CGE Modelling: A training material

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MENGTECH

MODELLING OF ENERGY TECHNOLOGIES PROSPECTIVE IN A GENERAL AND PARTIAL EQUILIBRIUM FRAMEWORK

Work Package 5

RESEARCH PROJECT N°20121

PARTNERS: CES KULEUVEN, CUB, ERASME, IER, LEPII, NTUA

Project funded
by the European Community
under the 6th Framework Programme (2006-2007)

Table of content

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Introduction

Computable General Equilibrium (CGE) modelling is an attempt to use general equilibrium theory as an operational tool in empirically oriented analyses of resource allocation and income distribution issues. Economic theory helps to understand conceptually the linkages between trade, income generation, employment, and the effect of government policies.

The distinguishing features of general equilibrium modelling derive from the Walrasian general economic equilibrium theory that considers the economy as a set of agents, interacting in several markets for an equal number of commodities under a given set of initial endowments and income distribution. Each agent defines individually his supply or demand behaviour by optimizing his own utility, profit or cost objectives. His decision yields a set of excess supply functions that fulfil the Walras law, i.e., the global identity of incomes and expenditures. Arrow and Debreu (1954), McKenzie (1954) and others have proved that under some general conditions, there exists a set of prices that bring supply and demand into equilibrium.

Computable general equilibrium (CGE) models turned the above theory into an operational model to be used for comparative static analysis. CGE models determine simultaneously changes in quantities of goods supplied and demanded, and their prices, in an aggregated multi-sectoral and multi-agent setup. Facilitated by the explicit representation of markets, the CGE models have been often extended beyond the original Walrasian framework to model market imperfections and other economic mechanisms that deviate from the original general equilibrium paradigm. For this and similar other reasons, some authors used the term “generalized equilibrium modelling” (Nesbitt, 1984) or “general equilibrium programming” (Zalai, 1982a) to underline the flexibility of the computable general equilibrium models.

Salient CGE models

CGE models have grown out of and combine different modelling traditions. The first CGE model, L. Johansen’s Multisectoral Growth (MSG) model (Johansen, 1960) was built for Norway. The MSG model was a combination of the dynamic Leontief-type (input-output) model with macroeconomic production and consumption functions, thus extending the input-output model with relative price driven substitution possibilities. Many models followed or were inspired later by Johansen’s pioneering work both in Norway (see, for example, Longva, Lorentsen and Olsen, 1985) and elsewhere (see, for example, the ORANI model in Australia, *Dixon et al.*, 1982).

In a related but somewhat different approach D. Jorgenson and his associates combined the input-output model with macro functions based on the econometric tradition (see, Hudson and Jorgenson, 1974 and 1977, Jorgenson, 1984, Jorgenson and Wilcoxen, 1990a and 1990b).

The work of Jorgenson has also inspired many modelling efforts, in which particular emphasis has been put to issues related to energy and environmental policy (see, for example, Bergman, 1988 and 1990, Capros and Ladoux, 1985, the OECD, 1994 GREEN model, Conrad and Henseler-Unger, 1986).

The 1970s and the 1980s witnessed a widespread use of CGE models for the analysis of economic development problems of the developing countries (see, e.g., Adelman and Robinson, 1978, Dervis, De Melo and Robinson, 1982, Devarajan, Lewis and Robinson, 1987). These models have enriched the CGE modelling tradition extending the focus of the previous models with elaborate treatment of foreign trade, income distribution and various policy instruments. Many of these models have further departed from the Walrasian concept by including “structuralist” features into the general equilibrium framework (see, for example Taylor and Black, 1974, Taylor and Lysy, 1979). Modellers associated with the World Bank have animated a large number of modelling projects. Their work contributed to the standardization of the CGE approach: data base cantered around the Social Accounting Matrix (see, Decaluwe and Martens, 1988) and computer software packages for handling CGE models such as like GAMS, HERCULES, MPS/CGE.

A significant source of inspiration for CGE modelling was the competitive general equilibrium interpretation of the primal-dual solutions to linear programming (LP) models of nation-wide resource allocation. LP models were extensively used in the 1960s and 1970s for economic policy analysis, both in the developing and the centrally planned economies. A distinct method that developed from that tradition was the activity analysis approach to CGE models (Ginsburgh and Waelbroeck, 1981). The development of the HUMUS model family has also taken this point of departure, interpreting CGE models as natural “general equilibrium” extensions of the LP programming-planning models (see, Zalai, 1984a).

Harberger’s (1962) early numerical two-sector model analyzing the incidence of taxation and the pioneering work of Scarf (1973) presenting the first constructive method for computing fixed points initiated another distinct trend of general equilibrium modelling. It oriented chiefly towards the study of tax policy and international trade (see, for example, Shoven and Whalley, 1972 and 1984, Scarf and Shoven, 1984, Fullerton, King, Shoven and Whalley, 1981, Pereira and Shoven, 1988). Shoven and Whalley provided a state-of-the-art methodology for model calibration and formulating multi-national market clearing mechanisms in a general equilibrium framework.

A more recent trend in computable general equilibrium modelling consists in incorporating an IS-LM mechanism (termed also macro-micro integration) which has been traditionally used in Keynesian models. Bourguignon, Branson and De Melo (1989) and others have proposed the ensuing hybrid models. These models often incorporate additional features that enhance their short or medium term analysis features, such as, for example, financial and monetary constraints and rigidities in wage setting.

Another recent development was the incorporation of economies of scale and non-competitive (oligopolistic) market structures into the CGE framework, in order to model the effects of trade liberalization and integration on micro efficiency. The forerunner of these models is a Harris's (1984) pioneering work and the partial equilibrium model of Smith and Venables (1988). In the 1990s several models carried further this line of research, including Harrison, Rutherford and Tarr (1994), Willenbockel (1994), Burniaux and Wealbroeck (1992), Capros *at al.* (1997).

Thus, in the last 20 years or so, an enormous number of practically useful CGE models have been developed to study a wide range of policy areas in which simpler, partial equilibrium tools would not be satisfactory. Equilibrium models have been used to study a variety of policy issues, including tax policies, development plans, agricultural programs, international trade, energy and environmental policies and so on. A range of mathematical formulations and model solution techniques has been used in these modelling experiences. The practice of model building itself became increasingly systematized, as reflected for instance, in the increasing use of standard and rather powerful packages such as GAMS.

The advantages of computable general equilibrium models for policy analysis compared to traditional macro-economic models are now widely admitted. The general equilibrium models allow for consistent comparative analysis of policy scenarios by standardizing their outcome around the concept of an equilibrium point, fulfilling the same consistency criteria. In addition, the computable general equilibrium models incorporate micro-economic mechanisms and institutional features within a consistent macro-economic framework, and avoid the representation of behaviour in reduced form. This allows analysis of structural change under a variety of assumptions.

Several surveys are available in various handbooks and journals from different points of time and focusing on models developed for one or other specific purpose. We call attention to a few of them. "*A Bibliography of CGE Models Applied to Environmental Issues*" by Adkins and Garbaccio (1992) contains most of the relevant literature up to the beginning of 1990s. In their conceptual, theoretical review, "*CGE Modeling of Environmental Policy and Resource Management*" Bergman and Henrekson (2003) provide a more up-to-date account on the area that is of special concern of our study: analysis of the interrelation of energy, environment and economy. Table 1 lists the models and their main characteristics they have covered in their review. It provides a useful quick orientation of the most known modelling experiments and their characteristics for the interested reader. Ghersi and Toman (2003) have also put together a very informative summary of 15 models in their paper, "*Modeling Challenges in Analyzing Greenhouse Gas Trading*" (see Table 2). Finally, Francois and Reinert eds. (1997/98) annually update their table contained in "*Applied Methods for Trade Policy Analysis: A Handbook*", the last update (2004) of their summary table available on the WEB is reproduced as Table 3.

TABLE 1: KEY CHARACTERISTICS OF SELECTED GLOBAL AND REGIONAL E3 CGE MODELS ***(BERGMAN AND HENREKSON, 2003)***

Model	Reference	Regions	Sectors per region	Dynamics	Energy sector	Backstop technology	Technological change	Environmental benefits
WW	Whalley and Wigle (1991)	6	3	Static	Top-down	No	None	Yes
GREEN	Burniaux et.al. (1992)	12	11	Quasi-dynamic	Top-down	Yes	AEEI	No
Global 2100	Manne and Richels (1992)	5	2	Fully dynamic	Bottom-up	Yes	AEEI	No
12RT	Manne (1993)	12	2	Fully dynamic	Bottom-up	Yes	AEEI	No
CRTM	Rutherford (1992)	5	3	Quasi-dynamic	Bottom-up	Yes	AEEI	No
G-Cubed	McKibbin et.al. (1995)	8	12	Fully dynamic	Top-down	No	None	No
MIT-EPPA	Yang et.al. (1996)	12	8	Quasi-dynamic	Top-down	Yes	AEEI	No
RICE	Nordhaus and Yang (1996)	13	1	Fully dynamic	Energy a single prod. sector	Yes	AEEI	Yes
IIAM	Harrison and Rutherford (1997)	5	2	Fully dynamic	Top down	No	None	No
ÚR	Babiker et.al. (1997)	26	13	Static	Top down	No	None	No

MS-MRT	Bernstein et.al. (1999)	10	6	Fully dynamic	Top down	Yes	AEEI	No
AIM	Kainuma et.al. (1999)	21	11	Quasi-dynamic	Top-down	No	AEEI	No
WorldScan	Bollen et.al. (1999)	13	11	Quasi-dynamic	Top-down	No	None	No
GEM-E3	Capros et.al (1995) (14 EU Member States and ROW)	15	18	Quasi-dynamic	Top-down	No	AEEI	Yes
BFR	Böhringer et.al. (1998) (Germany, France, UK, Italy, Spain, Denmark and ROW)	7	23	Static	Top-down	No	-	No
HRW	Harrison et.al. (1989) (US, Japan, France, Italy, UK, Ireland Germany, Netherlands, Belgium, Denmark, and ROW)	11	6	Static	Top-down	No	-	Yes

Table 2: Summary of Various Models (Ghersi and Toman, Appendix)

		Goulder Goulder (1995)	**EPPA (version 1.6)** Yang et al. (1996) Jacoby et al. (1999)	**MARKAL-MACRO** Hamilton et al. (1992)
Equity issues	regions	1 (U.S.)	12 (global)	1 (U.S.)
	sectors households	13 (6 energy-related) 1 infinitely-lived representative household	10 (7 energy-related) 1 myopic representative household	infinitely-lived single agent economy
	other	n.a.	n.a.	bottom-up energy module
Technical change	in energy	carbon liquid backstop, available 2010	• carbon liquid backstop, available 2000 • carbon-free electric backstop, av. 2000 • global constant AEEI in all non-energy sectors • global constant efficiency improvement for oil and gas supplies	AEEI differing in energy demands
	other	n.a.	n.a.	n.a.
Carbon trade	modeling	as a carbon tax (Bovenberg et al., 2000)	n.a.	as a carbon tax
	market powers	n.a.	n.a.	n.a.
	supplementarity	n.a.	n.a.	n.a.
	geographic restrictions	n.a.	no trading, Annex 1	n.a.
	CDM	n.a.	n.a.	n.a.
International linkage	Trade	• Armington specification for all goods except oil and gas Heckscher-Ohlin • zero balance constraint every period	• Armington specification for all goods except oil and gas Heckscher-Ohlin • zero balance constraint after 4 periods	n.a.
	Finance	n.a.	n.a.	n.a.

		MERGE Manne et al. (1995, 1999), http://www.stanford.edu/group/MERGE/	SGM MacCracken et al. (1999) Edmonds (1995)	G-CUBED McKibbin et al. (1995, 1999)
Equity issues	regions	5 (global), 9 (global) in MERGE 3.0	12 (global)	8 (global)
	sectors households	infinitely-lived single agent economy	13 (11 energy-related) 1 myopic (?) representative household	12 (5 energy-related) 1 hybrid representative household
	other	9 electric, 9 non-electric energy supplies in energy module	n.a.	n.a.
Technical change	in energy	• 2 carbon-free electric backstops, av. 2010 (low cost) and 2020 (high cost) • carbon liquid backstop (high price) • global constant AEEI in the aggregated sector	sector-specific exogenous growth in total productivity rate for energy sectors	• global constant AEEI • region-specific exogenous growth in total productivity rate for energy sectors
	other	n.a.	sector-specific exogenous growth in total productivity rate	region-specific exogenous growth in total productivity rate
Carbon trade	modeling	regional endowments are traded on an interregional market	as a carbon tax harmonized within trading limits	regional endowments are auctioned then traded on an interregional market
	market powers supplementarity	buyer's and seller's market cap on trade (33% of targeted reductions for net buyers)	seller's market cap on trade (10% of targeted reductions for net buyers, exact compensation in actual domestic efforts for net sellers)	n.a. n.a.
	geographic restrictions	no trading, Annex 1, global trading	no trading, double bubble, Annex 1, global trading	no trading, double bubble, Annex 1, global trading
	CDM	supplies an exogenous 15% of observed global trading transactions	global trading is provided as a limit of its benefits	n.a.
International linkage	Trade	• oil, gas, coal, and the single output, plus energy-intensive goods (EIG) in MERGE 3.0 are perfectly substitutable • carbon permits are perfectly substitutable • zero balance constraint every period • international transport priced • S/D ratio of domestic EIG provide assessment of trade impacts n.a.	• all goods perfectly substitutable except distributed gas nontradable • possibility of fixed quantities or prices • zero balance constraint after a few periods n.a.	Armington specification for all goods, with sensitivity analysis on the elasticities global investment market, perfect in OECD, constrained elsewhere
	Finance	n.a.	n.a.	n.a.

		RICE-99 Nordhaus et al. (1999a, b)	**FUND (version 1.6)** Tol (1999)	**GRAPE** Kurosawa et al. (1999)
Equity issues	regions	13 (global)	9 (global)	10 (global)
	sectors households	infinitely-lived single agent economy	non-overlapping generations single agent economy	infinitely-lived single agent economy
	other	n.a.	n.a.	bottom-up energy module
Technical change	in energy	• carbon-free energy backstop (high price) • region-specific A Carbon EI in the aggregated sectors	• global AEEI in the aggregated sector • global A Carbon EI in the aggregated sector	• AEEI in the aggregated sector • oil substitutes in transports av. 2010 • nuclear substitute available 2050
	other	region-specific exogenous growth in total factor productivity in the aggregated sector	n.a.	n.a.
Carbon trade	modeling	as a carbon tax harmonized within trading limits	as cooperation in a game-theoretic sense: sum of the welfares of the trading regions is maximized with actual regional reductions as control variables	as a carbon tax harmonized within trading limits, with a constant unit transaction cost of 1990$10 a ton
	market powers supplementarity	n.a.n.a.	n.a.cap on trade (10% of targeted reductions for net buyers, for net sellers, and for both jointly)	n.a. n.a.
	geographic restrictions	no trading, OECD, Annex 1, global trading	no trading, double bubble, Annex 1, Annex 1 and Asia, global trading	no trading, Annex 1, global trading
	CDM	n.a.	n.a.	global trading has emissions outside Annex 1 constrained to their no-trading level; CDM is not explicitly mentioned
International linkage	trade	n.a. except single output in compensation of permits	n.a.	• in single output • in energy products in the bottom-up energy module
	finance	n.a.	n.a.	n.a.

		WORLDSCAN Bollen et al. (1999), http://www.cpb.nl/nl/pub/pubs/bijzonder_20/	AIM Kainuma et al. (1999) http://www-cger.nies.go.jp/ipcc/aim/	MS-MRT Bernstein et al. (1999)
Equity issues	regions	13 (global)	21 (global)	10 (global)
	sectors	11 (4 energy-related)	11 (7 energy-related)	6 (4 energy-related)
	households	overlapping generations	1 myopic representative household	1 infinitely-lived representative household
	other?	• high and low-skilled labour • region-specific unformal (low-productivity) sectors	n.a.	n.a.
Technical change	in energy other	n.a. region- and sector-specific exogenous growth in factors productivity rate	• global constant AEEI • global constant A Carbon EI n.a.	• carbon-free backstop (high price) • AEEI growth in total factor productivity, endogenous returns on capital
Carbon trade	modeling	as a carbon tax harmonized within trading limits	regional endowments are traded on an interregional market	regional endowments are traded on an interregional market
	market powers supplementarity	n.a.cap on trade (10, 15 and 25% of targeted reductions for net buyers, and for net sellers)	n.a.n.a.	seller's market • cap on trade • ban on "hot air"
	geographic restrictions	no trading, double bubble, Annex 1, global trading	no trading, double bubble, Annex 1, global trading	no trading, Annex 1, global trading
	CDM	• financing of retrofit projects following a cost-benefit analysis with additionality constraint (cf. text) • exogenous 5% of targeted reductions	as global trading with emissions outside Annex 1 constrained to their BAU level	supplies an exogenous 15% of observed global trading transactions
International linkage	trade	Armington specification for all goods turning to Heckscher-Ohlin in the long-run	• all foreign goods perfectly substitutable • Armington specification for domestic and aggregated foreign goods	• Armington specification for all goods except oil, electricity nontradable • trade balanced over time horizon • study of terms-of-trade variations
	finance	Imperfect global investment market	perfect global investment market	• zeroed on the growth path • perfect mobility of capital

		GTEM Tulpulé et al. (1999) http://www.abare.gov.au/pdf/gtem.doc	OXFORD Cooper et al. (1999)	CETA Peck and Teisberg (1992, 1999)
Equity issues	regions	18 (global)	22 (mostly OECD), key macro variables for 50 more	2 (global)
	sectors households	16 (5 energy-related) 1 myopic representative households	infinitely-lived single agent economy	Infinitely-lived single agent economy
	other	saving decisions (forward-looking) are disaggregated in age groups	6 energy supplies, 4 energy demands in energy module for 8 regions	7 electric, 5 nonelectric energy supplies in energy module
Technical change	in energy	endogenous	n.a.	• nonelectric and electric carbon-free backstops (high prices) • global constant AEEI in aggregated sector
	other	endogenous	region-specific growth in total factor productivity, exogenous trend corrected by energy prices ("crowding-out wise")	n.a.
Carbon trade	modeling	as a carbon tax harmonized within trading limits, with impact on GNP	as a carbon tax harmonized within trading limits, with impact on GNP	Regional endowments are traded on an interregional market
	market powers	n.a.	n.a.	n.a.
	supplementarity	n.a.	n.a.	n.a.
	geographic restrictions	no trading, double bubble, Annex 1	no trading, double bubble without trade in the EU, Annex 1	Annex 1, global trading
	CDM	n.a.	n.a.	n.a.
International linkage	trade	• Armington specification for all goods • international transport priced	Armington specification for the single output	Carbon permits, the nonenergy good, oil and gas, and synthetic fuel are perfectly substitutable
	finance	imperfect global investment market	perfect global investment market	n.a.

Table 3: Calibration-based numerical trade policy models (Francois and Reiner, 2004)

Description	Software required	Partial or general equilibrium	Single or multi-region
Partial equilibrium models			
The GSIM model: (Global SIMulation model) included in the World Bank's WITS package for tariff and trade analysis, along with a short background technical paper from Francois and Hall (2002). This is a global, multi-region, partial-equilibrium model. ? GSIM4x4.XLS (a compact example model with up to 4 regions, and only import tariffs) ? GSIM25x25.XLS (a large template, for up to 25 regions, with tariffs, export taxes/subsidies, and production subsidies).	Excel	PE	MR
Perfect substitutes trade model: from Francois and Hall, Chapter 5 in Applied Methods for Trade Policy Analysis: A Handbook, J.F. Francois and K.A. Reinert, Cambridge University Press, 1997-1998.	Excel	PE	SR
Imperfect substitutes trade model: from Francois and Hall, Chapter 5 in Applied Methods for Trade Policy Analysis: A Handbook, J.F. Francois and K.A. Reinert, Cambridge University Press, 1997-1998.	Excel	PE	SR
Anti-Dumping &tc: The USITC's set of COMPAS models (including some documentation on the spreadsheet). These are used (or have been and sometimes are, depending on the political relevance of economics for any given fair trade investigation) for antidumping and countervailing duty investigations, for assessment of injury.	Excel	PE	SR
SWOPSIM: from Chapter 8 in Applied Methods for Trade Policy Analysis: A Handbook, J.F. Francois and K.A. Reinert, Cambridge University Press, 1997-1998.	Excel	PE	MR
General equilibrium Excel® models			
123 Model: Excel implementation Devarajan et al 1997, Chapter 6 in Applied Methods for Trade Policy Analysis: A Handbook, J.F. Francois and K.A. Reinert, Cambridge University Press, 1997-1998.	Excel	GE	SR
123 model for Egypt: Excel implementation with Egyptian data for 1998, J. Francois (2001).	Excel	GE	SR
123 model in steady-state: a steady-state extension (combining Chapters 6 and 12 from Applied Methods for Trade Policy Analysis: A Handbook, J.F. Francois and K.A. Reinert, Cambridge University Press, 1997-1998.	Excel	GE	SR
GE Armington model: General equilibrium extension of the Imperfect substitutes trade model from Francois and Hall,	Excel	GE	SR

with a short background technical paper from Francois and Hall (1998).			
Shipping model: International trade with imperfect competition in shipping, from Francois and Wooton (2001), “Market Structure, Trade Liberalization, and the GATS,” European Journal of Political Economy.	Excel	GE	MR
GTAP: Dominique van der Mennsbrugghe’s spreadsheet implementation of the GTAP model.	Excel	GE	MR
General equilibrium models in GAMS® and GAUSS®			
GTAP-E: A GAMS implementation of a model of international trade that includes carbon emissions. You will need access to the GTAP database (not provided here). This version is for GTAP4, which is benchmarked to 1995.	GAMS	GE	MR
IFPRI standard model: This is the standard model developed by Sherman Robinson *et al*. when at IFRPI.	GAMS	GE	SR
123 Model: GAMS implementation of the 123 model from Devarajan *et al* 1997, Chapter 6 in Applied Methods for Trade Policy Analysis: A Handbook, J.F. Francois and K.A. Reinert, Cambridge University Press, 1997-1998	GAMS	GE	SR
Single region U.S. CGE model: (from Chapter 7 of J.F. Francois and K.A. Reinert, Cambridge University Press, 1997-1998.)	GAMS	GE	SR
Transition dynamics: a model of the Austrian economy from Chapter 13 in Applied Methods for Trade Policy Analysis: A Handbook, J.F. Francois and K.A. Reinert, Cambridge University Press, 1997-1998.	GAUSS	GE	SR
Labor markets in GE: from Chapter 14 in Applied Methods for Trade Policy Analysis: A Handbook, J.F. Francois and K.A. Reinert, Cambridge University Press, 1997-1998.	GAMS	GE	SR
The Small model: a 3-region model implemented in MPSGE. This is the same aggregation used for the “SIMPLE” model below. HTML-based documentation is available.	GAMS	GE	SR
The Large Model: a multi-region, MPSGE-based model used while I was at the GATT/WTO to assess the Uruguay Round agreements. This model is SAM-based (our global SAM is included). This was published, in various forms, e.g,: Francois, J.F. B.J. McDonald, and H. Nordstrom (1996), "The Uruguay Round: A Numerically Based Qualitative Assessment," in W. Martin and A Winters, eds., The Uruguay Round and Developing Countries, Cambridge Univesity Press. This includes a number of features that were innovative for CGE models once upon a time: (1) explicit quotas (included in the file), (2) global monopolistic competition, and (3) steady-state investment effects (included in the file, and called “numeric ballistics” by Glenn Harrison when he first commented back in 1993).	GAMS	GE	MR

General equilibrium models in GEMPACK®

GTAP model (old version): from Chapter 9 in Applied Methods for Trade Policy Analysis: A Handbook, J.F. Francois and K.A. Reinert, Cambridge University Press, 1997-1998. This is a multi-region CGE model. Further documentation is available from Chapter 2 of Hertel, T. (1996), Global Trade Analysis, Cambridge University Press.	GEMPACK	GE	MR
Steady-state investment effects: from Chapter 12 in Applied Methods for Trade Policy Analysis: A Handbook, J.F. Francois and K.A. Reinert, Cambridge University Press, 1997-1998.	GEMPACK	GE	MR
Imperfect competition in GTAP: from J. Francois, Scale Economies and Imperfect Competition in the GTAP Model, GTAP Technical Paper No. 14, 1998.	GEMPACK	GE	MR
Capital accumulation in GTAP: from J. Francois, B. McDonald, and H. Nordstrom, , Liberalization and Capital Accumulation in the GTAP Model, GTAP Technical Paper No. 07, 1996.	GEMPACK	GE	MR
The SIMPLE model: This is a self-contained (i.e. executable) somewhat-dated version of the GTAP model. It includes scale economies, imperfect competition, nested- and non-nested import demand, rigid wages, and some capital mobility treatment. The idea is to follow a single experiment across different model features. HTML-based documentation is available. These examples are built on the same dataset as the "Small" model in GAMS/MPSGE linked above.	GEMPACK	GE	MR

Purpose and organization of the monograph

This monograph is part of the effort to increase the capacity to apply multisectoral models for economic policy analysis, especially lacking in the New EU Member States, where such models have been missing from both university curricula and practice. As a result, there are hardly any experts in these countries knowledgeable or experienced in multisectoral modelling methodology widely used in a variety of areas of economic policy analysis in other parts of the world. The proper use of CGE models requires substantial knowledge and skills in several fields, including economic theory, statistics and computation techniques. The monograph was tailored first of all to the needs of students, research assistants and modellers coming from this environment, but might be useful additional reading for other interested beginners in the field elsewhere too. The monograph is supplemented by various computer programmes to provide numerical illustration and models to the themes presented.

This training material has been organized into chapters and sections as follows. The first chapter reviews and deepens the reader's knowledge of the theoretical and methodical foundations of general equilibrium models necessary for the proper understanding of the strengths and weaknesses of general equilibrium analysis. It is especially designed for those who have not studied in a systematic fashion mathematical economics and the history of economic thoughts.

It starts with two abstract models developed by Walras to explain concepts such as Walras law, price homogeneity, counting equations and macro-closure that reappear in the CGE models. The use of simpler models facilitates the understanding of those concepts and provides a useful introduction to the art of CGE model building. We also touch upon the issue of the use of weak inequalities and complementary restrictions in the market clearing conditions as well as the structural and reduced forms of the models by invoking the Cassel and Schlesinger-Wald variants of the first model of Walras.

The models presented above were stylized theoretical models not intended for practical application. Unlike those models Leontief developed the first applied general equilibrium model for policy analysis, the model of interindustrial or input-output analysis. In the third section the basic concepts and equilibrium conditions of Leontief's model are introduced and discussed.

The early models of general equilibrium were holistic, macroeconomic models, not using any behavioural explanation for the determination of the choice of technology or final use. The pioneering works of Hicks and Samuelson filled this gap by merging the holistic, macro-economic framework with the neoclassical theories of firms and consumers. That approach became the framework of modern applied general equilibrium models. Section 1.4 describes the main components and conditions of general equilibrium in a model based on micro-economic foundations (differentiable production and utility functions).

The functions used in the neoclassical GE model however are not easy to estimate in the practice. Modellers therefore had to circumvent often the problem by using an alternative representation of technology and preferences based on the use of fixed coefficients and linear relationships. Koopmans and Kantorovich were awarded for laying down the theoretical and methodological foundations of applied linear economic models. In section 1.5 we discuss the basic concepts and theorems of linear activity analyses, which form the basis of the linear programming (LP) approach used extensively in the analysis of resource allocation. The nation-wide LP models can be interpreted as linear GE model. We illustrate that point by presenting a Koopmans–Kantorovich type model of general equilibrium that is the linear equivalent of the Hicks–Samuelson GE model presented in the previous section.

In section 1.6 we pave further the way leading to modern CGE models by presenting a stylized version of the first applied general equilibrium model developed by Leif Johansen. This model is a combination of Leontief-type input-output model with macroeconomic production and consumption functions, thus an input-output model extended with relative price driven substitution possibilities. Many models followed or were inspired later by Johansen's pioneering work and retained its original structure. In order to keep the model transparent, we present a prototype version of the model with no foreign trade and taxes, and with no income redistribution.

Finally, the last section summarizes the models presented and discussed in order to ease their survey and comparison.

The chapter is accompanied by two numerical examples and exercise package. The first is an Excel realization of the Cassel-model assuming 2 factors and 3 products (Cassel-2x3.xls and Cassel-2x3.doc). This exercise illustrates how the neoclassical theory works in practice and how the equilibrium solution depends on the parameters of the model. The second is an Excel illustration of the simplified Johansen CGE model (Johansen-DinLeo.xls). This program demonstrates also how simpler CGE models can be solved by iterating only with a few variables and repeatedly solving a set of simultaneous equations.

In the second chapter the main variants of applied multisectoral models macroeconomic models, the input-output, linear programming and computable general equilibrium approach discussed and compared with each other. Special emphasis is laid on their close similarity and features which link them together. The systematic review and discussion of the alternative macroeconomic multisectoral models reveals their common features and differences. The comparative analysis of the various model types is a crucial step in the explanation of the CGE approach especially for modellers coming from the former socialist countries. It enables the reader (student) to understand better the general philosophy lying behind the computable general equilibrium approach and models. In the training seminar we could see that it was considered to be very useful even for students who have acquired already some experience in building and running CGE models.

In the first section the basic concept and content of the statistical input-output tables (one of the main data source of the CGE models), and their relation to Leontief's static input-output model are reviewed. Next we discuss the alternative ways in which foreign trade can be represented in the I-O tables. The partial closure possibilities and possible extensions of the input-output models prepare the ground for the presentation of complex volume and price models, which reappear as product balance and equilibrium pricing conditions in the computable general equilibrium models as well, not only un the pure input-output models.

The second section deals with the optimal resource allocation models taking the form of a linear or nonlinear programming problem and based on an input-output technology. By means of simple models we discuss in brief how optimal resource allocation models can be used for economic policy analysis. We point out to the basic problem of the linear programming approach, namely that these models tend to produce unrealistic overspecialized solutions. The only way to constrain overspecialization in a linear model is the introduction of ad hoc bounds on certain groups of variables, which in turn distorts the shadow prices of the commodities and resources on the other hand.

We demonstrate next that the use of rigid bounds is equivalent to assume less then perfect substitution between certain pairs of commodities in use or production represented by piece-wise linear isoquants or indifference curves. Switching to smooth, differentiable curves one can arrive at a nonlinear version of the same resource allocation problem that uses 'flexible' rather than rigid bounds to constrain specialization. The nonlinear model produces much more meaningful prices for the commodities and resources appearing in the model.

It is pointed out that the first order necessary conditions of the optimal solutions of a nonlinear resource allocation model – using appropriate functional forms – resemble the conditions of general equilibrium. There can be only a few conditions which will have to be revised and changed in order to get the conditions of a perfect competitive equilibrium. Introducing taxes and subsidies in appropriate places into the set of equilibrium conditions one can arrive at a stylized CGE model.

In the third section the main building blocks of the CGE models are introduced and discussed: the commodity structure, representation of technology and production decisions, the representation of exports and imports, income distribution and final demand (consumption and investment), and market clearing equilibrium conditions. We close the section and the chapter by discussing the CES functional forms typically used in CGE models, and their calibration procedure.

This chapter is also accompanied by exercise programmes and materials. To facilitate the understanding of the characteristics of the programming models we developed an Excel program (LP-2x2-6eset-CES.xls), which computes and graphically displays the feasible set, the main functions and the optimal solution. The use of a CES welfare function helps also the user to understand the nature and role of CES functions in CGE-models. The program also illustrates the sensitivity analysis of the multisectoral macroeconomic models.

Another exercise possibility is provided by a GAMS model that distinguishes 3 sectors and 10 household groups, and was calibrated using Hungarian data for 1998 (MultHH-opt-scen.GMS). The model, by setting appropriately the value of certain parameters, can be used for the solution of both an NLP and CGE variant of the same problem and for the comparison of their behaviour. We have also developed an Excel interface for this GAMS program, which can present and compare the results of up to 7 simulation runs in a transparent Excel format. The GAMS code of this program provides thus a useful exercise that teaches the user how to present model results in Excel.

Having prepared the ground in this way, in chapter 3 we turn to the detailed presentation of the specific features of a typical GEM-E3 model in the third chapter. We go through one by one the issues related to the assumed household's, firm's and government's behaviour, domestic demand and trade flows, the equilibrium pricing identities, the representation of income distribution and redistribution, the market clearing conditions. We pay special attention to the issues related to model calibration and use in economic policy analysis.

Extensions of the GEM-E3 model include the generalization of the household utility function to take into account of the geographic variety of consumer goods, imperfect competition, the financial module determining the general price level, etc. In the training material we presented only the environmental module and discussed the possibility of representing private consumption and income generation with multiple households.

Chapter 4 describes two important extensions of the GEM-E3 model. In the first section the environment module is introduced, which represents the effects of different environmental policies on the economy and the state of the environment. It concentrates on three important environmental problems: (*i*) global warming (*ii*) problems related to the deposition of acidifying emissions, and (*iii*) ambient air quality.

Next, the three components of the environmental module are described in details. Namely, the *"behavioural"* module representing the effects of different policy instruments on the behaviour of the economic agents, a *"state of the environment"* module, and the "policy-support component", including the policy instruments related to environmental policy.

The three mechanisms that affect the level of actual emissions in the model: (*i*) end-of-pipe abatement technologies (*ii*) Substitution of fuels, and (*iii*) production or demand restructuring between sectors and countries are also explained in some details.

The presentation of the GEM-E3 model's environmental module was accompanied by a model developed for Hungary, which was based on the extension of the GEM-E3 environmental module.

The second section describes the problems that arise, when multiple households and their relationships with the labour market and income distribution are represented in a CGE model. After discussing the various socio-economic group-formation criteria, we present a neoclassical quasi-dynamic CGE-model. The model was calibrated for Hungary and distinguishes 3 sectors and 10 household groups (MULTHH.GMS program). The model contains group specific human capital (accumulated by the "productive" use of the household expenditure) and group specific financial wealth. CET-labour supply functions and alternative closures rules increase the flexibility of the model in policy analysis. This program allows also for useful practical exercises.

The fifth chapter is devoted to the issues related to the statistical background of the GEM-E3 models. In order to calibrate the parameters of a GEM-E3 model one has to compile benchmark year data on the production technology (incl. emission of air pollutants), consumption patterns, taxes, income distribution, savings and final demands. These data can be derived from various sources. The following data, their availability and method of estimation or derivation is discussed in details in this chapter:

- I-O tables and their supplementary tables, the import matrix and the matrix of indirect taxes and subsidies, import duty matrix;
- foreign trade matrices (exports and imports by commodity groups and partner countries);
- consumption transformation matrix;
- investment matrix;
- income distribution data, such as like the value added and its primary distribution /wages, social security, production taxes, production subsidies, operating surplus, income-expenditure balance sheets;
- environmental data needed for GEM-E3, for example, emission coefficients per type of activity for the pollutants considered in the model, marginal abatement cost functions for some of the pollutants, coefficients representing pollutants' transformation and transportation between countries, damage per pollutant and its monetary valuation;

- auxiliary data, such as like factor endowment data, interest rates, inflation rate in the base year, demographic data, foreign tourists domestic consumption expenditure by supplier branches and the related VAT and consumption tax, energy balance sheets, energy taxes, stocks of energy consuming durable goods, share of gasoline and gas-oil within motor-fuel demand, share of non-energetic use of the energy carriers, etc.

The GEM-E3 model distinguishes 18 branches, 13 consumption categories, the list of which and their content can be seen in the tables of Appendix 1 of the training material. The method of reclassification (aggregation) from the original break-downs to the desired break-downs can be found in Appendix 2. Since most of the above data enter into a SAM (Social Accounting Matrix) scheme, designed specially for the GEM-E3 model, this chapter gives a detailed description of the SAM and instructions how to fill it with the available data.

A separate appendix contains an extract of the SNA 1968 volume's method for the compilation of the Input-Output tables from the so called 'Make' and 'Use' tables. To illustrate this method in a simplified case, in this chapter we present an Excel-worksheet (MakeUse.XLS file) elaborated for the Hungarian CGE model. Several special programmes developed for these purposes of estimating missing data, reconciling and aggregating the available data (e.g., the flexible and general aggregation-reclassification program or the 'additive-RAS' algorithm) are also presented.

Chapter 6 describes step by step and in great details how the Hungarian data were compiled for the GEM-E3 model in order to illustrate the whole process. The compilation of the income distribution block of the SAM is discussed in the greatest detail, drawing useful conclusions for a similar process for other countries. This special emphasis is justified by the fact that the data availability and methodology of the income distribution is rather different across countries, so it is important to demonstrate how we can overcome these problems especially in new EU-countries where income distributional data are the least accurate and detailed.

Chapter 7 outlines the model implementation process. The latest versions of the GEM-E3 model involve systems of about 60,000 non-linear equations per time period. The GEM-E3 model has been successfully transformed into a mixed complementarity model and solved in GAMS using the PATH solver. Previous attempts to solve the model in other solution algorithms (as with MINOS and CONOPT) have been unsuccessful mainly due to the model's large size and complexity. The PATH solver on the other hand, has been successful in solving very large scale models and through the complementarity approach that it uses, enables the expansion of GEM-E3 to include inequalities and a separate optimisation energy sub-module.

The www.gams.com website contains the documentation and the system files of the GAMS package. The GAMS is a rather efficient and model-builder friendly software to handle and solve large nonlinear models with 'well-behaving' (twice differentiable, etc.) functions in its equations. Several sample programs are used to explain and illustrate the structure of the GAMS programmes and highlight the main syntactic rules which are important from the point of view of the GEM-E3 model's program.

1. Salient models of general equilibrium

1.1. The static Walras–Cassel model of general equilibrium

Walras modelled the exchange of commodities at macro level, as if it would take place only between two agents, one representing the households and the other the firms (producers). The exchange taking place between various households or producers is thus left out of consideration. Households own all stocks and resources, including the factors of production, and demand produced goods for consumption. At a given set of product and production factor prices they decide on the supply of factors and the demand for goods. Walras assumes that their choice can be represented by two sets of demand-and-supply functions: *demand functions* for the produced goods, $v_i(\mathbf{p}, \mathbf{r})$, $i = 1, 2,\ldots, n$ and *supply functions* of the factors of production, $s_k(\mathbf{p}, \mathbf{r})$, $k = 1, 2,\ldots, s$. The nomenclature used is the following: n is the number of products (final goods), s is the number of primary resources, $\mathbf{p} = (p_i)$ and $\mathbf{r} = (r_k)$ are the price vectors of products and primary resources, respectively.

Walras assumes that the household's demand and supply functions are *homogeneous* of degree zero (only price ratios matter) and always fulfil the *budget constraint*, that is, the total value of demand equals that of supply (the so called *Walras's law*):

$$\sum_i p_i \cdot v_i(\mathbf{p}, \mathbf{r}) = \sum_k r_k \cdot s_k(\mathbf{p}, \mathbf{r}).$$

Firms possess nothing, but merely organize production by demanding factors from households and supplying produced goods. Production technology is represented by fixed d_{kj} coefficients, which indicate how much factor k is used (on average) to produce one unit of final output j. At any given set of prices, firms produce only those commodities, the prices of which cover or exceed their cost of production. Walras assumes that no profit can be earned in perfectly competitive equilibrium, for any profit would lead to bidding up the prices of some factors of production. In equilibrium, therefore, prices have to meet the requirements of the so-called *non-profit* pricing rule:

$$p_j = r_1 \cdot d_{1j} + r_2 \cdot d_{2j} + \ldots + r_s \cdot d_{sj}, \qquad j = 1, 2,\ldots, n. \qquad \text{(WS-1)}$$

where the variables are

p_i unit price of product i ($i = 1, 2,\ldots, n$),

r_k unit price of factor k ($k = 1, 2,\ldots, s$).

The further conditions of general equilibrium in the static Walras model are the supply-demand equations on the product markets:

$$v_i(\mathbf{p}, \mathbf{r}) = y_i, \qquad i = 1, 2,\ldots, n, \qquad \text{(WS-2)}$$

where y_i is the (final) output of product i ($i = 1, 2,\ldots, n$),
and on the factor markets:

$$d_{k1} \cdot y_1 + d_{k2} \cdot y_2 + \ldots + d_{kn} \cdot y_n = s_k(\mathbf{p}, \mathbf{r}), \qquad k = 1, 2,\ldots, s. \qquad \text{(WS-3)}$$

The three sets of equations contain $2 \times n + s$ variables (unknowns). However, because of the assumed *homogeneity* of the demand and supply functions and the price formation rule, the price level is left undetermined by the above set of equations. It can be assigned any positive value, e.g., by selecting some commodity as *numeraire* good and setting its price to one. The total number of equations seems to exceed that of unknowns, and the model over-determined therefore.

We can however remove one equation by Walras's Law. Multiplying the equations with their complementing variables, y_j, p_i and r_k, respectively and summing them up, after some rearrangement we get:

$$\sum_i p_i \cdot v_i(\mathbf{p}, \mathbf{r}) = \sum_k r_k \cdot s_k(\mathbf{p}, \mathbf{r}),$$

which is the above introduced Walras's Law? In other words: if all markets clear but one, then that last one will have to clear too. Thus, we can drop one of the market-clearing conditions out of the model. Thus, the number of equations becomes also $2 \times n + s - 1$, equal to the number of the unknowns.

Although the equality of the number of equations and variables is neither necessary, nor sufficient condition for solution to exist, for Walras and his contemporaries this guaranteed that the model was consistent with the concept of equilibrium. This type of *equation counting* was not meant to prove the existence of equilibrium, as it is often falsely interpreted nowadays, but enough to prepare the ground for parametricising and calibrating the model in such a way that its solution would replicate the observed state of the economy concerned.

Cassel's variant of the static Walras model

The Swedish economist, Cassel (1918) illustrated with almost the same set of equations the concept of general equilibrium as Walras, apparently independently from him. He ignored factor supply functions and assumed that all factors are supplied inelastically by agents, or better to say, by nature (primary resources). The final demand for produced goods in his formulation was function of their prices alone:

$$p_j = r_1 \cdot d_{1j} + r_2 \cdot d_{2j} + \ldots + r_s \cdot d_{sj}, \qquad j = 1, 2, \ldots, n. \qquad \text{(C-1)}$$

$$y_i(\mathbf{p}) = y_i, \qquad i = 1, 2, \ldots, n, \qquad \text{(C-2)}$$

$$d_{k1} \cdot y_1 + d_{k2} \cdot y_2 + \ldots + d_{kn} \cdot y_n = s_k, \qquad k = 1, 2, \ldots, s. \qquad \text{(C-3)}$$

This set of conditions can be derived from a model consisting of equations (C-1), (C-2'), (C-3) and (C-4), where

$$y_i(\mathbf{p}, e) = y_i, \qquad i = 1, 2, \ldots, n, \qquad \text{(C-2')}$$

$$e = \sum_k r_k \cdot s_k, \qquad \text{(C-4)}$$

where e is the expenditure (income) of the housholds and the $y_i(\mathbf{p}, e)$ demand functions are homogeneous of degree zero and satisfy a generalized form of Walras's law: $\sum_i p_i \cdot y_i(\mathbf{p}, e) = e$. Setting $e = 1$ as numaraire, and dropping (C-4) we arrive at Cassel's form.

This model can easily be reduced further. Equations (C-1) define the p_j product prices as functions of the factor prices. Substituting the p_j variables with the resulting $p_j(\mathbf{r})$ functions in equation (C-2) we can express all y_i as functions of $\mathbf{r}$. Finally, substituting y_i variables in equation (C-3) by the resulting $y_i(\mathbf{r})$ functions the demand for factors can be expressed as functions of their own prices: $d_k(\mathbf{r})$. As a result, the conditions of equilibrium can be reduced to the market clearing equations of the production factors alone:

$$d_k(r_1, r_2, \ldots r_s) = s_k.$$

Cassel's model played crucial role in the later development of general equilibrium models. For a mathematician it was clear that the existence of solution of the above equation systems was far from a trivial mathematical problem that could be checked simply by counting equations. The examination of this problem made some Viennese scholars interested in this problem, and it was a variant of Cassel's model, proposed by *Schlesinger* (1935), within which *Wald* (1935) proved rigorously for the first time the existence of general equilibrium. In order to overcome some mathematical problems (negative factor prices) they used inequalities and complementary slackness conditions instead of the original equations in prescribing the equilibrium conditions on the factor markets:

$$d_{k1}\cdot y_1 + d_{k2}\cdot y_2 + \ldots + d_{kn}\cdot y_n \leqq s_k, \qquad k = 1, 2, \ldots, s, \qquad \text{(C-3a)}$$

i.e., in equilibrium there can be no excess demand, but

$$r_k\cdot(d_{k1}\cdot y_1 + d_{k2}\cdot y_2 + \ldots + d_{kn}\cdot y_n) = r_k\cdot s_k, \qquad k = 1, 2, \ldots, s. \qquad \text{(C-3b)}$$

the price of the oversupplied factor must be zero (rule of free goods).

Using matrix algebraic notation, $\mathbf{r} = (r_k)$, $\mathbf{D} = (d_{kj})$ and so on, we can rewrite the general equilibrium conditions of the Schlesinger–Wald model as follows:

$$\mathbf{r} \geqq \mathbf{0}, \quad \mathbf{p} = \mathbf{p}(\mathbf{y}), \quad \mathbf{p} = \mathbf{rD}, \quad \mathbf{Dy} \leqq \mathbf{s}, \quad \mathbf{rDy} = \mathbf{rs}.$$

1.2. The periodic model of Walras with capital goods

Decreasing the level of abstraction Walras introduced discrete time intervals and capital goods in the second version of his general equilibrium model. The formulation of that model gave rise to the *problem of closure*, a problem which is present in the typical CGE models as well. We will present a slightly more generalized version of this model. The goods serving for final consumption and investment will not necessarily be distinct commodities in our model. There will be three kinds of goods in our model:

- *final products* (goods produced but used only for consumption or investment in the given time period),
- *capital stocks* (final goods accumulated in the previous periods, physically the same as their currently produced counterparts) and
- *primary* (non-produced) *factors of production*.

Let us decompose final use (y_i) into consumption (v_i) and investment (z_i), and introduce, in addition to the variables and parameters of the static model the following ones:

b_{ij} the input coefficients of the capital goods,

k_{i0} the accumulated stock of capital good i (the supply of capital goods),

q_i the unit price (cost) of capital good i,

r_i^a the rate of amortization of capital good i,

π_i the net rate of return on capital good i.

Following Walras, we can formulate the necessary conditions of general equilibrium in this model as follows:

$$q_j = (r_j^{\text{a}} + \pi_j)\cdot p_j, \qquad j = 1, 2,\ldots, n, \qquad \text{(WP-1)}$$

$$p_j = r_1\cdot d_{1j} + r_2\cdot d_{2j} +\ldots+ r_s\cdot d_s + q_1\cdot b_{1j} + q_2\cdot b_{2j} +\ldots+ q_n\cdot b_{nj}, \qquad j = 1, 2,\ldots, n, \qquad \text{(WP-2)}$$

$$v_i(\mathbf{p}, \mathbf{q}, \mathbf{r}) + z_i = y_i, \qquad i = 1, 2,\ldots, n, \qquad \text{(WP-3)}$$

$$d_{k1}\cdot y_1 + d_{k2}\cdot y_2 +\ldots+ d_{kn}\cdot y_n = s_k(\mathbf{p}, \mathbf{q}, \mathbf{r}), \qquad k = 1, 2,\ldots, s, \qquad \text{(WP-4)}$$

$$b_{i1}\cdot y_1 + b_{i2}\cdot y_2 +\ldots+ b_{in}\cdot y_n = k_{i0}, \qquad i = 1, 2,\ldots, n, \qquad \text{(WP-5)}$$

where $v_i(\mathbf{p}, \mathbf{q}, \mathbf{r})$ is the consumers' demand function for good i, and $s_k(\mathbf{p}, \mathbf{q}, \mathbf{r})$ is the supply function of the kth primary factor of production, and $\mathbf{p} = (p_i)$, $\mathbf{q} = (q_i)$ and $\mathbf{r} = (r_k)$.

