ldots, Y_n, \widehat{CVaR}_p(X)), \\
\mathbf{b} &\circeq (VaR_{p-\frac{1}{n}}(X), VaR_p(X), \ldots, VaR_{\frac{n-1}{n}}(X), CVaR_p(X)), \\
\mathbf{x} &\circeq (x_0, x_1, ..., x_{n(1-p)}, y), \\
g(\mathbf{x}) &\circeq \frac{\sqrt{1-p}(y-c)}{\sqrt{\frac{\sum_{i=1}^{n(1-p)}(x_i-y)^2}{n(1-p)} + p(y-x_0)^2}}.
\end{aligned}$$

Assuming that the null hypothesis is true, the following two equalities trivially hold:

$$\begin{aligned}
g(\mathbf{b}) &= \mathbf{0}, \\
Z &= \sqrt{n}(g(\boldsymbol{\theta}) - g(\mathbf{b})). \qquad (3)
\end{aligned}$$

Additionally, the following equalities may easily be verified:

$$\begin{aligned}
\left.\frac{\partial g(\mathbf{x})}{\partial x_j}\right|_{\mathbf{x}=\mathbf{b}} &= 0 \quad (j = 0, 1, \ldots, n(1-p)), \qquad (4) \\
\left.\frac{\partial g(\mathbf{x})}{\partial y}\right|_{\mathbf{x}=\mathbf{b}} &= \frac{\sqrt{1-p}}{\sqrt{\frac{\sum_{i=np+1}^{n}(VaR_{\frac{i-1}{n}}(X) - CVaR_p(X))^2}{n(1-p)} + p(CVaR_p(X) - VaR_{p-\frac{1}{n}}(X))^2}}. \qquad (5)
\end{aligned}$$

The assumptions of the delta method hold due to the asymptotic properties of sample quantiles and tail sample means described in Section 2 and the continuous differentiability of the function g, and the variance of the tail sample mean exists due to the existence of a finite third moment, so it follows from Formulas (1), (2), (3), (4) and (5) that

$$\begin{aligned}
& \lim_{n\to\infty} Var(Z) = \\
= \ & \nabla g(\mathbf{b})^T \mathbf{V} \nabla g(\mathbf{b}) = \\
= \ & \left(\frac{\partial g(\mathbf{x})}{\partial y}\bigg|_{\mathbf{x}=\mathbf{b}} \right)^2 \lim_{n\to\infty} Var(\sqrt{n}\widehat{CVaR}_p(X)) = \\
= \ & \frac{Var(X|X > VaR_p(X)) + p(CVaR_p(X) - VaR_p(X))^2}{\frac{1}{n(1-p)}\sum_{i=np+1}^{n}(VaR_{\frac{i-1}{n}}(X) - CVaR_p(X))^2 + p(CVaR_p(X) - VaR_{p-\frac{1}{n}}(X))^2}. \quad (6)
\end{aligned}$$

The first term in the denominator of (6) is a Riemann sum that converges to

$$\begin{aligned}
& \frac{1}{n(1-p)} \lim_{n\to\infty} \sum_{i=np+1}^{n} \left(F^{-1}\left(\frac{i-1}{n}\right) - CVaR_p(X) \right)^2 = \\
= \ & \frac{1}{1-p}\int_p^1 (F^{-1}(x) - CVaR_p(X))^2 dx = \\
= \ & \frac{1}{1-p}\int_{F^{-1}(p)}^{\infty} (t - CVaR_p(X))^2 f(t) dt = \\
= \ & Var(X|X > VaR_p(X)), \quad (7)
\end{aligned}$$

and it follows from the continuity of $F^{-1}(p) = VaR_p(X)$ that

$$\lim_{n\to\infty} VaR_{p-\frac{1}{n}}(X) = VaR_p(X). \quad (8)$$

Formulas (6), (7) and (8) imply that

$$\lim_{n\to\infty} Var(Z) = 1, \quad (9)$$

so $Z \xrightarrow{D} N(0,1)$ holds due to Formulas (2) and (9). □

4 Numerical experiments

4.1 The distribution of the test statistic and the relative frequency of Type I errors

To verify the construction of the test statistic, I generated 1,000 samples of $n = 10,000$ observations per sample, using antithetic variables (see (21)), from the following distributions, which are commonly used to model claims in property and casualty insurance:

- Gamma with $\alpha = 1$, $\beta = 1$ (or equivalently, Exponential with $\lambda = 1$),
- Log-normal with $\mu = 0$, $\sigma = 1$,
- Pareto with cdf $F(x) = 1 - \left(\frac{1}{1+x}\right)^4$ $(x \in \mathbb{R})$,

and performed the proposed two-sided test using CVaR confidence level $p = 0.95$ and test significance level $\alpha = 5\%$ in every sample. For every distribution, I set the parameter c equal to the theoretical CVaR of the distribution. I arrived at the following results:

	Exponential	Log-normal	Pareto
Mean	0.03	-0.01	0.03
Variance	0.99	1.01	1
Skewness	-0.05	0.04	-0.06
Excess kurtosis	0.02	-0.03	0.09
Sig. of Jarque-Bera test	81%	86%	63%
Relative frequency of rejecting H_0	4.91%	5.05%	5.08%

Table 1: Sample properties of Z for three underlying distributions

Table (1) numerically validates the result that the distribution of the test statistic Z is approximately standard normal under the null hypothesis in large enough samples. As I expected, the relative frequency of rejecting the null hypothesis (Type I error) was very close to the significance level $\alpha = 5\%$ for every distribution.

4.2 Statistical power

As a next simulation experiment, I fixed the assumed value of the CVaR at $c = 4$, and performed the previous experiment several times using several different theoretical CVaRs close to the assumed level for every distribution. I recorded the relative frequencies of rejected null hypotheses in the experiments in order to estimate the statistical power of the test.

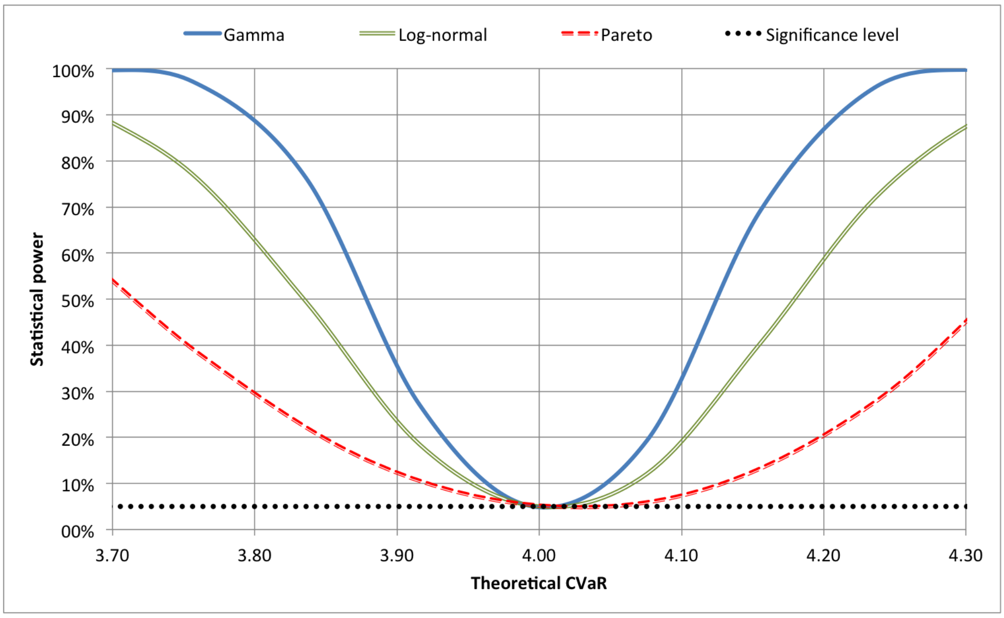

Figure 1: Statistical power for different values of the true CVaR and different underlying distributions

The results plotted in Figure 1 indicate that the power of the test strongly depends on the choice of the underlying distribution: it is highest for the Gamma distribution, which has the lightest tail, and lowest for the Pareto distribution, which has the heaviest tail out of the three selected loss distributions. It remains to be examined in a subsequent paper how the power of this test compares to that of alternative approaches.