Introducing appropriate vectors and matrices we can rewrite the entire system as

$$\mathbf{q} = \mathbf{p}\langle\mathbf{r}^{\text{a}} + \boldsymbol{\pi}\rangle \qquad \text{(WP-1a)}$$

$$\mathbf{p} = \mathbf{rD} + \mathbf{qB} \qquad \text{(WP-2a)}$$

$$\mathbf{v}(\mathbf{p}, \mathbf{q}, \mathbf{r}) + \mathbf{z} = \mathbf{y} \qquad \text{(WP-3a)}$$

$$\mathbf{Dy} = \mathbf{s}(\mathbf{p}, \mathbf{q}, \mathbf{r}) \qquad \text{(WP-4a)}$$

$$\mathbf{By} = \mathbf{k}_0 \qquad \text{(WP-5a)}$$

The first condition, (WP-1) is simply the definition of what is called nowadays the *Walras cost of capital* (amortization plus the net rate of return). The costs making up the price of the produced goods include now the cost of capital goods as well, as seen in (WP-2). Equations (WP-3) – (WP-1) represent the demand-supply equilibrium conditions on the markets of currently produced goods, primary factors of production and accumulated capital goods.

The number of the unknowns (y_i, z_i, p_j, q_j, π_i, r_k) in the above system of equations is ($5n$ + s), whereas the number of the equations is ($4n + s$). The system is thus underdetermined as yet, having n degrees of freedom. At the same time, investment demand has yet not been specified and the equality of the net rates of return on capital has not been postulated yet either that must be fulfilled in equilibrium. There are thus two competing sets of n additional equations to make the model mathematically well determined. Choosing any of them would let the values of the other set of variables determined by the model arbitrarily (as residuals), and not in accordance with the theoretical assumptions.

Walras was perfectly aware of the fact that the equality of the rates of return would require the harmonization of the accumulation of capital stocks with the demand for them. If he chose to close the model by prescribing the equality of the rates of return, nothing would ensure the above harmony. If, on the other hand, he specified the investment demand in one way or another, then nothing would guarantee the uniformity of the rates of return. The formal (mathematical) closure of the model leaves thus the model essentially open ended. This problem became to be known as *the issue of macro-closure*, which is present in any similar, finite period model with investment variables.[1]

Walras chose to close his model by prescribing the equality of the rates of return, i.e., substituting the equations (WP-1) with

$$q_j = (r_j^{\mathrm{a}} + \pi)\cdot p_j, \qquad j = 1, 2,\ldots, n. \qquad \text{(WP-1')}$$

and getting rid of the variables π_i. This reduces the number of unknowns and the degree of freedom to one, which can be eliminated by the numeraire, i.e. by fixing the price level (assuming, as usual, demand and supply functions homogeneous of degree zero). Walras left it simply for conviction that investments would adjust in such a way that maintained equilibrium on longer run and the equality of the rates of return. Walras's model with capital goods was essentially a static representation of long-term equilibrium. This is exactly the reason why the problem of macro-closure emerged in it.

1.3. The circularity of production and Leontief's model of general equilibrium

The models discussed above did not take into account the fact that 'commodities are produced by means of commodities' (*Sraffa*, 1960). In any given period a significant part of the total demand for produced goods is generated by the production itself. Not only by future production (through accumulation, as in the second model of Walras), but also and mainly by current production. The mutual interdependence of the various branches, the *circularity of production* creates in an economy a set of equilibrium conditions and linkages, which were neglected in the models discussed above.

It is easy to make up for this deficiency and introduce the intermediate use of the produced goods into the above models. Let a_{ij} denote the material input coefficients, the amount of produced good i used to produce one unit of good j and x_j the total production (output) of good j. (We will use y_j to denote total final demand, as before.)

The conditions of demand-supply equilibrium on the produced commodity markets will change as follows:

$$x_i = a_{i1}\cdot x_1 + a_{i2}\cdot x_2 + \ldots + a_{in}\cdot x_n + y_i, \quad \text{where } \ y_i = v_i(\mathbf{p}, \mathbf{q}, \mathbf{r}) + z_i$$

and the cost-of-production pricing rule will be modified accordingly too

$$p_j = p_1\cdot a_{1j} + \ldots + p_n\cdot a_{nj} + r_1\cdot d_{1j} + \ldots + r_s\cdot d_{sj} + q_1\cdot b_{1j} + \ldots + q_n\cdot b_{nj}.$$

The rest of the equilibrium conditions remain practically the same as before, except that factor demand depends now on total ($\mathbf{x}$) and not just final output ($\mathbf{y}$). Rewriting the conditions of equilibrium of the previous model we get the following set of equations:

$$\mathbf{q} = \mathbf{p}\langle\mathbf{r}^{\mathrm{a}} + \pi\cdot\mathbf{1}\rangle \qquad \text{(WL-1a)}$$

$$\mathbf{p} = \mathbf{pA} + \mathbf{rD} + \mathbf{qB} \qquad \text{(WL-2a)}$$

$$\mathbf{Ax} + \mathbf{v}(\mathbf{p}, \mathbf{q}, \mathbf{r}) + \mathbf{z} = \mathbf{x} \qquad \text{(WL-3a)}$$

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[1] The issue of macro-closure has been discussed in the literature on computable general equilibrium models, for example, by Dewatripont and Michel (1987) and Taylor et al. (1979).

$$\mathbf{Dx} = \mathbf{s}(\mathbf{p}, \mathbf{q}, \mathbf{r}) \tag{WL-4a}$$

$$\mathbf{Bx} = \mathbf{k}_0 \tag{WL-5a}$$

The modified set of the WL-equations yields nothing but the theoretical framework of *Leontief*'s (1928, 1941) static *input-output model*. Leontief's method of input-output analysis was designed for practical application and focused on the intersectoral linkages represented by the a_{in} input coefficients. Instead of using demand and supply functions he turned his model into a partial equilibrium model, in which *final demand* and *value added* became exogenous variables. Leontief's static input-output model consists of two sets of equations only:

$$x_i = a_{i1}\cdot x_1 + a_{i2}\cdot x_2 + \ldots + a_{in}\cdot x_n + y_i, \qquad i = 1, 2,\ldots, n, \tag{L-1}$$

$$p_j = p_1\cdot a_{1j} + p_2\cdot a_{2j} + \ldots + p_n\cdot a_{nj} + c_j, \qquad j = 1, 2,\ldots, n, \tag{L-2}$$

where c_i is the coefficient of value-added, which – unlike in the previous models – in addition to the cost of primary factors may contain pure profit as well. Rewriting the above equations into matrix algebraic forms one can see their perfect duality:

$$\mathbf{x} = \mathbf{Ax} + \mathbf{y} \tag{L-1a}$$

$$\mathbf{p} = \mathbf{pA} + \mathbf{c} \tag{L-2a}$$

Under normal conditions[2] $(\mathbf{I} - \mathbf{A})^{-1}$ exists and is non-negative, and the above two systems of equations can thus be rearranged and uniquely solved for the x_i and p_j unknowns:

$$\mathbf{x} = (\mathbf{I} - \mathbf{A})^{-1}\mathbf{y}, \tag{L-3}$$

$$\mathbf{p} = \mathbf{c}(\mathbf{I} - \mathbf{A})^{-1}, \tag{L-4}$$

where $\mathbf{I}$ stands for the identity matrix, and $(\mathbf{I} - \mathbf{A})^{-1}$ is the so called Leontief-inverse of matrix $\mathbf{A}$.

The Leontief-inverse matrix acts like a multiplier in the above equations, expanding and distributing the effect of any exogenous change in final demand and/or value added among various sectors. Unlike the model of Cassel, which is completely supply-driven (the given amounts of primary resources determine the level of economic activity), Leontief's model is fully demand-driven (the level of final demand determines the level of production). Its basic assumption is that the economy operates at less than full capacity and the changes in total output are not constrained by the availability of primary resources (labour and capital stocks). These seemingly simple forms can be developed into a variety of input-output models as we will illustrate it later.

1.4. The Paretian–Hicksian system of general equilibrium

The basic structure of the Paretian system is similar to the static Walras–Cassel model. Unlike, however, Walras and Cassel, *Pareto* (1906) represented the choice of consumers and producers not by demand and supply functions, but by optimal decisions derived from well-behaved (differentiable, strictly concave) utility and profit functions. Given a set of output and

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[2] They are usually referred to as the Simon–Hawkins or productivity conditions.

factor prices, households are assumed to choose their demand for produced goods and supply of factors via the unique solution of a utility-maximization problem, and firms to set their demand for factors and supply of produced goods via a profit-maximization problem. The rest of the model followed as in Walras's static model.

In the 1930s and 1940s this approach was further developed. The most influential among the contributors was *Hicks* (1939). Whereas the models of Walras and Cassel were basically macroeconomic, holistic constructs, their Paretian–Hicksian version is based on microeconomic foundations, due to the assumption of optimizing economic agents. This reformulation put the emphasis on efficiency of resource allocation and exhibits, as we will see, close similarity to the nation-wide models of optimal resource allocation.

Let us assume that there are m firms and the technology of firm j can be represented by the following differentiable production function:

$$F_j(\mathbf{t}^{(j)}) = F_j(t_1^j, t_2^j, \dots, t_i^j, \dots, t_n^j) = 0 \qquad (j = 1, 2, \dots, m), \qquad (1.4\text{-}1)$$

where $\mathbf{t}^{(j)} = (t_i^j)$ is the vector of net outputs, positive if output of good i exceeds its use, zero or negative otherwise.

Adapting this notational convention, the profit (net income) of production activity $\mathbf{t}^{(j)}$ earned at prices $\mathbf{p} = (p_i)$ can be expressed by the following scalar product:

$$\mathbf{pt}^{(j)} = p_1 \cdot t_1^j + p_2 \cdot t_2^j + \dots + p_i \cdot t_i^j + \dots + p_n \cdot t_n^j. \qquad (1.4\text{-}2)$$

The profit-maximization problem of firm j consists of the maximization of function (1.4-2) subject to the constraint given by (1.4-1). According the classical Lagrange method, the optimal solution must be a stationary point of the following Lagrange-function:

$$L_j(\mathbf{t}^{(j)}, \lambda_j) = \mathbf{pt}^{(j)} - \lambda_j \cdot F_j(\mathbf{t}^{(j)}).$$

Setting the

$$\frac{\partial L_j}{\partial t_i^j} = p_i - \lambda_j \cdot \frac{\partial F_j}{\partial t_i^j}$$

partial derivatives of the Lagrangian function equal to zero and replacing λ_j with its value given by the last equation,

$$\lambda_j = p_n : \frac{\partial F_j}{\partial t_n^j},$$

the necessary first order conditions for profit maximum can be expressed by the following set of equations:

$$\frac{F_{ji}}{F_{jn}} = \frac{p_i}{p_n} \qquad (i = 1, 2, \dots, n-1),$$

where $F_{ji} = \dfrac{\partial F_j}{\partial t_i^{\,j}}$ is the ith partial derivative of function F_j, and $-\dfrac{dt_n}{dt_i} = \dfrac{F_{ji}}{F_{jn}}$ represents the marginal rate of transformation between commodity n and i, which in optimum is equal to the ratio of their prices (the so called tangency condition).

Let us denote the utility function of the hth (h = 1,2, ... , k) household by

$$u_h(\mathbf{y}^{(h)}) = u_h(y_1^{\,h}, y_2^{\,h}, ..., y_i^{\,h}, ..., y_n^{\,h}),$$

and his disposable income by $e_h(\mathbf{p})$.

Household h faces the following optimization problem:

$$\max u_h(\mathbf{y}^{(h)}), \quad \text{s.t.} \quad e_h(\mathbf{p}) = \mathbf{p}\mathbf{y}^{(h)} = p_1y_1^{\,h} + p_2y_2^{\,h} + ... + p_iy_i^{\,h} + ... + p_ny_n^{\,h}.$$

The first order conditions for utility maximum can be expressed by the following set of equations:

$$-\frac{dy_n^{\,h}}{dy_i^{\,h}} = \frac{u_{hi}}{u_{hn}} = \frac{p_i}{p_n}, \qquad (i = 1, 2, ... , n-1),$$

where $u_{hi} = \dfrac{\partial u_h}{\partial y_i^{\,h}}$ is the ith partial derivative of function u_h, the marginal utility of good i and $-\dfrac{dy_n^{\,h}}{dy_i^{\,h}} = \dfrac{u_{hi}}{u_{hn}}$ represent the marginal rates of substitution between commodity n and i in the case of household h, which must be equal to the ratio of their prices (another set of tangency conditions).

Let us denote the vector of initial endowments owned by the households by vector **a**. The sum $\mathbf{a} + \sum_j \mathbf{t}^{(j)}$ defines the vector *net supply* of different goods. According to Walras's law the total value of this net supply, the sum of profits and the value of the endowments must be equal to the value of the net demand of the consumers:

$$e(\mathbf{p}) = \mathbf{p}(\mathbf{a} + \textstyle\sum_j \mathbf{t}^{(j)}) = \mathbf{p}\sum_h \mathbf{y}^{(h)} = \sum_h e_h(\mathbf{p}).$$

The fulfilment of this law can be secured if we assume that the net income of households is always equal to a given fraction of the total planned net supply: $e_h(\mathbf{p}) = \alpha_h \cdot e(\mathbf{p}) = \alpha_h \cdot \mathbf{p}(\mathbf{a} + \mathbf{t})$, where $\alpha_h \geqq 0$ and $\sum_h \alpha_h = 1$ represent the distributions share parameters.

Summing up: a general equilibrium is achieved if all markets clear, which means that prices are such that the demand for each good equals the supply of each good, i.e. there is a set of prices $\mathbf{p} = (p_1, p_2, ... , p_i, ... , p_n)$ such that:

PH(*a*): optimal producers' decisions (j = 1, 2, ... , m, $n\times m$ equations):

$$F_j(t_1^{\,j}, t_2^{\,j}, ..., t_i^{\,j}, ..., t_n^{\,j}) = 0,$$

$$-\frac{dt_n^{\,j}}{dt_i^{\,j}} = \frac{F_{ji}}{F_{jn}} = \frac{p_i}{p_n} \qquad (i = 1, 2, ... , n-1),$$

PH(*b*): optimal consumers' decisions (h = 1,2, ... , k, $n\times k$ equations):

$$p_1 y_1^{\,h} + p_2 y_2^{\,h} + \ldots + p_i y_i^{\,h} + \ldots + p_n y_n^{\,h} = \alpha_h \cdot \mathbf{p}\big(\mathbf{a} + \textstyle\sum_j \mathbf{t}^{(j)}\big)$$

$$-\frac{dy_n^{\,h}}{dy_i^{\,h}} = \frac{u_{hi}}{u_{hn}} = \frac{p_i}{p_n}, \qquad (i = 1, 2, \ldots, n-1),$$

HS(*c*): the demand for each good equals its supply (n equations):

$$a_i + t_i^1 + t_i^2 + \ldots + t_i^j + \ldots + t_i^m = y_i^1 + y_i^2 + \ldots + y_i^h + \ldots + y_i^k \qquad (i = 1, 2, \ldots, n).$$

Thus we have $n(m + k + 1)$ number of equations. Let us now turn to the establishing the number of unknowns. From firm j's optimization problem the unknowns are t_i^j for each good i = 1,..., n. As we have m firms, then we have $n\times m$ unknowns from the production side. Thus, as many unknowns as many equations characterize the optimal producers' decisions. From households' optimization problem the unknowns are y_i^h for each good and each household, that is $n\times k$ unknowns all together. Again, the number of unknowns matches number of equations in this exercise. And finally, each market-clearing equation can be matched by one commodity price. Thus, for the entire system there are equal number of unknowns and equations.

It is easy to check that the equations characterizing equilibrium depend only of the price ratios, their level can be set arbitrarily. We can thus set, say, the price of the last commodity to one (i.e. $p_n = 1$), whereby one of the unknowns will be eliminated. Thus the total number of equations seems to exceed that of the unknowns. But we should not forget Walras's law. Summing up the households' budget constraints and taking into account that $\sum_h \alpha_h = 1$, after some rearrangement we get:

$$\mathbf{pa} + \mathbf{pt}^{(1)} + \mathbf{pt}^{(2)} + \ldots + \mathbf{pt}^{(j)} + \ldots + \mathbf{pt}^{(m)} = \mathbf{py}^{(1)} + \mathbf{py}^{(2)} + \ldots + \mathbf{py}^{(h)} + \ldots + \mathbf{py}^{(k)},$$

wich is the same as the eqation we would get if we cross-multiplied each market-clearing equation by the price of the corresponding good and summed up them over commodities. We can thus remove one equation by Walras's law. The total number of equations is therefore equal to the total number of unknowns.

We should note that the above equations are first order necessary but not sufficient conditions for equilibrium in general and the equality of the numbers of unknowns and equations does not guaranty in general the existence of solutions. In any numerical exercise one has to make sure that the concrete forms of the production and utility functions are such that guaranty the fulfilment of the second order conditions of optima as well. The equality of the numbers of unknowns and equations is normally required for the local uniqueness and stability of the solution.

As we have seen, if we consider prices as parameters in the agents' maximization exercises, than the first order necessary conditions of optimum will define regular equation systems, see *PH*(*a*) and *PH*(*b*). Choosing well-behaving functional forms one can express the optimal solutions as functions of prices $\mathbf{p}$: $\mathbf{t}^{(j)}(\mathbf{p})$ net supply and $\mathrm{y}^{(h)}(\mathbf{p})$ net demand functions. Substituting these functions into the market-clearing equations we can reduce the conditions of general equilibrium to the equality of total net supply and total net demand on each good's market:

$$\mathbf{a} + \mathbf{t}^{(1)}(\mathbf{p}) + \mathbf{t}^{(2)}(\mathbf{p}) + \ldots + \mathbf{t}^{(m)}(\mathbf{p}) = \mathbf{y}^{(1)}(\mathbf{p}) + \mathbf{y}^{(2)}(\mathbf{p}) + \ldots + \mathbf{y}^{(k)}(\mathbf{p}).$$

These are formally nothing but the conditions of equilibrium used in the models of Walras and Cassel, in which the supply and demand functions are no longer *a priori* given macro-functions to be estimated, but the aggregates of the supply and demand functions of the individual agents, derived from their assumed optimal behaviour. As we will see later, in applied general equilibrium models we use this latter approach rather than econometrically estimated supply and demand functions, which is a task impossible in most cases.

EQUILIBRIUM, EFFICIENCY AND OPTIMALITY

As noted earlier, the significant novelty of the Paretian–Hicksian version of general equilibrium was the explicit introduction of optimizing economic agents. This made it possible to investigate the problems of efficiency and optimality within the framework of the general equilibrium model, not just the equality of supply and demand. Pareto introduced his well known *concept of efficiency*, which stated that a given allocation of economic resources is efficient if and only if there is no other feasible allocation such that would yield higher level of utility for at least one household and not lower for any one else.

In the case of the above model this can be interpreted and characterize as follows. The feasible set of allocations in our case means such collections of the variables $\mathbf{T} = (t_i^j)$, $\mathbf{Y} = (y_i^h)$ and $\mathbf{u} = (u_h)$ that satisfy the following resource and technological constraints:

$$\textstyle\sum_h y_i^h - \sum_j t_i^j = a_i \qquad (i = 1, \ldots, n); \qquad \text{(FA-1)}$$

$$F_j(\mathbf{t}^{(j)}) = 0 \qquad (j = 1, \ldots, m); \qquad \text{(FA-2)}$$

Each feasible allocation can be evaluated in terms of the utility levels, $u_h = u_h(\mathbf{y}^{(h)})$ it provides for the various households. From the definition of Pareto efficiency it follows that if a feasible allocation $\mathbf{T}^{\mathrm{e}}$, $\mathbf{Y}^{\mathrm{e}}$ resulting in utility levels $\mathbf{u}^{\mathrm{e}}$ is efficient than $t_i^{j\mathrm{e}}$ and $y_i^{h\mathrm{e}}$ satisfy constraints (FA-1) and (FA-2), and in addition there is no such allocation t_i^j and y_i^h that would satisfy these constrains and at the same time $u_h^{\mathrm{e}} \leqq u_h(\mathbf{y}^{(h)})$ weak inequality would hold for all h, and $u_h^{\mathrm{e}} < u_h(\mathbf{y}^{(h)})$ relation for at least one h.

Consider now the following constrained optimization problem (we put in parentheses on the left margin the assigned Lagrange multipliers):

$$\max u_1(\mathbf{y}^{(h)})$$

s.t.	(p_i)	$\sum_h y_i^h - \sum_j t_i^j = a_i$	$(i = 1, \ldots, n);$
	(λ_j)	$F_j(\mathbf{t}^{(j)}) = 0$	$(j = 1, \ldots, m);$
	(η_h)	$u_h^{\mathrm{e}} - u_h(\mathbf{y}^{(h)}) = 0$	$(h = 2, \ldots, k);$

From the definition of Pareto efficiency it follows that $t_i^{j\mathrm{e}}$ and $y_i^{h\mathrm{e}}$ is a feasible and in fact an optimal solution of the above maximization exercise, the Lagrangian function of which is

$$L(\mathbf{T}, \mathbf{Y}, \mathbf{p}, \lambda, \eta) = u_1(\mathbf{y}^{(h)}) - \sum_i p_i \cdot (\sum_h y_i^h - \sum_j t_i^j - a_i) - \sum_j \lambda_j \cdot F_j(\mathbf{t}^{(j)}) - \sum_{h=2}^{k} \eta_h \cdot (u_h^{\mathrm{e}} - u_h(\mathbf{y}^{(h)})).$$

Therefore they will satisfy the following first order conditions of constrained maximum:

$\partial L/\partial t_i^j$: $\quad p_i = \lambda_j \cdot F_{ji} \quad (i = 1, \ldots, n; j = 1, \ldots, m)$;

$\partial L/\partial y_1^h$: $\quad u_{1i} = p_i \quad (i = 1, \ldots, n)$;

$\partial L/\partial y_i^h$: $\quad \eta_h \cdot u_{hi} = p_i \quad (i = 1, \ldots, n; h = 2, \ldots, k)$;

As one can easily check these resource allocation feasibility and optimality constraints (more precisely, once the Lagrange multipliers are eliminated) are entailed by the necessary conditions of equilibrium, therefore, competitive equilibrium provides by force an efficient of allocation of the resources. This is the first statement of the well known theorem of welfare economics.

In the case of one (representative) household ($k = 1$) the conditions become simpler and at the same time the (single) utility function $u(\mathbf{y})$ provides a measure for nation-wide optimality or welfare, by which we can compare different allocations. In the case of multiple houshold one has to define an appropriate welfare function to aggregate (comeasure) the utility levels of the various households (household groups). In both cases we can express gains and losses in utility in terms of money, using either the Marshallian *consumers' surplus* or the Hicksian *equivalent* and/or *compesating income variation* formula, as we often do in applied models.

Using the welfare function approach we can re-establish the above statement of welfare economics. The maximization problem in this case is as follows:

$$\max W(u_1, u_2, \ldots, u_k)$$

s.t. $(p_i) \quad \sum_h y_i^h - \sum_j t_i^j = a_i \quad (i = 1, \ldots, n)$;

$(\lambda_j) \quad F_j(\mathbf{t}^{(j)}) = 0 \quad (j = 1, \ldots, m)$;

$(\eta_h) \quad u_h - u_h(\mathbf{y}^{(h)}) = 0 \quad (h = 1, \ldots, k)$;

The Lagrangian function of this problem takes the following form:

$$L(\mathbf{T}, \mathbf{Y}, \mathbf{u}, \mathbf{p}, \lambda, \eta) = W(\mathbf{u}) - \sum_i p_i \cdot (\sum_h y_i^h - \sum_j t_i^j - a_i) - \sum_j \lambda_j \cdot F_j(\mathbf{t}^{(j)}) - \sum_h \lambda_h \cdot (u_h - u_h(\mathbf{y}^{(h)})),$$

which yield the following set of first order conditions:

$\partial L/\partial t_i^j$: $\quad p_i = \eta_j \cdot F_{ji} \quad (i = 1, \ldots, n; j = 1, \ldots, m)$;

$\partial L/\partial y_i^h$: $\quad \lambda_h \cdot u_{hi} = p_i \quad (i = 1, \ldots, n; h = 1, \ldots, k)$;

$\partial L/\partial u_h$: $\quad W_h = \lambda_h \quad (h = 1, \ldots, k)$;

The first two sets of these conditions are the same as the necessary conditions of the previous maximization problem and their solutions differ only in terms of the numeraire.

1.5. A Koopmans–Kantorovich variant: a linear model based on fixed coefficients

ALTERNATIVE REPRESENTATIONS OF TECHNOLOGY

The early models of general equilibrium using fixed input coefficients represented production in an *ex post* manner, treating them simply as given average input coefficients, which could be observed once equilibrium is reached. Walras noted that they were in fact variables, depending on prices, but considered this fact a negligible technical detail, which can be ignored in an abstract model. In a later edition of his book he derived them from the marginal conditions of cost minimization. As for Leontief the use of fixed input coefficients was a pragmatic necessity dictated by the availability of statistical data as well as computational techniques and facilities.

It was, however, not so much the use of fixed coefficients that raised theoretical concerns, but rather the neglect of technological choice and joint production, the proper representation of the technology. Smooth classical production functions, allowing for substitutability between pairs of inputs and outputs in a wide range, offered an alternative and they became standard tools in neoclassical microeconomic theory. *Von Neumann* (1937), on the other hand, in his model of equilibrium growth demonstrated that the technology allowing for technological choice and joint production can be represented using fixed input-output coefficients.

It was *Koopmans* (1951) who laid down the axiomatic foundations of production theory, in general and the linear activity model, in particular. For Koopmans the choice between smooth, differentiable production functions or fixed coefficients was not a theoretical, but a practical problem, which should be governed by the purpose of the model (pure or applied), mathematical and computational algorithms and techniques, the availability of statistical data and so on.

In applied models of optimal resource allocation and choice of techniques, based on detailed representation of technology, the linear activity model combined with the method of linear programming proved to offer a more suitable approach than the models based on smooth, differentiable production functions. The former can be based on the knowledge of discrete technological variants, whereas the estimation of production functions is severely constrained in practice. The linear input-output programming approach dominated for many years the applied macroeconomic modelling for policy analysis.

The linear activity model rests on the assumption that the technology can be represented as the nonnegative combinations of finite number of elementary activities. Let us denote by vector $\tilde{\mathbf{a}}_j \in R^n$ the unit net input-output coefficients of the jth elementary activity, where $\tilde{a}_{ij}$ is positive if the output of good i exceeds its input, negative if its input is larger than its output, zero otherwise. Let us denote by $\mathbf{x} = (x_j)$ the vector the (nonnegative) levels of the various elementary activities. The technology defined by linear activity model is the following:

$$T = \{\mathbf{t}\colon \mathbf{t} = \sum_{j=1}^{m} x_j \cdot \tilde{\mathbf{a}}_j = \tilde{\mathbf{A}}\mathbf{x},\ \mathbf{x} \geqq \mathbf{0}\},$$

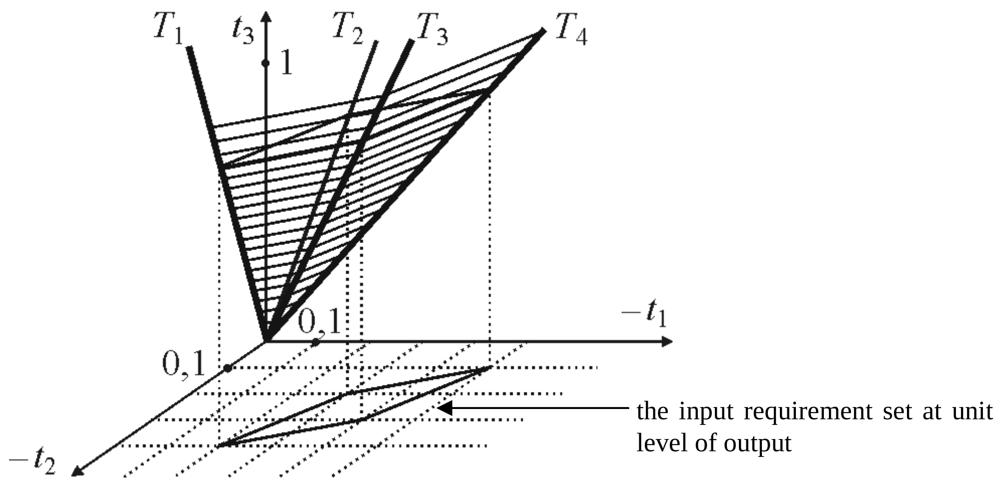
where $\tilde{\mathbf{A}}$ is the (unit) input-output coefficient matrix, where the unit levels of the elementary activities, and thus that of the unit coefficients, can be chosen arbitrarily.

The production set generated by the set of the following input-output coefficient matrix (two inputs and one output, four elementary activities) is illustrated on Figure 1.1 and 1.2.

$$\tilde{\mathbf{A}} = \begin{pmatrix} -0,2 & -0,3 & -0,4 & -0,5 \\ -0,4 & -0,2 & -0,3 & -0,1 \\ 1,0 & 1,0 & 1,0 & 1,0 \end{pmatrix}$$

Figure 1.1

The technological set defined by a matrix $\tilde{\mathbf{A}}$



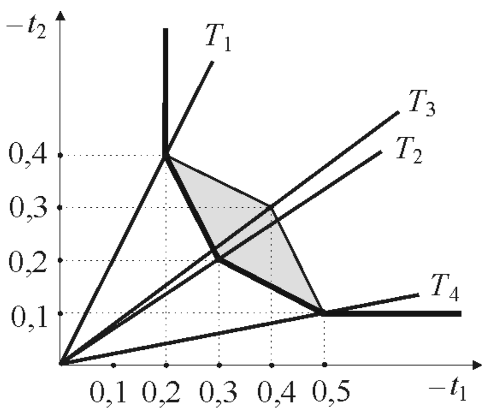
The assumptions of proportionality and additivity imply constant returns to scale and lack of production externalities, and that the technology is a convex polyhedral cone (see Figure 1.2). The production set generated by the linear activity model is a piecewise linear variant of the one defined as follows:

$$T = \{\mathbf{t}\text{: } \mathbf{t} = \sum_{j=1}^{m} \mathbf{t}^{(j)},\ F_j(\mathbf{t}^{(j)}) = 0\},$$

where F_j are the production functions used in the Paretian–Hicksian system of general equilibrium, and they are all homogenous of degree zero.

Figure 1.2

The input requirement set and the unit isoquant



A KOOPMANS–KANTOROVICH MODEL OF GENERAL EQUILIBRIUM

We will present now a completely linear variant of the Paretian–Hicksian system of general equilibrium based on the linear activity description of technology. As we will show it in the next chapter, we can easily solve this model by means of a linear programming problem.

The necessary conditions of equilibrium are as follows:

(E0) feasible activity levels and prices:

$$\mathbf{x}, \mathbf{y}, \mathbf{p} \geqq \mathbf{0},$$

(E1) producers maximize profit:

a) $\mathbf{p\tilde{A}} \leqq \mathbf{0}$, but b) $\mathbf{p\tilde{A}x} = \mathbf{0}$

(E2) consumers' choice and Walras's law:

a) $\mathbf{y} = \mathbf{y}(\mathbf{p}, e) = e \cdot \mathbf{c} / \mathbf{ps}^{\mathrm{y}}$, where

b) $e = \mathbf{pa}$,

(E3) all commodity markets are clearing:

a) $\mathbf{a} + \mathbf{\tilde{A}x} \geqq \mathbf{y}$, but b) $\mathbf{pa} + \mathbf{p\tilde{A}x} = \mathbf{py}$.

The commodity composition of consumption ($\mathbf{s}^{\mathrm{y}}$) is considered to be fixed in order to maintain the linearity of the equilibrium conditions, as much as possible. The notable exceptions are the so-called complementarity conditions E1/b and E3/b, which state that the profit is maximal (zero) in the case of activities used in equilibrium and that the price of the commodities in excess supply is zero in equilibrium.

Consumption is modelled here as if consumers' choice would be the outcome of an optimal decision in the case of a Leontief-type utility function:

$$u(y_1, y_2, \dots, y_n) = \min \{(y_1/s_1^{\mathrm{y}},\ y_2/s_2^{\mathrm{y}}, \dots, y_n/s_n^{\mathrm{y}}\}.$$

Condition E2/a is so far nonlinear, but we can linearize it by choosing the unit basket of consumption as numeraire, by setting its value to one, that is $\mathbf{ps}^{\mathrm{y}} = 1$, when we get

$$y_i(\mathbf{p}, e) = e \cdot s_i^{\mathrm{y}}.$$

It can be easily seen that the equilibrium conditions are equivalent to the optimality conditions of the following Kantorovich-type linear programming problem:

	<u>Primal problem</u>	<u>Dual problem</u>	
	$\mathbf{x} \geqq \mathbf{0},\ y \geqq 0$	$\mathbf{p} \geqq \mathbf{0}$	
($\mathbf{p}$)	$\mathbf{a} + \mathbf{\tilde{A}x} \geqq y \cdot \mathbf{s}^{\mathrm{y}}$	$\mathbf{p\tilde{A}} \leqq \mathbf{0}$	($\mathbf{x}$)
		$\mathbf{ps}^{\mathrm{y}} \geqq 1$	(y)
	$y \to \max!$	$\mathbf{pa} \to \min!$	

where $y = e / \mathbf{ps}^{\mathrm{y}}$ is the level of consumption.

Thus, we have again demonstrated the close connection that exists between the models of nation-wide optimal resource allocation and the macro-models of general equilibrium. As we will show it in the next chapter that close conceptual similarity led modellers for policy analysis to switch to computable general equilibrium from linear programming models in the second half of the 1990s.

1.6. A step towards computable models: Johansen's model of general equilibrium

In the previous section we illustrated the formulation of a general equilibrium model based on productions and utility functions, and the assumption of optimizing agents. For illustrative purposes we chose a rather general and abstract construct. In this section we will present a much more concrete specification, a model such that uses parameters, which one can be relatively easily estimated on the basis of available macro-statistical data.

As a matter of fact, we will construct a model very similar to the first CGE model, developed by Leif Johansen (1960) for Norway. This model is a combination of Leontief-type input-output model with macroeconomic production and consumption functions, thus an input-output model extended with relative price driven substitution possibilities. Many models followed or were inspired later by Johansen's pioneering work and retained its original structure. In order to keep the model simple, we will consider a *static and closed economy*, with no foreign trade.

THE REPRESENTATION OF PRODUCTION

There will be two types of commodities: outputs of n kinds of production sectors, and two kinds of primary resources (labour and capital) with *exogenously given supply* (L_0 and K_0). Each productions sector will be modelled as a representative firm. They use sectoral outputs in fixed proportions (a_{ij} unit input coefficients, as in the case of Leontief's model) and primary resources with variable *unit input coefficients* (l_j, k_j). The feasible combinations of these latter ones are defined by the following type of equations:

$$f_j(l_j, k_j) = 1$$

where each $f_j(l_j, k_j)$ is a linear homogeneous (constant return to scale) production function. Denoting in sector j the level of *output* by X_j, and the amount of *labour and capital* by L_j and K_j, respectively and the amount of *materials* originating from sector i by X_{ij} the complete specification of the production function of sector j will be the following, so-called *nested production function of Johansen-type*:

$$X_j = \min\left(\frac{X_{1j}}{a_{1j}},\cdots,\ \frac{X_{ij}}{a_{ij}},\cdots,\ \frac{X_{nj}}{a_{nj}},\ f_j\left(L_j,\ K_j\right)\right).$$

$f_j(L_j, K_j)$ is the partial capacity defined by L_j amount of labour and K_j amount of capital and its value can be interpreted as a measure of a *composite factor*, made up by them. Because of the linear homogeneity of function f_j, their minimal cost per unit of output (c_j) will be independent of the level of X_j and it can be determined by solving the following optimizing problem:

$$w \cdot l_j + q \cdot k_j \to \min! \qquad\qquad f_j(l_j, k_j) = 1,$$

where w and q is the unit cost (price) of labour and capital, respectively. The first order necessary conditions of this cost minimum can be derived from the following

$$L(l_j, k_j, \lambda_j) = w \cdot l_j + q \cdot k_j - \lambda_j \cdot \big(f_j(l_j, k_j) - 1 \big)$$

Lagrange-function and they will be the following:

$$w = \lambda_j \frac{\partial f_j}{\partial l_j}, \qquad q = \lambda_j \frac{\partial f_j}{\partial k_j}, \qquad f_j(l_j, k_j) = 1.$$

Because of the linear homogeneity of function f_j, and by virtue of Euler's Law we have

$$\lambda_j \left(\frac{\partial f_j}{\partial l_j} l_j + \frac{\partial f_j}{\partial k_j} k_j \right) = \lambda_j = w \cdot l_j + q \cdot k_j = c_j.$$

The value of the Lagrange-multiplier λ_j is in fact equal to the marginal composite cost of labour and capital, which in this case is the same as their average as well as their unit cost. In the case of well-behaved f_j functions the value of the three unknowns can be expressed as functions of the factor prices:

$$l_j = l_j(w, q), \qquad k_j = k_j(w, q), \qquad c_j = \lambda_j = \lambda_j(w, q),$$

which are equivalent representations of the first order necessary conditions. One can indeed choose among several alternative representations of the same conditions. Later, for example, we will use the following ones (two unit factor demand functions and their combined cost):

$$l_j = l_j(w, q), \qquad k_j = k_j(w, q), \qquad c_j = w \cdot l_j + q \cdot k_j.$$

If all factor prices, including sectoral goods, are all positive, than in a cost minimizing solution we will have

$$X_{1j} = a_{1j} \cdot X_j,\ X_{2j} = a_{2j} \cdot X_j,\ \dots,\ X_{nj} = a_{nj} \cdot X_j,\ L_j = l_j(w, q) \cdot X_j \text{ and } K_j = k_j(w, q) \cdot X_j,$$

thus the ratio of L_j and K_j is determined by the unit cost minimization problem.

In equilibrium the prices of the produced commodities and the composite labour-capital factor will be equal to their minimal cost. The price-equal-cost definition is not only the reflection of the neoclassical convention, but also the only prices, which are compatible with the assumption of constant returns to scale (linearly homogeneous) production functions.

In the case of sectoral output j its unit price, p_j can be determined by any of the following forms:

$$p_j = \Sigma_i\, p_i \cdot a_{ij} + w \cdot l_j + q \cdot k_j = \Sigma_i\, p_i \cdot a_{ij} + c_j,$$

which is formally the same as the price equation in Leontief's static model. Here, however, the unit value added, c_j is no longer exogenous but endogenous variable, and its value reflects changes in both the composition of labour and capital, and their relative scarcity.

Thus, the equilibrium values of the labour and capital coefficients as well as the prices of the sectoral commodities can be determined once we know the factor prices as follows:

(E1) $l_j = l_j(w, q), \qquad j = 1, 2, \dots, n,$

(E2) $k_j = k_j(w, q)$, $j = 1, 2, \dots, n$,

(E3) $p_j = \Sigma_i\, p_i \cdot a_{ij} + w \cdot l_j + q \cdot k_j$, $j = 1, 2, \dots, n$.

which contain $3n$ unknowns and equations in addition to factor prices w and q.

The level of the sectoral output can be adjusted to the size of demand, since the profit will be maximal (zero) at any production level at the above prices.

THE REPRESENTATION OF CONSUMPTION

Let us turn now our attention to the problem of final demand, which in our static model can be seen as household consumption. We will represent consumption as the decision of one agent, whose preferences can be represented by a so-called *Stone–Geary-type* utility function, which leads to a linear expenditure system. The utility function of this type will be illustrated on Figure 1.3 in the case of two commodities.

The problem *Stone* (1954) faced was that in the case of the CES (constant elasticity of substitution) utility functions typically used in applied models the price elasticity of the various goods was uniform. He wanted to stay with the simplest possible form (Cobb–Douglas function) and still apply different price elasticities. In order to achieve that he assumed for each commodity that part of the consumption (*committed consumption*) is fixed at a certain (realistically given) levels (c_1^0 and c_2^0) and only the expenditure left after their purchase (the variable or excess expenditure, $ev = e - p_1 \cdot c_1^0 - p_2 \cdot c_2^0$) is allocated between various goods according to a utility function. In other words, the utility function is defined only over the set of excess (variable) consumption ($y_i^{\mathrm{v}} = y_i - c_i^0$).

Let us denote the utility function given in terms of variable consumption by $u_{\mathrm{v}}(y_1^{\mathrm{v}}, y_2^{\mathrm{v}})$. Given the p_1 and p_2 commodity prices and excess expenditure ev, the conditional utility maximization problem will be the following:

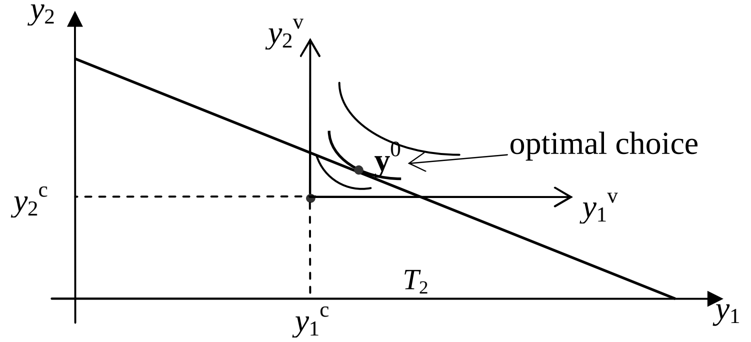
$$u_{\mathrm{v}}(y_1^{\mathrm{v}}, y_2^{\mathrm{v}}) \to \max! \qquad p_1 \cdot y_1^{\mathrm{v}} + p_2 \cdot y_2^{\mathrm{v}} = ev.$$

The corresponding Lagrangian function:

$$L(y_1^{\mathrm{v}}, y_2^{\mathrm{v}}, \lambda_{\mathrm{c}}) = u_{\mathrm{v}}(y_1^{\mathrm{v}}, y_2^{\mathrm{v}}) - \lambda_{\mathrm{c}} \cdot \big(p_1 \cdot y_1^{\mathrm{v}} + p_2 \cdot y_2^{\mathrm{v}} - ev\big).$$

Figure 1.3

Optimal choice in the case of committed and variable consumption



The first order necessary conditions of optimum are thus

$$\frac{\partial u_v}{\partial y_1^v} = \lambda_c \cdot p_1, \qquad \frac{\partial u_v}{\partial y_2^v} = \lambda_c \cdot p_2, \qquad p_1 \cdot y_1^v + p_2 \cdot y_2^v = ev,$$

where the optimal value of the Lagrange multiplier λ_c is the marginal utility of money and $p_{cv} = 1/\lambda_c$ is nothing but the minimal cost of reaching an additional unit of utility u_v. If the utility function is homogeneous of degree one, as we usually assume, it is the same as the average or unit utility of money. This means that in the optimal solution $u_v(y_1^v, y_2^v) = \lambda_c \cdot ev$, as it can be seen from the following identities (the first identity is implied by Euler's Law again):

$$\frac{\partial u_v}{\partial y_1^v} \cdot y_1^v + \frac{\partial u_v}{\partial y_2^v} \cdot y_2^v = v = \lambda_c \cdot (p_1 \cdot y_1^v + p_2 \cdot y_2^v) = \lambda_c \cdot ev.$$

In the case of homogeneous utility function the optimal structure of excess consumption will be independent of its level, so we may arrive at the optimal solution in a different way too. First we can determine the optimal structure of excess consumption by solving first the following utility maximization problem:

$$p_1 \cdot c_1^v + p_2 \cdot c_2^v \rightarrow \min! \qquad v(c_1^v, c_2^v) = 1,$$

the Lagrange multiplier of which – assuming that utility function is homogeneous of degree one – will be $p_{cv} = 1/\lambda_c$, which is the minimal cost at which a consumption bundle (s_1^v, s_2^v), yielding one unit of utility, can be purchased.

The cost minimizing bundle (s_1^v, s_2^v) can be interpreted as a composite good worth of one unit of utility, whose price is p_{cv}. ev/p_{cv} thus gives us the maximal level of utility that can be achieved from expenditure ev. Multiplying this by the above determined s_i^v coefficients, we can calculate the optimal level of the goods purchase in addition to committed consumption

$$y_i^v = s_i^v \cdot ev/p_{cv} = s_i^v \cdot y_v.$$

Following this route we have arrived at the demand system implied by the assumed preferences:

$$y_i^v(p_1, p_2, ev) = s_i^v(p_1, p_2) \cdot ev/p_{cv}(p_1, p_2),$$

or alternatively

$$y_i(p_1, p_2, e) = c_i^0 + s_i^v(p_1, p_2) \cdot (e - p_1 \cdot c_1^0 - p_2 \cdot c_2^0)/p_{cv}(p_1, p_2).$$

Let us now assume that the utility function is the following Cobb–Douglas type function:

$$u_v = y_1^{v\,\alpha_1} y_2^{v\,\alpha_2},$$

where $\alpha_1 + \alpha_2 = 1$, as Stone and Geary did. In such a case the implied demand system will be

$$y_i = c_i^0 + \alpha_i \cdot ev/p_i,$$

from which we get

$$p_i \cdot y_i = p_i \cdot c_i^0 + \alpha_i \cdot ev,$$

thus, the expenditure structure is linear. This explains the origin of the term *Linear Expenditure System*. But of course one must not insist on using Cobb–Douglas type function and can generalized the Stone–Geary approach. If, for example, one uses a Leontief-type utility function, which entails fixed s_i^{v} proportions, he will arrive at

$$y_i = c_i^0 + c_i^{\mathrm{v}} \cdot y_v,$$

a form that is often used in linear models of nation-wide optimal resource allocation.

We have not discussed yet the issue of the determination of consumers' expenditure e. In order to fulfil the requirements of Walras's law, we will assume here too that the households spend always as much at given prices as much the value of the initial endowments (labour and capital in this case) and the optimal level of profit is (zero in this case). Thus

$e = w \cdot L_0 + q \cdot K_0$ or alternatively $ev = w \cdot L_0 + q \cdot K_0 - p_1 \cdot c_1^0 - p_2 \cdot c_2^0$.

After all, the consumers' decisions can be represented in a compact way via the derived excess demand system

(E5) $y_i^{\mathrm{v}} = y_i^{\mathrm{v}}(p_1, p_2, \ldots, p_n, ev)$, $\quad i = 1, 2, \ldots, n$, where

(E4) $ev = w \cdot L_0 + q \cdot K_0 - \Sigma_i \, p_i \cdot c_i^0$,

which precedes in logical order the previous ones. These conditions add $n + 1$ new unknowns and equations to their already existing sets.

The above sets of equations fully describe the optimal choices of the representative economic agents and the formation of supply and demand. The market-clearing equations for sectoral outputs and primary inputs (n new unknowns and $n + 2$ additional equations) will make the set of necessary conditions of general equilibrium complete:

(E6) $x_i = \Sigma_j \, a_{ij} \cdot x_j + c_i^0 + y_i^{\mathrm{v}}$, $\quad i = 1, 2, \ldots, n$,

(E7) $\Sigma_j \, l_j \cdot x_j = L_0$,

(E8) $\Sigma_j \, k_j \cdot x_j = K_0$.

The market-clearing equations (E6) given for the sectoral outputs are again the same as those of the static Leontief model, except for the partly endogenous specification of final demand.

Table 1.4: Summary of the equations and variables

(E1)	$l_j = l_j(w, q)$	l_j	n
(E2)	$k_j = k_j(w, q)$	k_j	n
(E3)	$p_j = \Sigma_i \, p_i \cdot a_{ij} + w \cdot l_j + q \cdot k_j$	x_j	n
(E4)	$ev = w \cdot L_0 + q \cdot K_0 - \Sigma_i \, p_i \cdot c_i^0$	ev	1
(E5)	$y_i^{\mathrm{v}} = y_i^{\mathrm{v}}(p_1, p_2, \ldots, p_n, ev)$	y_i^{v}	n
(E6)	$x_i = \Sigma_j \, a_{ij} \cdot x_j + c_i^0 + y_i^{\mathrm{v}}$	p_i	n

(E7)	$\Sigma_j\, l_j \cdot x_j = L_0$	w	1
(E8)	$\Sigma_j\, k_j \cdot x_j = K_0$	q	1

In Table 1.4 we have summarized the equations and variables of the Johansen model of general equilibrium. To each equation we assigned a variable (see in the third column) and in the last column we put the number of the corresponding equation and variable kind. It makes it easy to check that the total number of equations is equal to the total number of unknowns, $5n + 3$. One can also show that from equations (E3), (E5) - (E8) we can derive (E4) by Walras's law, therefore we can eliminate one equation as before. On the other hand, all terms are homogeneous of degree zero in the prices and value terms, we can thus remove one unknown as well by setting the price level.

We can do the following. We remove the variable expenditure by setting its level equal to one, that is $ev = 1$. If the level of the committed consumption were all zero, than the demand function gained in this way, $y_i^{\mathrm{v}}(p_1, p_2, \ldots, p_n)$ would be the same as in Cassel's model. Let us eliminate, on the other hand equation (E4), which defined ev. In this way we arrive at a generalized version of the Cassel model, in which there are two primary factors with variable input cooefficients, the produced goods enter not only final consumption but also as factors of production with fixed input cooefficients.

Table 1.5: A solution scheme of the model

Equation	*Calculate*
$w = w_t,\ q = q_t$	
$l_j = l_j(w, q)$	l_j
$k_j = k_j(w, q)$	k_j
$p_j = \Sigma_i\, p_i \cdot a_{ij} + w \cdot l_j + q \cdot k_j$	p_j
$y_i^{\mathrm{v}} = y_i^{\mathrm{v}}(p_1, p_2, \ldots, p_n)$	y_i^{v}
$x_i = \Sigma_j\, a_{ij} \cdot x_j + c_i^0 + y_i^{\mathrm{v}}$	x_i
$L_{\mathrm{d}} = \Sigma_j\, l_j \cdot x_j = L_0$	L_{d}
$K_{\mathrm{d}} = \Sigma_j\, k_j \cdot x_j = K_0$	K_{d}
L_{d} ?? L_0, K_{d} ?? K_0 (if necessary, adjust factor prices and continue)	$w_{t+1},\ q_{t+1}$

We present the corrected system of equations in Table 1.5 and indicate a possible algorithm to solve the system. The algorithm does in fact reduce the solution of the model to the markets of the primary factors, as we have shown this possibility while discussing Cassel's model. We start the algorithm with some estimated primary factor prices (w and q), we solve sequentially for equilibrium in a recursive manner until we arrive at the calculation of factor demand. Then we check if their demand matches their supply, and increase or decrease their prices depending on the sign of their difference. If the case of well-behaved production and utility functions, one can design a simple heuristic iteration process to find the equilibrium value of the factor prices.

To show the close similarity of the Johansen and Cassel model we reduce further the equations to those of Cassel's.

$$x_i = \Sigma_j\, a_{ij}\cdot x_j + c_i^0 + y_i^{\mathrm{v}}(p_1, p_2, \dots, p_n),$$

$$\Sigma_j\, l_j(w, q)\cdot x_j = L_0,$$

$$\Sigma_j\, k_j(w, q)\cdot x_j = K_0,$$

$$p_j = \Sigma_i\, p_i\cdot a_{ij} + w\cdot l_j(w, q) + q\cdot k_j(w, q).$$

THE OPTIMAL RESOURCE ALLOCATION EQUIVALENT OF THE JOHANSEN MODEL

The equilibrium conditions of the Johansen model can be also easily reproduced from the necessary conditions of the following welfare maximizing resource allocation problem (on the left margin we put in parentheses the assigned Lagrange multipliers again):

$$\max y_{\mathrm{v}}$$

s.t.

$$(p_i)\qquad \Sigma_j\, a_{ij}\cdot x_j + c_i^0 + y_i^{\mathrm{v}} = x_i \qquad (i = 1, 2, \dots, n),$$

$$(c_j)\qquad x_j = f_j(L_j, K_j) \qquad (j = 1, 2, \dots, n),$$

$$(w)\qquad \Sigma_j\, L_j = L_0,$$

$$(q)\qquad \Sigma_j\, K_j = K_0,$$

$$(p_{\mathrm{cv}})\qquad y_{\mathrm{v}} = y_{\mathrm{v}}(y_1^{\mathrm{v}}, y_2^{\mathrm{v}}, \dots, y_n^{\mathrm{v}})$$

The Lagrangian function of the above problem is as follows:

$$L = y_{\mathrm{v}} - \Sigma_i\, p_i\cdot\left\{\Sigma_j\, a_{ij}\cdot x_j + c_i^0 + y_i^{\mathrm{v}} - x_i\right\} - \Sigma_j\, c_j\cdot\left\{x_j - f_j(L_j, K_j)\right\} -$$
$$w\cdot\left\{\Sigma_j\, L_j - L_0\right\} - q\cdot\left\{\Sigma_j\, K_j - K_0\right\} - p_{\mathrm{cv}}\cdot\left\{y_{\mathrm{v}} - y_{\mathrm{v}}(y_1^{\mathrm{v}}, y_2^{\mathrm{v}}, \dots, y_n^{\mathrm{v}})\right\}.$$

The partial derivatives provide as usual the further conditions of maximum:

$$\partial L/\partial y_{\mathrm{v}}:\qquad p_{\mathrm{cv}} = 1,$$

$$\partial L/\partial L_j:\qquad c_j\cdot\frac{\partial f_j}{\partial L_j} = w (j = 1, 2, \dots, n),$$

$$\partial L/\partial K_j:\qquad c_j\cdot\frac{\partial f_j}{\partial K_j} = q (j = 1, 2, \dots, n),$$

$$\partial L/\partial x_j:\qquad p_j = \Sigma_i\, p_i\cdot a_{ij} + c_j \qquad (j = 1, 2, \dots, n),$$

$$\partial L/\partial y_i^{\mathrm{v}}:\qquad p_{\mathrm{cv}}\cdot\frac{\partial y_{\mathrm{v}}}{\partial y_i^{\mathrm{v}}} = p_i (i = 1, 2, \dots, n).$$

One can easily verify that the necessary conditions of the above optimal resource allocation problem are equivalent to those of equilibrium. Consider, for example, the case of production. The second and the third set of the first order conditions are nothing but the necessary conditions of cost minimization. Because of the linear homogeneity of the production functions and Euler's theorem the following identities hold:

$$c_j \cdot \frac{\partial f_j}{\partial L_j} \cdot L_j + c_j \cdot \frac{\partial f_j}{\partial K_j} \cdot K_j = c_j \cdot x_j = w \cdot L_j + q_j \cdot K_j \ (= e),$$

from which, dividing both sides of the second and third equation by x_j, we get

$$c_j = w \cdot l_j + q_j \cdot k_j.$$

Thus, the fourth set of the first order conditions states that the prices of the sectoral commodities are equal to their cost of production, which means that production choices are maximizing the profit at prices p_j. In the same way we can show that

$$c_j \cdot x_j = w \cdot L_j + q_j \cdot K_j = \Sigma_i \, p_i \cdot y_i = e,$$

thus, Walras's law holds. Also, because of Euler's theorem

$$p_{\mathrm{cv}} \cdot \Sigma_i \frac{\partial y_{\mathrm{v}}}{\partial y_i^{\mathrm{v}}} \cdot y_i^{\mathrm{cv}} = p_{\mathrm{cv}} \cdot y_{\mathrm{v}} = \Sigma_i \, p_i \cdot y_i^{\mathrm{v}} \ (= ev).$$

This together with the last set of the first order conditions is nothing but the necessary conditions of utility maximization. Dividing both sides of the second and third equation in the above relation by y_{cv} we get

$$p_{\mathrm{cv}} = \Sigma_i \, p_i^{\mathrm{hm}} \cdot s_i^{\mathrm{v}},$$

indicating that p_{cv} can be interpreted as the cost-price of the composite consumption good (c_1^{v}, c_2^{v}, ... , c_n^{v}), $y_{\mathrm{v}}(c_1^{\mathrm{v}}, c_2^{\mathrm{v}}, \ldots, c_n^{\mathrm{v}}) = 1$, and this composite good plays the role of the numeraire in the above solution.

1.7. Summarizing the models presented

In Table 1.6 we present the models discussed in order to ease their survey and comparison. Wherever it was possible we used matrix algebraic notation for brevity, and omitted the lists of variables and potential sign restrictions.

Table 1.6: Comparing the various general equilibrium models

Walras I.	Cassel	Schlesinger–Wald
($\mathbf{x}$ =) $\mathbf{y} = \mathbf{v}(\mathbf{p}, \mathbf{r})$	($\mathbf{x}$ =) $\mathbf{y} = \mathbf{y}(\mathbf{p})$	($\mathbf{x}$ =) $\mathbf{p} = \mathbf{p}(\mathbf{y})$
$\mathbf{Dy} = \mathbf{s}(\mathbf{p}, \mathbf{r})$	$\mathbf{Dy} = \mathbf{s}$	$\mathbf{Dy} \leqq \mathbf{s}$, $\mathbf{r} \geqq \mathbf{0}$, de $\mathbf{rDy} = \mathbf{rs}$
$\mathbf{p} - \mathbf{rD} = \mathbf{0}$	$\mathbf{p} - \mathbf{rD} = \mathbf{0}$	$\mathbf{p} - \mathbf{rD} = \mathbf{0}$
price level: free	price level: $\mathbf{rs} = 1$	price level: $\mathbf{rs} = 1$

Walras II.	Walras–Leontief	Leontief (static model)
($\mathbf{x}$ =) $\mathbf{y} = \mathbf{v}(\mathbf{p}, \mathbf{q}, \mathbf{r}) + \mathbf{z}$	($\mathbf{v}$, $\mathbf{z}$, $\mathbf{r}$ and $\boldsymbol{\pi}$ exogenous)	($\mathbf{y}$ and $\mathbf{c}$ exogenous)
$\mathbf{Dy} = \mathbf{s}(\mathbf{p}, \mathbf{q}, \mathbf{r})$	$(\mathbf{I} - \mathbf{A})\mathbf{x} = \mathbf{v} + \mathbf{z}$	$(\mathbf{I} - \mathbf{A})\mathbf{x} = \mathbf{y}$
$\mathbf{By} = \mathbf{k}_0$		
$\mathbf{p} - \mathbf{rD} - \mathbf{qB} = \mathbf{0}$	$\mathbf{p}(\mathbf{I} - \mathbf{A}) - \mathbf{rD} - \mathbf{qB} = \mathbf{0}$	$\mathbf{p}(\mathbf{I} - \mathbf{A}) - \mathbf{c} = \mathbf{0}$
$\mathbf{q} = \mathbf{p}\langle\mathbf{r}^{\mathrm{a}} + \pi\mathbf{1}\rangle$	$\mathbf{q} = \mathbf{p}\langle\mathbf{r}^{\mathrm{a}} + \boldsymbol{\pi}\rangle$	
price level: free	price level: free	price level: set by $\mathbf{c}$

Paretian–Hicksian	Koopmans–Kantorovich	Johansen
$\mathbf{a} + \sum_j \mathbf{t}^{(j)} = \sum_h \mathbf{y}^{(h)}$	$\mathbf{y} = \mathbf{y}^{\mathrm{c}} + e\cdot\mathbf{c}^{\mathrm{v}}$	$\mathbf{y} = \mathbf{y}^{\mathrm{c}} + \mathbf{y}^{\mathrm{v}}(\mathbf{p})$
$F_j(\mathbf{t}^{(j)}) = 0$	$e = \mathbf{pa}$	$(\mathbf{I} - \mathbf{A})\mathbf{x} = \mathbf{y}$
$-\dfrac{dt_n^j}{dt_i^j} = \dfrac{F_{ji}}{F_{jn}} = \dfrac{p_i}{p_n}$	$\mathbf{a} + \tilde{\mathbf{A}}\mathbf{x} \geqq \mathbf{y}$, but	$\mathbf{l}(w, q)\mathbf{x} = L_0$,
$\mathbf{py}^{(h)} = \alpha_h\cdot\mathbf{p}(\mathbf{a} + \sum_j \mathbf{t}^{(j)})$	$\mathbf{pa} + \mathbf{p}\tilde{\mathbf{A}}\mathbf{x} = \mathbf{py}$	$\mathbf{k}(w, q)\mathbf{x} = K_0$
$-\dfrac{dy_n^h}{dy_i^h} = \dfrac{u_{hi}}{u_{hn}} = \dfrac{p_i}{p_n}$	$\mathbf{p}\tilde{\mathbf{A}} \leqq \mathbf{0}$, but	$\mathbf{p}(\mathbf{I} - \mathbf{A}) - w\cdot\mathbf{l}(w, q) -$
	$\mathbf{p}\tilde{\mathbf{A}}\mathbf{x} = \mathbf{0}$	$q\cdot\mathbf{k}(w, q) = \mathbf{0}$
price level: free ($p_n = 1$)	price level: $\mathbf{pc}^{\mathrm{v}} = 1$	price level: $ev = \mathbf{p}(\mathbf{a} - \mathbf{y}^{\mathrm{c}}) = 1$

We want add one final general remark to the presented models. Except for the models named after Schlesinger–Wald and Koopmans–Kantorovich, we assumed that all variables take positive values in equilibrium and therefore the optimality and equilibrium conditions can be given in the form as equation. In a more general setting we should have used inequalities with complementarity restrictions.

1.8. Illustrative programs

The special programs accompanying this training material contain a numerical example for the Cassel-model with 2 factors and 3 products in Excel (Cassel2x3.xls and its description in Cassel-2x3.doc), which can be solved both by the built-in SOLVER dialog box explicitly as a root of a one-variable polynomial equation of degree 3. This illustrates how neoclassical theory can be put to work and how the equilibrium solution depends on the choice of model parameters. It may also dissolve aversions about the “black-box” nature of the more complex CGE-models and the “mysterious” nature of their solution.

Interested reader may also find an illustrative Excel-program developed for the Johansen-type of CGE model (Johansen-DinLeo.xls). This program demonstrates how models can be solved by simple, heuristic iteration methods, using only a few iterandus (in our case only one factor price).

2. Applied multisectoral models: a comparative review

2.1. Applied input-output models

2.1.1. The input-output table and Leontief's static model

The statistical data base of input-output analysis is the input-output table, a macroeconomic accounting framework, which combines two sets of inter-sectoral balances (see Table 2.1). In the rows of the upper part of the table one finds a set of balances, which describe the sources and uses of sectoral outputs of a given period of time (typically a year). Each row corresponds to a particular sector (industry, commodity group), whereas the columns list the main areas of use, among them the sectors themselves. The intermediate uses of the sectoral outputs (x_{ij}, $i,j = 1, \dots , n$) are arranged into a quadratic matrix, and final uses into a rectangular one. Typical areas of final use (y_{il}) are private and public consumption, investments, (net) exports and change in stocks.

Table 2.1: The general scheme of the I-O tables

	Intermediate use (sectors)						Final use				Total
	1	2	...	j	...	n	1	2	...	r	source
1	x_{11}	x_{12}	...	x_{1j}	...	x_{1n}	y_{11}	y_{12}	...	y_{1r}	x_1
2	x_{21}	x_{22}	...	x_{2j}	...	x_{2n}	y_{21}	y_{22}	...	y_{2r}	x_2
.											.
i	x_{i1}	x_{i2}	...	x_{ij}	...	x_{in}	y_{i1}	y_{i2}	...	y_{ir}	x_i
.											.
n	x_{n1}	x_{n2}	...	x_{nj}	...	x_{nn}	y_{n1}	y_{n2}	...	y_{nr}	x_n
1	h_{11}	h_{12}	...	h_{1j}	...	h_{1n}					
2	h_{21}	h_{22}	...	h_{2j}	...	h_{2n}					
.											
m	h_{m1}	h_{m2}	...	h_{mj}	...	h_{mn}					
	x_1	x_2	...	x_j	...	x_n					

The first set of balances requirements state that the sum of the elements in a row (the total use) has to be equal to the total amount available, that is,

$$x_i = x_{i1} + x_{i2} + \dots + x_{in} + y_{i1} + y_{i2} + \dots + y_{ir} \qquad (i = 1, 2, \dots , n). \qquad (2.1\text{-}1)$$

Ex post, in a statistical table, this identity is always fulfilled by force, since the change in stocks creates balance between total source and use. In an *ex ante* analysis this set of equations becomes an equilibrium condition. Total source is either the total domestic output in the given period of time or the sum of domestic output and imports, depending on how one takes foreign trade into account (we will come back to this issue later).

Table 2.2: The I-O table in block form

	Sectors (as users)	Final use	Total source
Sectors (as producers)	$\mathbf{X} = (x_{ij})$	$\mathbf{Y} = (y_{il})$	$\mathbf{x} = (x_i)$
Value added	$\mathbf{H} = (h_{kj})$		
The value distributed	$\mathbf{x} = (x_j)$		

Looking at the <u>columns</u> of the sectors one will find the second set of accounting identities, which describe the composition of the cost of domestic production, adding to the cost of the consumed sectoral commodities (materials) various other cost and income components, so that the sum of the column will be equal to the value of the produced output:

$$x_j = x_{1j} + x_{2j} + \ldots + x_{nj} + h_{1j} + h_{2j} + \ldots + h_{mj} \qquad (j = 1, 2, \ldots, n). \qquad (2.1\text{-}2)$$

Typical components of the value added (h_{kj}) are amortization, the cost of labour and net operating surplus, and direct or indirect taxes, depending on the convention used. It is useful to remember the blocks of a typical I-O table (see Table 2.2) and the matrix algebraic representation of the two sets of balances.

$$\mathbf{x} = \mathbf{X1} + \mathbf{Y1} \qquad (2.1\text{-}3)$$

$$\mathbf{x} = \mathbf{1X} + \mathbf{1H} \qquad (2.1\text{-}4)$$

where **1** is a vector with elements all 1 (the summation vector).

Dividing the elements in the various columns by the corresponding column sum one can calculate the average input coefficients, for example,

$$\mathbf{A} = (a_{ij}) = \mathbf{X}\langle\mathbf{x}\rangle^{-1}, \quad \mathbf{C} = (c_{kj}) = \mathbf{H}\langle\mathbf{x}\rangle^{-1}, \quad \mathbf{S} = (s_{il}) = \mathbf{Y}\langle\mathbf{y}^0\rangle^{-1},$$

which can be arranged in similar table. ($\mathbf{y}^0 = \mathbf{1Y}$, where **1** is the summation vector, all ones and <**a**> denotes a diagonal matrix made up by vector **a**.)

A	**S**
C	0
1	1

The basic equations can be rewritten, using the above derived input coefficients, as follows:

$$\mathbf{x} = \mathbf{Ax} + \mathbf{Y1} = \mathbf{Ax} + \mathbf{y} \qquad (2.1\text{-}3)$$

$$\mathbf{1} = \mathbf{1A} + \mathbf{1C} = \mathbf{1A} + \mathbf{c} \qquad (2.1\text{-}4a)$$

The interpretation of the first equation is straightforward. As we can see, it is nothing but the equilibrium requirement related to the sectoral product balance in Leontief's static model. It is less clear, but the second equation can also be interpreted in terms of Leontief's model of general equilibrium. In a comparative static exercise the elements of the summation vector **1** can be interpreted as the base prices ($\mathbf{p} = \mathbf{1}$). Better to say, they are the base price indexes of the various sectoral commodities, assuming that the observed data represent an economy in equilibrium. We can thus rewrite this equation also into the form it has appeared in Leontief's model:

$$\mathbf{p} = \mathbf{pA} + \mathbf{c} = \mathbf{pA} + \mathbf{1C} \qquad (2.1\text{-}5)$$

Introducing the equivalent counterparts of the different indicators measured in physical units, for example, **q** for **x**, **v** for **y**, **R** for **A**, we can present their equivalence in a tabular form (see Table 2.3). The terms in equal position in the two tables are equal, for example: $\langle\mathbf{p}\rangle\mathbf{R}\langle\mathbf{q}\rangle = \mathbf{A}\langle\mathbf{x}\rangle$, $\langle\mathbf{p}\rangle\mathbf{q} = \mathbf{x}$. In rewriting the value added we have borrowed notations and concept from the previous chapter, thus matrix **D** contains the input coefficients of primary factors of production, vector **r** their prices, vector $\mathbf{c}^{\pi}$ the coefficients of the net operating surplus (profits).

Table 2.3: Input-output tables based on value and physical units

$\langle\mathbf{p}\rangle\mathbf{R}\langle\mathbf{q}\rangle$	$\langle\mathbf{p}\rangle\mathbf{v}$	$\langle\mathbf{p}\rangle\mathbf{q}$		$\mathbf{X} = \mathbf{A}\langle\mathbf{x}\rangle$	$\mathbf{y}$	$\mathbf{x}$
$\langle\mathbf{r}\rangle\mathbf{D}\langle\mathbf{q}\rangle$ $\mathbf{c}^{\pi}\langle\mathbf{q}\rangle$			=	$\mathbf{H} = \mathbf{C}\langle\mathbf{x}\rangle$		
$\mathbf{p}\langle\mathbf{q}\rangle$				$\mathbf{x}$		

From the above table one can also see the reason why the elements of vector **p** in equation (2.1.5) represent price indexes and not their absolute values.

2.1.2. Representation of foreign trade in the I-O tables

So far we have been dealing with closed economies and did not bother with exports, imports and the balance of trade. We can do that in an abstract model, but in applied models one can not ignore them. We will use the following notation:

- index of home origin *h*, index of foreign origin (import) *m*,

thus, for example, the intermediate use of the sectoral commodities (**X**) can be split up into domestically produced ($\mathbf{X}^{h}$) and imported part ($\mathbf{X}^{m}$). We will use similarly the **A**, $\mathbf{A}^{h}$ and $\mathbf{A}^{m}$ input coefficient matrices.

- index of domestic use d, index of terms related to export or foreign use e,

for example, total final use can be split up into domestic use and exports: $\mathbf{y} = \mathbf{y}^{d} + \mathbf{z}$. Distinguishing final use with respect to origin as well we can write: $\mathbf{y}^{hd}$ and $\mathbf{y}^{md}$, where $\mathbf{y}^{h} = \mathbf{y}^{hd} + \mathbf{z}$.

Total use of imported goods in production and their input coefficients can thus be defined as $\mathbf{x}^{m} = \mathbf{1X}^{m}$ and $\mathbf{a}^{m} = \mathbf{1A}^{m}$. Total import of various sectoral commodities, on the other hand, will be denoted by vector $\mathbf{m}$, where $\mathbf{m} = \mathbf{X}^{m}\mathbf{1} + \mathbf{y}^{md}$, and their sum is $m = \mathbf{1m}$. If we combine the two sources, domestic production and imports, we will denote their total by $\hat{\mathbf{x}} = \mathbf{x} + \mathbf{m}$.

The above examples illustrate the notation to be used, but their meaning will become clearer from the position they will hold in the various I-O tables which will be presented bellow. We will present four possible ways in which foreign trade can be incorporated into the scheme of an input-output table, without violating its basic convention, whereby the sectoral row and column totals must be equal.

In the first arrangement (I/O table of type A) we treat imports as if all imported commodities were perfect substitutes for their domestic sectoral output. Therefore, domestic production and imports are presented together in the upper part of the input-output table. Thus, the total amounts of the distributed sectoral commodities equal to $\mathbf{x} + \mathbf{m} = \hat{\mathbf{x}}$.

I/O table of type A

I-O table

$\mathbf{X}$	$\mathbf{y}^{d}$	$\mathbf{z}$	$\hat{\mathbf{x}}$
$\mathbf{h}$			
$(\mathbf{x})$	y_{d}	z	
$\mathbf{m}$			
$\hat{\mathbf{x}}$			$\hat{x}$

I-O coefficients

$\hat{\mathbf{A}}$	$\mathbf{s}^{d}$	$\mathbf{s}^{z}$
$\hat{\mathbf{c}}$		
$(\hat{\mathbf{s}}^{h})$	1	1
$\hat{\mathbf{s}}^{m}$		
$\mathbf{1}$		

In order to make the sectoral column sums to be equal to their row equivalent the amounts of imported commodities ($\mathbf{m}$) had to be added to the domestic output ($\mathbf{x}$) at the end of each column. At first glance this looks to be an extremely artificial and meaningless correction, especially the coefficients calculated from the above table. It can be shown, however, that this solution is not meaningless at all.

Let us write up first the commodity balance equations with the coefficients gained from I/O table of type A:

$$\hat{\mathbf{x}} = \hat{\mathbf{A}}\hat{\mathbf{x}} + \mathbf{y}^{d} + \mathbf{z},$$

by means of which we could estimate the likely effect of a change in final demand on the supply of sectoral commodities. This equation is equivalent the following two:

$$\mathbf{x} = \langle\hat{\mathbf{s}}^{\mathrm{h}}\rangle(\mathbf{Ax} + \mathbf{y}^{\mathrm{d}} + \mathbf{z}) = \hat{\mathbf{A}}^{\mathrm{h}}\mathbf{x} + \hat{\mathbf{y}}^{\mathrm{hd}} + \hat{\mathbf{z}}^{\mathrm{h}}$$

$$\mathbf{m} = \langle\hat{\mathbf{s}}^{\mathrm{m}}\rangle(\mathbf{Ax} + \mathbf{y}^{\mathrm{d}} + \mathbf{z}) = \hat{\mathbf{A}}^{\mathrm{m}}\mathbf{x} + \hat{\mathbf{y}}^{\mathrm{md}} + \hat{\mathbf{z}}^{\mathrm{m}},$$

where $\mathbf{A} = \mathbf{X}\langle\mathbf{x}\rangle^{-1}$, $\hat{\mathbf{A}}^{\mathrm{h}} = \langle\hat{\mathbf{s}}^{\mathrm{h}}\rangle\mathbf{A}$, $\hat{\mathbf{A}}^{\mathrm{m}} = \langle\hat{\mathbf{s}}^{\mathrm{m}}\rangle\mathbf{A}$ and so on.

From this transformation it turns out that the implicit assumption behind such calculation would be that share of domestic production ($\hat{\mathbf{s}}^{\mathrm{h}}$) and imports ($\hat{\mathbf{s}}^{\mathrm{m}}$) would remain the same after the changes take place, and this same composition would prevail in every area of use, including the exports as well. Notice that import would become endogenously determined in such a model.

As far as the pricing equation implied by the above table is concerned, i.e., equation

$$\hat{\mathbf{p}} = \hat{\mathbf{p}}\hat{\mathbf{A}} + p_{\mathrm{m}}\cdot\hat{\mathbf{s}}^{\mathrm{m}} + \hat{\mathbf{c}},$$

it is easy to see that it can also be rearranged in the following two sets of equations:

$$\hat{\mathbf{p}} = \mathbf{p}^{\mathrm{h}}\langle\hat{\mathbf{s}}^{\mathrm{h}}\rangle + p_{m}\cdot\hat{\mathbf{s}}^{\mathrm{m}}$$

$$\mathbf{p}^{\mathrm{h}} = \mathbf{p}^{\mathrm{h}}\hat{\mathbf{A}}^{\mathrm{h}} + p_{\mathrm{m}}\cdot\hat{\mathbf{a}}^{\mathrm{m}} + \mathbf{c},$$

where $\mathbf{p}^{\mathrm{h}}$ can be interpreted as the price index of the domestically produced sectoral commodities, p_{m} the average or general price index of the imports, and $\hat{\mathbf{a}}^{\mathrm{m}} = \mathbf{1}\hat{\mathbf{A}}^{\mathrm{m}} = \hat{\mathbf{s}}^{\mathrm{m}}\mathbf{A}$. Vector $\hat{\mathbf{p}}$ can thus be interpreted as an average users' cost-price of the sectoral commodities.

The second arrangement (I/O table of type B) rests on the opposite assumption: imports are perfect complements to the domestic output of the same sectoral origin. The upper part of the table contains only the commodity balances of domestic production. The import used in production ($\mathbf{x}^{\mathrm{m}} = \mathbf{1X}^{\mathrm{m}}$) or in final consumption (y_{m}) are represented in a separate row.

I/O table of type B

I-O table

$\mathbf{X}^{\mathrm{h}}$	$\mathbf{y}^{\mathrm{hd}}$	$\mathbf{z}$	$\mathbf{x}$
$\mathbf{x}^{\mathrm{m}}$	y_{m}	0	m
$\mathbf{h}$			
$\mathbf{x}$	y_{d}	z	

I-O coefficients

$\mathbf{A}^{\mathrm{h}}$	$\mathbf{s}^{\mathrm{hd}}$	$\mathbf{s}^{\mathrm{z}}$
$\mathbf{a}^{\mathrm{m}}$	s_{m}	0
$\mathbf{c}$		
1	1	1

Let us write up again the commodity balance equations with the coefficients gained from I/O table of type B:

$$\mathbf{x} = \mathbf{A}^{\mathrm{h}}\mathbf{x} + \mathbf{y}^{\mathrm{hd}} + \mathbf{z},$$

to which we can immediately add the following two equations:

$$m = \mathbf{a}^{\mathrm{m}}\mathbf{x} + y_{\mathrm{m}},$$

$$\mathbf{m} = \mathbf{A}^{\mathrm{m}}\mathbf{x} + \mathbf{y}^{\mathrm{hm}}.$$

The first equation can be used to estimate the likely effect of a change in the final demand for domestically produced sectoral commodities on their output. The second and the third equations will show the repercussive effect of the above change on the imports, where the use of imports in production is endogenously determined via **x**.

The interpretation of the

$$\mathbf{p}^{h} = \mathbf{p}^{h}\mathbf{A}^{h} + p_{m}\cdot\mathbf{a}^{m} + \mathbf{c}$$

price equation is straightforward. It can be used for tracing through the likely changes of the sectoral prices of domestically produced commodities resulting from an exogenous change in the cost of imports or some elements of value added.

A third version (I/O table of type C) is aimed at correcting the criticized problem of type A, that is, adding imports to domestic output to regain the equality of the last row and column in the table. Instead of that they propose to subtract imports from final demand, and make the sum of rows be equal to domestic production. In effect, instead of the exports the table contains only net exports ($\mathbf{z} - \mathbf{m}$).

I/O table of type C

I-O table

$\mathbf{X}$	$\mathbf{y}^{d}$	$\mathbf{z} - \mathbf{m}$	$\mathbf{x}$
$\mathbf{h}$	0		
$\mathbf{x}$	y_{d}		

I-O coefficients

$\mathbf{A}$	$\mathbf{s}^{d}$
$\mathbf{c}$	0
$\mathbf{1}$	1

As a result both exports and imports will be exogenous variables in the input-output analysis based on the coefficients gained from I/O table of type C, as can be seen from the basic equations:

$$\mathbf{x} = \mathbf{A}\mathbf{x} + \mathbf{y}^{d} + \mathbf{z} - \mathbf{m},$$

$$\mathbf{p} = \mathbf{p}\mathbf{A} + \mathbf{c}.$$

These equations are the straightforward extensions of the forms we got in the case of the closed model. It suffers, however, from the problem that imported goods are considered to be completely the same in all respect (quality, price) as their domestic counterparts. It provides no explanation for the size of foreign trade.

And finally, a fourth version (I/O table of type D) has been designed exactly with the purpose of making foreign trade, both exports and imports endogenous variables in the otherwise conventional input-output multiplier analysis. It is achieved by the introduction of an additional 'extern trade' sector, which purchases foreign currency by means of exports, which in turn finances imports. The total value of imports ($m = \mathbf{x}^{m}\mathbf{1} + y_{m}$) as a rule differs from the total value of exports ($z = \mathbf{1z}$). Their difference, the balance of trade ($m - z = d_{e}$) can be placed in different positions of the revised I-O table. We will show two possible arrangements here.

Both solutions are based on sound economic logic, both are meant to make up for the difference of produced and realized national income. It can be shown that using the first solution (D1), in which the balance of trade (d_e) appears as part of the value added, one assumes implicitly that the ratio of the balance of trade to total imports, that is, $c_e = d_e/m$, remains constant in the analysis. In the case of the second its absolute value should be set exogenously.

I/O table of type D

D1: I-O table

$\mathbf{X}^h$	$\mathbf{z}$	$\mathbf{y}^{hd}$	$\mathbf{X}$
$\mathbf{x}^m$	0	y_m	m
$\mathbf{h}$	d_e	0	h_d
$\mathbf{x}$	m	y_d	

I-O coefficients

$\mathbf{A}^h$	$\mathbf{r}^z$	$\mathbf{s}^{hd}$
$\mathbf{a}^m$	0	s_m
$\mathbf{c}$	c_e	0
$\mathbf{1}$	1	1

D2: I-O table

$\mathbf{X}^h$	$\mathbf{z}$	$\mathbf{y}^{hd}$	$\mathbf{x}$
$\mathbf{x}^m$	0	$y_m - d_e$	z
$\mathbf{h}$	0	0	h
$\mathbf{x}$	z	y_h	

I-O coefficients

$\mathbf{A}^h$	$\mathbf{s}^z$	$\mathbf{s}^{hd}$
$\mathbf{a}^m$	0	$s_m - s_{de}$
$\mathbf{c}$	0	0
$\mathbf{1}$	1	1

The basic equation of the conventional input-output multiplier analysis, assuming change in final domestic demand, takes the following form in the case of D1:

$$\begin{pmatrix} \Delta\mathbf{x} \\ \Delta m \end{pmatrix} = \begin{pmatrix} \mathbf{A}^h & \mathbf{r}^z \\ \mathbf{a}^m & 0 \end{pmatrix} \begin{pmatrix} \Delta\mathbf{x} \\ \Delta m \end{pmatrix} + \begin{pmatrix} \Delta\mathbf{y}^{hd} \\ \Delta y_m \end{pmatrix}$$

decomposing the equation we get:

$$\Delta\mathbf{x} = \mathbf{A}^h\Delta\mathbf{x} + \mathbf{r}^z\cdot\Delta m + \Delta\mathbf{y}^{hd}$$

$$\Delta m = \mathbf{a}^m\Delta\mathbf{x} + \Delta y_m.$$

These equations reveal the nature of this solution: the change in domestic final demand will not only directly effect domestic production, but indirectly too. It is assumed that export will change too, in order to restore the balance of trade which is upset by the change in imports. As a matter of fact, the change in the exports will be proportional to the change in the import, as can be seen from the first equation.

We can eliminate Δm by substituting its value with the expression standing on the right hand side of the second equation. In this way we may derive the condensed form equivalent of the augmented input-output model equation:

$$\Delta\mathbf{x} = (\mathbf{A}^{\mathrm{h}} + \mathbf{r}^{\mathrm{z}}\circ\mathbf{a}^{\mathrm{m}})\Delta\mathbf{x} + \mathbf{r}^{\mathrm{z}}\cdot\Delta y_{\mathrm{m}} + \Delta\mathbf{y}^{\mathrm{hd}} = \mathbf{A}^{\mathrm{z}}\Delta\mathbf{x} + \mathbf{r}^{\mathrm{z}}\cdot\Delta y_{\mathrm{m}} + \Delta\mathbf{y}^{\mathrm{hd}}.$$

where $\mathbf{r}^{\mathrm{z}}\circ\mathbf{a}^{\mathrm{m}}$ denotes the dyadic product of the two vectors. The matrix $\mathbf{A}^{\mathrm{z}} = (\mathbf{A}^{\mathrm{h}} + \mathbf{r}^{\mathrm{z}}\circ\mathbf{a}^{\mathrm{m}})$ is now of the same size as matrix $\mathbf{A}^{\mathrm{h}}$, containing larger elements. Thus if the changes effects only the final demand for domestic goods, its impact on domestic output can be estimited by solving the equation

$$\Delta\mathbf{x} = \mathbf{A}^{\mathrm{z}}\Delta\mathbf{x} + \Delta\mathbf{y}^{\mathrm{hd}},$$

which looks exactly like the basic equation, but the input-output coefficient matrix contains now elements that transmit the indirect effect of the changing export as well.

This modification of the input-output system will influence the form and content of the price multiplier associated with it too. On the basis of the above augmented input-output coefficient matrix we get the following price form:

$$(\mathbf{p}^{\mathrm{h}},\ p_{\mathrm{e}}) = (\mathbf{p}^{\mathrm{h}},\ p_{\mathrm{e}})\begin{pmatrix}\mathbf{A}^{\mathrm{h}} & \mathbf{r}^{\mathrm{z}} \\ \mathbf{a}^{\mathrm{m}} & 0\end{pmatrix} + (\mathbf{c},\ c_{\mathrm{e}}),$$

which can be decomposed as follows:

$$\mathbf{p}^{\mathrm{h}} = \mathbf{p}^{\mathrm{h}}\mathbf{A}^{\mathrm{h}} + p_{\mathrm{e}}\cdot\mathbf{a}^{\mathrm{m}} + \mathbf{c},$$

$$p_{\mathrm{e}} = \mathbf{p}^{\mathrm{h}}\mathbf{r}^{\mathrm{z}} + c_{\mathrm{e}},$$

where c_{e} denotes the balance of trade coefficient, which belongs to the column of the exports in the value added part of the modified input-output table.

In the definition of the domestic price indexes, unlike in its open version, the cost of import ($\mathbf{a}^{\mathrm{m}}$) is revaluated by the price index p_{e}. The value of this latter index, given by the second equation, reflects changes in the domestic cost of earning foreign exchange via exports.

2.1.3. Partial closure and extensions of the input-output models

The solution followed above is called *partial closure* in the input-output literature, which can be applied to other initially exogenous parts of the model, not accounted for in the chain of repercussion initiated by some external change. Take for example amortization, which is part of the exogenously given value added. One may want to take it into account that if output increases it will automatically generate an increase in investments proportional the increase of amortization, which represents the value of replacement investment. One may, thus, augment the model in the following way:

$$\begin{pmatrix}\Delta\mathbf{x} \\ \Delta\mathbf{x}^{\mathrm{b}}\end{pmatrix} = \begin{pmatrix}\mathbf{A} & \mathbf{B} \\ <\mathbf{c}^{\mathrm{a}}> & \mathbf{0}\end{pmatrix}\begin{pmatrix}\Delta\mathbf{x} \\ \Delta\mathbf{x}^{\mathrm{b}}\end{pmatrix} + \begin{pmatrix}\Delta\mathbf{y}^{\mathrm{e}} \\ \mathbf{0}\end{pmatrix}$$

where $\mathbf{c}^{\mathrm{a}}$ is the row vector of the amortization coefficients, $\mathbf{y}^{\mathrm{e}}$ the exogenous part of final demand, $\Delta\mathbf{x}^{\mathrm{b}}$ the sectoral level of amortization generated by the change in the level of production, and **B** the matrix of investment coefficients (use of sectoral commodities for investments in different sectors). The meaning of the form becomes clearer if we decompose it:

$$\Delta\mathbf{x} = \mathbf{A}\Delta\mathbf{x} + \mathbf{B}\Delta\mathbf{x}^{b} + \Delta\mathbf{y}^{e},$$

$$\Delta\mathbf{x}^{b} = \langle\mathbf{c}^{a}\rangle\Delta\mathbf{x},$$

from which we can derive again the equivalent condensed form:

$$\Delta\mathbf{x} = (\mathbf{A} + \mathbf{B}\langle\mathbf{c}^{a}\rangle)\Delta\mathbf{x} + \Delta\mathbf{y}^{e} = \mathbf{A}^{d}\Delta\mathbf{x} + \Delta\mathbf{y}^{e}.$$

The modification of the input-output system

$$\left(\mathbf{p},\ \mathbf{p}^{b}\right) = \left(\mathbf{p}, \mathbf{p}^{b}\right)\begin{pmatrix} \mathbf{A} & \mathbf{B} \\ <\mathbf{c}^{a}> & \mathbf{0} \end{pmatrix} + \left(\mathbf{c}^{e},\ \mathbf{c}^{b}\right),$$

which can be decomposed as follows:

$$\mathbf{p} = \mathbf{pA} + \mathbf{p}^{b}\langle\mathbf{c}^{a}\rangle + \mathbf{c}^{r},$$

$$\mathbf{p}^{b} = \mathbf{pB} + \mathbf{c}^{b},$$

and $\mathbf{c}^{r}$ denotes the value added coefficients other than amortization and $\mathbf{c}^{b}$ the additional cost coefficients belonging to sectoral investments but not included into the commodity block of the table (e.g., taxes, when material cost is measured at base prices). $\mathbf{p}^{b}$ can be interpreted as the vector of price indexes of capital goods invested into various sectors, which are supposed to modify the value of amortization too ($\mathbf{p}^{b}\langle\mathbf{c}^{a}\rangle$).

By appropriate substitutions we can again eliminate $\mathbf{p}^{b}$ and the second equation to an equivalent condensed form:

$$\mathbf{p} = \mathbf{p}(\mathbf{A} + \mathbf{B}\langle\mathbf{c}^{a}\rangle) + \mathbf{c}^{b}\langle\mathbf{c}^{a}\rangle + \mathbf{c}^{r} = \mathbf{pA}^{d} + \mathbf{c}^{b}\langle\mathbf{c}^{a}\rangle + \mathbf{c}^{r},$$

Partial closure thus extends the scope of the endogenously treated phenomena in the conventional input-output multiplier analyses. By the latter we mean the analysis which rests on a given input-output coefficient matrix and its Leontief inverse, and on a simple matrix-vector multiplication. For example, in last case on the following formulas:

$$\Delta\mathbf{x} = (\mathbf{I} - \mathbf{A}^{d})^{-1}\Delta\mathbf{y}^{e} \qquad \mathbf{p} = (\mathbf{c}^{b}\langle\mathbf{c}^{a}\rangle + \mathbf{c}^{r})(\mathbf{I} - \mathbf{A}^{d})^{-1}.$$

We could also see that the decomposed (*structural*) schemes are much more transparent then the augmented or the condensed (*reduced*) multiplier forms. Their only advantage was the computational convenience (a simple matrix-vector multiplication), which mattered in the early years of input-output analysis, but are no longer required.

2.1.4. Applied input-output volume models

We provide two examples to illustrate how one can formulate somewhat more complex input-output models in their structural form, based on the knowledge of coefficients gained from statistical input-output tables and potentially from other statistical sources. The various balance and functional equations of the input-output model will reappear in the computable general equilibrium models as well, thus, this illustration paves the way for the better understanding the latter models as well.

Table 2.4: The assumed structure and content of the input-output table

absolute values

	$\mathbf{X}^h$	$\mathbf{y}^{ch}$	$\mathbf{y}^{gh}$	$\mathbf{Y}^{vh}$	$\mathbf{z}$	$\mathbf{y}^{kh}$	$\mathbf{x}$
	$\mathbf{X}^m$	$\mathbf{y}^{cm}$	$\mathbf{y}^{gm}$	$\mathbf{Y}^{vm}$	$\mathbf{0}$	$\mathbf{y}^{km}$	$\mathbf{m}$
	$\mathbf{t}^{am}$	t_{cm}	t_{gm}	$\mathbf{t}^{vm}$	0	t_{km}	t_m
	$\mathbf{t}^a$	t_c	t_g	$\mathbf{t}^v$	$-t_z$	t_k	t
	$\mathbf{d}$ $\mathbf{w}$ $\mathbf{t}^w$ $\boldsymbol{\pi}$ $\mathbf{t}^x$						
Σ	$\mathbf{x}$	y_c	y_g	$\mathbf{y}^v$	z	y_k	

coefficients

$\mathbf{A}^h$	$\mathbf{s}^{ch}$	$\mathbf{s}^{gh}$	$\mathbf{B}^h$	$\mathbf{s}^z$	$\mathbf{b}^{kh}$
$\mathbf{A}^m$	$\mathbf{s}^{cm}$	$\mathbf{s}^{gm}$	$\mathbf{B}^m$	$\mathbf{0}$	$\mathbf{b}^{km}$
$\boldsymbol{\tau}^{am}$	τ_{cm}	τ_{gm}	$\boldsymbol{\tau}^{vm}$	0	τ_{km}
$\boldsymbol{\tau}^a$	τ_c	τ_g	$\boldsymbol{\tau}^v$	$-\tau_z$	τ_k
$\mathbf{c}^a$ $\mathbf{c}^w$ $\boldsymbol{\tau}^w$ $\mathbf{c}^\pi$ $\boldsymbol{\tau}^x$					
1	1	1	**1**	1	1

(We advise the reader to come back to that table when he or she feels lost in the midst of the – at the first glance – somewhat complicated notations.)

Table 2.4 lists the data, both the absolute values and the coefficients (the absolute values divided by the appropriate column sum) in an input-output table format. The first two blocks row wise describe the sectoral product balances, i.e., distribution the sectoral products available from home production ($\mathbf{x}$, index *h*) or import ($\mathbf{m}$, index *m*), where the notations are as follows:

$\mathbf{X}^h$, $\mathbf{X}^m$, $\mathbf{A}^h$, $\mathbf{a}^m$	intermediate use and their input coefficients
$\mathbf{y}^{ch}$, $\mathbf{y}^{cm}$, $\mathbf{s}^{ch}$, $\mathbf{s}^{cm}$	personal consumption and their coefficients
$\mathbf{y}^{gh}$, $\mathbf{y}^{gm}$, $\mathbf{s}^{gh}$, $\mathbf{s}^{gm}$	public consumption and their coefficients
$\mathbf{Y}^{vh}$, $\mathbf{Y}^{vm}$, $\mathbf{B}^h$, $\mathbf{B}^m$	sectoral investment and their coefficients
$\mathbf{y}^{kh}$, $\mathbf{y}^{km}$, $\mathbf{b}^{kh}$, $\mathbf{b}^{km}$	change in stocks and their coefficients

Following the two blocks one can find two rows. The first contains the import tariffs ($\mathbf{t}^{am}$, t_{cm}, t_{gm} …; $\boldsymbol{\tau}^{am}$, τ_{cm}, τ_{gm} …), because we assume here that the volume of import is measured at their world market prices, converted to local currency. Therefore, the *total value of imports*, $m = \mathbf{m1}$ measures the amount of foreign currency needed for their purchase, converted to local currency. The second row contains net indirect taxes and subsidies ($\mathbf{t}^a$, t_c, t_g …; $\boldsymbol{\tau}^a$, τ_c, τ_g …). The volume of exports ($\mathbf{z}$) is assumed to show the revenue of the producers, which may be higher than the actual price paid by the foreign buyers. Their difference, the export subsidy (t_z) appears also in this block (with negative sign). By this correction the column *total exports*, *z* becomes equal to the foreign currency earned by them, converted to local currency. The difference of *m* and *z* measures thus the foreign trade deficit (d_e).

Finally, the remaining block contains the value added items and their coefficients:

$\mathbf{d}$ and $\mathbf{c}^{a}$	amortization
$\mathbf{w}$, $\mathbf{t}^{w}$ and $\mathbf{c}^{w}$, $\boldsymbol{\tau}^{w}$	wages and wage surcharges
$\boldsymbol{\pi}$ and $\mathbf{c}^{\pi}$	net operating surplus (profit)
$\mathbf{t}^{x}$, $\boldsymbol{\tau}^{x}$	production taxes/subsidies and their coefficients

Based on the above data we can formulate alternative structural models that can be used for comparative static analysis. For example, we can split up final consumption in such a way that makes it easy to introduce exogenous or endogenous variables into an input-output model:

$$\mathbf{y}^{dh} = \mathbf{y}^{h0} + \mathbf{s}^{vh}\cdot y_{cv} + \mathbf{s}^{gh}\cdot y_{g} + \mathbf{B}\mathbf{y}^{v},$$

$$\mathbf{y}^{dm} = \mathbf{y}^{m0} + \mathbf{s}^{vm}\cdot \boldsymbol{y}_{cv} + \mathbf{s}^{gm}\cdot \boldsymbol{y}_{g} + \mathbf{B}^{m}\mathbf{y}^{v},$$

where the new symbols are as follows:

$\mathbf{y}^{h0}$, $\mathbf{y}^{m0}$	exogenously fixed part of final demand (cf. committed consumption)
y_{cv}	level of variable (personal) consumption
$\mathbf{s}^{vh}$, $\mathbf{s}^{vm}$	unit coefficients of (variable) personal consumption (if different from $\mathbf{s}^{ch}$, $\mathbf{s}^{cm}$)
y_{g}	level of public consumption
$\mathbf{y}^{v}$	level of sectoral investment

We can split up (gross) investment into replacement and net investment ($\mathbf{y}^{v} = \mathbf{y}^{rv} + \mathbf{y}^{nv}$) and the former can be made endogenous by means of the following definitional equation:

$$\mathbf{y}^{rv} = \langle\mathbf{r}^{a}\rangle\langle\mathbf{k}\rangle\mathbf{x},$$

where $\mathbf{r}^{a}\langle\mathbf{k}\rangle = \mathbf{c}^{a}$, $\mathbf{k}$ is the vector of capital coefficients per unit of output and $\mathbf{r}^{a}$ is the vector of the rates of amortization, as before.