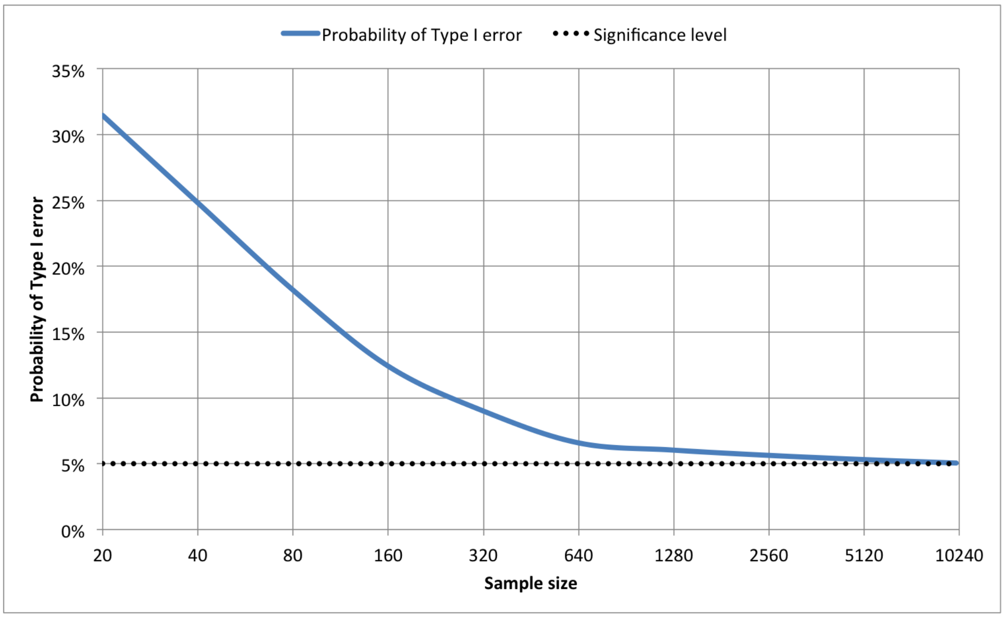

Figure 2: The probability of Type I errors for different sample sizes

For the normal and uniform distributions (not plotted), which have lighter tails than the Gamma distribution, even higher values of power were obtained in the experiments. In the case of uniform distributions, the power was nearly 100% even for hardly noticeable deviations from the assumed value of the CVaR.

4.3 Small-sample bias

Finally, I examined the effect of the sample size on the probability of Type I errors by performing the initial experiment for the Gamma distribution with the theoretical CVaR being equal to the assumed one. I repeated the experiment using several different sample sizes and recorded the relative frequencies of Type I errors. The results of this experiment plotted in Figure 2 indicate that the asymptotic approximation in the case of an underlying Gamma distribution is sufficiently precise for samples of at least 1000 observations. Therefore, the test in its presented form seems to be appropriate for markedly large samples, which are nevertheless abundant in insurance: e.g. the portfolio of all property insurance policies of an insurer. For smaller samples, the bias in the probability of Type I errors should be taken into account. I shall present an analysis of the case of smaller samples in a subsequent paper.

5 Applications

Assuming that insurance companies measure their risks associated with individual insurance policies in terms of CVaR as proposed in e.g. (1), (7), (8) and (11), an estimated CVaR for a portfolio of policies may be determined for a specific period (e.g. on a monthly or yearly basis). As new claim statistics become available in the future, the insurance company may be interested in the question whether it is reasonable to revise past risk estimates. If these risk estimates are used to determine capital requirements and the test proposed in this paper detects a significant change in the underlying risk then the company may revise its capital requirement and modify its current amount of solvency capital accordingly. If no significant change is detected then the capital requirement may remain unchanged. As mature property, life and health insurance portfolios typically number at least several thousand policies, the asymptotic standard normal approximation of the distribution of the test statistic is supposed to work sufficiently well for this problem.

Another possible application is capital requirement calculation using stochastic simulation, e.g. in life insurance, where deaths, disabilities and lapses may need to be simulated. In this case, the asymptotic approximation may be made arbitrarily precise by increasing the number of simulation runs. Besides the comparison of the performance of competing approaches and the handling of small-sample bias, actuarial applications concerning real-life numerical data will be presented in a subsequent paper.

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