Next we may assume that the level of variable personal consumption changes in proportion to wages, where the propensity to consume (φ) is a potential exogenous variable:

$$y_{cv} = \varphi\cdot\mathbf{c}^{w}\mathbf{x}.$$

Summing up, we have so far defined the following sets of equations:

$$\mathbf{x} = \mathbf{A}^{h}\mathbf{x} + \mathbf{y}^{h0} + \mathbf{s}^{vh}\cdot y_{cv} + \mathbf{s}^{gh}\cdot y_{g} + \mathbf{B}^{h}\mathbf{y}^{rv} + \mathbf{B}^{h}\mathbf{y}^{nv} + \mathbf{z} \tag{2.1-6}$$

$$\mathbf{m} = \mathbf{A}^{m}\mathbf{x} + \mathbf{y}^{m0} + \varphi\cdot\mathbf{s}^{vm}\cdot y_{cv} + \mathbf{s}^{gm}\cdot y_{g} + \mathbf{B}^{m}\mathbf{y}^{rv} + \mathbf{B}^{m}\mathbf{y}^{nv} \tag{2.1-7}$$

$$\mathbf{y}^{rv} = \langle\mathbf{r}^{a}\rangle\langle\mathbf{k}\rangle\mathbf{x} \tag{2.1-8}$$

$$y_{cv} = \varphi\cdot\mathbf{c}^{w}\mathbf{x} \tag{2.1-9}$$

We have altogether $3n + 1$ number of equations. If we chose $\mathbf{x}$, $\mathbf{m}$, $\mathbf{y}^{rv}$ and y_{cv} as unknown (endogenous) variables, the number of which is also $3n + 1$, we would arrive at a well defined model, which could be solved once the values of the parameters and potential exogenous variables (first of all $\mathbf{z}$, y_{g}, φ, $\mathbf{y}^{nv}$, $\mathbf{y}^{h0}$ and $\mathbf{y}^{m0}$) are given. We could thus run comparative static simulations to test the likely effect of their changes on the endogenous variables.

Eliminating the unknowns other than $\mathbf{x}$ we reduce the set of equations to n equations, containing only the sectoral levels of output as variables:

$$\mathbf{x} = (\mathbf{A}^{h} + \mathbf{B}^{h}\langle\mathbf{r}^{a}\rangle\langle\mathbf{k}\rangle + \varphi\cdot\mathbf{s}^{vh}\circ\mathbf{c}^{w})\mathbf{x} + \mathbf{y}^{h0} + \mathbf{s}^{gh}\cdot y_{g} + \mathbf{B}^{h}\mathbf{y}^{nv} + \mathbf{z},$$

the solution of which is

$$\mathbf{x} = (\mathbf{I} - \mathbf{A}^{h} - \mathbf{B}^{h}\langle\mathbf{r}^{a}\rangle\langle\mathbf{k}\rangle - \varphi\cdot\mathbf{s}^{vh}\circ\mathbf{c}^{w})^{-1}(\mathbf{y}^{h0} + \mathbf{s}^{gh}\cdot y_{g} + \mathbf{B}^{h}\mathbf{y}^{nv} + \mathbf{z}),$$

where the coefficient matrix

$$(\mathbf{A}^{h} + \mathbf{B}^{h}\langle\mathbf{r}^{a}\rangle\langle\mathbf{k}\rangle + \varphi\cdot\mathbf{s}^{vh}\circ\mathbf{c}^{w})$$

is nothing but the coefficients of a partially closed input-output table. The structural form given by equations (2.1-6) - (2.1-9) is, however, a more transparent presentation of the model, than its reduced form.

The special advantage of the structural form is that it makes easy to redefine the model. We may for example introduce further variables and add further equations to the above core model. For example, the definition of the balance of trade

$$d_{e} = \mathbf{p}^{wm}\mathbf{u} - \mathbf{p}^{we}\mathbf{z}, \tag{2.1-10}$$

where $\mathbf{p}^{wm}$ and $\mathbf{p}^{we}$ are the exogenously given price indexes of imports and exports. As long as we consider d_e an endogenous variable this equation would be just an epilogue added to the rest of the equations, since d_e does not appear in them.

One can also define the demand for labour and capital

$$l_{d} = \mathbf{lx} \tag{2.1-11}$$

$$k_{d} = \mathbf{kx} \tag{2.1-12}$$

or the level of total exports and imports

$$z = \mathbf{1z} - t_{z} \tag{2.1-13}$$

$$m = \mathbf{m1}. \tag{2.1-14}$$

As long as we do not change the rest of the model, the value of the new variables (d_e, l_d, k_d, z, m) can be simply calculated after we have solved already the model given by equations (2.1-6) - (2.1-9), since none of the new variables appears in them. We can, nevertheless, revise the model with respect to variables considered to be endogenous or exogenous. We might want, for example fix the value of k, interpreting it as capital constraint, at the cost of freeing some variable that was exogenous so far (e.g. φ or y_g). We might want to fix the structure of exports ($\mathbf{z} = z\cdot\mathbf{s}^{z}$) and make its level depend on the volume of imports ($\mathbf{z} = m\cdot\mathbf{r}^{z}$).

All these changes would imply another model structure and require another type of solution. One more reason why one might prefer structural to reduced form. Consider, for example, a variant of the model consisting of (2.1-6a), (2.1-7) - (2.1-9), (2.1-10a), where

$$\mathbf{x} = \mathbf{A}^{h}\mathbf{x} + \mathbf{y}^{h0} + \mathbf{s}^{vh}\cdot y_{cv} + \mathbf{s}^{gh}\cdot y_{g} + \mathbf{B}^{h}\mathbf{y}^{rv} + \mathbf{B}^{h}\mathbf{y}^{nv} + z\cdot\mathbf{s}^{z} \tag{2.1-6a}$$

$$\mathbf{p}^{wm}\mathbf{u} - z\cdot\mathbf{p}^{we}\mathbf{s}^{z} = d_{e}. \tag{2.1-10a}$$

In this variant of the model we handle foreign trade (exports and imports) in a similar way as in the analysis based on an input-output model of type D2. One could define a variant analogous with an input-output model of type D1 by the set of equations (2.1-6b), (2.1-7) - (2.1-9), (2.1-10a) and (2.1-14), in which d_e becomes endogenous variable too, moving in proportion to the level of imports and exports, leaving thus their observed proportion (s_{de}) unchanged.

$$\mathbf{x} = \mathbf{A}^h\mathbf{x} + \mathbf{y}^{h0} + \mathbf{s}^{vh}\cdot y_{cv} + \mathbf{s}^{gh}\cdot y_g + \mathbf{B}^h\mathbf{y}^{rv} + \mathbf{B}^h\mathbf{y}^{nv} + m\cdot\mathbf{r}^z, \qquad (2.1\text{-}6b)$$

$$s_{de}\cdot m = d_e. \qquad (2.1\text{-}10b)$$

The possibilities to form models for economic policy analysis on the basis of statistical input-output tables is wide enough, nevertheless, constrained especially by the rigidity of the linear forms. For example, they do not allow for making some of the coefficients dependent on price changes. The price models based on the input-output tables are developed with no reference to the volume side either. Let us turn now our attention to applied input-output price models.

2.1.5. Applied input-output price models

Input-output tables can be compiled using two different price concepts: at users' prices and/or at net (producers') prices, expressing material cost net of indirect taxes/subsidies. Table 2.4 presented in the previous section was supposed to be compiled at net prices, that is, the value of the domestic goods expressed in producers' prices, and that of the imported goods in their cost of purchase, not including import tariffs. This is why a separate row had to be added to the table, which contained the sum of indirect taxes, subsidies and import tariffs. If we started from a table compiled at users' prices, we should try to separate, as accurately as auxiliary data allow for, the various components that make up unit prices. This will be discussed in more details later, in Chapter 5.

As pointed out earlier, the column sum identities of the first block of the input-output tables, i.e. the equations

$$\mathbf{1A}^h + \mathbf{1A}^m + \boldsymbol{\tau}^{am} + \boldsymbol{\tau}^a + \mathbf{c}^a + \mathbf{c}^w + \boldsymbol{\tau}^w + \mathbf{c}^\pi + \boldsymbol{\tau}^x = \mathbf{1}$$

reflect the basic price accounting identity: prices (revenues) equal costs. In other words, they define the base price levels of the domestically produced sectoral outputs. The summation vectors (**1**) represent in the above equations indeed the base price levels, all assumed to be equal to 1, because the volume of the sectoral outputs are measured with their values expressed at base prices.

So, one can easily revise the above price accounting identity by explicitly indicating the price indexes that are hidden in them. Let us introduce in the first step the domestic and import price indexes of the sectoral outputs, $\mathbf{p}^h = (p_i^h)$ and $\mathbf{p}^m = (p_i^m)$, which are equal to **1** in the base case. Assuming that $\mathbf{p}^m$ contains import tariffs as well, one gets the following form:

$$\mathbf{p}^h = \mathbf{p}^h\mathbf{A}^h + \mathbf{p}^m\mathbf{A}^m + \boldsymbol{\tau}^a + \mathbf{c}^a + \mathbf{c}^w + \boldsymbol{\tau}^w + \mathbf{c}^\pi + \boldsymbol{\tau}^x, \qquad (2.1\text{-}15)$$

where the values in $\mathbf{p}^h$ will be taken to be endogenous, whereas those in $\mathbf{p}^m$ and the rest of the components exogenous variables in the price model. Note that the domestic price of the imported sectoral commodities (elements of vector $\mathbf{p}^{m}$) is assumed to include the import tariffs as well. This is why they do not appear explicitly in equation (2.1-15).

With the model defined in this way we can evaluate the likely effect of expected changes in the exogenously treated (tax and value added items) variables on domestic sectoral price levels. This is the basic idea lying behind the input-output price models. The above model is just their simplest version. One can get more complex and complicated models by introducing further exogenous explanatory variables.

For example, the consumers' price index (p_c) can be defined, based on data from Table 2.4 as follows

$$p_c = \mathbf{p}^h\mathbf{s}^{ch} + \mathbf{p}^m\mathbf{s}^{cm} + \tau_c. \tag{2.1-16}$$

Observe, that in the base case, when $\mathbf{p}^h = \mathbf{p}^m = \mathbf{1}$, the value of p_c is also 1, and the above equation coincides with that defining the column sum identity of private consumption. We can use the consumers' price index (p_c) defined in this way to revaluate the wage level in case of assumed changes in the price levels of the sectoral commodities. If one introduces this price index into equation (2.1-15), as an index variable that valorises wages in an endogenous way, he will get the following set of equations:

$$\mathbf{p}^h = \mathbf{p}^h\mathbf{A}^h + \mathbf{p}^m\mathbf{A}^m + \boldsymbol{\tau}^a + \mathbf{c}^a + p_c\cdot\mathbf{c}^w + \boldsymbol{\tau}^w + \mathbf{c}^{\pi} + \boldsymbol{\tau}^x. \tag{2.1-17}$$

Changing some elements of $\mathbf{p}^m$, for example, and solving equations (2.1-16) and (2.1-17) for variables $\mathbf{p}^h$ and p_c, one can estimate the likely effect of expected changes in import prices on domestic sectoral price levels, assuming that the wages and salaries will also increase in proportion to the consumers' price index. We pause here for a moment and show that the reduced form of the above simple price model can be derived from a partially closed input-output table, in which wages are expressed in terms of private consumption. Eliminating p_c we get

$$\mathbf{p}^h = \mathbf{p}^h(\mathbf{A}^h + \mathbf{s}^{ch}\circ\mathbf{c}^w) + \mathbf{p}^m(\mathbf{A}^m + \mathbf{s}^{cm}\circ\mathbf{c}^w) + \boldsymbol{\tau}^a + \mathbf{c}^a + \tau_c\cdot\mathbf{c}^w + \boldsymbol{\tau}^w + \mathbf{c}^{\pi} + \boldsymbol{\tau}^x,$$

from which, after simple rearrangement, we can calculate the value of vector $\mathbf{p}^h$ as

$$\mathbf{p}^h = \{\mathbf{p}^m(\mathbf{A}^m + \mathbf{s}^{cm}\circ\mathbf{c}^w) + \boldsymbol{\tau}^a + \mathbf{c}^a + \tau_c\cdot\mathbf{c}^w + \boldsymbol{\tau}^w + \mathbf{c}^{\pi} + \boldsymbol{\tau}^x\}(\mathbf{I} - \mathbf{A}^h - \mathbf{s}^{ch}\circ\mathbf{c}^w)^{-1}.$$

The input-output coefficient matrix in the above equations, ($\mathbf{A}^h + \mathbf{s}^{ch}\circ\mathbf{c}^w$) is nothing but the coefficients gained from a partially closed input-output table, in which wages ($\mathbf{c}^w$) are represented by consumption basket units, composed as ($\mathbf{s}^{ch}$, $\mathbf{s}^{cm}$, τ_c).

A somewhat more complex version of this model can be formulated by the following set of equations:

$$\mathbf{p}^h = \mathbf{p}^h\mathbf{A}^h + p_z\cdot\mathbf{p}^m\mathbf{A}^m + \boldsymbol{\tau}^a + \mathbf{p}^b\langle\mathbf{c}^a\rangle + p_c\cdot(\mathbf{c}^w + \boldsymbol{\tau}^w) + \boldsymbol{\tau}^x + \mathbf{c}^{\pi}, \tag{PM-I-1}$$

$$p_c = \mathbf{p}^h\mathbf{s}^{ch} + p_z\cdot\mathbf{p}^m\mathbf{s}^{cm} + \tau_c, \tag{PM-I-2}$$

$$\mathbf{p}^b = \mathbf{p}^h\mathbf{B}^h + p_z\cdot\mathbf{p}^m\mathbf{B}^m + \boldsymbol{\tau}^v, \tag{PM-I-3}$$

$$p_z = \mathbf{p}^h\mathbf{s}^z - \tau_z, \tag{PM-I-4}$$

In this model not only the wages but the surcharges on wages are also valorised by the consumer price index, $p_c \cdot (\mathbf{c}^w + \boldsymbol{\tau}^w)$, taking into account their *ad valorem* nature. Also, we have introduced the investment price indexes ($\mathbf{p}^b$) to reflect the revaluation of the investment goods and thus the value of amortization. Finally, the export price index, p_z has been introduced, which can be interpreted as the cost index of purchasing foreign currency via exports. If domestic prices increase, p_z will increase too, raising automatically the cost of imports as well. It acts thus as an endogenous exchange rate index that neutralizes, in a sense, the impact of the domestic price level on the position of the trade balance.

MORE COMPLEX INPUT-OUTPUT PRICE MODELS

The price models presented so far were using only data available from an I-O table, and they were structural variants of some straightforward multipliers, which could be composed on the basis of partially closed input-output tables. These models can be augmented and refined.

In more complex models, depending on the availability of auxiliary data and the purpose of investigation, we can for example represent material cost, i.e. the total cost of the commodities used in various sectors of production or areas of final use in different ways. The options differ from each other with respect to the treatment of indirect taxes/subsidies (which make the users' prices different from the producers' prices) and imports.

a) Representing the users' cost of commodity inputs

Consider first the issue of indirect taxes and subsidies. In the statistical input-output tables expressed in base prices (like in Table 2.4) only their sum is presented in one raw, thus, we can calculate and use only the average tax or subsidy coefficients ($\boldsymbol{\tau}^a$, τ_c, τ_g, $\boldsymbol{\tau}^v$, $-\tau_z$, τ_k) paid or received by the specific users.[3] In the statistical input-output tables expressed in users prices the indirect taxes/subsidies are represented, inseparably from the base price, in the cost of materials. In any case, one needs access to a rectangular tax table in order to define a model in which indirect taxes/subsidies are represented in sectoral details. In order to keep the number of parameters and/or variables of the model relatively small, even if we have at our disposal an estimated tax table, we would not represent specific tax rate in each cell of use. Either we use average tax rates calculated for given areas of material use (i.e., column wise) or the average tax rates calculated for given type of sectoral commodity (i.e., row wise).

Another issue related to taxes and subsidies is the question of valorisation. Most of the indirect taxes are of *ad valorem* type. This means that they change in proportion to the value of the material cost measured at base prices. In the above (PM-I) price model, however, most of the tax/subsidy coefficients remained nominal, not valorised. This is the inherent weakness of the simple input-output price models, which can be easily eliminated in the refined models. Let us take the example of the investment price indexes to illustrate the alternative ways in which indirect taxes/subsidies can be represented in valorised (real rather than nominal) form. Using

[3] In this subsection and only here we will notations τ, τ' and τ'' to distinguish from each other the tax coefficient calculated from an input-output table (τ), the average *ad valorem* tax rate in a given area of use (τ'') and the average *ad valorem* tax rate imposed on some kind of sectoral commodity (τ').

nominal (unvalorised) tax coefficients the definition of the investment price index takes the following form:

$$\mathbf{p}^{\mathrm{b}} = \mathbf{p}^{\mathrm{h}}\mathbf{B}^{\mathrm{h}} + \mathbf{p}^{\mathrm{m}}\mathbf{B}^{\mathrm{m}} + \boldsymbol{\tau}^{\mathrm{v}}.$$

If we know only the average tax coefficients, i.e., the vector $\boldsymbol{\tau}^{\mathrm{v}}$, we can equivalently use any of the two following forms to valorise the indirect taxes:

A) $\mathbf{p}^{\mathrm{b}} = \mathbf{p}^{\mathrm{h}}\mathbf{B}^{\mathrm{h}} + \mathbf{p}^{\mathrm{m}}\mathbf{B}^{\mathrm{m}} + \mathbf{p}^{\mathrm{b}}\langle\boldsymbol{\tau}^{\mathrm{v}}\rangle$, that is, $p_j^{\mathrm{b}} = \Sigma_i\,(p_i^{\mathrm{h}}\cdot b_{ij}^{\mathrm{h}} + p_i^{\mathrm{m}}\cdot b_{ij}^{\mathrm{m}}) + p_j^{\mathrm{b}}\cdot \tau_j^{\mathrm{v}}$, or

A') $\mathbf{p}^{\mathrm{b}} = (\mathbf{p}^{\mathrm{h}}\mathbf{B}^{\mathrm{h}} + \mathbf{p}^{\mathrm{m}}\mathbf{B}^{\mathrm{m}})\langle\mathbf{1} + \boldsymbol{\tau}''^{\mathrm{v}}\rangle$, that is, $p_j^{\mathrm{b}} = (1 + \tau''^{\mathrm{v}}_j)\cdot\Sigma_i\,(p_i^{\mathrm{h}}\cdot b_{ij}^{\mathrm{h}} + p_i^{\mathrm{m}}\cdot b_{ij}^{\mathrm{m}})$,

where $\tau''^{\mathrm{v}}_j = \tau_j^{\mathrm{v}}/(1 + \tau_j^{\mathrm{v}})$, that is, using matrix algebraic notation: $\boldsymbol{\tau}''^{\mathrm{v}} = \boldsymbol{\tau}^{\mathrm{v}}\langle\mathbf{1} - \boldsymbol{\tau}^{\mathrm{v}}\rangle^{-1}$.

If we know the average tax rates ($\boldsymbol{\tau}'^{\mathrm{v}}$), which are imposed on various sectoral commodities used for investment, we may assume that the same rate applies in each sector (i.e., in each cell in a given raw of the investment matrix). In such a case the definition of the investment price indexes will be as follows:

B) $\mathbf{p}^{\mathrm{b}} = (\mathbf{p}^{\mathrm{h}}\langle\mathbf{1}+\boldsymbol{\tau}'^{\mathrm{v}}\rangle\mathbf{B}^{\mathrm{h}} + \mathbf{p}^{\mathrm{m}}\langle\mathbf{1}+\boldsymbol{\tau}'^{\mathrm{v}}\rangle\mathbf{B}^{\mathrm{m}})$, that is, $p_j^{\mathrm{b}} = \Sigma_i\,(1 + \tau'^{\mathrm{v}}_i)\cdot(p_i^{\mathrm{h}}\cdot b_{ij}^{\mathrm{h}} + p_i^{\mathrm{m}}\cdot b_{ij}^{\mathrm{m}})$,

It is often the case that one has access only to an input-output table of type A and there is no separate rectangular import table available. Even in such a case one may separate the use of domestic and imported commodities, making a bold assumption that their ratio is the same in each area of use. Let $\mathbf{s}^{\mathrm{h}} = (s_i^{\mathrm{h}})$ and $\mathbf{s}^{\mathrm{m}} = (s_i^{\mathrm{m}})$ be the vectors of shares of domestic supply $(x_i - z_i)$ and imports (m_i) in total domestic supply of various sectoral goods $(x_i - z_i + u_i)$, i.e.,

$$s_i^{\mathrm{h}} = (x_i - z_i)/(x_i - z_i + u_i) \quad \text{and} \quad s_i^{\mathrm{m}} = u_i/(x_i - z_i + u_i).$$

With the above share coefficients one can hypothetically split up the commodities used into domestic and imported parts. For example, in the case of the production and investment input coefficient matrixes, matrixes $\mathbf{A}$ and $\mathbf{B}$ can be split up as follows:

$$\mathbf{A}^{\mathrm{h}} = \langle\mathbf{s}^{\mathrm{h}}\rangle\mathbf{A} \text{ and } \mathbf{A}^{\mathrm{m}} = \langle\mathbf{s}^{\mathrm{m}}\rangle\mathbf{A}, \qquad \mathbf{B}^{\mathrm{h}} = \langle\mathbf{s}^{\mathrm{h}}\rangle\mathbf{B} \text{ and } \mathbf{B}^{\mathrm{m}} = \langle\mathbf{s}^{\mathrm{m}}\rangle\mathbf{B},$$

and the missing real $\mathbf{A}^{\mathrm{h}}$, $\mathbf{A}^{\mathrm{m}}$, $\mathbf{B}^{\mathrm{h}}$ and $\mathbf{B}^{\mathrm{m}}$ coefficients can be approximated by their above rough estimates.

This solution implies yet another possible definition of the investment price indexes, which is based on formula B) and the above estimates of $\mathbf{B}^{\mathrm{h}}$ and $\mathbf{B}^{\mathrm{m}}$:

C) $\mathbf{p}^{\mathrm{b}} = (\mathbf{p}^{\mathrm{h}}\langle\mathbf{1}+\boldsymbol{\tau}'^{\mathrm{v}}\rangle\mathbf{B}^{\mathrm{h}} + \mathbf{p}^{\mathrm{m}}\langle\mathbf{1}+\boldsymbol{\tau}'^{\mathrm{v}}\rangle\mathbf{B}^{\mathrm{m}}) = \mathbf{p}^{\mathrm{hm}}\langle\mathbf{1} + \boldsymbol{\tau}'^{\mathrm{v}}\rangle\mathbf{B}$,

where

$$\mathbf{p}^{\mathrm{hm}} = \mathbf{p}^{\mathrm{h}}\langle\mathbf{s}^{\mathrm{h}}\rangle + \mathbf{p}^{\mathrm{m}}\langle\mathbf{s}^{\mathrm{m}}\rangle, \text{ that is, } p_i^{\mathrm{hm}} = p_i^{\mathrm{h}}\cdot s_i^{\mathrm{h}} + p_i^{\mathrm{m}}\cdot s_i^{\mathrm{m}},$$

the vector of average price indexes of the domestically produced and imported cost. They are weighted averages, as a matter of fact, where the weights are the share coefficients $\mathbf{s}^{\mathrm{h}}$ and $\mathbf{s}^{\mathrm{m}}$. This solution rests on the assumption, as noted earlier, that each user gets domestic and imported commodities in the same composition, as if they used a special composite commodity.

In the absence of an investment matrix, one may simply use the average investment price index, defined by the average sectoral composition of investments, for example, as

$$p_{\mathrm{b}} = (\mathbf{p}^{\mathrm{h}}\mathbf{b}^{\mathrm{h}} + \mathbf{p}^{\mathrm{m}}\mathbf{b}^{\mathrm{m}})(1 + \tau_{\mathrm{v}}),\ \text{or}\ (\mathbf{p}^{\mathrm{h}}<\mathbf{1}+\boldsymbol{\tau}'^{\mathrm{v}}>\mathbf{b}^{\mathrm{h}} + \mathbf{p}^{\mathrm{m}}<\mathbf{1}+\boldsymbol{\tau}'^{\mathrm{v}}>\mathbf{b}^{\mathrm{m}})\ \text{or}\ \mathbf{p}^{\mathrm{hm}}<\mathbf{1}+\boldsymbol{\tau}'^{\mathrm{v}}>\mathbf{b}.$$

Mutatis mutandis the same possibilities are open for representing the material cost of production, which – depending on the availability of date – can be expressed by any of the following forms:

$$\begin{array}{lll}
(\mathbf{p}^{\mathrm{h}}\mathbf{A}^{\mathrm{h}} + \mathbf{p}^{\mathrm{m}}\mathbf{A}^{\mathrm{m}})<\mathbf{1} + \boldsymbol{\tau}''^{\mathrm{a}}>, & \text{that is,} & (1+\tau''^{\mathrm{a}}_j)\cdot\Sigma_i\,(p_i^{\mathrm{h}}\cdot a_{ij}^{\mathrm{h}} + p_i^{\mathrm{m}}\cdot a_{ij}^{\mathrm{m}}),\\
(\mathbf{p}^{\mathrm{h}}<\mathbf{1}+\boldsymbol{\tau}'^{\mathrm{a}}>\mathbf{A}^{\mathrm{h}} + \mathbf{p}^{\mathrm{m}}<\mathbf{1}+\boldsymbol{\tau}'^{\mathrm{a}}>\mathbf{A}^{\mathrm{m}}), & \text{that is,} & \Sigma_i\,(1+\tau'^{\mathrm{a}}_i)\cdot(p_i^{\mathrm{h}}\cdot a_{ij}^{\mathrm{h}} + p_i^{\mathrm{m}}\cdot a_{ij}^{\mathrm{m}}),\\
\mathbf{p}^{\mathrm{hm}}<\mathbf{1} + \boldsymbol{\tau}'^{\mathrm{a}}>\mathbf{A}, & \text{that is,} & \Sigma_i\,(1+\tau'^{\mathrm{a}}_i)\cdot p_i^{\mathrm{hm}}\cdot a_{ij},
\end{array}$$

where $\tau''^{\mathrm{a}}_j = \tau^{\mathrm{a}}_j/(\Sigma_i(p_i^{\mathrm{h}}\cdot a_{ij}^{\mathrm{h}} + p_i^{\mathrm{m}}\cdot a_{ij}^{\mathrm{m}})$ and τ'^{a}_i is the tax imposed on sectoral commodity i used in production. The *users' price of commodity i* is usually taken to be the same across all uses except in private consumption, where additional taxes/subsidies modify it.

b) Representing foreign trade prices

If sufficiently detailed foreign trade information is available, one can define the indexes of the imported sectoral commodities as

$$\mathbf{p}^{\mathrm{m}} = v\cdot\mathbf{p}^{\mathrm{wm}}<\mathbf{1} + \boldsymbol{\tau}'^{\mathrm{m}}>,\ \text{that is,}\ \ p_i^{\mathrm{m}} = (1+\tau'^{\mathrm{m}}_i)\cdot v\cdot p_i^{\mathrm{wm}},$$

where v is the exchange rate, $\boldsymbol{\tau}'^{\mathrm{m}} = (\tau'^{\mathrm{m}}_i)$ are the average *ad valorem* import tariff rates across sectors of origin.

Table 2.4 contained only the sums of the import tariffs paid by the different users, ($\mathbf{t}^{\mathrm{am}}$, t_{cm}, t_{gm} …; $\boldsymbol{\tau}^{\mathrm{am}}$, τ_{cm}, τ_{gm} …). If we have only these data, we can only calculate *ad valorem* rates of import tariffs for the different users of the imported commodities, ($\boldsymbol{\tau}''^{\mathrm{am}}$, τ''_{cm}, τ''_{gm} …). In such a case we could define the $\mathbf{p}^{\mathrm{m}}$ import prices as we assumed above. Instead of the term $\mathbf{p}^{\mathrm{m}}\mathbf{A}^{\mathrm{m}}$, for example, we would have to use $v\cdot\mathbf{p}^{\mathrm{wm}}\mathbf{A}^{\mathrm{m}}<\mathbf{1} + \boldsymbol{\tau}''^{\mathrm{m}}>$ to represent the value of the imported commodities used in the different sectors of production, $v\cdot\mathbf{p}^{\mathrm{wm}}\mathbf{B}^{\mathrm{m}}<\mathbf{1} + \boldsymbol{\tau}''^{\mathrm{vm}}>$ in investment and so on.

One can define the domestic price indexes of exported commodities in a way similar to the case of imports. In case we know the subsidy rates commodity by commodity we can define

$$\mathbf{p}^{\mathrm{e}} = v\cdot\mathbf{p}^{\mathrm{we}}<\mathbf{1} + \boldsymbol{\tau}'^{\mathrm{e}}>,\ \text{that is,}\ \ p_i^{\mathrm{e}} = (1+\tau'^{\mathrm{e}}_i)\cdot v\cdot p_i^{\mathrm{we}},$$

where $\mathbf{p}^{\mathrm{we}} = (p_i^{\mathrm{we}})$ is the vector of world market price indexes of the sectoral commodities, and $\boldsymbol{\tau}'^{\mathrm{e}} = (\tau'^{\mathrm{e}}_i)$ that of the *ad valorem* export subsidy ratios.

Setting $v = p_i^{\mathrm{m}} = p_i^{\mathrm{e}} = 1$ for the base, the corresponding values of the world market price indexes (p_i^{wm}, p_i^{we}) can be calculated from their above definitions as

$$p_i^{\mathrm{wm}} = 1/(1+\tau_i^{\mathrm{m}}),\quad p_i^{\mathrm{we}} = 1/(1+\tau_i^{\mathrm{e}}).$$

From the above definitions it follows that the export prices, calculated from the model assuming changes in the exogenous variables, may be different from the $\mathbf{p}^{\mathrm{h}} = (p_i^{\mathrm{h}})$ domestic prices, despite the fact that in principle they should be regarded homogenous goods. Therefore, the unit value of the output, the price of the domestic/export composite, must be calculated as the weighted average of the prices achieved on the two markets, where the weights (s_j^{d}, s_j^{e}) are the observed market shares:

$$\mathbf{p}^{\mathrm{a}} = \mathbf{p}^{\mathrm{h}}\langle\mathbf{s}^{\mathrm{d}}\rangle + \mathbf{p}^{\mathrm{e}}\langle\mathbf{s}^{\mathrm{e}}\rangle, \text{ that is, } p_j^{\mathrm{a}} = p_j^{\mathrm{h}}\cdot s_j^{\mathrm{d}} + p_j^{\mathrm{e}}\cdot s_j^{\mathrm{e}}.$$

c) <u>Representing the components of value added</u>

One may also introduce further explanatory variables into the definition of the *labour cost* and define it as $\mathbf{w}\langle\mathbf{l}\rangle$, where $\mathbf{l} = (l_j)$ is the vector of labour input coefficients, and $\mathbf{w}$ is the unit cost of labour defined as

$$\mathbf{w} = w\cdot\mathbf{d}^{\mathrm{w}}\langle\mathbf{1} + \boldsymbol{\tau}'^{\mathrm{w}}\rangle, \text{ that is, } w_j = (1+\tau_j'^{\mathrm{w}})\cdot w\cdot d_j^{\mathrm{w}},$$

where w is the average (general) wage level (w = 1 in the base), the vector $\mathbf{d}^{\mathrm{w}} = (d_j^{\mathrm{w}})$ contains the average sectoral wage coefficients, whereas $\boldsymbol{\tau}'^{\mathrm{w}} = (\tau_j'^{\mathrm{w}})$ the wage tax rates.

The treatment of *net operating surplus* deserves special attention. In theory, this should be interpreted as net return on capital, which in equilibrium is uniform across sectors. In practice, however, the net returns on capital exhibit lasting sectoral variations. One could follow the solution used by L. Johansen to overcome this problem in his pioneering CGE model. He used a variable to indicate the general rate of net return on capital (π), yet he differentiated the rates among sectors by multiplying it with their observed differences in the base case, represented here by vector $\mathbf{d}^{\pi}$. Adopting this solution one may redefine Walras's cost of capital as indicated by the following form:

$$\mathbf{q} = \mathbf{p}^{\mathrm{b}}\langle\mathbf{r}^{\mathrm{a}} + \pi\cdot\mathbf{d}^{\pi}\rangle, \text{ that is, } q_j = p_j^{\mathrm{b}}\cdot(r_j^{\mathrm{a}} + \pi\cdot d_j^{\pi}),$$

where the vector $\mathbf{r}^{\mathrm{a}} = (r_j^{\mathrm{a}})$ contains the rates of amortization as before.

The price indexes in $\mathbf{p}^{\mathrm{b}}$ serve here for the revaluation of capital used in the different sectors (k_j capital/output coefficients). $p_j^{\mathrm{b}} = 1$ and $\pi\cdot d_j^{\pi}\cdot k_j = \pi_j$ in the base. Once the capital/output coefficients are known and the base value (π_{a}) of the general rate of net return is set (usually to 1), the value of parameters $\mathbf{d}^{\pi} = (d_j^{\pi})$ can be uniquely determined. One can assign the average rate of return as base value to π_{a}, too. The capital cost component in the price formation equation will thus take the form $\mathbf{q}\langle\mathbf{k}\rangle = \mathbf{p}^{\mathrm{b}}\langle\mathbf{r}^{\mathrm{a}} + \pi\cdot\mathbf{d}^{\pi}\rangle\langle\mathbf{k}\rangle$.

d) <u>An input-output price model based on Table 2.4</u>

We will first list the equations of a price model, whose parameters are calibrated only using data in Table 2.4. We will assign a set of endogenous variables to each set of equations of the model (on the left hand side), which makes it easy to check the equality of the numbers of variables and equations.

$$(\mathbf{w}) \qquad \mathbf{w} = w\cdot\mathbf{d}^{\mathrm{w}}\langle\mathbf{1} + \boldsymbol{\tau}''^{\mathrm{w}}\rangle \qquad \text{(PM-II-1)}$$

$$(\mathbf{q}) \qquad \mathbf{q} = \mathbf{p}^{\mathrm{b}}\langle\mathbf{r}^{\mathrm{a}} + \pi\cdot\mathbf{d}^{\pi}\rangle \qquad \text{(PM-II-2)}$$

$$(\mathbf{p}^{\mathrm{b}}) \qquad \mathbf{p}^{\mathrm{b}} = \mathbf{p}\mathbf{B}^{\mathrm{h}} + v\cdot\mathbf{p}^{\mathrm{wm}}\mathbf{B}^{\mathrm{m}}\langle\mathbf{1} + \boldsymbol{\tau}''^{\mathrm{vm}}\rangle + \mathbf{p}^{\mathrm{b}}\langle\boldsymbol{\tau}^{\mathrm{v}}\rangle \qquad \text{(PM-II-3)}$$

$$(\mathbf{p}^{\mathrm{h}}) \qquad \mathbf{p}^{\mathrm{h}} = (\mathbf{p}^{\mathrm{h}}\mathbf{A}^{\mathrm{h}} + v\cdot\mathbf{p}^{\mathrm{wm}}\mathbf{A}^{\mathrm{m}}\langle\mathbf{1} + \boldsymbol{\tau}''^{\mathrm{m}}\rangle)\langle\mathbf{1} + \boldsymbol{\tau}''^{\mathrm{a}}\rangle + \mathbf{w}\langle\mathbf{l}\rangle + \mathbf{q}\langle\mathbf{k}\rangle + \mathbf{p}^{\mathrm{h}}\langle\boldsymbol{\tau}^{\mathrm{x}}\rangle \qquad \text{(PM-II-4)}$$

In this model the general wage level (w), the rate of net return (π) and the exchange rate (v) are exogenous variables. We could, for example, estimate with this model the likely effects of their change on the sectoral price levels. In this definition it is implicitly assumed that producers continue to receive the same prices for their products on both home and foreign markets ($\mathbf{p}^{e} = \mathbf{p}^{h}$). It is, therefore, implicitly also assumed, that if world market prices or the exchange rates changed, the export subsidies would adjust. Their rates are thus endogenous variables too, which can be determined by solving the following set of equations for $\boldsymbol{\tau}'^{e}$.

$$(\boldsymbol{\tau}'^{e}) \qquad \mathbf{p}^{h} = v \cdot \mathbf{p}^{we} \langle \mathbf{1} + \boldsymbol{\tau}'^{e} \rangle \tag{PM-II-5}$$

This additional set of equations are separable from the above ones, form an epilogue, since the $\boldsymbol{\tau}'^{e}$ variables play no role in the other equations. One could also add further equations, as part of the epilogue. For example, we can calculate the average consumers' price index in the following way:

$$(p_{c}) \qquad p_{c} = \mathbf{p}^{h}\mathbf{s}^{ch} + v \cdot \mathbf{p}^{wm}\mathbf{s}^{cm} \cdot (1 + \tau''_{cm}) + p_{c} \cdot \tau_{c}, \tag{PM-II-6}$$

where τ_{c} is the net tax/subsidy coefficient directly taken over from the input-output table, and the general rate of import tariffs on consumption, which can be calculated from the coefficient τ_{cm} of the same table. By means of the consumers' price index, we can determine the general real wage index (ω) as

$$(\omega) \qquad \omega = w/p_{c}. \tag{PM-II-7}$$

In a similar way, we can calculate the price index of the exported goods by the following form:

$$(p_{e}) \qquad p_{z} = \mathbf{p}^{h}\mathbf{s}^{z} + p_{z} \cdot \tau_{z}, \tag{PM-II-8}$$

which is the average cost of earning one unit of foreign exchange. This could be used to define a real exchange rate (υ) as follows:

$$(\upsilon) \qquad \upsilon = v/p_{z}. \tag{PM-II-9}$$

We introduced $\boldsymbol{\tau}'^{e}$, p_{c}, p_{z}, ω and υ as endogenous variables therefore the equations which define them form part of the epilogue. We could, however, redefine the role of the real wage rate or the real exchange rate, turning them into exogenous variables instead of their nominal counterparts, w and v. That would, of course, take their equations out of the epilogue, and place them into the simultaneous set of core equations. As a result, we would get a variant of the model (PM-II).

This change would affect profoundly the model, because – as one can check easily – the resulting model will be homogenous of degree zero with respect to prices and other nominal cost and value items. Thus, their general level is undetermined, in other words, can be set freely, for example, by choosing $v = 1$. But this means that the number of the variables is less by one than the number of equations. As a result, the profit rate (π) (or some other formerly exogenous parameter) can no longer remain exogenous variable either, it must be freed.

By the technique of substitution and elimination one can reduce the model we got after these changes to a set of equations of the following form:

$$\mathbf{p}^{h} = \mathbf{p}^{h}\mathbf{S}(\omega, \pi, \upsilon),$$

where matrix **S** represents the parameters of the reduced form, which depend on the values of ω, π and υ. Because, as one can see, this form is also homogenous in terms of the prices, we can choose freely only two of the above three variables. This phenomenon will reappear in the CGE models as well.

d) The input-output pricing block of a typical CGE model

Below we present an input-output price model that forms a part of a typical CGE model as well. The only new symbol in this model is $\boldsymbol{\tau}'^{\mathrm{u}}$, the vector of the commodity specific excise tax rates imposed on every user, except private consumers, in which case the tax rates are different ($\boldsymbol{\tau}'^{\mathrm{c}}$), due to special consumption tax/subsidy provisions, including VAT.

$(\mathbf{p}^{\mathrm{m}})$ $$\mathbf{p}^{\mathrm{m}} = v\cdot\mathbf{p}^{\mathrm{wm}}\langle\mathbf{1}+\boldsymbol{\tau}'^{\mathrm{m}}\rangle \qquad \text{(PM-III-1)}$$

$(\mathbf{p}^{\mathrm{e}})$ $$\mathbf{p}^{\mathrm{e}} = v\cdot\mathbf{p}^{\mathrm{we}}\langle\mathbf{1}+\boldsymbol{\tau}'^{\mathrm{e}}\rangle \qquad \text{(PM-III-2)}$$

$(\mathbf{p}^{\mathrm{h}})$ $$\mathbf{p}^{\mathrm{a}} = \mathbf{p}^{\mathrm{h}}\langle\mathbf{s}^{\mathrm{d}}\rangle + \mathbf{p}^{\mathrm{e}}\langle\mathbf{s}^{\mathrm{e}}\rangle \qquad \text{(PM-III-3)}$$

$(\mathbf{w})$ $$\mathbf{w} = w\cdot\mathbf{d}^{\mathrm{w}}\langle\mathbf{1}+\boldsymbol{\tau}'^{\mathrm{w}}\rangle \qquad \text{(PM-III-4)}$$

$(\mathbf{q})$ $$\mathbf{q} = \mathbf{p}^{\mathrm{b}}\langle\mathbf{r}^{\mathrm{a}} + \pi\cdot\mathbf{d}^{\pi}\rangle \qquad \text{(PM-III-5)}$$

$(\mathbf{p}^{\mathrm{b}})$ $$\mathbf{p}^{\mathrm{b}} = \mathbf{p}^{\mathrm{hm}}\langle\mathbf{1}+\boldsymbol{\tau}'^{\mathrm{u}}\rangle\mathbf{B} \qquad \text{(PM-III-6)}$$

$(\mathbf{p}^{\mathrm{a}})$ $$\mathbf{p}^{\mathrm{a}} = \mathbf{p}^{\mathrm{hm}}\langle\mathbf{1}+\boldsymbol{\tau}'^{\mathrm{u}}\rangle\mathbf{A} + \mathbf{w}\langle\mathbf{l}\rangle + \mathbf{q}\langle\mathbf{k}\rangle + \mathbf{p}^{\mathrm{a}}\langle\boldsymbol{\tau}'^{\mathrm{x}}\rangle \qquad \text{(PM-III-7)}$$

$(\mathbf{p}^{\mathrm{hm}})$ $$\mathbf{p}^{\mathrm{hm}} = \mathbf{p}^{\mathrm{h}}\langle\mathbf{s}^{\mathrm{h}}\rangle + \mathbf{p}^{\mathrm{m}}\langle\mathbf{s}^{\mathrm{m}}\rangle \qquad \text{(PM-III-8)}$$

$(\mathbf{p}^{\mathrm{c}})$ $$\mathbf{p}^{\mathrm{c}} = \mathbf{p}^{\mathrm{hm}}\langle\mathbf{1}+\boldsymbol{\tau}'^{\mathrm{c}}\rangle\mathbf{s}^{\mathrm{c}}, \qquad \text{(PM-III-9)}$$

(p_{c}) $$p_{\mathrm{c}} = \mathbf{p}^{\mathrm{c}}\mathbf{s}^{\mathrm{c}}, \qquad \text{(PM-III-10)}$$

To end this section we list the equations of the last model in scalar form as well, for later reference, to ease for the reader the comparison of these equations with the price formation equations typically present in the programming and CGE models.

(p_i^{m}) $$p_i^{\mathrm{m}} = (1+\tau_i'^{\mathrm{m}})\cdot v\cdot p_i^{\mathrm{wm}}$$

(p_i^{e}) $$p_i^{\mathrm{e}} = (1+\tau_i'^{\mathrm{e}})\cdot v\cdot p_i^{\mathrm{we}}$$

(p_i^{a}) $$p_i^{\mathrm{a}} = p_i^{\mathrm{h}}\cdot s_i^{\mathrm{d}} + p_i^{\mathrm{e}}\cdot s_i^{\mathrm{e}}$$

(w_j) $$w_j = (1+\tau_j'^{\mathrm{w}})\cdot w\cdot d_j^{\mathrm{w}}$$

(q_j) $$q_j = p_j^{\mathrm{b}}\cdot(r_j^{\mathrm{a}} + \pi\cdot d_j^{\pi})$$

(p_j^{b}) $$p_j^{\mathrm{b}} = \Sigma_i\,(1+\tau_i'^{\mathrm{u}})\cdot p_i^{\mathrm{hm}}\cdot b_{ij}$$

(p_j^{a}) $$p_j^{\mathrm{a}} = \Sigma_i\,(1+\tau_i'^{\mathrm{u}})\cdot p_i^{\mathrm{hm}}\cdot a_{ij} + w_j\cdot l_j + q_j\cdot k_j + p_j^{\mathrm{a}}\cdot\tau_j^{\mathrm{x}} =$$
$$(1+\tau_j'^{\mathrm{x}})\cdot\left(\Sigma_i\,(1+\tau_i^{\mathrm{u}})\cdot p_i^{\mathrm{hm}}\cdot a_{ij} + w_j\cdot l_j + q_j\cdot k_j\right)$$

(p_i^{hm}) $$p_i^{\mathrm{hm}} = p_i^{\mathrm{h}}\cdot s_i^{\mathrm{h}} + p_i^{\mathrm{m}}\cdot s_i^{\mathrm{m}}$$

(p_i^{c}) $$p_i^{\mathrm{c}} = (1+\tau_i^{\mathrm{c}})\cdot p_i^{\mathrm{hm}}$$

(p_{c}) $$p_{\mathrm{c}} = \Sigma_i\, p_i^{\mathrm{c}}\cdot s_i^{\mathrm{c}}$$

2.2. Multisectoral resource allocation models: optimum *versus* equilibrium

In this section we will briefly review the once dominant linear programming approach applied to planning and policy analysis. Our aim is partly to remind the reader of the main concepts and methods of the programming approach as applied to macroeconomic policy analysis. More importantly, we want to call attention to the fundamental methodological connections that link together the applied macroeconomic linear or nonlinear programming models, on the one hand and the general equilibrium models, on the other. Presenting a series of multisectoral models, starting with simple linear models, we will gradually shift to their nonlinear versions just in order to arrive at their computable general equilibrium version.

One of the lessons that should be learned form this exercise is to dismiss the claim or criticism that the computable general equilibrium are applicable only to completely or almost completely perfect market economies. They are instead special nonlinear macroeconomic resource allocation models that borrow adjustment mechanisms from microeconomic theory on practical grounds.

Most of the symbols that will be used have already been introduced previously, and we will not repeat their definitions, which will make our presentation easier. In order to make the models more readily understandable and comparable similar models presented before, we will present some of the models in scalar as well as matrix algebraic form.

2.2.1. Linear optimal resource allocation models for economic policy analysis

Let us start with a rather simple model that is based on input-output technology (Leontief) and in which the structure of final demand ($\mathbf{s}^y$) is assumed to be fixed (Kantorovich). So it could be named a Leontief–Kantorovich model. The problem is to find an optimal allocation of resources that provides the highest level (y) of final consumption at given level of primary resources. First we take the example of a closed economy with no foreign trade and having only two primary resources, labour and capital.

We will use the following convention to represent the primal (P) and the corresponding dual (D) problem:

(LP-2.2-1)	(P)	(D)	
	$\mathbf{x} \geqq \mathbf{0},\ y \geqq 0$	$\mathbf{p} \geqq \mathbf{0},\ w,\ q \geqq 0$	
($\mathbf{p}$)	$\mathbf{Ax} + y\cdot\mathbf{s}^y \leqq \mathbf{x}$	$\mathbf{p} \leqq \mathbf{pA} + w\cdot\mathbf{l} + q\cdot\mathbf{k}$	($\mathbf{x}$)
(w)	$\mathbf{lx} \leqq L_0$	$1 \leqq \mathbf{ps}^y$	(y)
(q)	$\mathbf{kx} \leqq K_0$		
	$y \to \max!$	$w\cdot L_0 + q\cdot K_0 \to \min!$	

With scalar algebraic notation:

(LP-2.2-1a)	(P)	(D)
	$x_j,\ y \geqq 0$	$p_j,\ w,\ q,\ v \geqq 0$

$$
\begin{array}{llll}
(p_i) & \Sigma_j\, a_{ij}\cdot x_j + y\cdot s_i^{\mathrm{y}} \leqq x_i & p_j \leqq \Sigma_i\, p_i\cdot a_{ij} + w\cdot l_j + q\cdot k_j & (x_j) \\
(w) & \Sigma_j\, l_j\cdot x_j \leqq L_0 & 1 \leqq \Sigma_i\, p_i\cdot s_i^{\mathrm{y}} & (y) \\
(q) & \Sigma_j\, k_j\cdot x_j \leqq K_0 & & \\
 & y \to \max! & w\cdot L_0 + q\cdot K_0 \to \min! &
\end{array}
$$

On the left and the right margins we have indicated the complementary variables assigned to the constraints. Just to remind the reader, in the optimal solution the complementary slackness conditions must be satisfied, and the optimal values of the dual variables can be interpreted as (*shadow*) *prices* of the respective goods, whose balance requirement is represented by the given constraint.

Under normal assumptions (productive $\mathbf{A}$, $\mathbf{s}^{\mathrm{y}}$, $\mathbf{l}$, $\mathbf{k}$, L_0 and K_0 positive) unique optimal solution will exist, and

$$\mathbf{x}^0, \mathbf{p}^0 > \mathbf{0},\ \ y^0 > 0, \quad \mathbf{x}^0 = y^0\cdot(\mathbf{I}-\mathbf{A})^{-1}\mathbf{s}^{\mathrm{y}}, \quad \mathbf{p}^0 = (w^0\cdot\mathbf{l} + q^0\cdot\mathbf{k})(\mathbf{I}-\mathbf{A})^{-1} \text{ and } \mathbf{p}^0\mathbf{s}^{\mathrm{y}} = 1$$

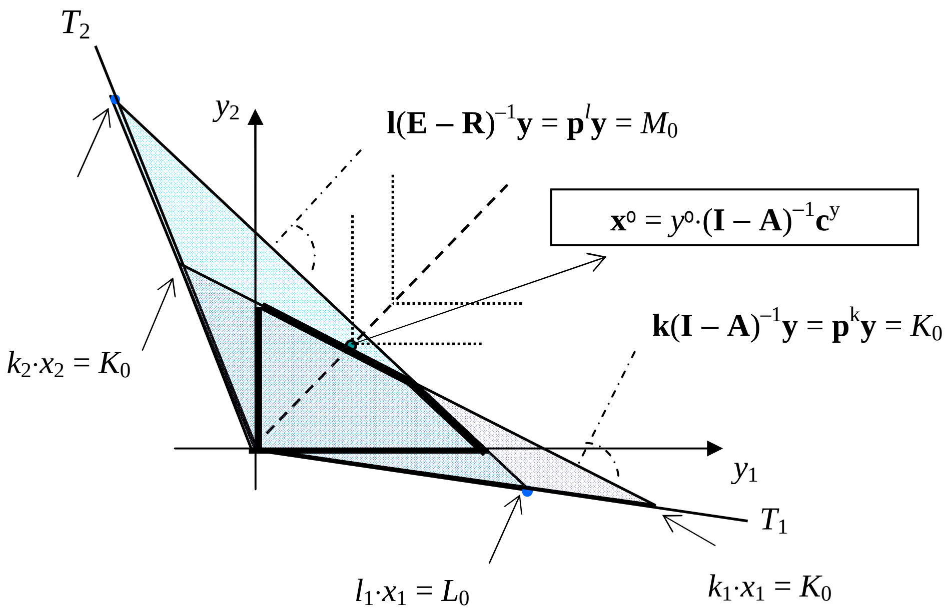
It can be easily seen that this is nothing but the *general equilibrium* of the given Leontief–Kantorovich economy. The numeraire, setting the price level, is $\mathbf{s}^{\mathrm{y}}$, the unit consumption basket ($\mathbf{p}^0\mathbf{s}^{\mathrm{y}} = 1$). The equality if the optimal values of the objective functions

$$y = y\cdot\mathbf{p}^0\mathbf{s}^{\mathrm{y}} = w^0\cdot L_0 + q^0\cdot K_0$$

ensures the fulfilment of Walras's law, that is, the value of final demand will be equal to the value of the primary resources. In fact, it a special case of Koopmans–Kantorovich model presented in the previous chapter, we will therefore leave its further analysis for the reader.

Figure 2.1

The optimal solution of the LP problem 2.2-1 in the case of two goods



From the perspective of the input-output models, $\mathbf{x}^0$ and $\mathbf{p}^0$ are the solutions of the following sets of input-output equations:

$$\mathbf{x} = \mathbf{A}\mathbf{x} + y{\cdot}\mathbf{s}^{\mathrm{y}} \quad \text{and} \quad \mathbf{p} = \mathbf{p}\mathbf{A} + w{\cdot}\mathbf{l} + q{\cdot}\mathbf{k}$$

They are, however, different from Leontief's original model in that both final demand and the prices of primary resources (value added) are now endogenous variables, whereas their values were exogenously given in Leontief's model. As a matter of fact, as a rule, only one price of the primary resources will assume positive value (see Figure 2.1, where we assumed that only the capital constraint is binding).

Let us now introduce the foreign trade possibilities into our model. As will be seen it alters qualitatively the solutions, and we bump into the problem of over-specialization, which is a typical phenomenon of the linear resource allocation models.

(LP-2.2-2)

$$\begin{array}{llcl}
 & (P) & (D) & \\
 & \mathbf{x}, \mathbf{m}, \mathbf{z} \geqq \mathbf{0},\ y \geqq 0 & \mathbf{p} \geqq \mathbf{0},\ w, q, v \geqq 0 & \\
(\mathbf{p}) & \mathbf{A}\mathbf{x} + y{\cdot}\mathbf{s}^{\mathrm{y}} + \mathbf{z} \leqq \mathbf{x} + \mathbf{m} & \mathbf{p} \leqq \mathbf{p}\mathbf{A} + w{\cdot}\mathbf{l} + q{\cdot}\mathbf{k} & (\mathbf{x}) \\
(w) & \mathbf{l}\mathbf{x} \leqq L_0 & \mathbf{p} \leqq v{\cdot}\mathbf{p}^{\mathrm{wm}} & (\mathbf{m}) \\
(q) & \mathbf{k}\mathbf{x} \leqq K_0 & \mathbf{p} \geqq v{\cdot}\mathbf{p}^{\mathrm{we}} & (\mathbf{z}) \\
(v) & \mathbf{p}^{\mathrm{wm}}\mathbf{m} - \mathbf{p}^{\mathrm{we}}\mathbf{z} \leqq d_{\mathrm{e}} & \mathbf{p}\mathbf{s}^{\mathrm{y}} \geqq 1 & (y) \\
 & y \to \max! & w{\cdot}L_0 + q{\cdot}K_0 + v{\cdot}d_{\mathrm{e}} \to \min! &
\end{array}$$

where notations $\to$ max! and $\to$ min! indicate the maximand or minimand objective functions.

Using scalar algebraic notation we rewrite the problems as follows:

(LP-2.2-2a)

$$\begin{array}{llcl}
 & (P) & (D) & \\
 & x_j, m_i, z_i, y \geqq 0 & p_j, w, q, v \geqq 0 & \\
(p_i) & \Sigma_j\, a_{ij}{\cdot}x_j + y{\cdot}s_i^{\mathrm{y}} + z_i \leqq x_i + m_i & p_j \leqq \Sigma_i\, p_i{\cdot}a_{ij} + w{\cdot}l_j + q{\cdot}k_j & (x_j) \\
(w) & \Sigma_j\, l_j{\cdot}x_j \leqq L_0 & p_i \leqq v{\cdot}p_i^{\mathrm{wm}} & (m_i) \\
(q) & \Sigma_j\, k_j{\cdot}x_j \leqq K_0 & v{\cdot}p_i^{\mathrm{we}} \leqq p_i & (z_i) \\
(v) & \Sigma_i\, (p_i^{\mathrm{wm}}{\cdot}m_i - p_i^{\mathrm{we}}{\cdot}z_i) \leqq d_{\mathrm{e}} & 1 \leqq \Sigma_i\, p_i{\cdot}s_i^{\mathrm{y}} & (y) \\
 & y \to \max! & w{\cdot}L_0 + q{\cdot}K_0 + v{\cdot}d_{\mathrm{e}} \to \min! &
\end{array}$$

We assume that $0 < p_i^{\mathrm{we}} \leqq p_i^{\mathrm{we}}$. Under normal condition we may expect that in the optimal solution all primal constraints will be binding. The commodity prices will be all positive, and as can be seen from the dual conditions they will assume values somewhere between the world market export and import prices, in line with the theories on small open economies. The shadow price of the foreign trade constraint (v) can be interpreted as the optimal rate of foreign exchange. The equality of the optimal values of the objective functions

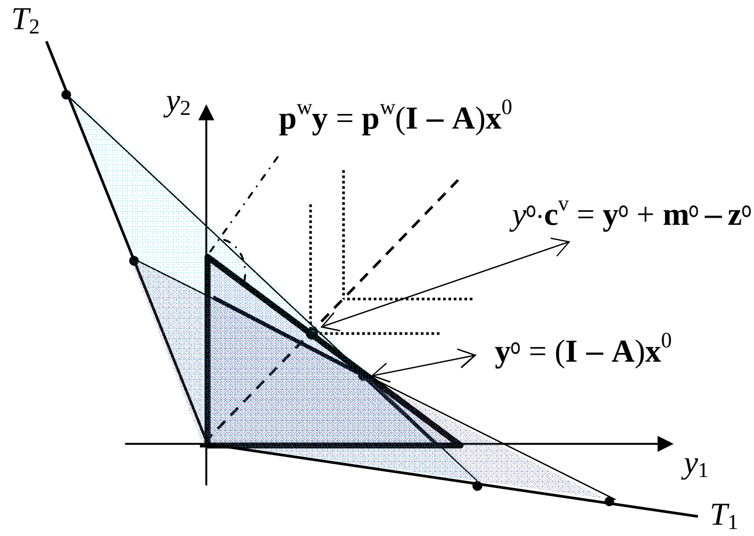
$$y = y{\cdot}\mathbf{p}^0\mathbf{s}^{\mathrm{y}} = w^0{\cdot}L_0 + q^0{\cdot}K_0 + v^0{\cdot}d_{\mathrm{e}}$$

ensures the fulfilment of Walras's law in this case too. In fact, this solution is similar to the one we have seen in the analysis done with the input-output of type D1, where the foreign trade deficit ($d_e > 0$) or surplus ($d_e < 0$) increased or decreased the level of domestic expenditure also.

The opportunity to trade makes it thus possible for the economy to exhaust both primary resources. It introduces, on the other hand, the possibility to specialize. As a matter of fact, it can be shown, that in the case of unique optimal solution at most to sectors will have positive output. Figure 2.2 illustrates the optimal solution in the case of two sectors, assuming that $\mathbf{p}^{wm} = \mathbf{p}^{we} = \mathbf{p}^{w}$.

Figure 2.2

The optimal solution of the LP problem 2.2-2 in the case of two goods



2.2.2. Ad hoc bounds in linear models to constrain overspecialization

In the applied linear programming models designed for policy analysis this possibility of overspecialization was a bothering fact (it challenged the relevance of the models) and therefore the modellers tried to avoid by introducing upper and/or lower bounds on some key variables. We will illustrate this technique and its consequences in our model by introducing upper and lower bounds on the volume of exports

$$\mathbf{z}^l \leqq \mathbf{z} \leqq \mathbf{z}^u, \text{ that is, } z_i^l \leqq z_i \leqq z_i^u,$$

and on the ratios of import/domestic supply:

$$\mathbf{r}^{ml}\langle\mathbf{x}^h\rangle \leqq \mathbf{m} \leqq \mathbf{r}^{mu}\langle\mathbf{x}^h\rangle, \text{ that is, } r_i^{ml}x_i^h \leqq m_i \leqq r_i^{ml}x_i^h,$$

where $x_i^h = x_i - z_i$ as before, and $r_i^m = m_i / x_i^h$.

As a result we get the following, slightly modified version of the previous LP problem.

(LP-2.2-3a)	(P)	(D)	
	$x_j^{\mathrm{h}}, x_j, m_i, z_j, y \geqq 0$	$p_j^{\mathrm{h}}, p_i^{\mathrm{hm}}, w, q, v, \tau_i^{\mathrm{ml}}, \tau_i^{\mathrm{mu}}, \tau_i^{\mathrm{el}}, \tau_i^{\mathrm{eu}} \geqq 0$	
(p_j^{h})	$x_j^{\mathrm{h}} + z_j \leqq x_j$	$p_j^{\mathrm{h}} \leqq \Sigma_i\, p_i^{\mathrm{hm}} \cdot a_{ij} + w \cdot l_j + q \cdot k_j$	(x_j)
(p_i^{hm})	$\Sigma_j\, a_{ij} \cdot x_j + y \cdot s_i^{\mathrm{y}} \leqq x_i^{\mathrm{h}} + m_i$	$1 \leqq \Sigma_i\, p_i^{\mathrm{hm}} \cdot s_i^{\mathrm{y}}$	(y)
(w)	$\Sigma_j\, l_j \cdot x_j \leqq L_0$	$p_i^{\mathrm{hm}} + \tau_i^{\mathrm{mu}} \cdot r_i^{\mathrm{mu}} - \tau_i^{\mathrm{ml}} \cdot r_i^{\mathrm{ml}} \leqq p_i^{\mathrm{h}}$	(x_i^{h})
(q)	$\Sigma_j\, k_j \cdot x_j \leqq K_0$	$p_i^{\mathrm{hm}} \leqq v \cdot p_i^{\mathrm{wm}} + \tau_i^{\mathrm{mu}} - \tau_i^{\mathrm{ml}}$	(m_i)
(v)	$\Sigma_i\, (p_i^{\mathrm{wm}} \cdot m_i - p_i^{\mathrm{we}} \cdot z_i) \leqq d_{\mathrm{e}}$	$v \cdot p_i^{\mathrm{we}} - \tau_i^{\mathrm{eu}} + \tau_i^{\mathrm{el}} \leqq p_i^{\mathrm{h}}$	(z_i)
(τ_i^{ml})	$r_i^{\mathrm{ml}} x_i^{\mathrm{h}} \leqq m_i$		
(τ_i^{mu})	$m_i \leqq r_i^{\mathrm{mu}} x_i^{\mathrm{h}}$		
(τ_i^{el})	$z_i^{l} \leqq z_i$		
(τ_i^{eu})	$z_i \leqq z_i^{u}$		
	$y \to \max!$	$w \cdot L_0 + q \cdot K_0 + v \cdot d_{\mathrm{e}} + \tau_i^{\mathrm{eu}} \cdot z_i^{u} - \tau_i^{\mathrm{el}} \cdot z_i^{l} \to \min!$	

What becomes apparent at the first glance that there is a price for keeping all output, export and import levels all positive, not falling too far from their observable levels (for the sake of simplicity we assumed that all the sectoral commodities were traded in the base). The dual problem became much less transparent and it is less obvious how one can interpret its optimal solution. Since all the primal variables are positive, the dual constraints will be fulfilled in the form of equations. The first two sets

$$p_j^{\mathrm{h}} = \Sigma_i\, p_i^{\mathrm{hm}} \cdot a_{ij} + w \cdot l_j + q \cdot k_j$$

$$\Sigma_i\, p_i^{\mathrm{hm}} \cdot s_i^{\mathrm{y}} = 1$$

are basically the same as before, except for the notation. The first primal constraint belongs to the domestic output, the second to domestic (composite) supply. This is why we assigned to them the dual variables p_j^{h} and p_i^{hm}, in line with notation introduced in the previous section.

The meaning of the

$$p_i^{\mathrm{hm}} = p_i^{\mathrm{h}} - \tau_i^{\mathrm{mu}} \cdot r_i^{\mathrm{mu}} + \tau_i^{\mathrm{ml}} \cdot r_i^{\mathrm{ml}} = v \cdot p_i^{\mathrm{wm}} + \tau_i^{\mathrm{mu}} - \tau_i^{\mathrm{ml}}$$

equations can be deciphered on the basis of the following economic reasoning.

Observe that the formulation of the problem implicitly assumes, that domestic outputs and imports are perfect substitutes, therefore, their prices should be equal in perfect equilibrium. And in fact, both should be sold at prices p_i^{hm}. However, the purchasing price of the imports is $v \cdot p_i^{\mathrm{wm}}$, whereas the producers' price of domestic output is p_i^{h}, and they will be different, as a rule. The shadow prices assigned to the individual constraints, confining their ratios into the given range, generate such taxes/subsidies that equalize them. If the lower limit is binding, which indicates that the import is more expensive than the domestic production, than τ_i^{ml} will be positive and it will lower the domestic sales price of the import and increase that of the domestic output in order to equalize them.

It can be shown that in effect p_i^{hm} will be equal to the weighted average of the component prices,

$$p_i^{\mathrm{hm}} = p_i^{\mathrm{h}} \cdot s_i^{\mathrm{h}} + v \cdot p_i^{\mathrm{wm}} \cdot s_i^{\mathrm{m}},$$

just the way as we defined the price of the domestic/import composite it in the previous section. Observe, however, that here the prices are equalized at the same time, and the taxes/subsidies serve for this purpose. In the previous section we did not assume that, we just calculated the average price. Thus, according to the logic of the model these taxes/subsidies just redistribute income among the users of the commodity, so it will not affect the total available net income, which is given by the objective function of the dual problem as before. The equality of the optimal values of the two objective functions ensures the fulfilment of Walras's law in this case too:

$$y = y \cdot \mathbf{p}^0 \mathbf{s}^{\mathrm{y}} = w^0 \cdot L_0 + q^0 \cdot K_0 + v^0 \cdot d_{\mathrm{e}} + \tau_i^{\mathrm{eu}} \cdot z_i^{u} - \tau_i^{\mathrm{el}} \cdot z_i^{l}.$$

As we can see, these taxes/subsidies do not appear in the net income, unlike those related to the regulations of the export prices, which follows a different logic. Since the prices of the domestic products is determined by their cost, to make the producers sell on both the domestic and the foreign market, the export price have to be made equal to the former. This is exactly the meaning of the dual equation

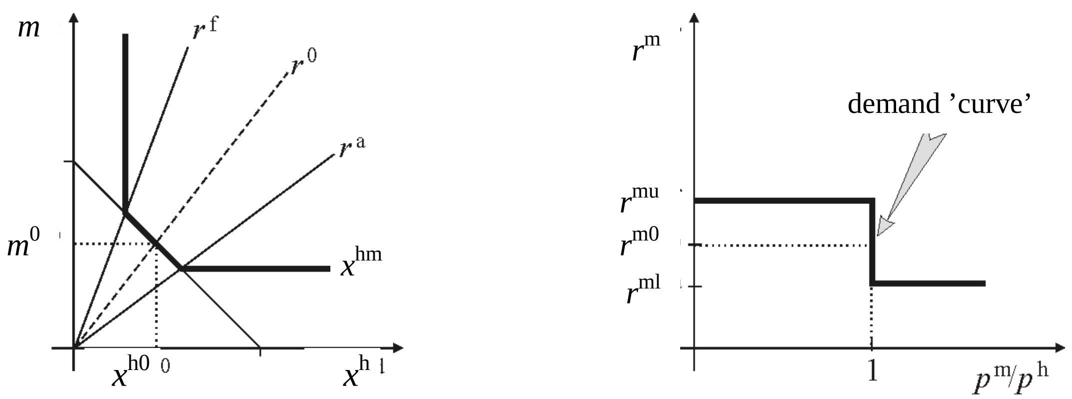
$$v \cdot p_i^{\mathrm{we}} - \tau_i^{\mathrm{eu}} + \tau_i^{\mathrm{el}} = p_i^{\mathrm{h}}.$$

The producers sell their products at prices p_i^{h}, whereas the foreign buyers pay $v \cdot p_i^{\mathrm{we}}$. The price differences (taxes/subsidies) can be interpreted here as income transfers between the domestic tax authority and the foreign buyers, which modify the level of domestic income.

We can illustrate the logic followed in the case of import constraint on Figure 2.3.

Figure 2.3

The logic and working of the import constraints in the LP problem 2.2-3



the level of use value

the graph of the demand implied

The above graph should be all familiar from the microeconomics textbooks, except for the use of piece-wise linear rather than smooth indifference and demand curves. The algebraic representation of the problem illustrated on the graphs is the following

$$\min p_i^{\mathrm{h}}\cdot x_i^{\mathrm{h}} + p_i^{\mathrm{m}}\cdot m_i, \quad \text{s.t.} \quad x_i^{\mathrm{h}} + m_i = x_i^{\mathrm{hm}}, \quad r_i^{\mathrm{ma}}\cdot x_i^{\mathrm{h}} \leqq m_i \leqq r_i^{\mathrm{mf}}\cdot x_i^{\mathrm{h}},$$

and $x^{\mathrm{h}0}$, m^0 and r^0 on the graph represent the observed (base, not the optimal!) values of the variables. Note also, that p_i^{h}, p_i^{m} and x_i^{hm} are considered here to be parameters, unlike in the model itself.

The original indifference curves were linear, in line with the implicit assumption that the domestically produced and the imported goods are perfect substitutes. What the individual bounds do is they turn this relationship into less then perfect substitutability. In a rather rigid manner: they are perfect substitutes between the given bounds, and perfect complements beyond them.

Consider also the case, when domestic outputs and imports are assumed to be perfect complements, that is, their ratio is fixed: $r_i^{\mathrm{m}0} = m_i/x_i^{\mathrm{h}}$. In the above linear model this would mean that $r_i^{\mathrm{ma}} = r_i^{\mathrm{mf}} = r_i^{\mathrm{m}0}$. Modifying the problem accordingly, we would only have one equality condition,

$$r_i^{\mathrm{m}0} x_i^{\mathrm{h}} = m_i$$

instead of the pair of inequality constraints,

$$r_i^{\mathrm{ml}} x_i^{\mathrm{h}} \leqq m_i, \quad \text{and} \qquad m_i \leqq r_i^{\mathrm{mu}} x_i^{\mathrm{h}},$$

and the sign of the dual variable assigned to it (τ_i^{m}) would be undetermined, and the corresponding dual constraints would take the following forms:

$$p_i^{\mathrm{hm}} \pm \tau_i^{\mathrm{m}}\cdot r_i^{\mathrm{m}0} \leqq p_i^{\mathrm{h}} \qquad (x_i^{\mathrm{h}})$$

$$p_i^{\mathrm{hm}} \leqq v\cdot p_i^{\mathrm{wm}} \pm \tau_i^{\mathrm{m}} \qquad (m_i)$$

The optimality conditions would, thus, only slightly change. What we wanted to illustrate, as a matter of fact, was that letting the ratio of import/domestic supply (r_i^{m}) move within some bounds could also be interpreted as relaxing a former assumption of perfect complementarity.

2.2.3. Flexible versus rigid individual bounds: nonlinear approach

One may rightly ask, would it be a better solution to introduce imperfect substitutability by a smooth relationship, as it is usually assumed in microeconomics. The corresponding graph, the equivalent of Figure 2.3 is illustrated on Figure 2.4.

The introduction of smooth substitution possibility would have the same effect as the individual bounds, and in a flexible way. The larger is the difference between their shadow prices, the further the ratio of the two components may depart from their observed value (r^0). Unlike in the case of the rigid bounds, where it jumps to the lower or to the upper bound, whenever they are different. So it makes sense to experiment with such smooth curves.

All we have to do is to replace the constraints

$$\Sigma_j\, a_{ij}\cdot x_j + y\cdot s_i^{\mathrm{y}} \leqq x_i^{\mathrm{h}} + m_i$$

$r_i^{\mathrm{ml}} \cdot x_i^{\mathrm{h}} \leqq m_i$

$m_i \leqq r_i^{\mathrm{mu}} \cdot x_i^{\mathrm{h}}$

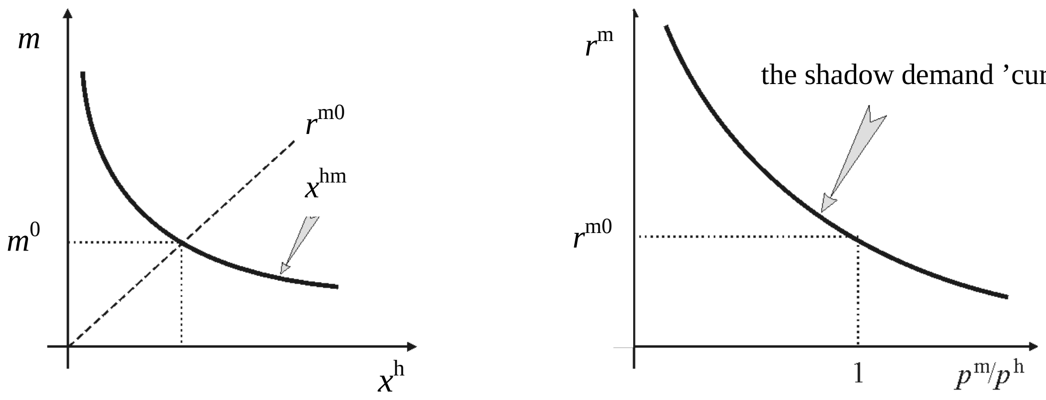
by the following one:

$$\Sigma_j\, a_{ij} \cdot x_j + y \cdot s_i^{\mathrm{y}} \leqq x_i^{\mathrm{hm}}(x_i^{\mathrm{h}},\, m_i),$$

where $x_i^{\mathrm{hm}}(x_i^{\mathrm{h}}, m_i)$ is an appropriately chosen smooth function that represents the substitution possibility assumed to exist between the two goods, taken to be less then perfect substitutes. And we will soon see, exactly this is what we do in the applied models of general equilibrium as well.

Figure 2.4

The logic and working of the flexible bounds



the level of use value ⠀⠀⠀⠀ the graph of the demand implied

The shadow demand curve expresses the ratio of the two components (r^{m}) as a function their price ratio ($p^{\mathrm{m}}/p^{\mathrm{h}}$): $r^{\mathrm{m}} = r^{\mathrm{m}}(p^{\mathrm{h}}, p^{\mathrm{m}})$. The elasticity of this function,

$$\sigma = \frac{\mathrm{d}(u/x^{\mathrm{h}})}{\mathrm{d}(p^{\mathrm{m}}/p^{\mathrm{h}})} : \frac{u/x^{\mathrm{h}}}{p^{\mathrm{m}}/p^{\mathrm{h}}} = \frac{\mathrm{dln}(u/x^{\mathrm{h}})}{\mathrm{dln}(p^{\mathrm{m}}/p^{\mathrm{h}})},$$

the direct (or Allen) elasticity of substitution of the $x_i^{\mathrm{hm}}(x_i^{\mathrm{h}}, m_i)$ function, which determines (together with the cost shares) the own and cross price elasticity of demand. In applied models imperfect substitutability is most often represented by CES (constant elasticity of substitution) functions, the parameters of which are easy to estimate, once we set the elasticity of substitution. The size of the elasticity is usually chosen on the bases of somewhat *ad hoc* expert judgment. But as long we want to use such functions to generate flexible bounds in applied resource allocation models, the *ad hoc* choice of the elasticity parameters is perhaps still superior to the *ad hoc* choice of rigid lower and upper bounds on certain variables.

If we carry out this replacement, the so far linear model of optimal resource allocation will become nonlinear. But this would not matter nowadays, since we have rather powerful algorithms and software that could solve a nonlinear programming model, as long as the nonlinear functions used in the model are well-behaved. So let us do that, and only in the case of the domestic/import supply, but in other parts of the model too. In the case of the ratio of domestic sales and exports ($r_j^{\mathrm{e}} = z_j/x_j^{\mathrm{h}}$) we can also use flexible bounds by means of properly chosen $x_j(x_j^{\mathrm{h}}, z_j)$ transformation functions.

We could and will go on and extend further the scope of using similar smooth functions elsewhere in the model too, for example, in the case of labour and capital, the composition of the personal consumption. But before we do that, we will illustrate with help of the yet simple enough model, how the conditions of optimality will be modified as a result of changing its specification. The table bellow contains the nonlinear version of the LP-2.2-3a model, where in the place of the dual conditions we put the first order necessary conditions of maximum derived by means of the Lagrange (or Kuhn–Tucker) method. Since we assume the observed values of all variables were and remain positive in the optimal solution (including the Lagrange multipliers), we may represent the conditions as equalities (in general, we should use in equalities and complementary slackness conditions as in the case of the LP problem).

NLP-2.2-1	(P)	(KTD)	
	$x_j^{\mathrm{h}}, x_j, m_i, z_j, y \geqq 0,$	$p_j^{\mathrm{a}}, p_i^{\mathrm{hm}}, w, q, v \geqq 0,$	
(p_j^{a})	$x_j(x_j^{\mathrm{h}}, z_j) = x_j,$	$p_j^{\mathrm{a}} = \Sigma_i\, p_i^{\mathrm{hm}} \cdot a_{ij} + w \cdot l_j + q \cdot k_j,$	(x_j)
(p_i^{hm})	$\Sigma_j\, a_{ij} \cdot x_j + y \cdot s_i^{\mathrm{y}} = x_i^{\mathrm{hm}}(x_i^{\mathrm{h}}, m_i),$	$p_i^{\mathrm{hm}} \cdot \dfrac{\partial x_i^{\mathrm{hm}}}{\partial x_i^{\mathrm{h}}} = p_i^{\mathrm{a}} \cdot \dfrac{\partial x_i}{\partial x_i^{\mathrm{h}}},$	(x_i^{h})
(w)	$\Sigma_j\, l_j \cdot x_j = L_0,$	$p_i^{\mathrm{hm}} \cdot \dfrac{\partial x_i^{\mathrm{hm}}}{\partial m_i} = v \cdot p_i^{\mathrm{wm}},$	(m_i)
(q)	$\Sigma_j\, k_j \cdot x_j = K_0,$	$v \cdot p_j^{\mathrm{we}} = p_j^{\mathrm{a}} \cdot \dfrac{\partial x_j}{\partial z_i},$	(z_i)
(v)	$\Sigma_i\, (p_i^{\mathrm{wm}} \cdot m_i - p_i^{\mathrm{we}} \cdot z_i) = d_{\mathrm{e}},$	$1 = \Sigma_i\, p_i^{\mathrm{hm}} \cdot s_i^{\mathrm{y}}.$	(y)
	$y \to \max!$		

The optimality conditions that appearing in the right column of the table can be derived from the differentiation of the following Lagrangian function:

$$L = y - \Sigma_j\, p_j^{\mathrm{a}} \cdot \{x_j(x_j^{\mathrm{h}}, z_j) - x_j\} - \Sigma_i\, p_i^{\mathrm{hm}} \cdot \{\Sigma_j\, a_{ij} \cdot x_j + y \cdot s_i^{\mathrm{y}} - x_i^{\mathrm{hm}}(x_i^{\mathrm{h}}, m_i)\} -$$
$$- w \cdot \{\Sigma_j\, l_j \cdot x_j - L_0\} - q \cdot \{\Sigma_j\, k_j \cdot x_j - K_0\} - v \cdot \{\Sigma_i\, (p_i^{\mathrm{wm}} \cdot m_i - p_i^{\mathrm{we}} \cdot z_i) - d_{\mathrm{e}}\}.$$

We have changed the notation of the dual variable (the Lagrange multiplier) assigned to the first constraint, because here the output is also a composite good, and its shadow price will be interpreted as its equilibrium cost-price, as it will become clear later. In order to be able to decipher the meaning of the dual prices and constraints, we introduce a few auxiliary symbols (p_i^{h}, p_i^{m} and p_i^{e}) by means of the following definitions (the second and the third elements of the equations are taken from the optimality conditions):

$$p_i^{\mathrm{h}} = p_i^{\mathrm{hm}} \cdot \frac{\partial x_i^{\mathrm{hm}}}{\partial x_i^{\mathrm{h}}} = p_i^{\mathrm{a}} \cdot \frac{\partial x_i}{\partial x_i^{\mathrm{h}}}, \qquad p_i^{\mathrm{m}} = v \cdot p_i^{\mathrm{wm}} = p_i^{\mathrm{hm}} \cdot \frac{\partial x_i^{\mathrm{hm}}}{\partial m_i}, \qquad p_i^{\mathrm{e}} = v \cdot p_i^{\mathrm{we}} = p_i^{\mathrm{a}} \cdot \frac{\partial x_i}{\partial z_i}.$$

The names given suggest already in advance their intended meaning. We will show that p_i^{h}, p_i^{m} and p_i^{e} can indeed be interpreted as the equilibrium price of the domestically produced and sold, imported and exported product variety of the same sectoral origin, which are assumed to be imperfect substitutes.

One can easily show that the dual conditions are the same as the ones which characterize the optimal solutions of the following constrained cost minimization and revenue maximization problems:

$$\min\ p_i^{\mathrm{h}} \cdot x_i^{\mathrm{h}} + p_i^{\mathrm{m}} \cdot m_i, \ \text{ s.t. } \ x_i^{\mathrm{hm}} = x_i^{\mathrm{hm}}(x_i^{\mathrm{h}}, m_i),$$

$$\max\ p_j^{\mathrm{h}} \cdot x_j^{\mathrm{d}} + p_j^{\mathrm{e}} \cdot z_j \ \text{ s.t. } \ x_j = x_j(x_j^{\mathrm{d}}, z_j),$$

where the variables are x_i^{h}, m_i and p_i^{hm} Lagrange multiplier in the first problem, and x_j^{d}, z_j and p_j^{a} Lagrange multiplier in the second problem. (Notice that in the programming model both the domestic demand for (x_i^{h}) and the supply of home produced commodities (x_i^{d}) are denoted by the same variable (x_i^{h}), whereby we implicitly assume the fulfilment of the $x_i^{\mathrm{h}} = x_i^{\mathrm{d}}$ equilibrium condition. We will switch to this notation in the following discussion as well.) Let us show it for the first case, where the Lagrangian functions takes the following form:

$$L(x_i^{\mathrm{h}}, m_i, p_i^{\mathrm{hm}}) = p_i^{\mathrm{h}} \cdot x_i^{\mathrm{h}} + p_i^{\mathrm{m}} \cdot m_i - p_i^{\mathrm{hm}} \cdot \{x_i^{\mathrm{hm}}(x_i^{\mathrm{h}}, m_i) - x_i^{\mathrm{hm}}\}.$$

$$\partial L/\partial x_i^{\mathrm{h}}: \qquad p_i^{\mathrm{hm}} \cdot \frac{\partial x_i^{\mathrm{hm}}}{\partial x_i^{\mathrm{h}}} = p_i^{\mathrm{h}},$$

$$\partial L/\partial m_i: \qquad p_i^{\mathrm{hm}} \cdot \frac{\partial x_i^{\mathrm{hm}}}{\partial m_i} = p_i^{\mathrm{m}},$$

where in case of linearly homogeneous x_i^{hm} function, that we assume, by force of Euler's theorem we have

$$p_i^{\mathrm{hm}} \cdot \left(\frac{\partial x_i^{\mathrm{hm}}}{\partial x_i^{\mathrm{h}}} \cdot x_i^{\mathrm{h}} + \frac{\partial x_i^{\mathrm{hm}}}{\partial m_i} \cdot m_i \right) = p_i^{\mathrm{hm}} \cdot x_i^{\mathrm{hm}} = p_i^{\mathrm{h}} \cdot x_i^{\mathrm{h}} + p_i^{\mathrm{m}} \cdot m_i,$$

from which we get

$$p_i^{\mathrm{hm}} = p_i^{\mathrm{h}} \cdot s_i^{\mathrm{h}} + p_i^{\mathrm{m}} \cdot s_i^{\mathrm{m}}.$$

Thus, as in the case of the linear programming solution, the optimal the shadow, that is the equilibrium cost-price of the domestic/import composite commodity will be the weighted averages of component prices, where s_i^{h} and s_i^{m} are functions of the prices, homogenous of degree zero:

$$s_i^{\mathrm{h}}(p_i^{\mathrm{h}}, p_i^{\mathrm{m}}) \ \text{ and } \ s_i^{\mathrm{m}}(p_i^{\mathrm{h}}, p_i^{\mathrm{m}}).$$

However, unlike in the linear case, the values of s_i^{h} and s_i^{m} are no longer constants, and they are not simply shares in a linear composition. Since, in this case, the amount of the composite commodity is not the algebraic sum, the components, the domestic (x_i^{h}) and the import (m_i) supply, but their nonlinear aggregate: $x_i^{\mathrm{hm}} = x_i^{\mathrm{hm}}(x_i^{\mathrm{h}}, m_i)$, where x_i^{hm} measures the joint use value of the two components. The values of s_i^{h} and s_i^{m} are the cost minimizing levels of the components making up at least one unit use value. In the jargon of competitive equilibrium, they could be interpreted as *demand functions*. $s_i^{\mathrm{h}} + s_i^{\mathrm{m}} > 1$, as a rule, except for the base equilibrium at unit level prices, when $s_i^{\mathrm{h}} + s_i^{\mathrm{m}} = 1$, as the share coefficients in the linear case

Following the same line of argument we can show that the first order conditions in the case of the constrained revenue maximization problem are the following:

$$\partial L/\partial x_j^{\mathrm{h}}: \quad p_j^{\mathrm{a}}\cdot\frac{\partial x_j}{\partial x_j^{\mathrm{h}}} = p_j^{\mathrm{h}},$$

$$\partial L/\partial m_i: \quad p_i^{\mathrm{a}}\cdot\frac{\partial x_j}{\partial z_j} = p_j^{\mathrm{e}},$$

$$p_j^{\mathrm{a}}\cdot\Big(\frac{\partial x_j}{\partial x_j^{\mathrm{h}}}\cdot x_j^{\mathrm{h}} + \frac{\partial x_j}{\partial z_j}\cdot z_j\Big) = p_j^{\mathrm{a}}\cdot x_j = p_j^{\mathrm{h}}\cdot x_j^{\mathrm{h}} + p_j^{\mathrm{e}}\cdot z_j,$$

$$p_j^{\mathrm{a}} = p_j^{\mathrm{h}}\cdot s_j^{\mathrm{d}} + p_j^{\mathrm{e}}\cdot s_j^{\mathrm{e}},$$

as before, where s_j^{d} and s_j^{e} are again functions of the prices, homogenous of degree zero:

$$s_j^{\mathrm{d}}(p_j^{\mathrm{h}}, p_j^{\mathrm{e}}) \text{ and } s_j^{\mathrm{e}}(p_j^{\mathrm{h}}, p_j^{\mathrm{e}}),$$

which can be interpreted as supply functions of a firm in a perfectly competitive market.

Next, note that there are three unknowns in both optimization problems and same number equations representing the first order necessary conditions of the optimal solutions. If the chosen functions are well-behaved and simple, we can explicitly solve analytically the model before any calculation, and express the values of the unknown variables (e.g., x_i^{h}, m_i and p_i^{hm}) as functions of the variables (e.g. p_i^{h}, p_i^{m} and x_i^{hm}) assumed to be known and the parameters of the substitution functions. Thus, if we wish, we can derive the closed analytical forms of the following solution functions:

$$x_i^{\mathrm{h}} = x_i^{\mathrm{h}}(p_i^{\mathrm{h}}, p_i^{\mathrm{m}}, x_i^{\mathrm{hm}}),\;\; m_i = m_i(p_i^{\mathrm{h}}, p_i^{\mathrm{m}}, x_i^{\mathrm{hm}}) \text{ and } p_i^{\mathrm{hm}} = p_i^{\mathrm{hm}}(p_i^{\mathrm{h}}, p_i^{\mathrm{m}}, x_i^{\mathrm{hm}}),$$

the first two of which are derived demand functions, the third a price index aggregator.

We can in fact arrive at the optimal values of the variables in various ways. Because of the assumed first order homogeneity of the substitution (aggregation) functions, the optimal ratios, such as $s_i^{\mathrm{h}} = x_i^{\mathrm{h}}/x_i^{\mathrm{hm}}$, $m_i = m_i/x_i^{\mathrm{hm}}$ and $r_i^{\mathrm{hm}} = m_i/x_i^{\mathrm{h}} = s_i^{\mathrm{m}}/s_i^{\mathrm{h}}$, in the first case, depend only on the price ratios, and not on the level of x_i^{hm}. This also means that we can replace the first order optimality conditions with equivalent alternative forms, which may be more familiar for the user or the reader. For example, the following three sets of the dual equations,

$$p_i^{\mathrm{hm}}\cdot\frac{\partial x_i^{\mathrm{hm}}}{\partial x_i^{\mathrm{h}}} = p_i^{\mathrm{a}}\cdot\frac{\partial x_i}{\partial x_i^{\mathrm{h}}}\;, \quad p_i^{\mathrm{hm}}\cdot\frac{\partial x_i^{\mathrm{hm}}}{\partial m_i} = v\cdot p_i^{\mathrm{wm}}\,, \quad v\cdot p_j^{\mathrm{we}} = p_j^{\mathrm{a}}\cdot\frac{\partial x_j}{\partial z_j}$$

upon introducing the three sets of new variables p_i^{h}, p_i^{m} and p_i^{e}, could be equivalently represented by the following six sets of equations:

$$p_i^{\mathrm{m}} = v \cdot p_i^{\mathrm{wm}}, \qquad p_i^{\mathrm{hm}} = p_i^{\mathrm{h}} \cdot s_i^{\mathrm{h}}(p_i^{\mathrm{h}}, p_i^{\mathrm{m}}) + p_i^{\mathrm{m}} \cdot s_i^{\mathrm{m}}(p_i^{\mathrm{h}}, p_i^{\mathrm{m}}), \qquad m_i = r_i^{\mathrm{hm}}(p_i^{\mathrm{h}}, p_i^{\mathrm{m}}) \cdot x_i^{\mathrm{h}},$$

$$p_j^{\mathrm{e}} = v \cdot p_j^{\mathrm{we}}, \qquad p_j^{\mathrm{a}} = p_j^{\mathrm{h}} \cdot s_j^{\mathrm{d}}(p_j^{\mathrm{h}}, p_j^{\mathrm{e}}) + p_j^{\mathrm{e}} \cdot s_j^{\mathrm{e}}(p_j^{\mathrm{h}}, p_j^{\mathrm{e}}), \qquad z_j = r_j^{\mathrm{de}}(p_j^{\mathrm{h}}, p_j^{\mathrm{e}}) \cdot x_j^{\mathrm{h}},$$

where $r_i^{\mathrm{hm}}(p_i^{\mathrm{h}}, p_i^{\mathrm{m}}) = s_i^{\mathrm{m}}(p_i^{\mathrm{h}}, p_i^{\mathrm{m}})/s_i^{\mathrm{h}}(p_i^{\mathrm{h}}, p_i^{\mathrm{m}})$, and the s_i^{h} and s_i^{m} demand functions are derived from solving parametrically the following cost minimization problem:

$$\min p_i^{\mathrm{h}} \cdot s_i^{\mathrm{h}} + p_i^{\mathrm{m}} \cdot s_i^{\mathrm{m}}, \text{ s.t. } x_i^{\mathrm{hm}}(s_i^{\mathrm{h}}, s_i^{\mathrm{m}}) = 1,$$

and $r_j^{\mathrm{de}}(p_j^{\mathrm{h}}, p_j^{\mathrm{e}}) = s_j^{\mathrm{e}}(p_j^{\mathrm{h}}, p_j^{\mathrm{e}})/s_j^{\mathrm{d}}(p_j^{\mathrm{h}}, p_j^{\mathrm{e}})$, and the s_j^{d} and s_j^{e} supply functions are derived from solving the following revenue maximization problem:

$$\max p_j^{\mathrm{h}} \cdot s_j^{\mathrm{d}} + p_j^{\mathrm{e}} \cdot s_j^{\mathrm{e}} \text{ s.t. } x_j(s_j^{\mathrm{d}}, s_j^{\mathrm{e}}) = 1.$$

Such a formulation of the dual condition of the optimal solution would be more familiar for a former student of economics than the original ones. They are also closer to the forms input-output models.

Let us now make some steps forward and make use of the possibility provided by the use of nonlinear functional forms in our resource allocation model. Let us first of all separate, as we did in the case of the applied input-output models, the main components of final use,

$$y_i = y_i^{\mathrm{cv}} + g_i \cdot y_{\mathrm{g}} + \Sigma_j\, b_{ij} \cdot y_j^{\mathrm{b}} + c_i^0,$$

where c_i^0 denotes the fixed (‘committed’) part, and the sectoral levels of gross investments (y_j^{b}) are defined as before:

$$y_j^{\mathrm{b}} = r_j^{\mathrm{a}} \cdot K_j + y_j^{\mathrm{bn}}.$$

Since our model is nonlinear, we have to insist no longer on using fixed coefficients in describing the commodity composition of personal consumption. We may allow for some degree of substitutability by introducing the following,

$$y_{\mathrm{cv}} = y_{\mathrm{v}}(y_1^{\mathrm{cv}}, y_2^{\mathrm{cv}}, \dots, y_n^{\mathrm{cv}}) = y_{\mathrm{cv}} \cdot y_{\mathrm{v}}(c_1^{\mathrm{cv}}, c_2^{\mathrm{cv}}, \dots, c_n^{\mathrm{cv}}).$$

linear homogenous utility (welfare) function to determine variable consumption y_i^{cv}, where the variables s_i^{cv} take the place of the unit coefficients and the consumption level, since $y_i^{\mathrm{cv}} = y_{\mathrm{cv}} \cdot s_i^{\mathrm{cv}}$.

At prices p_i^{hm} and expenditure level ev the constrained utility maximum problem and the optimality conditions take the following forms:

$$\max y_{\mathrm{v}}(y_1^{\mathrm{cv}}, y_2^{\mathrm{cv}}, \dots, y_n^{\mathrm{cv}}), \quad \text{s.t. } \Sigma_i\, p_i^{\mathrm{hm}} \cdot y_i^{\mathrm{cv}} = ev,$$

$$p_{\mathrm{cv}} \cdot \frac{\partial y_{\mathrm{v}}}{\partial y_i^{\mathrm{cv}}} = p_i^{\mathrm{hm}},$$

where p_{cv} is the Lagrange multiplier. We expect and will see these latter conditions to appear among the dual constraints of the optimality conditions.

Production technology will be represented by Johansen type production functions, which were introduced in the previous chapter. We will thus allow for substitution between labour and capital, by using smooth, well-behaved production functions given in the forms of

$x_j = f_j(L_j, K_j)$.

At factor prices w_j and q_j the optimality conditions of cost minimum are as follows:

$$c_j \cdot \frac{\partial f_j}{\partial L_j} = w_j$$

$$c_j \cdot \frac{\partial f_j}{\partial K_j} = q_j,$$

where c_j's are the Lagrange multipliers (the unit cost of the composite labour-capital input). They will also appear among the necessary conditions of the optimal macroeconomic resource allocation.

As long as functions f_j are homogeneous of degree one and well-behaved, as we usually assume, one can derive the identities

$$c_j \cdot \frac{\partial f_j}{\partial L_j} \cdot L_j + c_j \cdot \frac{\partial f_j}{\partial K_j} \cdot K_j = c_j \cdot x_j = w_j \cdot L_j + q_j \cdot K_j, \text{ that is}$$

$$c_j = w_j \cdot l_j + q_j \cdot k_j,$$

where $l_j = L_j/x_j$, $k_j = K_j/x_j$, and their values can be determined by the following unit factor demand functions:

$l_j = l_j(w_j, q_j)$ and $k_j = k_j(w_j, q_j)$.

What concerns the flexible bounds one can use in the case of exports, we have in fact two possibilities. One of them is the $x_j(x_j^{\mathrm{d}}, z_j)$ transformation functions, used in the previous model, which put limits on the movement of the export volume from the supply side. Instead of or together with it, we can also use smooth export demand functions, which can also effectively constrain their levels. We will introduce them in the form of indirect demand functions, $p_i^{\mathrm{we}}(z_i)$, which define their external market price in the foreign trade balance. As a result we will end up with the following model and its optimality conditions. We will discuss later the potential side effects of this solution.

As a result of introducing the suggested changes into our optimal resource allocation model, we will arrive at the following nonlinear programming model and optimality conditions. Note, that y_{g} and y_j^{bn} are treated as exogenous variables in this version of the model. In the column on the left we find the resource allocation constraints ($4n + 4$ numbers of equations in terms of $8n + 1$ number of variables) and the assigned Lagrange multipliers. On the right, we have listed the Lagrange (or Kuhn–Tucker) first order necessary conditions of the optimal solution ($8n + 1$ number of equations in terms of $4n + 4$ numbers of variables). We have attached to each of them the primal variable according to which we differentiated the Lagrangian function in order to derive them. The total number of unknowns and equations in the Lagrange (Kuhn–Tucker) conditions of the optimal solution are thus $12n + 5$, arranged into 17 blocks.

NLP-2.2-2	(P)	(KTD)	
	$x_j^{\mathrm{h}}, x_j, y_i^{\mathrm{cv}}, y_j^{\mathrm{b}}, m_i, z_j, L_j, K_j, y_{\mathrm{cv}} \geqq 0$	$p_j^{\mathrm{a}}, p_i^{\mathrm{hm}}, c_j, w, \rho, v, p_{\mathrm{cv}} \geqq 0$	
(p_j^{a})	$x_j(x_j^{\mathrm{h}}, z_j) = x_j$	$p_j^{\mathrm{a}} = \Sigma_i\, p_i^{\mathrm{hm}} \cdot a_{ij} + c_j$	(x_j)
(p_i^{hm})	$\Sigma_j\, a_{ij} \cdot x_j + y_i^{\mathrm{cv}} + g_i \cdot y_{\mathrm{g}} + \Sigma_j\, b_{ij} \cdot y_j^{\mathrm{b}} + c_i^0 = x_i^{\mathrm{hm}}(x_i^{\mathrm{h}}, m_i)$	$p_i^{\mathrm{hm}} \cdot \dfrac{\partial x_i^{\mathrm{hm}}}{\partial x_i^{\mathrm{h}}} = p_i^{\mathrm{a}} \cdot \dfrac{\partial x_i}{\partial x_i^{\mathrm{h}}}$	(x_i^{h})
(p_j^{b})	$r_j^{\mathrm{a}} \cdot K_j + y_j^{\mathrm{bn}} = y_j^{\mathrm{b}}$	$p_j^{\mathrm{b}} = \Sigma_i\, p_i^{\mathrm{hm}} \cdot b_{ij}$	(y_j^{b})
(w)	$\Sigma_j\, L_j = L_0$	$p_i^{\mathrm{hm}} \cdot \dfrac{\partial x_i^{\mathrm{hm}}}{\partial m_i} = v \cdot p_i^{\mathrm{wm}}$	(m_i)
(ρ)	$\Sigma_j\, K_j = K_0$	$v \cdot \left(p_j^{\mathrm{we}} + \dfrac{\partial p_j^{\mathrm{we}}}{\partial z_j} \cdot z_j\right) =$	(z_j)
(c_j)	$x_j = f_j(L_j, K_j)$	$c_j \cdot \dfrac{\partial f_j}{\partial L_j} = w$	(L_j)
(v)	$\Sigma_i\, \left(p_i^{\mathrm{wm}} \cdot m_i - p_i^{\mathrm{we}}(z_i) \cdot z_i\right) = d_{\mathrm{e}}$	$c \cdot \dfrac{\partial f_j}{\partial K_j} = p_j^{\mathrm{b}} \cdot r_j^{\mathrm{a}} + \rho$	(K_j)
(p_{cv})	$y_{\mathrm{cv}} = y_{\mathrm{v}}(y_1^{\mathrm{cv}}, y_2^{\mathrm{cv}}, \ldots, y_n^{\mathrm{cv}})$	$p_{\mathrm{cv}} \cdot \dfrac{\partial y_{\mathrm{v}}}{\partial y_i^{\mathrm{cv}}} = p_i^{\mathrm{hm}}$	(y_i^{cv})
	$y_{\mathrm{cv}} \to \max!$	$1 = p_{\mathrm{cv}}$	(y_{cv})

As before, it will be useful to introduce some auxiliary variables, which make it easier to interpret the solution. Thus, we may introduce again the symbols p_i^{h}, p_i^{m} and p_i^{e} with slightly different definitions as above (the second and the third members of the equations are again implied by the second, fourth and fifth optimality conditions):

$$p_i^{\mathrm{h}} = p_i^{\mathrm{hm}} \cdot \frac{\partial x_i^{\mathrm{hm}}}{\partial x_i^{\mathrm{h}}} = p_i^{\mathrm{a}} \cdot \frac{\partial x_i}{\partial x_i^{\mathrm{h}}},\ \ p_i^{\mathrm{m}} = v \cdot p_i^{\mathrm{wm}} = p_i^{\mathrm{hm}} \cdot \frac{\partial x_i^{\mathrm{hm}}}{\partial m_i},\ p_i^{\mathrm{e}} = v \cdot \left(p_i^{\mathrm{we}} + \frac{\partial p_i^{\mathrm{we}}}{\partial z_i} \cdot z_i\right) = p_i^{\mathrm{a}} \cdot \frac{\partial x_i}{\partial z_i}\ .$$

In addition to these, we introduce also q_j, l_j and k_j as

$$q_j = p_j^{\mathrm{b}} \cdot r_j^{\mathrm{a}} + \rho,\ \ l_j = L_j / x_j,\ \ k_j = K_j / x_j.$$

Their meaning is suggested by the chosen notation. They are in fact the shadow prices of the capital goods in the different sectors. They differ from each other only to the extent the amortization rates (r_j^{a}) and the prices of the composite sectoral capital goods (p_j^{b}) are different. The above definition of the cost of capital is slightly different from Walras's, which would be

$$q_j = (r_j^{\mathrm{a}} + \pi_j) \cdot p_j^{\mathrm{b}}.$$

The reason behind this difference is that in the capital constraint we treat capital as a homogeneous factor, which takes up a uniform net shadow rate of return, ρ. Unlike its counterpart in Walras's definition, the uniform rate of return is defined here in relation to the physical volume of capital (K_j) used and not on its value ($p_j^{\mathrm{b}} \cdot K_j$). In Walras's definition this latter is assumed to be uniform in competitive equilibrium, whereas here they are different, as a rule, since $\pi_j = \rho / p_j^{\mathrm{b}}$.

What explains this strange logic is the somewhat contradictory treatment of capital goods in the above model. In the capital constrain capital is considered to be freely mobile across sectors, which would imply uniform composition (b_i), price ($p_{\mathrm{b}} = \Sigma_j\, p_i^{\mathrm{hm}} \cdot b_i$) and a uniform rate of return (ρ on the physical volume and $\pi = \rho / p_{\mathrm{b}}$ on the value of capital used).

We could dissolve this contradiction in two ways. The first possibility is to enforce fully the assumption that capital is a homogeneous good and revise the definition accordingly, that is, replace the y_j^{b} variables with a single $\mathbf{y}^{\mathrm{v}}$ scalar, and the sectorally different b_{ij} investment coefficients with b_i's, as suggested above. The other possibility is to treat capital as sector specific goods in all its appearance, thus, replace the single capital constrain with sector specific constraints: $K_j = K_{j0}$. As a result of this solution the net return both on the physical volume (ρ_j) and the value of the capital (π_j) would be different in the various sectors in general. This latter differentiation would, however, violate the requirement of competitive equilibrium. Neither treatment provides, thus, a fully satisfactory solution. The root of this dilemma lies basically in the problem of macro-closure, discussed earlier.

Another questionable feature of the above model is the derived definition of the shadow price of exports:

$$p_i^{\mathrm{e}} = v \cdot \left(p_i^{\mathrm{we}} + \frac{\partial p_i^{\mathrm{we}}}{\partial z_i} \cdot z_i \right) = (1 + 1/\varepsilon_i) \cdot v \cdot p_i^{\mathrm{we}},$$

where ε_i is the price elasticity of export demand. Since under normal conditions the sign of the latter is negative, the term $1/\varepsilon_i$ can be interpreted as a tax rate applied on incomes earned via exports. This solution is well known in international trade theory and they are called *optimal tariffs*. The theory calls attention to the possibility that the introduction of such tariffs could make price-taking producers behave collectively as a monopoly. Nevertheless, it would not be reasonable to use such an assumption in a macroeconomic resource allocation model.

2.2.4. Conclusions: towards the computable general equilibrium models

By means of introducing auxiliary variables and mathematically equivalent alternative forms, the conditions of optimal resource allocation can be rearranged into an alternative specification. We have chosen such an alternative set of equations, which will be easy to compare with the input-output volume and price models. Their comparison reveals the real nature of not only the optimal resource allocation model chosen, but also of the typical computable equilibrium models, the structure and the underlying logic of which is basically the same. The system will consist of $20n + 5$ unknowns and equal number of equations. The variables and parameters of the derived equation system can be classified in the following way:

Endogenous variables:	real (volumes)	$x_j, x_i^{\mathrm{hm}}, x_j^{\mathrm{h}}, z_j, m_i,$	($5n$)
($20n$+5)	real (structural)	$s_i^{\mathrm{h}}, s_i^{\mathrm{m}}, s_j^{\mathrm{d}}, s_j^{\mathrm{e}}, p_i^{\mathrm{we}}, l_j, k_j, s_i^{\mathrm{cv}}, y_{\mathrm{cv}},$	($8n$+1)
	nominal (value)	$p_i^{\mathrm{h}}, p_j^{\mathrm{a}}, p_i^{\mathrm{m}}, p_i^{\mathrm{e}}, p_i^{\mathrm{hm}}, p_j^{\mathrm{b}}, q_j, w, \rho, v, p_{\mathrm{cv}}.$	($7n$+4)

Potential (endogenous or exogenous) variables: y_{g}, y_j^{bn}, c_i^0, p_i^{wm}, L_0, K_0 and d_{e}.

Parameters: a_{ij}, b_{ij}, r_j^{a}, g_i and the parameters of the various functions.

We will group the conditions characterizing the optimal solution of the problem NLP-2.2-2 similarly into three categories:

A) balances and definitions of volume categories (x_j, x_i^{hm}, x_j^{h}, z_j, m_i),

B) balances and definitions of price categories (p_i^{h}, p_j^{a}, p_i^{m}, p_i^{e}, p_i^{hm}, p_j^{b}, q_j, p_{cv}, w, ρ, v),

C) definitions of structural parameters s_i^{h}, s_i^{m}, s_j^{d}, s_j^{e}, p_i^{we}, l_j, k_j, s_i^{cv}.

A) BALANCES AND DEFINITIONS OF THE VOLUME CATEGORIES ($5n + 3$):

We can further divide this category into two subgroups. (On the left hand side we will assign list numbers to the equation blocks, whereas, on the right hand side, endogenous variables, that will make it easier to count the number of unknowns and equations, and check the regularity of the equation system.)

A1) Sectoral commodity balances and their components ($5n$):

(P1)	$x_i^{\mathrm{hm}} = \Sigma_j (a_{ij} + b_{ij}\cdot r_j^{\mathrm{a}}\cdot k_j)\cdot x_j + s_i^{\mathrm{cv}}\cdot y_{\mathrm{cv}} + g_i\cdot y_{\mathrm{g}} + \Sigma_j b_{ij}\cdot y_j^{\mathrm{bn}} + c_i^0$	(x_i^{hm})
(P2)	$x_j = x_j(x_j^{\mathrm{h}}, z_j)$	(x_j)
(P3)	$x_i^{\mathrm{hm}} = x_i^{\mathrm{hm}}(x_i^{\mathrm{h}}, m_i)$	(x_i^{h})
(P4)	$z_j = x_j^{\mathrm{h}}\cdot s_j^{\mathrm{e}}/s_j^{\mathrm{d}}$	(z_j)
(P5)	$m_i = x_i^{\mathrm{h}}\cdot s_i^{\mathrm{m}}/s_i^{\mathrm{h}}$	(m_i)

This subgroup defines an extended input-output volume model, in which both imports and exports are treated as endogenous variables by setting them proportional to domestic supply. We have discussed earlier an almost identical linear version of this model, in which we had $x_j = x_j^{\mathrm{h}} + z_j$ and $x_i^{\mathrm{hm}} = x_i^{\mathrm{h}} + m_i$ (it was different from this one only in the way exports were made endogenous variables).

In a conventional linear input-output volume model only x_j, x_i^{hm}, x_j^{h}, z_j and m_i would be considered endogenous variables, y_{cv} would be, thus, in addition to y_{g}, y_j^{bn} and c_i^0 also exogenous variable, whereas s_i^{h}, s_i^{m}, s_j^{d}, s_j^{e} and s_i^{cv} constant parameters, in addition to a_{ij}, b_{ij}, r_j^{a} and g_i. In such a setup, the equation system, which consists of the same number ($5n$) of unknowns and equations, is a well determined macro-model. Under normal conditions would expect it to have a unique solution, thus, the values of x_j^{h}, z_i and m_i would also be determined in it. Consequently, the value of the expressions on the right hand side of following balance equations (labour, capital and foreign currency demand) would be also determined by them:

A2) Balances of the primary resources (3):

(P6)	$\Sigma_j\, l_j\cdot x_j = L_0$	(w, L_0)
(P7)	$\Sigma_j\, k_j\cdot x_j = K_0$	(ρ, K_0)
(P8)	$\Sigma_i \left(p_i^{\mathrm{wm}}\cdot m_i - p_i^{\mathrm{we}}\cdot z_i\right) = d_{\mathrm{e}}$,	(v, d_{e})

where p_i^{we}, l_j, k_j would also be constant parameters in a conventional input-output model.

The natural closure of the extended input-output system, defined by the parameters and unknowns, and equations (P1) – (P8), would be to treate L_0, K_0 and d_{e} as endogenous demand variables (this is why we have also assigned these variables to the equations). The extended input-output model with such a closure would be final demand driven. We have discussed the closure possibilities of such a model in connection with the applied linear input-output volume models. We have pointed out that one can not expect that the above equation system, in which two or more of the supply constraints would be fixed, would have a sensible solution.

We could choose, in general, only one out of L_0, K_0 and d_{e} as exogenous supply variable and turn, in exchange, the level of some final demand (e.g., y_{cv}, as in the optimal resource allocation model) into endogenous variable. With such a change, the demand driven model would become supply driven, as the optimal resource allocation model. In the latter model all the three, that is, L_0, K_0 and d_{e} were considered to be fixed (exogenous supply variables). This is why we would assign w, ρ and v as complementary variables in the optimal resource allocation model to the same equations, in which their values reflect their relative scarcity.

B) PRICE IDENTITIES AND EQUATIONS ($7n$+1):

This block contains shadow price identities and equations, derived from the dual optimum conditions. Most of them can be interpreted and take the form of cost-price definitions, as we have discussed already.

(P9)	$p_j^{\mathrm{a}} = \Sigma_i\, p_i^{\mathrm{hm}} \cdot a_{ij} + w \cdot l_j + q_j \cdot k_j$	(p_j^{h})
(P10)	$p_j^{\mathrm{a}} = p_j^{\mathrm{h}} \cdot s_j^{\mathrm{d}} + p_j^{\mathrm{e}} \cdot s_j^{\mathrm{e}}$	(p_j^{a})
(P11)	$p_i^{\mathrm{m}} = v \cdot p_i^{\mathrm{wm}}$	(p_i^{m})
(P12)	$p_j^{\mathrm{e}} = (1 + 1/\varepsilon_j) \cdot v \cdot p_j^{\mathrm{we}}$	(p_j^{e})
(P13)	$p_i^{\mathrm{hm}} = p_i^{\mathrm{h}} \cdot s_i^{\mathrm{h}} + p_i^{\mathrm{m}} \cdot s_i^{\mathrm{m}}$	(p_i^{hm})
(P14)	$p_j^{\mathrm{b}} = \Sigma_i\, p_i^{\mathrm{hm}} \cdot b_{ij}$	(p_j^{b})
(P15)	$q_j = p_j^{\mathrm{b}} \cdot r_j^{\mathrm{a}} + \rho$	(q_j)
(P16)	$p_{\mathrm{cv}} = \Sigma_i\, p_i^{\mathrm{hm}} \cdot s_i^{\mathrm{cv}}$.	(p_{cv})

It should not be surprising at all that the $7n$+1equations (P9) – (P16) define a complete, well defined extended input-output model in terms of the $7n$+1 number of unknowns, p_i^{h}, p_j^{a}, p_i^{m}, p_j^{e}, p_i^{hm}, p_j^{b}, q_j and p_{cv}, listed on the right hand side in brackets. In a conventional input-output price model p_j^{we}, w, ρ and v would be, of course, exogenous variables, and s_i^{h}, s_i^{m}, s_j^{d}, s_j^{e}, s_i^{cv} and ε_j constant parameters, in addition to a_{ij}, b_{ij}, r_j^{a}.

C) EQUATIONS OF STRUCTURAL VARIABLES AND WORLD MARKET EXPORT PRICES (8N):

The last block contains equations that define the required proportions between various volume and price variables, which were also derived mainly from the dual optimality conditions. The coefficients setting these proportions would be typically fixed in a linear input-output or programming model of resource allocation. The constraint defining the inverse export demand function seems to behave like a cuckoo's egg among the other variables. But it could be rearranged into such an alternative form that would also define a proportion, namely that of the domestic export to the offer of foreign competitors.

As we have demonstrated, these constraints can be interpreted as setting flexible upper and lower bounds on specific variables to confine their departure from their observed values into reasonable ranges. In addition to that these proportion variables *link together* the first two, *the volume and price blocks* of equations, which would be completely independent from each other, were these proportions exogenously fixed, as in the input-output models.

(P17)	$s_i^{\mathrm{h}} = s_i^{\mathrm{h}}(p_i^{\mathrm{h}}, p_i^{\mathrm{m}})$	(s_i^{h})
(P18)	$s_i^{\mathrm{m}} = s_i^{\mathrm{m}}(p_i^{\mathrm{h}}, p_i^{\mathrm{m}})$	(s_i^{m})
(P19)	$s_j^{\mathrm{d}} = s_j^{\mathrm{d}}(p_j^{\mathrm{h}}, p_j^{\mathrm{e}})$	(s_j^{d})
(P20)	$s_j^{\mathrm{e}} = s_j^{\mathrm{e}}(p_j^{\mathrm{h}}, p_j^{\mathrm{e}})$	(s_j^{e})
(P21)	$p_i^{\mathrm{we}} = p_i^{\mathrm{we}}(\mathrm{z}_i)$	(p_i^{we})
(P22)	$s_i^{\mathrm{cv}} = c_i^{\mathrm{v}}(p_1^{\mathrm{hm}}, p_2^{\mathrm{hm}}, \ldots, p_n^{\mathrm{hm}})$	(s_i^{cv})
(P23)	$l_j = l_j(w, q_j)$	(l_j)
(P24)	$k_j = k_j(w, q_j)$	(k_j)

Counting equations and variables

We have listed above $20n+4$ equations altogether, whereas there are $20n+5$ variables. It seems as if the equation system is yet not fully determined. Checking the variables assigned to the constraints listed, we can see that the only variable, which does not have a counterpart equation, is the level of variable (private) consumption, y_{cv}. It can be shown that the rest of the equations will uniquely define its value by force of Walras's law.

One could introduce one more variable, *ev* as the expenditure spent on variable consumption and add one more equation, the expenditure constraint, to the system in one of the following forms

$$p_{\mathrm{cv}} \cdot y_{\mathrm{cv}} = \Sigma_i\, p_i^{\mathrm{hm}} \cdot s_i^{\mathrm{cv}} \cdot y_{\mathrm{cv}} = ev \quad \text{or} \quad \Sigma_i\, p_i^{\mathrm{hm}} \cdot s_i^{\mathrm{cv}} \cdot y_{\mathrm{cv}} = ev. \qquad (y_{\mathrm{cv}})$$

In such a specification *ev* could be taken as the undefined variable, whose value is set by Walras's law, as in the case of the Johansen model.

In any case, there would be one more unknown than equations. One can, however, easily check, here again, that all equations are independent of the general level of the price and value terms (i.e., homogenous in prices), therefore we can fix the price level by setting the value of one them, for example set $p_{\mathrm{cv}} = \Sigma_i\, p_i^{\mathrm{hm}} \cdot s_i^{\mathrm{cv}} = 1$, as in the programming model, or $ev = 1$, as the Johansen model). That would make the equations system well determined (regular).

Summary and conclusions

It is worth summing up what we have demonstrated. We have seen, first of all, that the feasibility conditions of the resource allocation problem are nothing but the conventional macroeconomic accounting identities: balance requirements described in terms of supply-demand equations for sectoral commodities, primary resources and foreign currency (see, balance of trade). They are the same conditions that should appear in any nation-wide model, especially in those, which are built upon the input-output tables.

The number of the equations defined by these constraints is relatively small compared to the number of the potential unknowns. For example, in the case of NLP-2.2-2, the primal (physical) resource allocation constraints consisted of $4n + 3$ equations, expressed in terms of $8n$ variables (ignoring y_{cv} and its definition, which could have been put directly into the objective function). Unlike, thus, in an input-output model, there will be a large degree of freedom left by the primal resource constraints and balance requirements. (In the case of the input-output models we reduced the degree of freedom to zero by fixing the value of many potential variables exogenously). In other words, the set of feasible resource allocation patterns will be quite large. We try to reduce this set to a single point by optimizing an appropriate welfare function over it. If there is a single solution, the model can be used for comparative static exercises, that is, compare solutions received by assigning different values to specific exogenous variables.

We have demonstrated that the above ‘regularisation’ of the set of feasible allocation patterns, i.e., its representation by just one salient point, is equivalent with complementing the set of primal variables and equations with appropriate dual variables and constraints, which would together define a regular equation system. The numerical solution of the optimizing model would, however, not be able to replicate the observed values of the variables, even if the benchmark data set were consistent with the implicitly assumed optimizing behaviour. For the simple reason that the necessary conditions of the optimal solution are not all reflecting the actual rules of accounting, especially not in the case of the pricing rules, which ignore taxes and subsidies. The programming approach corresponds to the world of perfect competition.

It is important to note in this connection that that the majority of the dual variables and conditions follow quite closely the conventional accounting principles. So, if one changed the specification of the unrealistic dual conditions, he could achieve both goals, i.e., the feasible set would become ‘regularised’ and the solution of the equation would replicate the observed values of the endogenous variables. As a matter of fact, as we will soon show it, this is exactly the purpose of *model calibration* in the case of the computable general equilibrium models. (Calibration means the adjustment of parameter values of the model until the output from the model matches an observed set of data.)

What makes the general equilibrium approach feasible is the way we introduced another set of dual variables and constraints (various proportions), which so as to set flexible upper and lower bounds on specific variables. As we have seen, they could be derived from the optimizing behaviour of representative agents (producers, consumers, foreign buyers) put in charge to make analogous decisions. This is in fact their usual interpretation based on neoclassical economics. Our above demonstration should have, however, convinced the reader that it would be more proper to view these dual equations as describing the behaviour of the structural variables, rather than of some mysterious representative agents.

This would not contradict the fact that we borrowed concepts and tools for our macro model from microeconomic theories. It will not make the indisputably macro model, built up from macroeconomic aggregates, into a microeconomic construct either. What justifies the use of microeconomic rules of behaviour in a genuinely macroeconomic model is that they provide a convenient way to link together, and at the same time limit the movement of certain macroeconomic variables, as we have pointed out (see, flexible bounds).

2.3. The concept and the main building blocks of the CGE models

2.3.1. From programming to applied equilibrium model

As noted before, the necessary conditions of the optimal solution are not all reflecting the actual rules of accounting, especially not in the case of the pricing rules, which ignore taxes and subsidies. The only place they have appeared was the formation of the export prices (see, optimal tariffs), the use of which in an applied model would be unrealistic. Because of the aggregate representation of labour and capital the wage rate and the net rate of return are uniform across sectors, whereas empirical data make it clear they are sector specific. Also, as we have seen, the net rate of return is proportional with the physical amount and not the value of capital, as its theoretical concept would imply.

These and similar other potential shortages of the condition of optimality can be taken care simply by modifying the equation system derived from the optimality conditions in accordance with economic theory or observed practices. The solution of the revised equation system will, of course, be no longer the optimal solution of the programming model. As a matter of fact, what we would like to achieve is that the model solution would reproduce the observed values of the endogenous variables, if the values of the parameters and exogenous variables were set according to their observed values too, and the unobservable parameters of functions x_j, x_i^{hm} and f_j were calibrated in such a way that the choices at base prices would show consistent with the assumed optimizing behaviour. The solution obtained would thus look like a state of equilibrium, distorted by the presence of taxes and subsidies and other imperfections, for example, on the factor markets. This is the idea behind the *computable general equilibrium approach*.

We will introduce, to this end, various rates of indirect and direct taxes/subsidies, including VAT taxes, in the equations and terms, which define the rules of price and cost formation. Instead of the cost based producers' and supplier's prices we will use, where ever appropriate, users' prices, which include the net result of taxes and subsidies. We assume that the indirect taxes/subsidies applied to commodity use can be classified into three categories of use, use in production and public consumption, in private consumption and in investments. Their rates may differ across these categories, but they are the same within these categories for the same commodity. The tax/subsidy rates and the revised prices and costs will be as follows (the same symbols will be used as before, where ever it is possible).

Use in production and public consumption, rates: $\boldsymbol{\tau}^{\mathrm{u}} = (\tau_i^{\mathrm{u}})$, users' prices:

$p_i^{\mathrm{u}} = (1+\tau_i^{\mathrm{u}})\cdot p_i^{\mathrm{hm}}$,

Private consumption, rates: $\boldsymbol{\tau}^{\mathrm{c}} = (\tau_i^{\mathrm{c}})$, consumers' prices:

$p_i^{\rm c} = (1+\tau_i^{\rm c})\cdot p_i^{\rm hm}$,

Investment inputs, rates: $\boldsymbol{\tau}^{\rm v} = (\tau_i^{\rm b})$, users' prices in investment,

$p_i^{\rm b} = (1+\tau_i^{\rm b})\cdot p_i^{\rm hm}$,

Export taxes/subsidies $\boldsymbol{\tau}^{\rm e} = (\tau_i^{\rm e})$, export prices

$p_i^{\rm e} = (1+\tau_i^{\rm e})\cdot v\cdot p_i^{\rm we}$,

Import duty rates: $\boldsymbol{\tau}^{\rm m} = (\tau_i^{\rm m})$, import prices:

$p_i^{\rm m} = (1+\tau_i^{\rm m})\cdot v\cdot p_i^{\rm wm}$,

Production taxes/subsidies, rates: $\boldsymbol{\tau}^{\rm x} = (\tau_i^{\rm x})$, producers' prices

$p_j^{\rm a} = (1+\tau_j^{\rm x})\cdot p_j^{\rm ac}$,

Social security contribution, rates: $\boldsymbol{\tau}^{\rm w} = (\tau_i^{\rm w})$, labour cost

$w_j = (1+\tau_j^{\rm w})\cdot w\cdot d_j^{\rm w}$,

The revised cost of capital:

$q_j = p_j^{\rm b}\cdot(r_j^{\rm a} + \pi\cdot d_j^{\pi})$.

In order to see the differences, we will compare the original (optimality) and the revised (equilibrium) price equations. The latter equations will be the same, as will be seen, as the ones introduced already in the case of the input-output price models. The tax/subsidy rates will be, as a rule, considered to be as parameters, but in certain exercises they could become exogenous or even endogenous variables.

The revised conditions:		The original optimality conditions:	
(P9')	$p_j^{\rm a} = (1+\tau_j^{\rm x})\cdot\big(\Sigma_i\,(1+\tau_i^{\rm u})\cdot p_i^{\rm hm}\cdot a_{ij} + w_j\cdot l_j + q_j\cdot k_j\big)$,	(P9)	$p_j^{\rm a} = \Sigma_i\, p_i^{\rm hm}\cdot a_{ij} + w\cdot l_j + q_j\cdot k_j$,
(P11')	$p_i^{\rm m} = (1+\tau_i^{\rm m})\cdot v\cdot p_i^{\rm wm}$,	(P11)	$p_i^{\rm m} = v\cdot p_i^{\rm wm}$,
(P12')	$p_i^{\rm e} = (1+\tau_i^{\rm e})\cdot v\cdot p_i^{\rm we}(z_i)$,	(P12)	$p_i^{\rm e} = (1 + 1/\varepsilon_i)\cdot v\cdot p_i^{\rm we}(z_i)$.
(P14')	$p_j^{\rm b} = \Sigma_i\,(1+\tau_i^{\rm b})\cdot p_i^{\rm hm}\cdot b_{ij}$,	(P14)	$p_j^{\rm b} = \Sigma_i\, p_i^{\rm hm}\cdot b_{ij}$,
(P15')	$q_j = p_j^{\rm b}\cdot(r_j^{\rm a} + \pi\cdot d_j^{\pi})$,	(P15)	$q_j = p_j^{\rm b}\cdot r_j^{\rm a} + \rho$,
(P16')	$\Sigma_i\, p_i^{\rm c}\cdot s_i^{\rm cv} = p_{\rm cv}$,	(P16)	$\Sigma_i\, p_i^{\rm hm}\cdot s_i^{\rm cv} = 1\ (= p_{\rm cv})$,
(P22')	$s_i^{\rm cv} = c_i^{\rm v}(p_1^{\rm c}, p_2^{\rm c}, \ldots, p_n^{\rm c})$,	(P22)	$s_i^{\rm cv} = c_i^{\rm v}(p_1^{\rm hm}, p_2^{\rm hm}, \ldots, p_n^{\rm hm})$,
(P23')	$l_j = l_j(w_j, q_j)$	(P23)	$l_j = l_j(w, q_j)$
(P24')	$k_j = k_j(w_j, q_j)$	(P24)	$k_j = k_j(w, q_j)$
(P25')	$w_j = (1+\tau_j^{\rm w})\cdot w\cdot d_j^{\rm w}$,	wage rate was uniform before,	
(P26')	$p_i^{\rm c} = (1+\tau_i^{\rm c})\cdot p_i^{\rm hm}$,	no special consumers' prices before.	

The reformulation of the producers' prices allows us to depart further from the idea of perfect competition. One might want to redefine operating surplus as differentiated profit mark-ups (π_j^c) instead of the differentiated net rates of return on capital. As a result, one would redefine producers' prices as follows:

(P10'') $p_j^a = \Sigma_i(1+\tau_i^u)\cdot p_i^{hm}\cdot a_{ij} + w_j\cdot l_j + q_j\cdot k_j + p_j^a\cdot(\pi_j^c + \tau_j^x)$,

where we can make further choices. In the case of pure mark-up profits the cost of capital should be redefined as $q_j = p_j^b\cdot(r_j^a + \pi\cdot d_j^\pi)$, whereas the profit mark-ups as $\pi_j^c = \pi\cdot d_j^\pi$. This would leave the number of variables unchanged, and one can show that π would reflect changes in the relative scarcity of the factor constraints and foreign currency as before. In another solution π_j^c could be introduced as parameters, and retain the definition of the cost of capital as $q_j = p_j^b\cdot(r_j^a + \pi\cdot d_j^\pi)$. This would result in a lower base value of the net rate of return and a narrower range for π to vary in general.

2.3.2. A stylised CGE model based on problem 2.2-2

The previous section has paved the way for reformulating the optimum conditions of the studied optimal resource allocation problem into a set of equilibrium conditions. We will use the equations characterizing the optimum conditions, revised in the way suggested above, organized into the same 26 blocks as before. We will simply tell the story of the origin of these conditions using the language and terminology of general equilibrium theory, and following its logic, rebuild the model using the typical building blocks of the CGE models.

It will be ease to identify the equations, and the following summary table will help the reader to recall the meaning of the 27 blocks of core variables used. We call them core variables, because the 26 blocks of equations define yet not a complete stylised CGE model, and we will have to introduce further equations and variables to complete the model.

<u>Endogenous variables of the core model</u>

x_j	(composite) production levels	p_i^h	producers' prices of domestic sales
x_j^h	production supplied on domestic markets	p_j^a	average producers' prices
z_j	volume of exports	p_i^m	domestic prices of imports
s_j^d	Share coefficients of domestic sales	p_i^{we}	world market prices of exports
s_j^e	share coefficients of exports	p_i^e	domestic prices of exports
x_i^{hm}	(composite) supply on domestic markets	p_i^{hm}	average users' prices net of taxes
m_i	volume of imports	p_i^c	consumers' prices
s_i^h	share coefficients of domestic supply	p_{cv}	consumers' price index
s_i^m	share coefficients of imports	p_j^b	prices of capital goods
s_i^{cv}	structure of variable consumption	q_j	cost of capital
y_{cv}	level of variable consumption	w_j	cost of labour

l_j	labour input coefficients	w	general level of wages
k_j	capital input coefficients	π	net rate of return on capital
		v	exchange rate

PRIMARY GOODS, DOMESTICALLY PRODUCED AND IMPORTED COMMODITIES

There are two types of commodities: n kinds of goods produced in n sectors, and two kinds of primary resources (labour and capital) with exogenously given supply (L_0 and K_0). It is assumed that all users of the various commodities minimize their costs. Domestic products compete with imported commodities of the same sort in each sector, and that they are imperfect substitutes of each other, making up a special composite commodity (the so-called Armington assumption). The joint use value, that is, the volume of the domestic/import composite commodity of sectoral origin i is given as

(P3) $\quad x_i^{\mathrm{hm}} = x_i^{\mathrm{hm}}(x_i^{\mathrm{h}}, m_i), \qquad j = 1, 2, \ldots, n, \qquad$ (E1)

where x_i^{hm} is a linear homogeneous (aggregator) function, and x_i^{h} and m_i the amount of the domestically produced and imported component, respectively.

Domestic outputs as well as imported goods are assumed to be composed of homogenous commodities themselves. x_i^{hm} is a linear homogeneous function and all users to minimize their cost, therefore each user will use domestic and imported goods of the same variety in the same proportion: $m_i/x_i^{\mathrm{h}} = s_i^{\mathrm{m}}/s_i^{\mathrm{h}}$, where $x_i^{\mathrm{hm}}(s_i^{\mathrm{h}}, s_i^{\mathrm{m}}) = 1$.

The unit share coefficients, s_i^{h} and s_i^{m}, are determined by the relevant cost minimizing exercise. We assume that they are uniquely determined and their values can be expressed in closed forms, by the following *unit demand functions* (homogeneous of degree zero):

(P17) $\quad s_i^{\mathrm{h}} = s_i^{\mathrm{h}}(p_i^{\mathrm{h}}, p_i^{\mathrm{m}}), \qquad i = 1, 2, \ldots, n, \qquad$ (E2)

(P18) $\quad s_i^{\mathrm{m}} = s_i^{\mathrm{m}}(p_i^{\mathrm{h}}, p_i^{\mathrm{m}}), \qquad i = 1, 2, \ldots, n, \qquad$ (E3)

where p_i^{h} and p_i^{m} are the component prices, given in the cost minimizing exercise, and

(P11') $\quad p_i^{\mathrm{m}} = (1+\tau_i^{\mathrm{m}})\cdot v\cdot p_i^{\mathrm{wm}}, \qquad i = 1, 2, \ldots, n. \qquad$ (E4)

Based on the above forms, the ratio of imports to domestic supply can be defined as follows:

(P5) $\quad m_i = m_i(p_i^{\mathrm{h}}, p_i^{\mathrm{m}}, x_i^{\mathrm{h}}) = x_i^{\mathrm{h}}\cdot s_i^{\mathrm{m}}/s_i^{\mathrm{h}}, \qquad i = 1, 2, \ldots, n. \qquad$ (E5)

The demand for sectoral goods is always given in terms of the domestic/import composite commodity. In the case of linear homogeneous production (aggregation) functions, the only prices, which are compatible with the assumption of profit maximization and equilibrium, are such that are equal to their minimal cost. Therefore, the *unit price of the domestic/import composite* good of sector origin i (p_i^{hm}) must also be equal to their minimal combine cost (the value of the Lagrange multiplier in the cost minimizing problem):

(P13) $\quad p_i^{\mathrm{hm}} = p_i^{\mathrm{h}}\cdot s_i^{\mathrm{h}} + p_i^{\mathrm{m}}\cdot s_i^{\mathrm{m}}, \qquad i = 1, 2, \ldots, n. \qquad$ (E6)

THE REPRESENTATION OF PRODUCTION AND EXPORTS

Production is organized into n sectors, each producing jointly two commodities of the same sector origin, close, but not perfect substitutes for each other. One is supplied on domestic, the other on foreign markets. The joint level of the output in sector j is given by the set of

(P2) $\qquad x_j = x_j(x_j^{\mathrm{h}}, z_j), \qquad j = 1, 2, \dots, n, \qquad$ (E7)

linear homogeneous transformation (aggregation) function, and it can be interpreted as measuring the volume of a special home/export composite good. One unit of such composite commodity can be achieved by various combinations of the two goods, which must satisfy the condition $x_i(s_i^{\mathrm{d}}, s_i^{\mathrm{e}}) = 1$.

The home/export composite commodities are produced by means of domestic/import composite commodities (a_{ij} fixed unit input coefficients), and primary resources (l_j and k_j variable unit input coefficients). Various combinations of labour and capital can provide the same level of capacity, defined by $f_j(K_j, L_j)$, linear homogeneous (constant return to scale) production functions. The scale of these functions is set in such a way that the production of one unit home/export composite good requires (at least) one unit of the composite factor:

$$f_j(l_j, k_j) = 1, \qquad j = 1, 2, \dots, n.$$

The overall production function and capacity constraint in sector j takes, thus, the following form:

$$x_j(x_j^{\mathrm{h}}, z_j) = x_j = \min\left(\frac{X_{1j}}{a_{1j}}, \cdots, \frac{X_{ij}}{a_{ij}}, \cdots, \frac{X_{nj}}{a_{nj}}, f_j(L_j, K_j)\right).$$

Production sectors are assumed to operate as price taking and profit maximizing firms, which maximize revenues and minimize costs, *ceteris paribus*. Thus, deciding on how large part of their output will be supplied on the domestic and how large on foreign markets, they solve a revenue maximizing exercise taking the prices offered on the two markets (p_j^{h} and p_j^{e}) as given. We assume that these problems, too, have always unique solutions, and the revenue maximizing compositions, at unit level of the outputs, can be expressed in closed forms, by the following *supply functions* (homogeneous of degree zero):

(P19) $\qquad s_j^{\mathrm{d}} = s_j^{\mathrm{d}}(p_j^{\mathrm{h}}, p_j^{\mathrm{e}}), \qquad j = 1, 2, \dots, n, \qquad$ (E8)

(P20) $\qquad s_j^{\mathrm{e}} = s_j^{\mathrm{e}}(p_j^{\mathrm{h}}, p_j^{\mathrm{e}}), \qquad j = 1, 2, \dots, n, \qquad$ (E9)

where p_j^{h} and p_j^{m} are the component prices, given in the cost minimizing exercise, and

(P12’) $\qquad p_j^{\mathrm{e}} = (1+\tau_j^{\mathrm{e}})\cdot v\cdot p_j^{\mathrm{we}}, \qquad j = 1, 2, \dots, n, \qquad$ (E10)

and the world market price of exports is given by the following inverse demand functions:

(P21) $\qquad p_j^{\mathrm{we}} = p_j^{\mathrm{we}}(z_j), \qquad j = 1, 2, \dots, n. \qquad$ (E11)

Based on the above forms, the ratio of exports to domestic supply can be defined as follows:

(P4) $\qquad z_j = x_j^{\mathrm{h}}\cdot s_j^{\mathrm{e}}/s_j^{\mathrm{d}}, \qquad j = 1, 2, \dots, n, \qquad$ (E12)

The revenue achieved in sector j by selling one unit of output in optimal proportions can be expressed as

(P10) $p_j^{\mathrm{a}} = p_j^{\mathrm{h}} \cdot s_j^{\mathrm{d}} + p_j^{\mathrm{e}} \cdot s_j^{\mathrm{e}}$, $j = 1, 2, \ldots, n$, (E13)

which is equal to the value of the Lagrange multiplier in the revenue maximizing problem, and it defines the *unit price of the domestic/export composite* output in sector j (p_j^{a}).

Similarly, producers choose such combination of labour and capital that minimizes their joint cost. The labour and capital demand per unit of output in sector j is, thus, determined by a cost minimization exercise, in which the unit costs (prices) of labour and capital are given by the following forms:

(P25') $w_j = (1+\tau_j^{\mathrm{w}}) \cdot w \cdot d_j^{\mathrm{w}}$, $j = 1, 2, \ldots, n$, (E14)

(P15') $q_j = p_j^{\mathrm{b}} \cdot (r_j^{\mathrm{a}} + \pi \cdot d_j^{\pi})$, $j = 1, 2, \ldots, n$. (E15)

Here again, it is assumed that functions f_j are all well-behaved and the cost minimizing *demand for labour and capital* at unit level of the output can be expressed as functions (homogeneous of degree zero) of the factor prices:

(P23') $l_j = l_j(w_j, q_j)$, $j = 1, 2, \ldots, n$, (E16)

(P24') $k_j = k_j(w_j, q_j)$, $j = 1, 2, \ldots, n$. (E17)

The unit cost of the output in sector j will be, thus, as follows:

$$p_j^{\mathrm{ac}} = \Sigma_i (1+\tau_i^{\mathrm{u}}) \cdot p_i^{\mathrm{hm}} \cdot a_{ij} + w_j \cdot l_j + q_j \cdot k_j, \qquad j = 1, 2, \ldots, n,$$

where the value of l_j and k_j is given by the above unit level factor demand functions. Apart from the producers' taxes and revenues, the unit price of the domestic/export composite output in equilibrium must be equal to its cost, therefore the following equilibrium pricing condition must hold:

(P9') $p_j^{\mathrm{a}} = (1+\tau_j^{\mathrm{x}}) \cdot \left(\Sigma_i (1+\tau_i^{\mathrm{u}}) \cdot p_i^{\mathrm{hm}} \cdot a_{ij} + w_j \cdot l_j + q_j \cdot k_j\right)$, $j = 1, 2, \ldots, n$. (E18)

THE REPRESENTATION OF FINAL DEMAND

Final demand for the domestic/import composite commodities will be grouped into the following components: (variable) private consumption, public consumption, replacement and net investments and the rest in the following way:

$$FD_i = s_i^{\mathrm{cv}} \cdot y_{\mathrm{cv}} + g_i \cdot y_{\mathrm{g}} + \Sigma_j\, b_{ij} \cdot (y_j^{\mathrm{br}} + y_j^{\mathrm{bn}}) + c_i^0.$$

In the current specification of the model, only the variable part of private consumption (y_{cv} and s_i^{cv} variables) and the commodities used for replacement investments (y_j^{br} variables) will be treated as endogenous variable. The category rest includes, thus, committed consumption as well in addition to the change in stock, if we employ, as usual, a utility function of Stone–Geary type to explain private consumption decisions.

The levels of the replacement investments in the different sectors (y_j^{br}) will be defined by the depreciation of the capital used, that is, $y_j^{\mathrm{br}} = r_j^{\mathrm{a}} \cdot k_j \cdot x_j$. The equilibrium prices of the sector specific capital goods (p_j^{b}), which appear in the definition of the cost of capital, are equal to the costs of their formation via the investments, that is,

(P14') $p_j^{\mathrm{b}} = \Sigma_i\, (1+\tau_i^{\mathrm{b}})\cdot p_i^{\mathrm{hm}}\cdot b_{ij},$ $j = 1, 2, \ldots, n.$ (E19)

Consumption decisions are assumed to imitate that of a representative household, having a (linear homogeneous) utility function, which is defined over the set of variable consumption only: $y_{\mathrm{v}}(y_1^{\mathrm{cv}}, y_2^{\mathrm{cv}}, \ldots, y_n^{\mathrm{cv}})$. The utility level, $y_{\mathrm{cv}} = y_{\mathrm{v}}(y_1^{\mathrm{cv}}, y_2^{\mathrm{cv}}, \ldots, y_n^{\mathrm{cv}})$ can be interpreted again as defining the volume of a composite consumption good represented by the basket of (y_1^{cv}, y_2^{cv}, ... , y_n^{cv}). The optimal composition of this basket (s_i^{cv}) is determined by the cost minimizing choice at unit level of utility. It is assumed again the cost minimizing solution is unique and yields the following demand functions:

(P22') $s_i^{\mathrm{cv}} = c_i^{\mathrm{v}}(p_1^{\mathrm{c}}, p_2^{\mathrm{c}}, \ldots, p_n^{\mathrm{c}}),$ $i = 1, 2, \ldots, n.$ (E20)

where the set of the consumers' prices are given as:

(P26') $p_i^{\mathrm{c}} = (1+\tau_i^{\mathrm{c}})\cdot p_i^{\mathrm{hm}},$ $i = 1, 2, \ldots, n.$ (E21)

The optimal consumption basket will be the numeraire, and its price index is given as

(P16') $\Sigma_i\, p_i^{\mathrm{c}}\cdot s_i^{\mathrm{cv}} = p_{\mathrm{cv}}\ (= 1),$ (E22)

where p_{cv} is equal to the value of the Lagrange multiplier used in the above cost minimization exercise, and it will be set to be equal to one.

MARKET CLEARING CONDITIONS AND THE CURRENT ACCOUNT

The above definitions of the behavioural rules and optimal decisions determine the supply and demand of the various goods represented in the model. We can now formulate the market clearing conditions of general equilibrium.

(P1) $x_i^{\mathrm{hm}} = \Sigma_j\, (a_{ij} + b_{ij}\cdot r_j^{\mathrm{a}}\cdot k_j)\cdot x_j + s_i^{\mathrm{cv}}\cdot y_{\mathrm{cv}} + g_i\cdot y_{\mathrm{g}} + \Sigma_j\, b_{ij}\cdot y_j^{\mathrm{bn}} + c_i^0,$ $i = 1, 2, \ldots, n.$ (E23)

(P6) $\Sigma_j\, l_j\cdot x_j = L_0,$ (E24)

(P7) $\Sigma_j\, k_j\cdot x_j = K_0,$ (E25)

(P8) $\Sigma_i\, \left(p_i^{\mathrm{wm}}\cdot m_i - p_i^{\mathrm{we}}\cdot z_i\right) = d_{\mathrm{e}}.$ (E26)

The last equation represents the current account (the balance of trade). If its balance, d_{e} is exogenously set, as in this specification, it behaves as a special commodity (foreign currency) with limited external supply. Therefore, the price associated with it (v) reflects the scarcity of this resource relative to the other primary resources and the final goals, as we have seen it in the programming version of the same model. The only variable part of final demand that can adjust to the production capacity determined by the technology and the available stock of resources (labour, capital and currency, in this case) is variable personal consumption. As we have seen it in the programming version of the model, the utility function of the representative households takes in fact the place of a welfare function to be maximized.

2.3.3. Counting equations and variables and closing the CGE model

We have above formulated set of equations similar to those representing the optimal solution of the nation-wide resource allocation model NLP-2.2-2. They are different from each other in as much we have used the revised forms of the equations suggested above. This is also the reason that we have $2n$ more equations and variables than before. The total number of equations is $22n+4$, and we have $22n+5$ variables. All equations are homogenous in prices, therefore we can fix the price level by setting the value of one them, for example, $p_{\mathrm{cv}} = 1$. Thus, the number of unknowns is equal to that of the equations, the equations system is regular.

Although we could expect the derived system of equations to have solution, the general equilibrium model is not yet complete and its specification provides opportunities for revision. Consider first of all that y_{g} and y_j^{bn} are exogenous variables and y_{cv} is set by Walras's law, rather than by means of utility maximization subject to budget constraints. There are no budget considerations introduced explicitly into the model, despite the fact that the distribution and redistribution of incomes is an important constraint as well as means in generating final demand that matches total supply in a monetary economy.

Let us try to construct the budgets of the *economic agents* represented in one way or another in our model. They are the *private households*, in charge of making the private consumption and savings decisions, the *government*, who decides on public consumption and budget deficit or surplus, the *firms* (production sectors), who can be charged to make the investment decisions as well in addition to production decisions, and the *foreigners*, who represent the rest of the world.

One of the special advantages of the computable general equilibrium models is that they can cover all the major aspects of public finance including all substantial taxes, social policy transfers, public expenditures and deficit financing instruments. The models contain, usually in considerable detail, the process of income distribution and redistribution, which takes place via various channels. The primary incomes received can be interpreted in broader sense than usual, to include not only wages and gross operating surplus (labour and capital), various taxes, for example on wages, consumption, imports and exports. Secondary income generation takes place in the form of transfers between the above mentioned various agents.

Most of the transfers are assumed to be proportional to some activity levels, represented in the model, and are assumed to be set in real (valorised) terms so as to maintain the price homogeneity of the model. Here we will represent them only by their net outcome ($\pm\, tr^{\mathrm{a}}$) in the budgets of the economic agents, which eventually define the net monetary savings (S^{a}) as the difference between disposable incomes and expenditures.

We will, thus, introduce for each agent a function, $tr^{\mathrm{a}}(\cdot)$, representing the net result, the positive or negative *balance of the transfers* taking place between them. The net transfer functions depend on specific endogenous variables, which determine the levels of the various activities that form the basis of the transfers. The sum of the transfers is by definition zero, that is,

$$tr^{h}(\cdot) + \Sigma_j\, tr_j^{f}(\cdot) + tr^{g}(\cdot) + tr^{rw}(\cdot) = 0.$$

We have to introduce also n+3 additional variables, S^{h}, S_j^{f}, S^{g} and S^{w}, to represent the net monetary saving position (savings or borrowings) of the *households*, the *firms* (production sectors), the *government* and the *foreigners*.

We will also add the same number of equations to the system, to define the budget conditions of the various agents that must be satisfied by the feasible solutions. The primary incomes received and the net result of the transfers will be presented on the left hand sides of the budgets, which defines thus the total income available for the given economic agent. On the right hand side one will find the expenditures and the net result of savings/borrowings.

To keep the presentation simple, we assume that the whole of c_i^0 consists of committed private consumption. The *budget of the private households* will thus be as follows (wages plus/minus transfers equal to consumption expenditure plus net savings):

$$\Sigma_j\, w\cdot d_j^{w}\cdot l_j\cdot x_j \pm tr^{h}(\cdot) = \Sigma_i\, p_i^{c}\cdot(c_i^{0} + y_{cv}\cdot c_i^{v}) + S^{h}. \qquad \text{(E27)}$$

The *budget of production sector j* will take the following form (amortization plus net operating surplus plus/minus transfers equal to investment expenditure plus net savings, typically minus borrowings):

$$p_j^{b}\cdot(r_j^{a} + \pi\cdot d_j^{\pi})\cdot k_j\cdot x_j \pm tr_j^{f}(\cdot) = p_j^{b}\cdot y_j^{bn} + S_j^{f}, \qquad j = 1, 2, \ldots, n. \qquad \text{(E28)}$$

The *government* collects direct and indirect taxes, gives subsidies, redistributes income via transfers, finances public expenditure and the resulting balance will be equal to the budget deficit or surplus:

$$\Sigma_j\{\Sigma_i\,\tau_i^{u}\cdot p_i^{hm}\cdot a_{ij} + \tau_j^{w}\cdot w\cdot d_j^{w}\cdot l_j + p_j^{a}\cdot\tau_j^{x}\}\cdot x_j + \Sigma_j\Sigma_i\,\tau_i^{b}\cdot p_i^{hm}\cdot b_{ij}\cdot y_j^{bn} + \Sigma_i\{\tau_i^{c}\cdot p_i^{hm}\cdot(c_i^{0} + y_{cv}\cdot c_i^{v}) +$$

$$\tau_i^{u}\cdot p_i^{hm}\cdot a_{ij} + \tau_i^{b}\cdot p_i^{hm}\cdot b_{ij}\cdot y_j^{b} + \tau_i^{m}\cdot v\cdot p_i^{wm}\cdot m_i - \tau_i^{e}\cdot v\cdot p_i^{we}\cdot z_i\} \pm tr^{g}(\cdot) = \Sigma_i\, p_i^{u}\cdot g_i\cdot y_g + S^{g}. \qquad \text{(E29)}$$

Finally, the *budget of the foreigners* (rest of the world)

$$\Sigma_i\, v\cdot p_i^{wm}\cdot m_i + tr^{rw}(\cdot) = \Sigma_i\, v\cdot p_i^{we}\cdot z_i + S^{rw}, \qquad \text{(E30)}$$

which could also be written as the sum of the current account and monetary transfers, which result in the balance of payments, represented by S^{w}:

$$v\cdot d_e + tr^{w}(\cdot) = S^{w}. \qquad \text{(E30')}$$

The sum of the net savings/borrowings should be zero, that is,

$$S^{h} + \Sigma_j\, S_j^{f} + S^{g} + S^{rw}(\cdot) = 0.$$

It can be shown that equations (E1) - (E26) imply the basic accounting identity that states that the value of final demand is equal to the sum of primary incomes, as required by Walras's law. The same identity is implied by equations (E27) - (E30), if we assume that total savings and borrowings match each other. The above equilibrium condition will, thus, be automatically satisfied, so there is no need to introduce it separately.

As we have discussed it, equations (E1) - (E26) can be solved for the variables contained by them. Once we know their solution, equations (E27) - (E30) can be independently solved for the net savings/borrowings variables. This shows the specificity of the *implied macro-closure* of the CGE model specification derived, step-by-step, from the macro-programming model 2.2-2. Namely, *savings adjust to the structure of final demand*, which is set basically by fiat, externally, by considering public consumption ($\mathbf{y}^{g}$) and net investments ($\mathbf{y}^{nv}$) to be exogenously given. This choice of endogenous and exogenous macro variables can be called a *programming macro-closure*.

It is, however, far not clear, which potential variables should be treated as exogenous and which as endogenous variables in the model. As a matter of fact, exogenous variables are partly used to counterbalance the shortage of the static model (see the discussion of the problem of macro-closure in the case of Walras's second model), partly the lack of well tested theories to describe the complex interdependence of the main economic variables. Each exogenous variable represents in a sense an equation missing from the model. As a matter of fact, when we set the value of certain variables exogenously, we make a conditional ('what if□) forecast in terms of the endogenous variables. By the same token, choosing one or another plausible specification possibility, we fix some conditions as corner stone for our analysis.

The above uncertainty and certain arbitrariness involved in the choice of specification can be counterbalanced by using alternative assumptions and test the robustness of the conditional forecasts. With each specification option we can generate an internally consistent forecast for the endogenous variables. In this way, we can derive "packages", that indicate alternative, possible and consistent changes in macroeconomic variables (see, *Zalai et al.*, 2002 for more details on this).

We will illustrate the macro-closure possibilities by some characteristic macro-closure options (see *Dewatripont and Michel*, 1987, *Lysy*, 1983, *Taylor*, 1983 and 1990 on a theoretical discussion of closure options). Let us take first the example of public consumption, the level of which (y_g) is usually set exogenously in the CGE models, because it is decided by economic policy makers in a way, which is difficult to model. Nevertheless, an alternative variable that could be exogenously set instead of y_g, is public deficit (S^g), which has become a growing concern in many countries nowadays. Yet another option is to fix both macro-variables, that is, both y_g and S^g, and free the general level of some tax rates (e.g., social security contribution, τ_j^w), by means of which the government collects income. That could to bring public budget into the required balance.

In the given specification of the model, there was no behavioural equation that would have explained the amount of the net savings of the households (S^g). It was rather adjusting to the level of private consumption. One might want to introduce such behavioural equation, for example, by assuming constant propensity to save or using the so-called *ELES* (Extended Linear Expenditure System) model, which would derive the level of savings from the optimizing behaviour of the representative household. That would require, in either case, the introduction of additional equation into the model, which in turn would necessitate to free some variable, considered to be exogenous so far, that would bring the households budget into balance instead of savings. A new endogenous variable to play this role could be, for example, the general level of investments.

The general level of investments could be made endogenous also by fixing the level of some other components of savings instead. That would shift the Keynesian macro-closure towards a neoclassical one, in which investments adjust to savings. As a matter of fact, a major source of savings, the balance of payments, is to a large extent determined exogenously, as one can see it from equation (E30'), in which d_e is considered to be an exogenous variable. Again, to shift further towards a Keynesian macro-closure, one might make d_e endogenous and fix the nominal exchange rate (v) instead exogenously.

In typical neoclassical models, the supply of labour would be made an endogenous variable, the level of which would be also defined by the optimizing choice of the representative household. A radical shift in the direction of the Keynesian world would be to fix the *nominal wage level* (w) and let the labour market move out of equilibrium. This could be made technically feasible by introducing a new variable, the labour utilization index (l_u) and replacing L_0 in the labour market clearing condition with $l_u \cdot L_0$. That would, of course, completely change the original meaning of that condition. It would simply set the level of the labour utilization index (l_u), which could be seen as an indicator of disequilibrium or tension on the labour market, rather than a resource constraint as before.

Another variant of this closure would be fixing the *real wage rate* instead, that could make more sense, especially in economies with strong labour unions. Instead of the real wage rate one could fix the level of utility function (the real value of consumption), that Taylor would classify as a *Marxian macro-closure*.

One may find difficult to justify the assumption of fixed capital stock, the scarcity of which determines the rate of return on capital, in a CGE model meant to generate a longer term perspective scenario. One could instead fix the rate of return and introduce a capital utilization index (k_u) as in the case of labour above.

As can be seen from the above examples, the macro-closure problem is closely related to the mechanism that sets the proportions between the main components of final demand, such as the general level of private and public consumption, investments and net exports, on the one hand, and the level of the key variables that determine the distribution of the national income, such as the general rate of wages, return on capital and foreign exchange. The above sets of variables compete with each other, that is, they can increase only at the expense of each other, because the overall level of the national income (net product) is determined practically by the available stocks of primary resources in our model. The two sets of macro variables are connected to each other through the income (re)distribution rules, which should secure that the demand generated by disposable income matches the emerging supply.

Some CGE models attempt to integrate the microeconomic general equilibrium models with macroeconomic IS-LM mechanism (termed as macro-micro integration), which has been traditionally used in Keynesian models (see for example *Bourguignon, Branson, DeMelo*, 1989, *Capros et al.*, 1990). These hybrid models are designed to overcome the limitation of arbitrary closure rules, which must otherwise be adopted. In addition, due to the introduction of financial market mechanisms and related structural adjustment, they allow to set the level of prices as well.

2.3.4. Notes on the calibration of substitution functions used in the CGE model

General equilibrium is about supply and demand, and relative price driven changes. The economic agents are assumed minimize cost and maximize their profit or utility. Comparative static analysis requires the model to possess locally stable solution, which in turn implies mathematical restrictions on the forms of the production and utility functions, and other possible functions that are assumed to direct and limit choices flexibly between alternatives.

Substitutability between goods in production or in consumption is a key element that makes adjustment to changing prices or, the other way around, price adjustments to changing volumes of supply or demand feasible, and in this way the emergence of an equilibrium solution possible. The representative firms and households react on prices (including taxes) and adjust their mix of inputs and outputs, or consumer goods by substituting away from the relatively more expensive input or good.

The representation of technologies and preferences with well-behaved functions, and the assumption of rational (optimizing) representative agents provide a pragmatic approach to model changes in aggregate macroeconomic variables, as we have discussed it in the previous chapter. Even if one treated this assumption as real, observable and testable behaviour, the theoretical structure of applied general equilibrium models is far too rich to allow for the proper estimation of their parameters by means of econometric techniques (the information required for the direct econometric estimation is incomplete, scattered, does not show enough variation, and is utterly unreliable).

The issue related to the use of well-behaved functions and their elasticities is essentially twofold. First, it is the question of the choice of functional forms to describe the assumed substitution possibilities (constant or variable elasticities of substitution), second the way one sets the values of the parameters of the chosen functions (statistical estimation versus calibration).

The elasticity is a mathematical concept related to differentiable functions. In general, if $y = f(x)$, the elasticity of y with respect to x measures in percentages the change in y induced by an infinitely small change in x (the logarithmic derivative of the f function with respect to x):

$$\sigma = \frac{\partial f}{\partial x} : \frac{y}{x} = \frac{\mathrm{d}y}{\mathrm{d}x} : \frac{y}{x} = \frac{\mathrm{d}\ln y}{\mathrm{d}\ln x}.$$

The *direct partial elasticity of substitution*, for example, in the case of the production functions characterizes the ease at which factor j can compensate a change in the amount of factor i, *ceteris* paribus, that is, when the output and all other factors are fixed. It measures the curvature of the partial isoquant (in the case of a utility function the indifference curve). The concept of direct elasticity assumes that the ratio of the two factors ($r_{ji} = x_j/x_i$) is a function of the slope of the isoquant ($f_j/f_i = \mathrm{d}x_i/\mathrm{d}x_j$, the marginal rate of substitution, which is equal to the ratio of the factor prices in the optimal solution), and it is the elasticity of that function:

$$\sigma = \frac{\partial(x_j / x_i)}{\partial(f_j / f_i)} : \frac{x_j / x_i}{f_j / f_i} = \frac{\partial r_{ji}}{\partial(p_j / p_i)} : \frac{r_{ji}}{p_j / p_i}.$$

Another concept is the *Hicks-Allen partial elasticity of substitution*, which can be viewed as normalized price elasticity. It is defined as the ratio of the price elasticity of factor demand and the factor's cost share. It measures by how much the demand for factor j changes in response to a change in the price of factor i, when output and the prices of all the other factors are taken as given (where i and j can be the same).

The general form of the *constant-elasticity-of-substitution* (CES) functions can be written as

$$y = f(\mathbf{x}) = (A_1 x_1^{-\beta} + A_2 x_2^{-\beta} + \ldots + A_n x_n^{-\beta})^{-\frac{1}{\beta}},$$

where A_i are called share parameters. The direct elasticity of substitution between any pair of factors is $\sigma = 1/(1+\beta)$.

At given the factor prices the derived (minimum) unit cost function is given also by a CES form, which is dual to the original (the production) function:

$$c(\mathbf{w}) = (A_1^{\sigma} w_1^{1-\sigma} + A_2^{\sigma} w_2^{1-\sigma} + \ldots + A_n^{\sigma} w_n^{1-\sigma})^{\frac{1}{1-\sigma}} = \left(\sum_{j=1}^{n} A_j^{\sigma} w_j^{1-\sigma} \right)^{\frac{1}{1-\sigma}}.$$

The so-called *Shephard lemma* states that the unit factor demand function is nothing but the derivative of that cost function with respect to the price of the given factor:

$$a_i = a_i(\mathbf{w}) = A_i^{\sigma} \left(\frac{c}{w_i} \right)^{\sigma},$$

where c is the value of the minimum cost, equal to the cost-price of the output in equilibrium.

A) SUBSTITUTION ELASTICITIES IN THE SECTORAL PRODUCTION FUNCTIONS

CGE models focusing on environmental and energy policy issues need to have an elaborated treatment of the demand for energy resources and energy intensive sectoral goods. That determines to a large extent the sectoral break-down used in such models, and has certain implications for the specification of production functions, as well. Production sectors that are fossil fuel intensive may consist of sub-sectors that differ significantly from this point of view. This is clearly the case for the electricity sector where the output can be produced both by fossil fuel intensive technologies such as coal and oil power, and fossil fuel free technologies such as hydroelectric power and nuclear power. In order to capture these substitution possibilities in a realistic way the technological constraints of the electricity sector, or the entire energy sector, is sometimes represented by a separate sub-model rather than by a standard neoclassical production function (bottom-up approach).

The sectoral production functions basically define substitution possibilities between explicitly defined input factors. In CGE models focused on environmental policies distinguish not only between capital, labour, non-energy intermediate inputs and energy, but also between fossil and non-fossil energy. Often it is also convenient to distinguish between fuels and electricity. In some CGE models instead of the production function its dual, the cost functions appears. This is typically the case if flexible form (translog or generalized Leontief) functions are used, the parameters of which are econometrically estimated. However, lack of data often prevents econometric estimation of sectoral cost functions. Instead, both the nesting structure of the production functions and the adopted numerical values are based on literature surveys of relevant econometric studies.

The constant elasticity of substitution (CES) form is convenient especially because it uses but a few parameters and can be relatively easily handled mathematically. The particular disadvantage of the basic CES function is that the direct elasticity of substitution between any pair of inputs is the same, as we have seen. One can allow the elasticity of substitution to vary between certain separable groups of inputs by using nested functions. The existing literature on econometric studies of production does not lead to definite conclusions about the most appropriate nesting structure. However, in most models fuels and electricity are combined in a CES function with a relatively high elasticity of substitution. The input "fuels" is often defined as a CES-aggregate of different type fossil and non-fossil fuels. The elasticities of substitution between different types of fuels are usually taken to be relatively high.

Treating goods produced for domestic (X^d) and foreign (X^e) markets also as imperfect substitutes, represented by a constant-elasticity-of-transformation (CET) function, a typical and quite general representation of the technology in a model of GEM-E3 type would be of the following structure:

$$X^p(X^d,\ X^e) = X^c\big(L,\ M,\ Q\{K,\ EN[E,\ F(F_1,\ F_2, \ldots\ , F_k)]\}\big)$$

Thus, fuels (F), which is a CES aggregate of k different types, and electricity (E) are combined in a CES aggregate that defines a composite energy good (EN). The composite energy input is then combined with capital in a CES aggregate of capital-energy (Q). Finally, the composite capital-energy input Q is combined with labour (L) and materials (M) that determines the capacity of the given amount of production factors (X^c) expressed in terms of the composite output. X^p, on the other hand, combines the capacity need of the two types of products.

In the simplest case, when the production function is given by the form of $X\{L,\ KE(K,\ E)\}$, with substitution parameters σ_{VA} (outer function) and $\sigma_{KE\text{-}DE}$ (inner function is). The Hicks-Allen elasticity of substitution between energy and capital can be calculated by means of Keller's formula (*Keller*, 1980, p. 83) as

$$\sigma_{KE\text{-}HA} = (\sigma_{KE\text{-}DE} - \sigma_{\mathrm{VA}})/S_{KE} + \sigma_{\mathrm{VA}} = (\sigma_{KE\text{-}DE} - S_L\sigma_{\mathrm{VA}})/S_{KE}\,.$$

We have noted also that the Hicks-Allen elasticity is the ratio of the price elasticity and the cost share of the input, whose price has changed: $\sigma_{KE\text{-}HA} = \varepsilon_{E/Pk}/S_K$, where S_{KE} and S_K is the cost share of the KE-composite and K in the value-added (outer nest), respectively. If the level of the composite capital/energy input remained constant instead of the level of output, the resulting elasticity of substitution would be $\sigma_{KE\text{-}DE}$, which is often denoted simply by σ_{KE}. It is important to note that if $\sigma_{KE\text{-}DE}$ is smaller than σ_{VA}, then $\sigma_{KE\text{-}HA}$ can be negative, implying complementarity between *K* and *E* in the outer nest, while *K* and *E* are substitutes in the inner nest (in the *KE* composite).

To calculate the elasticity of substitution between any two inputs *n* and *m* at a particular level *L* in the nested-CES structure, the formula derived by Keller takes the following form:

$$\sigma_{nm} = \sigma_{n,h} S_{n,h}^{-1} - \sum_{s=l+1}^{h} \sigma_{n,s} [S_{n,s-1}^{-1} - S_{n,s}^{-1}]$$

where *l* represents the lowest level in the nested-CES structure at which a component exists, associated with both the *n* and the *m* inputs (the lowest common level) and *h* is the highest level in the nested structure at which the elasticity σ_{nm} is calculated, and the cost share $S_{n,s}$ is defined by

$$S_{n,s} = \sum_{i \in n} S_i \text{ ,}$$

that is, the sum of all the cost shares associated with the aggregate input *n* at level *l*, or, in other words, the cost share of the input component *n*.

In some models capital and labour rather than capital and energy are combined, and one can find further varieties in the literature. Which particular structure should be used to represent the substitution possibilities between alternative fuels (inter-fuel substitution) and between the energy aggregate as a whole and other primary factors, such as labour and capital (fuel-factor substitution)? In particular, the question of energy-capital complementarity or substitutability is a major issue in the literature. The econometric evidence is conflicting. Some studies indicate that capital and energy are substitutes at the relevant level of aggregation, while others suggest that capital and energy are complements. Most CGE models assume that capital and energy are substitutes, although the elasticity of substitution between capital and energy is generally taken to be quite low.

The issue of energy-capital complementarity or substitutability (whether output produced goes up or down after an increase in the energy price, indicated by σ_{EK}) may turn out to be a crucial one in determining the direction of the adjustment of aggregate output following energy price changes. Despite the importance of the σ_{EK} parameter, *empirical* estimates of this parameter must overcome many difficulties. Table 2.5 bellow gives some indicative values of elasticities used in various empirical studies. It can be seen from this table that both the sign and magnitude of the σ_{EK} parameter varies significantly between different studies.

Table 2.5: Some estimates of the Partial Hicks-Allen Elasticities of Substitution (σ) and Factor Shares (α)

	US	US	US	Europe	Australia
	Berndt-Wood (1975)	Kulatilaka (1980)	Pindyck (1979)	Pyndyck (1979)	Truong (1985)
σ_{KK}	-8.8	-2.75	-1.66	-0.98	-16.46
σ_{LL}	-1.5	-0.22	-1.19	-0.82	-1.388
σ_{EE}	-10.7	-2.70	-24.21	-13.16	-19.60

σ_{MM}	-0.39				-0.222
σ_{KL}	1.01	0.69	1.41	0.69	1.02
σ_{KE}	-3.5	-1.09	1.77	0.60	-2.95
σ_{LE}	0.68	0.61	0.05	1.13	1.77
σ_{KM}	0.49				0.78
σ_{LM}	0.61				0.42
σ_{EM}	0.75				0.17
α_L	0.289	0.76	0.478	0.526	0.263
α_E	0.044	0.10	0.032	0.055	0.023
α_K	0.046	0.14	0.488	0.409	0.044
α_M	0.619				0.67

K = Capital, *L*= Labour, *E* = Energy, *M*= Material.

Source: Burniaux – Truong (2002), originally Vinals (1984), and Truong (1985).

B) THE PARAMETERS OF THE DEMAND SYSTEMS OF HOUSEHOLDS

Consumers' demand is represented by one or several utility maximizing representative households in the CGE models. The most widely used is the Linear Expenditure System (LES) or its extended version (ELES). The LES demand system is derived, as we have seen, on the basis of maximizing a Cobb-Douglas utility function defined over excess consumption, i.e., demand in excess to the so-called committed consumption (Stone-Geary preferences). This representation of consumption allows also for the differentiation of the price elasticities (as the nesting structures) among various goods, despite the use of a linear homogenous utility function.

In its extended version not only the demand for ordinary goods but also that for reserved income (saving) is derived through preferences. Yet another extension of the preference approach includes leisure among the consumption categories, and in this way, utility maximization yields also the supply of labour. The demand for sectoral (produced and/or imported) goods is assumed to be separable from other goods and their preference ordering is represented by a separate nest in the utility function. The structure of this (sub-)utility function is often similar to that part of the production functions that relates to sectoral goods (materials/energy).

A further extension of the LES approach is the introduction of consumption categories (wants), which are served by various bundles of sectoral commodities. Following such an approach the utility functions are defined in the space of wants, and the demand for wants is then translated by means of (so-called Lancaster) conversion matrices into demands for sectoral goods. In multi-period GEM-E3 type models special consideration is given to the consumption and accumulation of durable goods, and to the consumption of non-durable goods linked to durable ones.

C) THE ARMINGTON ELASTICITIES (SUBSTITUTION POSSIBILITIES VIA FOREIGN TRADE)

The so-called Armington elasticities, used typically in CGE models, refer simply to the elasticities of substitution between domestically produced and imported goods. In an open economy, each commodity can be differentiated according to its source of production: domestic or foreign (import). In Armington's approach these goods are assumed to be differentiated products, less the perfect substitutes. In this way one can model intra-sectoral trade in a theoretically consistent manner.

Domestic absorption in the CGE models is given in terms of demand for domestic/imported composite goods, the actual mix of which is determined by their relative prices and the degree of their substitutability, captured by the Armington elasticities. The higher is the value of this parameter, the closer substitutes the domestically produced and imported goods are. In multiregional models Armington elasticities constitute a significant subset of the parameter space of the demand system. They play an especially important role in the analyses of the economic effects of trade policies. When the tariff applied to imports of a particular commodity changes, it directly affects the domestic price of the imported commodity, and indirectly the price of the domestically produced commodity and domestic resource allocation effects. Such changes in trade policy have, thus, an effect on the structure of domestic production, the size of which depends on the degree of substitutability between domestically produced and imported commodities.

Multiregional models differentiate imports also by the region of origin. The empirical literature, however, concentrates on the differentiation between domestic supplies and imports, rather than on the differentiation among import supplies. This approach is followed also by the CGE models, which apply in most cases two-level nested CES functions. The upper nest defines the degree of substitutability between domestic production and the composite of imported goods. The lower nest does the same between imports coming from different regions. This may seem an oversimplification, but in view of the enormous difficulties to obtain accurate enough statistics on foreign trade, the difference between domestic and imported goods seems likely to be greater than the differences among imports coming from different countries.

In many CGE models using nested import structure the "rule of two" is applied, by which the Armington elasticity of substitution across imports by sources (lower nest) is set equal to twice of the elasticity of substitution between domestic goods and imports (upper nest). We can see this rule of two at work in the table below, sampling elasticity values taken from the GTAP6 database.

Table 2.6: Some elasticity values taken from the GTAP6 database

Sector	Import-domestic	Import-import
Wheat	4,45	8,90
Vegetables, fruit, nuts	1,85	3,70
Forestry	2,50	5,00

Fishing	1,25	2,50
Coal	3,05	6,10
Oil	5,20	10,40
Gas	17,20	34,40
Sugar	2,70	5,40
Textiles	3,75	7,50
Leather products	4,05	8,10
Wood products	3,40	6,80
Paper products, publishing	2,95	5,90
Chemical, rubber, plastic products	3,30	6,60
Metal products	3,75	7,50
Motor vehicles and parts	2,80	5,60
Electricity	2,80	5,60
Construction	1,90	3,80
Trade	1,90	3,80
Transport	1,90	3,80
Communication	1,90	3,80

THE PRACTICE OF CALIBRATING THE APPLIED SUBSTITUTION FUNCTIONS

The real use of CGE models is in counterfactual analysis that is based on coherent theoretical framework (conditional insights based on theories with numbers, indications of the relative orders of magnitude for possible policy adjustments). This analysis consists of the following, equally crucial steps:

step 1: choice of appropriate model (specification, alternative theoretical structures),

step 2: construction of consistent equilibrium data set (benchmarking, data derived from several sources of information (extensive use of RAS, Row and Column Scaling technique),

step 3: calibration, i.e., setting of specified parameters to replicate a benchmark data set,

step 4: consistency check and preparation of the base scenario,

step 5: counterfactual simulation and analysis.

A variety of approaches have been used to obtain parameters for CGE models. By far the most common approach is to specify fairly parsimonious functional forms, obtain necessary behavioural parameters from the micro-econometric literature (or other sources), and then calibrate the remaining parameters such that the model perfectly reproduces a base year data set. Calibration means the setting of specified parameters to replicate a benchmark data set, making use of the equations characterizing an equilibrium solution (solution of the model for the unknown parameters), set out by Mansur and Whalley (1984). The calibration results have therefore only conditional predictive power and can not be used for forecasting. The model is conditional on the

a) choice from the alternative theoretical structures (see for example the issue of macro-closure),

b) choice of some key parameters (non-existent or contradictory estimates provided by the literature),

c) selection of assumptions employed in the base scenario.

This approach has the distinct advantage of not requiring time series data and leaving estimation issues to the econometricians, and of imposing the full set of general equilibrium constraints. On the other hand, it makes limited use of the historical record and provides no statistical basis for judging the robustness of estimated parameters. Therefore other, more ambitious, approaches to parameter estimation and/or model validation have been attempted.

The CGE models embody three types of information: analytical, functional and numerical. The analytical structure is the background theoretical material which identifies the variables of interest and posits their causal relations. The functional structure is the mathematical representation of the analytical material, and consists of the algebraic equations which make up the actual model. The numerical structure consists of the signs and magnitudes of the coefficients in the equations which form the functional structure. The econometric critique is not directed at the analytical structure of these models, which is based on the neoclassical canon, but it is directed at the functional and numerical structures of calibrated CGE models.

Calibration usually follows a method that includes the interaction of a strict theoretical structure with the observed benchmark data that are assumed to represent equilibrium solutions at the base prices. The elasticities that indicate the degree of response to changes in relative prices are often borrowed from independent databases, usually from other similar models, which include micro-econometric estimates on each of the required elasticities. Once the elasticity has been chosen, one can easily identify the share and scale parameters of the CES functions, which are in line with the assumption that the base data reflect equilibrium.

It can be shown that the cost shares of the various components (S_i) in the optimal solution must satisfy the following condition:

$$S_i = \frac{w_i x_i}{c} = \frac{A_i^{\sigma} w_i^{1-\sigma}}{A_1^{\sigma} w_1^{1-\sigma} + A_2^{\sigma} w_2^{1-\sigma} + \ldots + A_n^{\sigma} w_n^{1-\sigma}}.$$

Thus, if the prices (**w**) and the elasticity of substitution (σ) is known, the above equations uniquely define the proportions of the share parameters. Their scale (the scale parameter) can then be adjusted so that the value of the function will be equal to its observed magnitude.

In the case of the nested functions, the volume of the lower level composite goods can be freely chosen. More precisely, one can freely choose either the level of the unit base cost-price of the composite good (c) or their composite volume, $x_{\mathrm{EK}} = x_{\mathrm{EK}}(x_{\mathrm{E}}, x_{\mathrm{K}})$, because their product must be equal to their observed joint cost in the base case, which is given by $c \cdot x_{\mathrm{EK}} = w_{\mathrm{E}}^0 \cdot x_{\mathrm{E}}^0 + w_{\mathrm{K}}^0 \cdot x_{\mathrm{K}}^0$.

For example, the unit level of the composite labour/capital factor, the level of $F(L, K)$ can be set by the $X^0 = F(L^0, K^0)$ equation, whereby we get $f(l, k) = 1$ for the unit composite coefficient, as we did in the case of the Johansen technology, and $c = w^0 \cdot l^0 + q^0 \cdot k^0 = c^0$ in the base. Take as another example the case of the composite domestic/home supply. Suppose, the base prices of the components were equal to one in the base ($p_i^{h0} = p_i^{m0} = 1$). It appears natural to set the base price level of their unit composite good to one ($p_i^{hm0} = 1$), too. In such a case the level of the composite good, $x_i^{hm} = x_i^{hm}(x_i^h, m_i)$ will be set by the following condition: $x_i^{hm}(x_i^{h0}, m_i^0) = x_i^{hm0} = x_i^{h0} + m_i^0$.

2.4. Illustrative programs

The special programs accompanying this training material contain illustrative numerical examples for the models covered in this chapter too. An Excel program (LP-2x2-6cases-CES.xls) facilitates the understanding of the main characteristics of the programming models. The program automatically computes and graphically displays the feasible set and the optimal solution of the resource allocation model. The welfare function used is a generalized CES-function, thus the program illustrates also the nature and role of CES functions in CGE-models. The program simultaneously shows the solutions of a closed and open model, comparing their results and showing how the comparative advantages improve welfare.

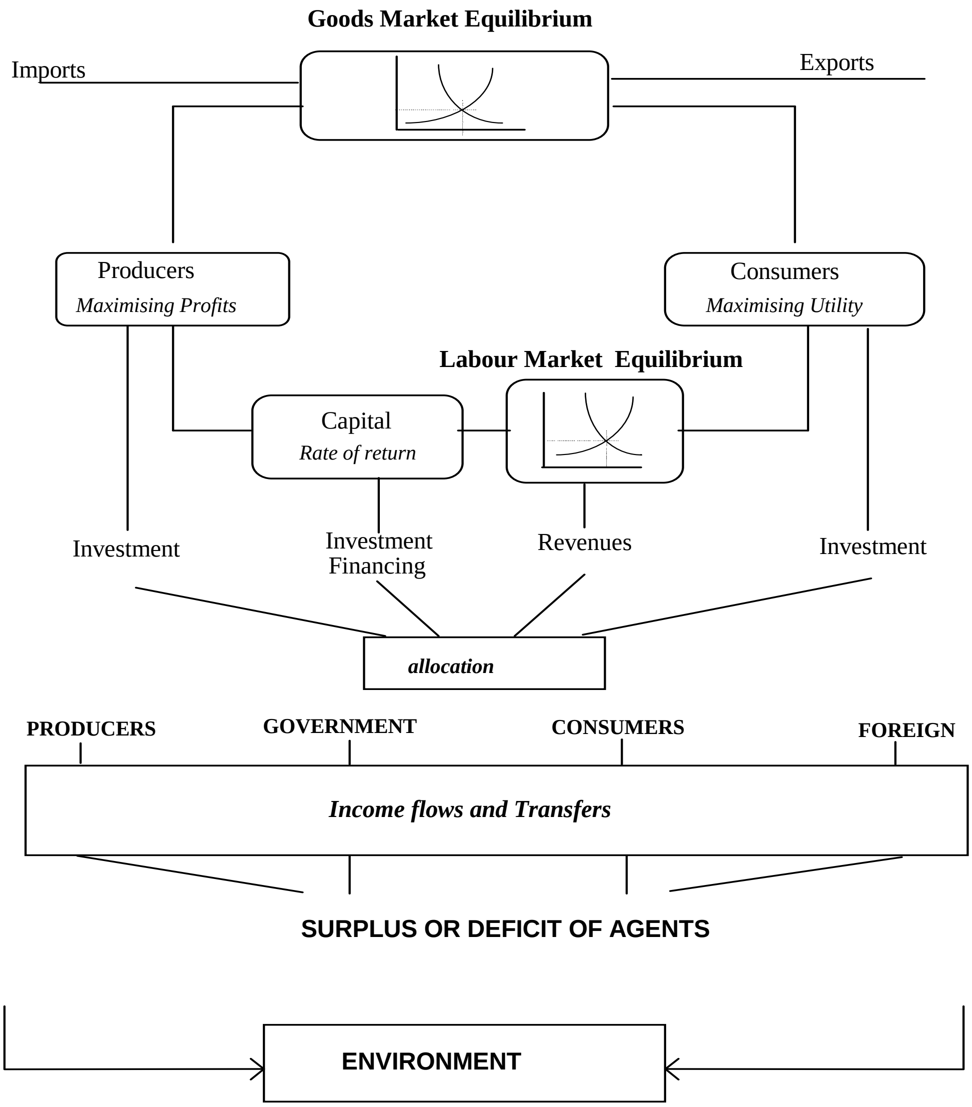
Another package, a program written in GAMS and related files (PROJECT.ZIP, MultHH-opt-scen) compares the behaviour of the NLP and CGE models. (See Appendix 6 for the derivation of the necessary conditions of the optimal solution of the NLP model, which are solved as those of a modified CGE model.) The model distinguishes 3 sectors and 10 household groups and is calibrated for Hungarian data for 1998, and is complemented with an Excel interface, which presents, summarizes and compares the results of up to 7 simulation runs in a transparent Excel sheet. The GAMS code of this program illustrates also the way how model results can be presented in Excel.

A flowchart in Appendix 7 demonstrates the interdependency of the individual blocks of a CGE model and shows how a particular macroeconomic closure defines the logic of the recursive computation of the variables. The example used is based on the structure of a CGE model (called HUMUS), developed originally for Hungary, and adaped for Austria (with 2000 as the benchmark year). See Balabanov, T. – Revesz, T. – Zalai, E. (2007).

3. The specific features of the *GEM-E3* model

The ***GEM-E3*** model (***G***eneral ***E***quilibrium ***M***odel for ***E***nergy-***E***conomy-***E***nvironment interactions) has been developed as a multinational collaboration project[4], partly funded by the Commission of the European Communities. The initial GEM-E3 European model has simultaneously represented fifteen European countries, linked through endogenous bilateral trade, and is being extended towards the associated countries and Switzerland. GEM-E3 aims at covering the interactions between the economy, the energy system and the environment. Figure 3.1 gives the basic scheme of the model.

Figure 3.1: The basic scheme of the GEM-E3 model



The model is simultaneously multinational and specific for each country/region. Although global, the model exhibits a sufficient degree of disaggregation concerning sectors, structural

[4] The GEM-E3 model was built under the auspices of European Commission (DG-Research) by a consortium involving BUES, ERASME, NTUA, KUL, PSI, ZEW and at the beginning of the project CORE, Univ. Strathclyde and CEA.

features of energy/environment and policy-oriented instruments (e.g. taxation). The model is multi-period, recursive over time. Technology progress is explicitly represented in the production function, either exogenous or endogenous. The model formulates pollution permits for atmospheric pollutants and flexibility instruments allowing for a variety options.

The ***GEM-E3*** model starts from the same basic structure as the standard *CGE models*. Following a micro-economic approach, it formulates the supply or demand behaviour of the economic agents regarding production, consumption, investment, employment and allocation of their financial assets. Prices are computed by the model as a result of supply and demand interactions in the markets. The current stream of CGE models, through its modular design, encompasses the whole area of modern economics going much beyond the standard neo-classical economics on which the first generation of CGE models was confined. This new generation of model design is the inspiration behind the development of the ***GEM-E3*** model.

The model is built on the basis of a Social Accounting Matrix by combining Input-Output tables with national accounts data. The specification of production and consumption follows the generalised Leontief type of models. Technical coefficients in production and demand are flexible in the sense that producers can alternate the mix of production not only regarding the primary production factors but also the intermediate goods. Production is modelled through KLEM (capital, labour, energy and materials) nested production functions, involving multiple factors (all intermediate products and two primary factors, capital and labour). Consumers can also decide the structure of their demand for goods and services. Their consumption mix is decided through a flexible expenditure system involving durable and non-durable goods.

Bilateral trade flows are also calibrated for each sector represented in the model, taking into account trade margins and transport costs. Consumption and investment is built around transformation matrices linking consumption by purpose to demand for goods and investment by origin to investment by destination. The model includes a very detailed treatment of taxation and trade. To this respect the model follows the methodology of the models that are developed to study tax policy and international trade.

Through its flexible formulation, it also enables the representation of hybrid or regulated situations, as well as perfect and imperfect competition. The current model version for example, incorporates sectors in which only a limited number of firms operate under oligopoly assumptions. Models with imperfect competition are a rather recent addition to the literature of CGE models.

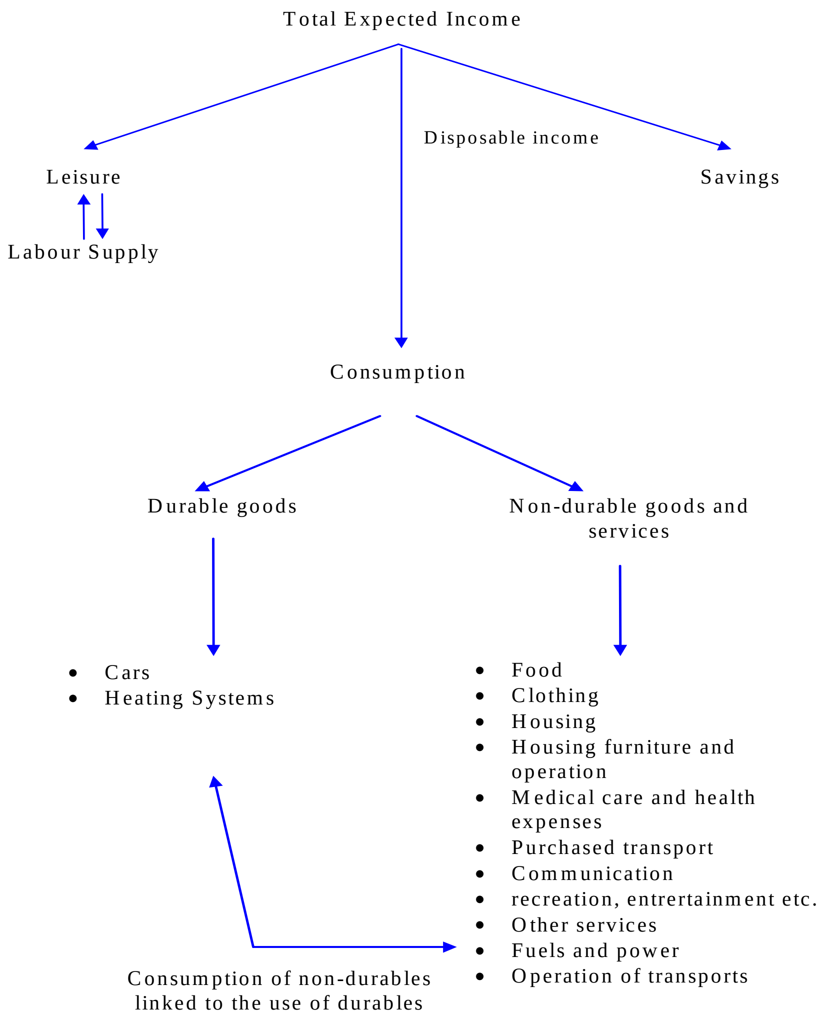
The ***GEM-E3*** model is built in a modular way around its central CGE core. The internalisation of environmental externalities is conveyed either through taxation or global system constraints, the shadow costs of which affect the decision of the economic agents. The current version of ***GEM-E3*** links global constraints to environmental emissions, changes in consumption or production patterns, external costs/benefits, taxation, pollution abatement investments and pollution permits. It evaluates the impact of policy changes on the environment by calculating the change in atmospheric emissions and damages and determines costs and benefits through an equivalent variation measurement of welfare.

The model is recursive over time, and is solved for each year following a time-forward path. The model is written as a mixed non-linear complementarity problem and is solved by using the PATH algorithm of the GAMS software. The main building blocks of the GEM-E3 model are specified as follows.

3.1. Household's behaviour

Private consumption decisions are derived from an intertemporal model of the household sector with two stages. In a first stage the households decide each year on the allocation of their expected resources between present and future consumption of goods and leisure, by maximising over their entire life horizon an intertemporal utility function subject to an intertemporal budget constraint defining total available resources. It is assumed that at the end of his life they will have no savings left. The utility function has as arguments consumption of goods and leisure. The specification of the first stage problem is based on a Stone-Geary utility function. The discrete approximation of this problem can readily be solved[5].

Figure 3.2: The consumption structure of the GEM-E3 model



[5] For a detailed presentation of the derivation of the demand functions using optimal control see C. Lluch (1973). A similar formulation can also be found in Jorgenson et. al (1977).

In the second stage households allocate their total consumption expenditure between expenditure on non-durable consumption categories (food, culture etc.) and services from durable goods (cars, heating systems, and electric appliances). In GEM-E3 the above general scheme is implemented with the structure as given in Figure 3.2.

Households, modelled through one representative consumer for each EU country, allocate in each period their total expected income between consumption of goods (both durables and non-durables) and services, leisure and savings in the first stage.

The Stone-Geary utility function, yielding a LES demand system is based on a Cobb–Douglas utility function and the maximisation problem is written[6]:

$$\text{Max } U = \sum_t (1+stp)^{-t} \cdot \left(BH \ln\left(HCDTOTV_t - CH \right) + BL \ln\left(LJV_t - CL \right) \right)$$

where $HCDTOTV$ represents the consumption of goods,

LJV the consumption of leisure,

stp the subjective discount rate of the households, or social time preference,

CH and CL the committed amount of consumption and leisure,

BH and BL the cost shares of consumption and leisure.

The expenditure choice is subject to the following budget constraint, which states that all available disposable income will be spent either now or some time in the future:

$$\sum_t (1+r)^{-t} \cdot \left(HCDTOT_t - PCI_t \cdot CH + PLJ_t \cdot LJV_t - PLJ_t \cdot CL \right)$$
$$= \sum_t (1+r)^{-t} \cdot \left(YTR_t + PLJ_t \cdot LTOT_t - PCI_t \cdot CH - PLJ_t \cdot CL \right)$$

where r is the nominal discount rate (parameter),

$PLJ \cdot LTOT$ is the value of the available time resources,

YTR is the total income of the households from sources other than wages (transfers).

The household behaviour is assumed to be formed as a sequential decision tree: based on assumptions about the future, the household decides the amount of leisure, by which they define their labour supply. Computing the Lagrangian of the above problem the first order conditions are obtained. These consist of the budget constraint, plus the two derived demand functions:

$$(1+stp)^{-t} \cdot BH / \left(HCDTOTV_t - CH \right) - \lambda \cdot (1+r)^{-t} \cdot PCI_t = 0$$

$$(1+stp)^{-t} \cdot BL / \left(LJV_t - CL \right) - \lambda \cdot (1+r)^{-t} \cdot PLJ_t = 0$$

the value of the Lagrange multiplier λ can be derived by summing up these equations over time, and substituting them into the budget constraint.

---

[6] Equations without numbering are not included in the model text, as they are only intermediate steps used for the derivation of other formulas.

Expressing now the above equations for the current time period ($t = 0$) and using the value of the multiplier, the two demand functions to be used in the model are obtained:

$$HCDTOTV = CH + \frac{stp}{rr}\frac{BH}{PCI}\left(YDISP + PLJ \cdot LJV - Obl\right) \tag{1}$$

$$LJV = CL + \frac{stp}{rr}\frac{BL}{PLJ \cdot}\left(YDISP + PLJ \cdot LJV - Obl\right) \tag{2}$$

where $Obl = PCI \cdot CH + PLJ \cdot CL$ is the value of committed consumption and rr the real discount rate.

Given the fact that the model is calibrated to a base year data set in which households have a positive savings rate, the computed *stp* is less than *rr*. The savings rate computed from the above is not fixed but rather depends on such factors as the social time preference, the real interest rate and the relative shares of consumption and leisure in total potential disposable income.

In the second stage, total consumption is further decomposed into demand for specific consumption goods. For this allocation an integrated model of consumer demand for non durables and durables, developed by Conrad and Schröder (1991) is implemented. The rationale behind the distinction between durables and non durables is the assumption that the households obtain utility from consuming a non-durable goods or services and from *using* durable goods. So for the latter the consumer has to decide on the desired stock of the durable good based not only on the relative purchase cost of the durable, but also on the cost of those goods that are needed in connection with the durable (as for example fuels for cars or for heating systems).

The consumer problem can be written as

$$Max\,Uc = \prod_{ND}(q_i - \gamma_i)^{\beta_i}\prod_{DG}(SDG_j^{fix} - \gamma_j)^{\beta_j}$$

subject to the constraint

$$HCDTOTV \cdot PC = \sum_{ND} p_i q_i + \sum_{DG}(p_j^u SDG_j^{fix} + p_j I_j)\,,$$

where Uc is the level of utility, PC is the consumption price, SDG is the stock of durables, γ is the minimum obliged consumption and β is the elasticity in private expenditure by category, non-durable goods and services are denoted by the index ND while durables by the index DG.

Under this specification, one can derive the following LES expenditure system for non durables:

$$HCNDTOT = E(U, p, SDG) = \sum_{ND} PC_{ND} \cdot \gamma_{ND} + Uc \cdot \prod_{DG}\left(SDG - \gamma_{DG}\right)^{-\beta_{DG}} \cdot \prod_{ND}\left(\frac{PC_{ND}}{\beta_{ND}}\right)^{\beta_{ND}}\,,$$

which gives the (minimum) expenditure on non durables given the stock of durables and the utility level U. We obtain the derived demand functions for the non-durable goods by differentiating the expenditure function (Shephard's lemma):

$$HCFV_{ND} = \gamma_{ND} + \left(\frac{\beta_{ND}}{PC_{ND}}\right) \cdot \left(HCNDTOT - \sum_{ND} PC_{ND} \cdot \gamma_{ND}\right)$$

where HCNDTOT is equal to E, the total expenditure on non durables.

The cost of using a durable is obtained by differentiating the above expenditure function with respect to the stock of each of the durables. This quantity represents the amount of non-durables that the consumer is willing to forsake for one extra unit of the particular durable:

$$\frac{\partial E}{\partial SDG} = -\frac{\beta_{DG}\left(HCNDTOT - \sum_{ND} PC_{ND} \cdot \gamma_{ND}\right)}{SDG - \gamma_{DG}}$$

The cost of operating the durables, that is, consumption of linked non durables is included in the user's cost of the durable $(PDUR)$:

$$PDUR_{DG} = PC_{DG}(rr + \delta_{DG}) + TX_{PROP,\,DG}(1 + rr) + \Sigma_{LND}\, \lambda_{LND,DG} PC_{LND,DG} \qquad (3)$$

where δ_{DG} is the replacement rate for durable goods,

TX is the property tax for the durables,

LND is the set defining all linked non-durable goods and

λ is the consumption of non durables per unit of durable.

The last part of the equation links non-durable goods to the use of durables, Energy being the main linked non-durable good. Consumption of energy does not affect the expenditure of durables through the change in preferences but rather through the additional burden in the user cost.

To calculate the desired stock levels of the durables, this quantity is set equal to the marginal cost of holding one more unit of a durable good for one period. The desired stock level of the durables is:

$$SDG = \gamma_{DG} + \left(\frac{\beta_{DG}}{PDUR_{DG}}\right) \cdot \left(HCNDTOT - \sum_{ND} PC_{ND} \cdot \gamma_{ND}\right) \qquad (4)$$

The demand for linked non-durable goods, coupled with the use of the durable is then:

$$LLNDC_{ND} = \sum_{DG} \lambda_{DG,ND} \cdot \left(\theta_{ND,DG} SDG\right) \qquad (5)$$

where λ_{DG} measures the proportion of the consumption of the linked non-durable good that is used along with the durable so as to provide positive service flow, $\theta_{ND,DG}$ represents the minimum consumption of the non-durable that is needed for a positive service flow to be created. If there is no need for non-durable good the $\theta_{ND,DG}$ in the first equation of the linked non-durables becomes zero. Therefore, we get:

$$HCFV_{ND} = CH_{ND} + \left(\frac{\beta_{ND}}{PC_{ND}}\right)\left[HCNDTOT - \Sigma_{ND}\, PC_{ND} \cdot \gamma_{ND}\right] + LLNDC_{ND}. \quad (6)$$

Total household's expenditure is then the sum of consumption (for non-linked non-durables) plus investment in durables plus consumption in non-durables used with durables.

$$HCDTOTV = HCNDTOT + \sum_{DG} HCFV + \sum_{ND} LLNDC \tag{7}$$

where $\sum_{DG} HCFV$ represents the change in stocks of durables or in other words, the net investment that is necessary to move towards the long run equilibrium durable goods levels. Assuming a rate of replacement δ, this investment is equal to:

$$HCFV_{DG} = SDG_{DG} - (1-\delta) \cdot SDG_{DG}\left[-1\right] \tag{8}$$

The demand for consumption categories is then transformed into demand for products through a consumption transformation matrix with fixed coefficients:

$$HC_j = \sum_{i=1}^{n+m} THV_{i,j} * HCFV_i$$

This equation determines the final consumption expenditure of the households. The consumption transformation matrix is also used to compute the consumption price as the weighted average of the consumers' prices of products in private consumption (PH):

$$PC_i = \sum_{j=1}^{l} THV_{j,i} * PH_j$$

A cost-of-living index can be then derived as the ratio of the value and the volume of consumption:

$$PCI = \frac{\sum_{i=1}^{n+m} PC_i * HCFV_i}{\sum_{i=1}^{n+m} HCFV_i}.$$

3.2. Firms' behaviour

Production functions in ***GEM-E3*** appear in the form of nested, constant return to scale CES functions. At the first level, production splits into two aggregates, one consisting of capital stock and the other of labour, materials, electricity and fuels. At the second level, the latter aggregate is further divided in their component parts. Figure 3.3 illustrates the nesting structure of the production functions.

The model considers 18 production sectors, each represented as a firm which decides on the supply of goods or services given their sales prices and the prices of production factors. The stock of capital is fixed within each period, the supply curve of the produced goods exhibits, therefore, decreasing return to scale[7].

The production function has the following form (for the1st nest):

[7] This description applies only to the case, where capital is assumed immobile across sectors and countries.

$$XD_{PR} = \left[\delta_{KAV,PR}^{\frac{1}{\sigma}} \cdot \left(KAV_{PR} \cdot e^{tgk_{PR} \cdot t} \right)^{\frac{\sigma-1}{\sigma}} + \delta_{LEM,PR}^{\frac{1}{\sigma}} \cdot LEM_{PR}^{\frac{\sigma-1}{\sigma}} \right]^{\frac{\sigma}{\sigma-1}}$$

where XD_{PR} is the level of domestic production,

KAV_{PR} is the amount of fixed capital stock,

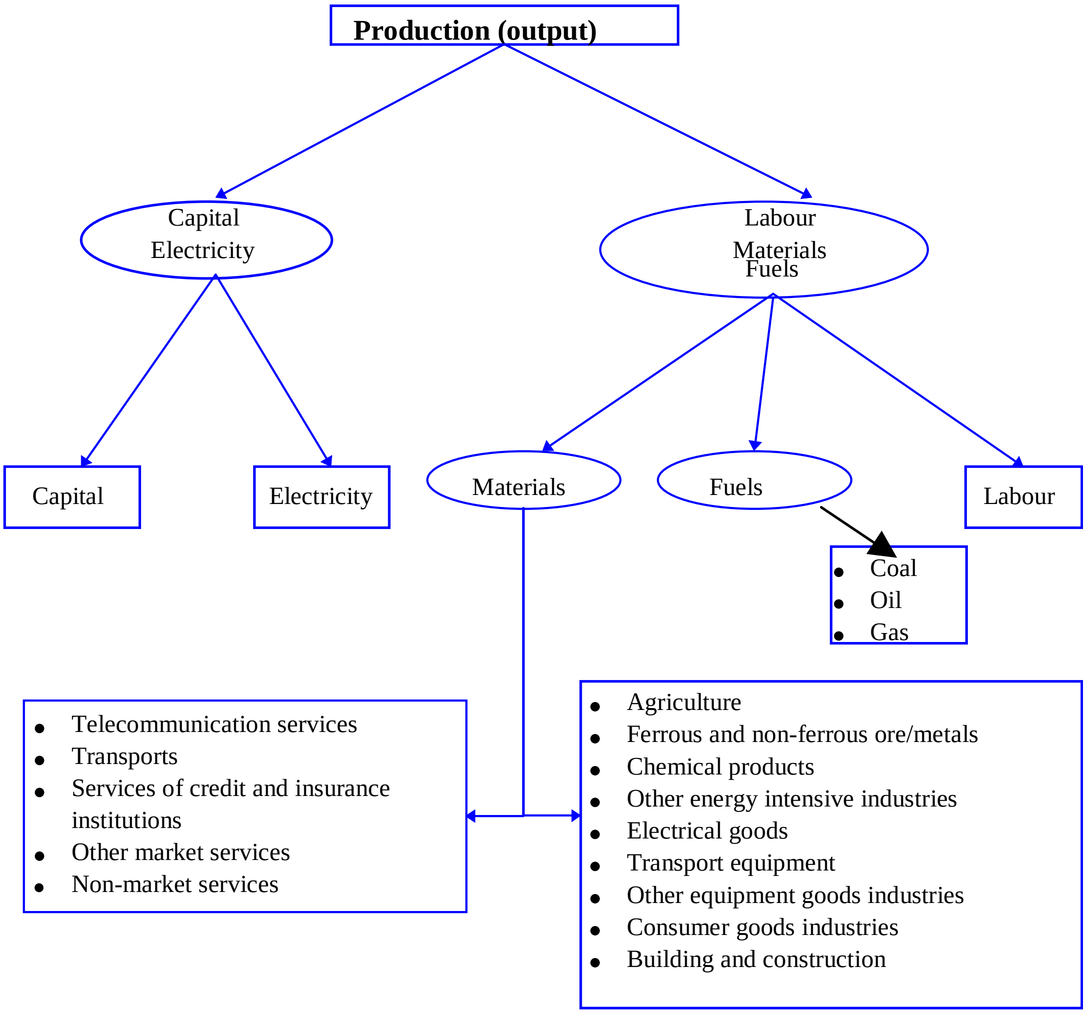
LEM_{PR} is the Labour-Energy-Materials composite factor of production,

σ is the elasticity of substitution between KAV_{PR} and LEM_{PR},

tgk is the technical progress of capital, whereas

$\delta_{KAV,PR}$ and $\delta_{LEM,PR}$ are scale parameters.

Figure 3.3: Production nesting scheme in the GEM-E3 model[8]



The dual function, representing the minimal unit production cost, can be expressed in the following way:

[8] Production factors are denoted by bold letters and are in rectangle. Round boxes represent intermediate bundles of goods with no physical relevance.

$$PD_{PR} = \left[\delta_{KAV,PR} \left(\frac{PK_{PR}}{e^{-tgk_{PR} \cdot t}} \right)^{1-\sigma} + \delta_{LEM,PR} \cdot PLEM_{PR}^{1-\sigma} \right]^{\frac{1}{1-\sigma}}$$

where PD_{PR} is the cost-price index in domestic production, and PK_{PR} and $PLEM_{PR}$ are the rate of return for fixed capital and the cost-price index of Labour-Energy-Materials composite factor, respectively.

The derived optimal factor demand function for the Labour-Energy-Materials bundle and desired capital stock are as follows:

$$LEM_{PR} = XD_{PR} \cdot \delta_{LEM,PR} \cdot \left(\frac{PD_{PR}}{PLEM_{PR}} \right)^{\sigma} \tag{9}$$

$$KAV_{PR} = XD_{PR} \cdot \delta_{KAV,PR} \cdot \left(\frac{PD_{PR}}{PK_{PR}} \right)^{\sigma} \cdot e^{tgk_{PR} \cdot t \cdot (\sigma - 1)} \tag{10}$$

The desired capital demand will be used in the capital market equilibrium equation, which derives the rate of return on capital, PK_{PR} as the equilibrium price that equalises demand and supply of capital.

Similar formulas can be derived for the other levels of the nesting scheme of the production function, always linking the demand for a factor at a lower level of the nesting scheme to the bundle to which it belongs, with different substitution elasticities at each level. In this way we can derive the cost-minimising demand for each production factor, which will be represented here in general functional forms only:

$$ENL_{PR} = f\left(XD_{PR}, \delta_{ENL,PR}, PEL_{PR}, PD_{PR}, e^{tge_{PR} \cdot t \cdot (\sigma_2 - 1)} \right) \tag{11}$$

$$LAV_{PR} = f\left(XD_{PR}, \delta_{LAV,PR}, PL_{PR}, PD_{PR}, e^{tgl \cdot t \cdot (\sigma_2 - 1)} \right) \tag{12}$$

$$IOVE_{BRE,PR} = f\left(XD_{PR}, \delta_{IOVE,PR}, PIO_{BRE}, PD_{PR}, e^{tgi \cdot t \cdot (\sigma_3 - 1)} \right) \tag{13}$$

$$IOVM_{BRM,PR} = f\left(XD_{PR}, \delta_{IOVM,PR}, PIO_{BRM}, PD_{PR}, e^{tgi \cdot t \cdot (\sigma_4 - 1)} \right) \tag{14}$$

where ENL_{PR} is the demand for electricity, PEL_{PR} is the corresponding cost-price index, tge_{PR} is the technical progress in energy use, LAV_{PR} is the labour demand, PL_{PR} is the unit cost of labour and tgl_{PR} is the technical progress of embodied in labour.

The last two equations represent the demand for intermediate consumption of commodity BR used in the production of sector PR ($IOVE_{BRE,PR}$ for energy and $IOVM_{BRM,PR}$ for material inputs) with PIO_{BR} being the unit cost of the intermediate good. The unit labour cost is a function of the average wage rate (*WR*) that reflects the relative scarcity of labour:

$$PL_{PR} = f(WR) \tag{15}$$

Under the above specification, the zero profit condition is always satisfied.

INVESTMENT DEMAND

The demand for capital goods, which fixes the investment demand, is derived from the cost minimizing decision of the producers, as described above. The long run cost of capital is given by Walras's definitions as

$$PK_{opt} = PINV \cdot (r + d),$$

where *PINV* is the price index of investments, *r* is the real rate of interest, and *d* the rate of amortization (depreciation). The desired capital for the following year (K_{fut}) is given as

$$\frac{K_{fut}}{Y_{\exp}} = \delta_{k,PR} \cdot \left(\frac{PD_{\exp}}{PK_{opt}} \right)^{\sigma}$$

The exact formulation of the capital demand function above, depends on the type of expectations that producers are assumed to have concerning the evolution of the economy and the future prices. In the ***GEM-E3***, these are linked to the expected rate of growth of the economy and the current price level. Although the exact formulation of the expectations affects the quantitative results of the simulations of ***GEM-E3***, the qualitative ones remain unaffected.

The comparison of the available stock of capital in the current year with the desired one determines the volume of investment decided by the firms. Given a partial adjustment mechanism and the fixed replacement rate d, the derived investment demand of the firm is

$$INVV_{PR} = m \times (K_{PR,fut} - (1-d)K_{PR,fixed})$$

or replacing future capital by the equation determining the desired capital

$$INVV_{PR} = m \times KAV_{PR} \cdot \left[\left(\frac{PK_{PR}}{PINV_{PR} \cdot (r+d)} \right)^{\sigma} \cdot (1 + STGR) \cdot e^{tgk(\sigma - 1)} - (1-d) \right] \tag{16}$$

where $PINV_{PR}$ is the price index of investments in sector *PR* and $STGR_{PR}$ is the expected growth rate of the sector.

The capital stock of the next period is given by the equation:

$$KAVC_{PR} = (1-d)^{T} \cdot KAV_{PR} + \left(\frac{1-(1-d)^{T}}{d} \right) \cdot INVV_{PR} \tag{17}$$

where T is the length of the period in the model.

The investment demand of each branch is transformed into a demand by product, through fixed coefficients, given by an investment matrix by product and sector. This and the government investments, which are exogenous in the model, define the total demand for investment goods.

DERIVED PRICING EQUATIONS

Firms supply their products on three market segments, namely, on the domestic market, for other EU countries and for the rest of the world. Firms do not differentiate their pricing according to market segments, but set a uniform price (PXD_{PR}) equal to unit cost (PD_{PR}) modified by the amount of production taxes or subsidies ($TXSUB_{PR}$):

$$PXD_{PR} = PD_{PR} \cdot \left(1 + TXSUB_{PR}\right) \tag{18}$$

$$PWE_{PR} = PD_{PR} \cdot \left(1 + TXSUB_{PR}\right) / EX \tag{19}$$

where $\mathrm{PWE}_{\mathrm{PR}}$ is domestic supply price of exports and EX is the rate of exchange.

3.3. Government's Behaviour

Public consumption decisions are exogenous in GEM-E3. Government final demand (GV) by product (index omitted) is obtained by applying fixed coefficients (t_G) to the exogenous volume of government consumption (G_C):

$$\mathrm{GV} = t_G \cdot G_C$$

Public investment, also exogenous in the model, is represented by a special branch of non market services. Transfers to the households are computed assuming an exogenous rate per head. As far as government's income is concerned, the model distinguishes between nine categories of receipts: indirect taxes, environmental taxes, direct taxes, value added taxes, production taxes/subsidies, social security contributions, import duties, foreign transfers and government firms. These receipts are coming from product sales (i.e. from production sectors) and from economic agents. The receipts from product sales in value (F_G), which include indirect taxes, the VAT, subsidies and duties, are computed from the corresponding receipts in value, given the tax base and the tax rate.

Import duties and production subsidies are defined as

$$F_{G,Duties} = t_{Duties} \cdot IMP,$$

$$F_{G,Subsidies} = t_{Subsidies} \cdot XD \cdot PD,$$

where IMP and XD denote the value of import and domestic production, respectively, t_{Duties} is import tariff s and $t_{Subsidies}$ are the subsidy rates.

Indirect taxes and VAT is defined by the following formulas:

$$F_{G,Ind.Tax} = \frac{t_{Ind.Tax}}{1 + t_{Ind.Tax}} * \left[\frac{C_H + C_G + I_H + I_G}{1 + t_{VAT}} + \sum I_C + \sum F_I + S \right]$$

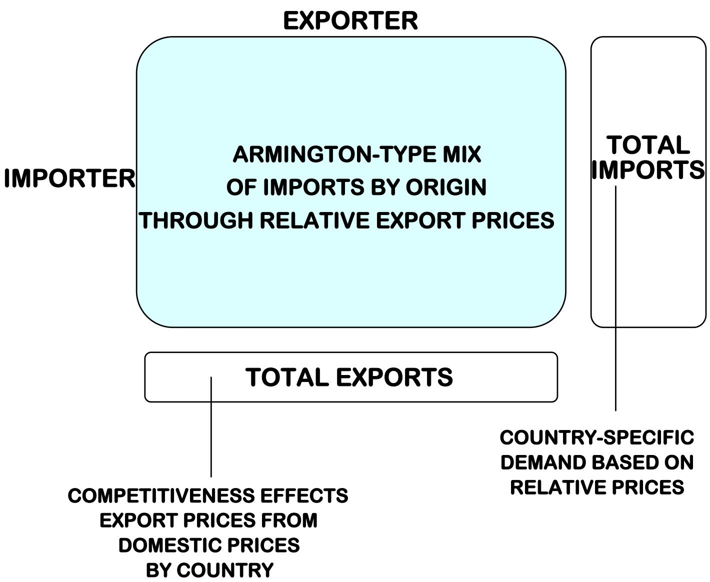
$$F_{G,VAT} = \frac{t_{VAT}}{1 + t_{VAT}} * \left(C_H + C_G + I_H + I_G\right)$$

The receipts from agents are computed from the tax base and the tax rate (social security contributions, direct taxation), share of government in total capital income (for government firms' income) or exogenous (transfers from and to the RW).

3.4. Domestic demand and trade flows

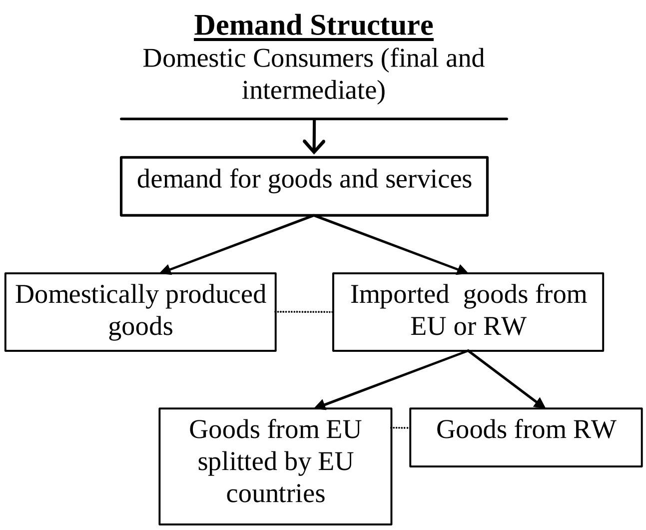
The demand of products by the consumers, the producers (for intermediate consumption and investment) and the public sector define the total domestic demand. This total demand is expressed for the domestic/import composite goods, following the Armington specification. The supplier of the composite good (domestic) seeks to minimise his cost and decides on the mix of imported and domestic products.

Figure 3.4: Trade matrix for EU and the rest of the world



The behaviour of the rest of the world (RW) is left exogenous: imports demanded by the rest of the world depend on export prices offered by the European Union countries, while exports from the rest of the world to the EU, i.e., the demand of the EU, are supplied by the RW flexibly at constant prices.

Figure 3.5: The demand structure in the GEM-E3 model



GEM-E3 employs a nested commodity aggregation hierarchy, in which branch's i total demand is modelled as demand for a composite good or quantity index Y_i, which is defined over demand for the domestically produced variant (XXD_i) and the aggregate import good (IMP_i). At a next level, demand for imports is allocated across imported goods by country of origin. Bilateral trade flows are thus treated endogenously in GEM-E3.

The cost-price of the domestic/import composite (Y_{PR}) good is determined by its minimum unit cost. This is formulated through a CES unit cost function, involving the selling price of the domestic good and the price of imported goods, which is taken from the second level Armington. The allocation of total demand between domestic and imported is determined by cost minimization subject to the following CES functional form (index *PR* refers to the sectoral origin of the goods all the way along):

$$Y_{PR} = AC_{PR}\left[\delta_{1,PR}\cdot\left(XXD_{PR}\right)^{\frac{\sigma_x-1}{\sigma_x}} + (1-\delta_{1,PR})\cdot\left(IMP_{PR}\right)^{\frac{\sigma_x-1}{\sigma_x}}\right]^{\frac{\sigma_x}{\sigma_x-1}}$$

where XXD_{PR} represents the demand for domestic production,

IMP_{PR} is the demand for imports,

$\delta_{1,PR}$ is a scale parameter,

σ_x is the elasticity of substitution between domestic and imported goods.

The corresponding dual form defines the prices of the domestic/import composite goods:

$$PY_{PR} = \frac{1}{AC_{PR}}\left[\delta_{1,PR}{}^{\sigma_x}\cdot PXD_{PR}^{(1-\sigma_x)} + (1-\delta_{1,PR})^{\sigma_x}\cdot PIMP_{PR}^{(1-\sigma_x)}\right]^{\frac{1}{1-\sigma_x}} \tag{20}$$

where PY_{PR} stands for the absorption price of composite good,

$PIMP_{PR}$ is the average price of imported goods of sector of origin PR,

PXD_{PR} is the price of the similar domestically produced goods,

AC is the scale parameter in the Armington substitution function.

The demand for domestic and imported goods can be derived by differentiating the above cost function with respect to the component prices (Shephard's lemma):

$$XXD_{PR} = Y_{PR}\cdot AC_{PR}{}^{(\sigma_x-1)}\cdot\delta_{1,PR}{}^{\sigma_x}\cdot\left(\frac{PY_{PR}}{PXD_{PR}}\right)^{\sigma_x} \tag{21}$$

$$IMP_{PR} = Y_{PR}\cdot AC_{PR}{}^{(\sigma_x-1)}\cdot(1-\delta_{1,PR})^{\sigma_x}\cdot\left(\frac{PY_{PR}}{PIMP_{PR}}\right)^{\sigma_x} \tag{22}$$

At the second level, import demand is allocated across countries of origin using again a CES functional form. In the equation below, *EU* and *CO* denote the countries. Index *EU* refers to European Union countries, while index *CO* also includes the rest of the world.

$$PIMP_{PR,EU} = \left[\sum_{CO} \beta^{\sigma_{xx}} \cdot PIMPO_{PR,EU,CO}^{(1-\sigma_{xx})} \right]^{\frac{1}{1-\sigma_{xx}}} \qquad (23)$$

where $PIMP_{PR,EU}$ denotes price of total imports demanded by country *EU*,

$PIMPO_{PR,EU,CO}$ denotes the *EU* import price of goods originating from country *CO*,

β is the share parameter for Armington and σ_{xx} is the elasticity of substitution.

Import prices are equal to the export prices set by the country of origin, multiplied by the appropriate import tariffs rate:

$PIMPO_{PR,EU,CO} = PWE_{PR,\ CO} \cdot EX_{CO}/EX_{EU} \cdot (1 + txduties_{PR,EU,CO})$.

The *EU*'s *import demand* for goods coming from country *CO* can be calculated by means of the following form:

$$IMPO_{PR,EU,CO} = IMP_{PR,EU} \frac{\partial PIMP_{PR,EU}}{\partial PIMP_{PR,EU,CO}} \qquad (24)$$

where $IMPO_{PR,EU,CO}$ denotes imports demanded by country *EU* from country *CO*.

Imports demanded by the rest of the world from the EU ($IMPO_{PR,RW,EU}$) are, on the other hand, determined as

$$IMPO_{PR,RW,EU} = \alpha_{RW} \cdot \left(\frac{PWEO_{PR,RW}}{PWEO_{PR,EU} EX_{EU}} \right)^{\varepsilon_{RW}} \qquad (25)$$

where α_{RW} is a scale parameter of export demand of the rest of the world,

$\mathrm{PWEO}_{\mathrm{PR,RW}}$ is the exogenous price set by the rest of the world, and

$PWEO_{PR,EU,RW}$ is the export price set by the *EU* to the rest of the world.

The export of goods of sector origin *PR* coming from country *CO* to the *EU* must, of course, be equal to their imports in the *EU* originating from country *CO*:

$$EXPO_{PR,CO,EU} = IMPO_{PR,EU,CO}, \qquad (26)$$

and

$$EXPOT_{PR,EU} = \sum_{CO} EXPO_{PR,EU.CO}, \qquad (27)$$

where $EXPOT_{PR,EU}$ is the total export of good *PR* from country *EU* and $EXPO_{PR,EU,CO}$ denote exports of good *PR* from country *EU* to country *CO*.

A trade flow from one country to another will thus be equal, by construction, to the inverse flow. The model ensures this symmetry in volume, value and price indexes. It is obvious, then, that the model guarantees (in any scenario run) all balance conditions concerning the trade with the rest of the world will be met.

3.5. Equilibrium pricing identities

The users' prices of the domestic/import composite commodities are derived from their cost-prices (Y_{PR}), by applying appropriate rules of taxation. Depending on the destination of a commodity, differentiated taxation may be applied, as for example indirect taxation τ_{PR} or VAT. The prices of goods at intermediate consumption are given by (28), while the prices of goods in final consumption are computed via (29) for households and by means of (30) for government. Finally, (31) defines the prices of goods used in investment formation.

$$PIO_{PR} = PY_{PR} \cdot (1 + \tau_{PR}), \tag{28}$$

$$PHC_{PR} = PY_{PR} \cdot (1 + \tau_{PR}) \cdot (1 + vat_{PR}), \tag{29}$$

where vat_{PR} is a rate of value added tax imposed on good PR.

$$PGC_{PR} = PY_{PR} \cdot (1 + \tau_{PR}), \tag{30}$$

$$PINVP_{PR} = PY_{PR} \cdot (1 + \tau_{PR}). \tag{31}$$

The unit cost of investment by sector of destination (owner) depends on its composition in investment goods (by sector of origin). This structure is represented by a set of fixed technical coefficients $tcf_{PR,BR}$:

$$PINV_{BR} = \sum_{PR} tcf_{PR,BR} \cdot PINVP_{PR} , \tag{32}$$

3.6. The income distribution and redistribution block

The formation of the flows in volume and their closure are fully defined at this stage. It is necessary, then, to formulate the income and transfer flows in value at the level of the Social Accounting Matrix and ensure the closure of the model, by verifying the Walras law.

CURRENT ACCOUNT

In some versions of the model the balance of payments is an endogenous variable, while the rate of exchange is kept fixed. An alternative approach, implemented in the GEM-E3 model as an alternative option, is to set the current account of the EU with the rest of the world (RW) (as a percentage of total EU imports from the RW) to a pre-specified value, in fact, to a time-series of set values, specified in the baseline scenario. The shadow price of this constraint will determine the shift in the rat of exchange endogenously.

THE SOCIAL ACCOUNTING MATRIX AND THE MONETARY POSITION OF THE ECONOMIC AGENTS

The real sector of the model is grouped within the framework of a Social Accounting Matrix - SAM (see Table 3.1), which ensures consistency and equilibrium of flows from production (branches) to the economic agents (sectors) and back to consumption.

The sources of income for consumers and producers are labour, capital rewarding and transfers. Respectively the sources of income for government are transfers and taxes. The agents use income for consumption or investment. Finally the surplus of deficit by agent equals net savings minus investment. To understand the notation used, consider a more detailed presentation of the SAM framework, in the table bellow.

Table 3.1: Framework of a Social Accounting Matrix (SAM)

EXPENDITURES

REVENUES		**Branches**	**Factors**	**Sectors**	**Investment & Stocks**	**Total expend.**
	Branches (producers)	intermediate consumption	0	demand for products by sector and exports	demand for products by sector	total demand
	Factors (labour and capital)	services in value added	0	income transfers from abroad	0	total factors
	Sectors (economic agents)	indirect taxes, VAT, subsidies, duties and imports	factor payments to sectors	income transfers between sectors	0	total sectors
	Gross Savings	0	0	total revenues minus investment and stocks	0	
	Total Revenues	total supply	total factors	total sectors	0	

Surplus or Deficit	0	0	lending (+ or -) capacity by sector	0

All elements of demand for products, in value, are computed by multiplying the results in volume with the corresponding price index. This is derived from the composite good price inflated by the appropriate taxes. As already mentioned, in the description of the government receipts, the indirect taxes are applied to all domestic consumption (intermediate and final), while the VAT tax is only imposed on the consumption and investment of the government and the households. For imports, the deflator equals the export price that the country of origin has set augmented by the import duties rate.

The income flows as in the SAM framework of the ***GEM-E3*** model are presented in detail below. The following indexes are used to denote, respectively, the economic agents and the factors of production: m:H (households), F (firms), G (government) and W (foreign); and n:K (capital) and L (labour).

- Revenues of sectors from branches (F_{SB}), which includes capital and labour income, government receipts (indirect taxes, subsidies and duties) and imports; government receipts are denoted in detail (by category) within the variable F_G (value). The revenues of the rest of the world from the branches, is of course equal to the net value of imports by branch.

- Revenues of sectors coming from other sectors (F_{SS}), which includes transfers, taxes, social benefits etc.; government receipts are further detailed within the variable F_{GS} (in value);

The most important of these are:

(i) the dividends the firms pay to the households ($F_{HF,}$), which is proportional to the net revenues of the firms

(ii) the social benefits that the government pays to the households ($F_{HG,}$), which depends on the number of employees by branch (N) and the rate of government payments to the unemployed (U)

(iii) the direct taxes on the firms ($F_{GS,F}$) which is again proportional to the net revenues of the firms (now excluding dividends) and the households($F_{GS,H}$), where the tax is proportional to their disposable income, and

(iv) the payments of individuals to the government for social security ($F_{GS,SS}$).

$$F_{HF} = t_{Dividend} \cdot \left(\sum Re\,venues_F - \sum Payments_F\right)$$

$$F_{HG} = t_{Benefits} \cdot \left(\sum N + t \cdot U\right)$$

$$F_{GS,F} = t_{Direct,F} \cdot \left(\sum Re\,venues_F - \sum Payments_F - F_{HF}\right)$$

$$F_{GS,H} = t_{Direct,H} \cdot Y_{Disposable}$$

$$F_{GS,SS} = t_{individuals} Income + t_{firms} ValueofLabourService$$

- Revenues of sectors coming from factors (F_{SF}), e.g. labour income of households;
- Revenues of factors coming from sectors (F_{FS}); this mainly concerns factor income from abroad;
- Flows considered as revenues of branches (in fact product demand) coming from sectors are detailed in: final consumption of products by sector in value (F_C), which includes exports, investment by product and sector in value (I) and stock variation in value (S);
- Flows considered as revenues of factors coming from branches represent the value added, in value;
- Flows from branches to branches are the values of intermediate consumption, as computed from the production behaviour of the firms;
- Flows from factors to factors and from factors to branches are equal to zero.
- The change in stocks is considered proportional to the volume of production for each branch.

The disposable income (Y_D) of households (domestic) is evaluated as their net earnings which comprises their receipts from branches, factors and sectors minus their payments to the sectors and the factors:

$$Y_D = \sum_i F_{HB_i} + \sum_n F_{HF} + \sum_m F_{HS} - \sum_n F_{FH} - \sum_m F_{SH}$$

- Income transfers and factor payments to or from abroad are set equal to exogenous variables
- Factor payments to sectors are coming from value added and distributed according to an exogenous structure:
- Firms disposable income (F_D) is computed as the sum of their income flows coming from branches factors and sectors:

$$F_D = \sum_B F_{FB} + \sum_F F_{FF} + \sum F_{FS}$$

The gross profits (G_P) are then computed by subtracting from the disposable income the payments of the firms to the households and the rest of the world, as follows

$$G_P = F_D - F_{HF} - F_{WF}$$

Public budget results are summarised by computing total government revenues (G_{REV}) and total government expenditures (G_{EXP}) which includes final consumption and investment of the government.

$$G_{REV} = \sum_i F_{GB} + \sum_i F_{GF} + \sum_i F_{GS}$$

$$G_{EXP} = \sum F_C + \sum F_{FS} + \sum F_{SS} + \sum I$$

GROSS SAVINGS AND THE CLOSURE RULE

Gross savings (*SA*) by sector are then computed as the difference between revenues (which consists of the receipts from the branches plus income from factors and sectors) and expenditures (which include final consumption and transfers to factors and sectors):

$$SA_m = \text{Revenues}_m - \text{Expenditures}_m,$$

where *m* stands for all the economic agents. The lending capacity, i.e., the net monetary position, positive or negative surplus (*SU)* of the various sectors can be given by subtracting investment (*I*) and stock variation (*S*) from gross savings:

$$SURPLUS_m = SA_m - I_m - S_m$$

If a monetary/financial sub-model is incorporated in the model, this identity is the starting point of the monetary/financial sub-model which, in fact, expands, the way the identity is satisfied.

$$SURPLUS_G =$$

$$\sum_{PR} \tau_{PR} \cdot \left[\sum_{BR} IOV_{PR,BR} + HC_{PR} + \sum_{BR} tcf \cdot INVV_{BR} + INVH_{PR} + INVG_{PR} \right]$$

$$+ \sum_{PR} vat_{PR} \cdot \left[(PY_{PR} + \tau_{PR}) \cdot HC_{PR} + (PINVH_{PR} + \tau_{PR}) \cdot INVH_{PR} \right]$$

$$+ \sum_{EU} TXDUT_{EU,CO} \cdot PIMPO_{PR,EU,CO} \cdot IMPO_{PR,EU} \tag{33}$$

$$+ \tau s \cdot \sum_{PR} PL_{PR} \cdot LAV_{PR} \;\; - HTRA - \sum_{PR} \left(PGC_{PR} \cdot GC_{PR} + PINVP_{PR} \cdot INVG_{PR} \right)$$

where τs is the rate of social security contribution,

HTRA denote income transfers from government to households,

GC_{PR} is government consumption,

PGC_{PR} is the price of government consumption, and

$SURPLUS_h \;\; \forall h = G, H, F, W$ denote the surplus or deficit of the agents.

$$\begin{aligned} SURPLUS_H &= HTRA + \sum_{PR} PL_{PR} \cdot LAV_{PR} \cdot (1 - \tau_i) - PC \cdot HCT \\ &- \sum_{PR} TINV_{PR} \cdot PINV_{PR} \cdot INVV_{PR} \end{aligned} \tag{34}$$

where $TINV$ investments financed by the households (dwellings),

HCT is total household consumption and *INVV* is value of investments.

$$SURPLUS_F = \sum_{PR} PK_{PR} \cdot KAV_{PR} - \sum_{PR} PINV_{PR} \cdot INVV_{PR} \tag{35}$$

$$SURPLUS_W = \sum_{PR} PIMPO_{PR} \cdot IMPO_{PR} - \sum_{PR} PEX_{PR} \cdot EXPOT_{PR} \tag{36}$$

The model is constructed in such a way that the sum of the net surpluses is zero, in other words Walras's law is satisfied. The definition of the prices ensures the consistency of the SAM, also in current currency, a fact, this is reflected in the above condition.[9]

3.7. Market clearing conditions

The equilibrium in the real part of the model is achieved simultaneously in the goods market and in the labour market. In the goods market a distinction is made between tradable and non tradable goods. For the tradable goods the equilibrium condition refers to the equality between the supply of the composite good, related to the Armington equation, and the domestic demand for the composite good. This equilibrium combined with the sales identity, guarantee that total resource and total use in value for each good are identical. For the non tradable, there

[9] Other aspects of the SAM (e.g., how one can fill in the SAM scheme with actual statistical data) will be discussed in section 5.4.

is no Armington assumption and the good is homogeneous. The equilibrium condition serves then to determine domestic production.

THE GOODS' MARKET

The equilibrium of the goods markets states that production must equal demand at the branch level. In the primal version, this condition serves to compute the unit cost of production (that is of course related to the selling price).

$$XD_{PR} = XXD_{PR} + EXPOT_{PR} \text{ serving to compute } PD_{PR} \tag{37}$$

In the dual version, as GEM-E3, this equation determines the total production, the dual price equation gives the production price and the equilibrium condition on the capital market determines the rate of return of capital.

In the case of capital mobility across branches and/or countries the equilibrium condition relates to the total country or EU capital stock: producers demand an amount of capital (as derived from their cost-minimising behaviour), while the total stock of capital available is fixed within the time period either at country or EU level The equilibrium of the market defines then the average uniform rate of return of capital across the area of capital mobility.

If capital is mobile across branches only then:

$$KAV\_Supply = \sum_{PR} KAV_{PR} \tag{38}$$

computing an average country-wide rate of return of capital, while if capital is perfectly mobile across countries as well:

$$KAV\_Supply = \sum_{EU} \sum_{PR} KAV_{PR} \tag{39}$$

where KAV_Supply is the total capital stock available, fixed within the time period.

THE LABOUR MARKET

For the labour market it is postulated that wage flexibility ensures full employment. On the demand side we have the labour demanded by firms (as derived from their production behaviour), while on the supply side we have the total available time resources of the households minus the households' desire for leisure (which is derived from the maximisation of their utility function). The equilibrium condition serves to compute the wage rate. In another version, wage rigidity can be assumed.

The equilibrium condition serves to determine the wage rate.

$$\sum_{PR} LAV_{PR} = TOTTIME - LJV \tag{40}$$

Being within a competitive equilibrium regime, the labour market is influenced by the slope of labour supply (as decided by households simultaneously with consumption and leisure). In this sense, the model assumes that the entire unemployment is voluntary. However, as the model assumes that, if the economic conditions are favourable, the households can supply more labour force, a relative high real wage elasticity of labour supply can reflect unemployment that prevails in European countries.

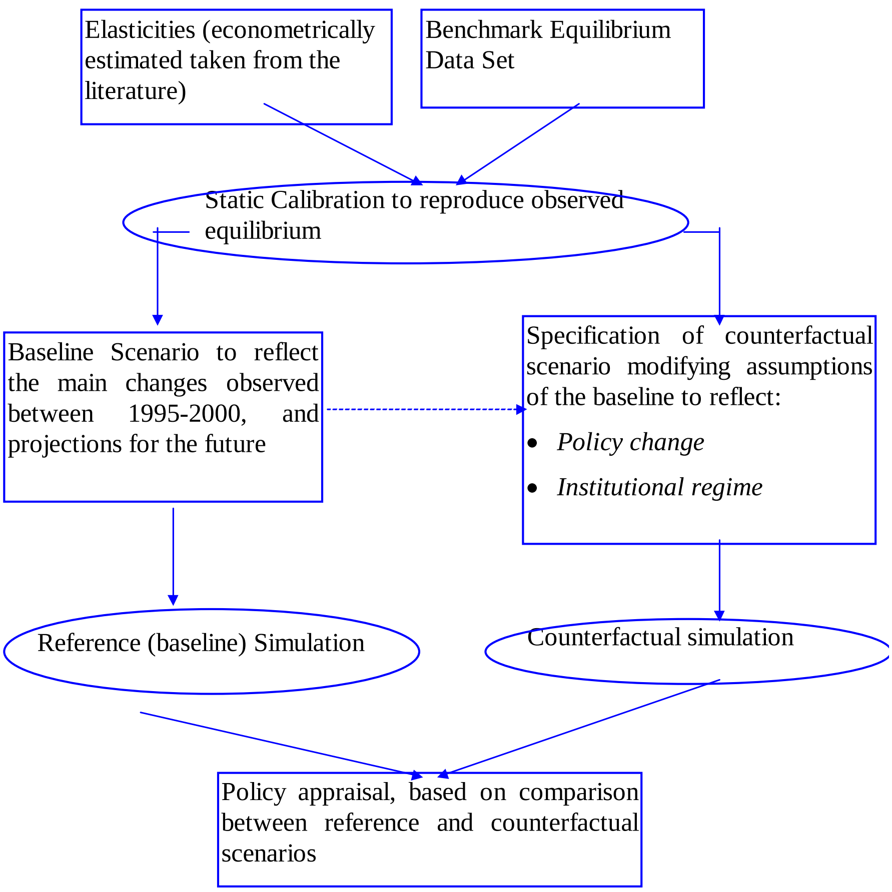
The labour supply is not totally elastic. This elasticity can be thought of, as representing the bargaining power of the already employed people. A high bargaining power would entail that an increase of the labour demand, would lead to an important increase in the wage rate, without any additional employment. The other extreme would be for the wage rate to remain constant and the employment to increase to cover the whole labour demand. The elasticity used in the model, falls between these two extremes.

Another market that can be activated in the model is the pollution permits market, which will be described in the section on the environmental module.

3.8. Model Calibration and Use

The first step for running the calibration procedure of the ***GEM-E3*** model, is to define values for the elasticities that determine all coefficients that do not correspond to directly observable variables and then to run the calibration procedure. This is written as a separate model and has a recursive structure. The base-year data used for calibration, correspond to monetary terms, therefore appropriate price indices are chosen to compute the corresponding volumes (quantities).

Figure 3.6: Using the *GEM-E3* model



Once the model is calibrated, the next step is to define a baseline scenario that starts from 1995, tries to reproduce as accurately as possible the last year for which observations are available (2000) and then gives some projections up to a future year which is the final year of the model simulation (usually 2030). This simulation defines the model ***baseline*** projection against which the policy simulations can be evaluated.

The "counterfactual" equilibria can be computed by running the model under assumptions that diverge from those of the baseline. This corresponds to scenario building. In this case, a scenario is defined as a set of changes of exogenous variables, for example a change in the tax rates. Changes of institutional regimes, that are expected to occur in the future, may be reflected by changing values of the appropriate elasticities and other model parameters that allow structural shifts (e.g. market regime). These changes are imposed in the baseline scenario thereby modifying it. To perform a counterfactual simulation it is not necessary to re-calibrate the model. The exact process of calibrating and running ***GEM-E3*** is illustrated in figure 3.

POLICY EVALUATION OF CHANGES IN CONSUMERS' WELFARE

Every policy simulation can be characterised by the implied equivalent variation change. The equivalent variation of a scenario, giving the index A in a policy simulation and B in the reference situation, is given as:

$$EV_t\left(U_t^A, U_t^B\right) = C_t^H\left(U_t^A, PCI_t^B, PLJ_t^B\right) - C_t^H\left(U_t^B, PCI_t^B, PLJ_t^B\right) \qquad (41)$$

for every time period t, where C_t^H is the expenditure function

$$C_t^H = \left(\frac{rr}{\rho}\right)\left(\frac{PC_t^B}{BH}\right)^{BH}\left(\frac{PLJ_t^B}{BL}\right)^{BL} U_t$$

Putting in base year prices and summing over the whole time period, the present value of the equivalent variation is obtained:

$$EV = \sum_{t=0}^{T}\left[\frac{1}{1+\rho}\cdot \Delta PC_t^B \cdot \Delta PLJ_t^B\left(CH_t^A - CH_T^B\right)\right] \qquad (42)$$

where

$$\Delta PCI_t^B = \left(\frac{PC_t^B}{PC_0^B}\right)^{\beta_H} \text{ and } \Delta PLJ_t^B = \left(\frac{PLJ_t^B}{PLJ_0^B}\right)^{\beta_L}$$

represent weighted changes in the consumer's price index and the valuation of leisure (equal to net wage rate) between the reference and the counterfactual simulation.

4. Extensions of the GEM-E3 core models

The above set equations represented the core of GEM-E3 model. The web-site in which various documents[10] deal with the elaborated extensions of the GEM-E3 model, include the generalization of the household utility function to take into account of the geographic variety of consumer goods, the imperfect competition, the financial module taking into account the portfolio decisions, etc. Of these extensions we present here only the environmental module. In addition to that we also discuss the possibility of representing private consumption and income generation with multiple households.

4.1. The environmental module

The objective of the environment module is to represent the effect of different environmental policies (*i*) on the EU economy and (*ii*) on the state of the environment. It concentrates on three important environmental problems: (*i*) global warming (*ii*) problems related to the deposition of acidifying emissions and (*iii*) ambient air quality linked to acidifying emissions and troposheric ozone concentration. Hence, we consider energy-related emissions of CO_2, NO_x, SO_2, VOC and particulates, which are the main source of air pollution. NO_x is almost exclusively generated by combustion process, whereas VOC's are only partly generated by energy using activities (refineries, combustion of motor fuels; other important sources of VOC's are the use of solvents in the metal industry and in different chemical products but are not considered here. For the problem of global warming, CO_2 is responsible for 60% of the radiative forcing (IPCC, 1990).

The environment module contains three components:

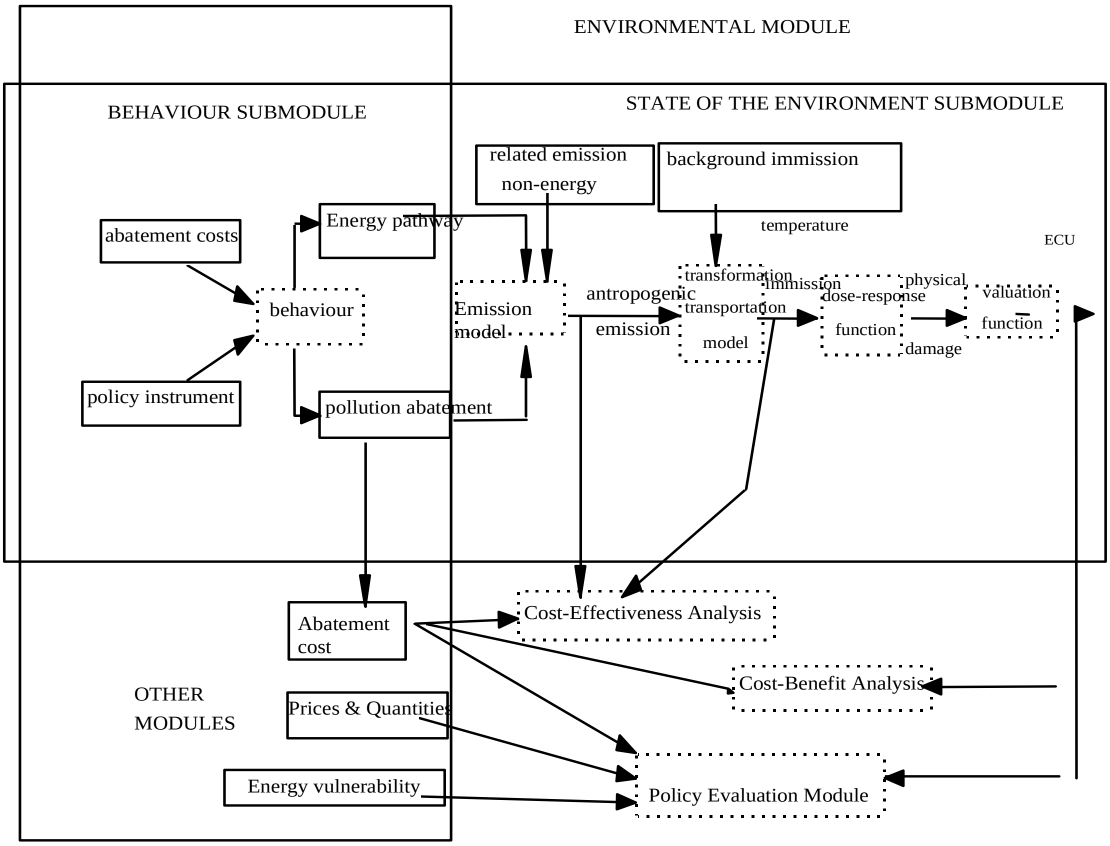
1. a *"behavioural"* module, which represents the effects of different policy instruments on the behaviour of the economic agents (e.g. additive (end-of-pipe) and integrated (substitution) abatement)
2. a *"state of the environment"* module, which uses all emission information and translates it into deposition, air-concentration and damage data. Depending on the version of the model, there is a feedback to the behaviour modules.
3. a "policy-support component", which includes representation of policy instruments related to environmental policy, such as taxation, tradable pollution permits and global constraint emissions; through policy instruments, emissions may influence on the behaviour of economic agents as formulated in the model.

The emission factors and other data related to the pollutants are differentiated by country, sector, fuel, and type of durable good (e.g. cars, heating systems). The links to inputs to production or consumption only concern the use and conversion of energy. Non-energy sources of emissions, like refinery and other processing are treated separately. To be able to evaluate excise taxes on energy, the energy content of fuels and electricity is also considered. For

[10] http://www.gem-e3.net/, with detailed description of the model at: http://www.gem-e3.net/download/GEMmodel.pdf, http://gem-e3.zew.de/, with reference manual of the model at: http://gem-e3.zew.de/geme3ref.pdf

private consumption the major links between energy inputs and consuming durable goods are specified as follows: cars and gasoline; heating systems and oil; coal, gas and electricity; electrical appliances and electricity.

Figure 4.1: Flow chart of the environmental module



The explicit formulation of a cost function in the supply side of the GEM-E3 model eases the representation of the effects of emission or energy based environmental policy instruments on economic behaviour. The costs induced by the environmental policy instruments act on top of production input costs. Derived demand for intermediate goods is derived from the unit cost function that takes these environmental costs into account. Similarly the demand of households for consumption categories is derived from the expenditure function, which is derived from utility maximisation. Hence, the environment-related policy instruments convey effects on prices and volumes of equilibrium.

The model takes into account the trans-boundary effects of emissions through transport coefficients, relating the emissions in one country to the deposition/concentration in other countries. For secondary pollutants as the tropospheric ozone, this formulation needs to consider the relation between the emission of primary pollutants (NOx emissions and VOC emissions for ozone) and the level of concentration of the secondary pollutants.

Damage estimates are computed for each country and for the EU as a whole, making the distinction between global warming, health damages and others. The data for damages per unit of emission, deposition or concentration and per person as well as their monetary valuation are based on the ExternE project of the EC Joule programme.

4.1.1. Mechanisms of emission reduction

There are three mechanisms that affect the level of actual emissions in the model:

- End-of-pipe abatement (SO2, NOx, VOC and PM): end-of-pipe abatement technologies are formulated explicitly through abatement cost functions associated to production sectors. These cost functions differ across sectors, durable goods, and pollutants but not between countries. It is assumed that these abatement technologies are available all over Europe at uniform costs. The data come from bottom-up studies. As the cost of abatement is an increasing function of the degree of abatement, the sectors and countries differ according to the country- or sector-specific abatement efforts already assumed to be undertaken in the base year.
- Substitution of fuels (all fuels): as the production of the sectors is specified through nested CES-functions, some degree of substitution between production factors is allowed. The demand for production inputs depends on relative factor prices and is therefore influenced by additional costs conveyed by environmental policy or constraints.
- Production or demand restructuring: in a general equilibrium framework sectors and countries are interdependent. Environmental constraints imply additional costs that differ across sectors or countries as they have different possibilities for substitution or abatement. This situation may further imply restructuring, for example by inducing a decline of a sector or a shift of demand to some countries.

4.1.2. The firm's behaviour

The abatement activities are modelled such as to increase the user cost of the energy in the decision process of the firm. When an environmental tax is imposed it is paid to the government by the branch causing the pollution. This has the following implications for the *energy price* modelling:

- the price of energy, inclusive abatement cost and taxes, is used in the decision by the firm on production factors; it represents the user's cost of energy;
- the price of energy, exclusive taxes and abatement cost, is used to value the delivery of the energy sectors to the other sectors;
- a price for the abatement cost per unit of energy has been defined, because the abatement cost is defined in constant price.

In modelling the *abatement activities*, the instalment of abatement technologies is treated as an input for the firms and not as an investment. This formulation is simple and the abatement costs do not increase directly GDP as it would if modelled as investment. For the latter purpose a depreciation and replacement mechanism would have to be introduced. The input demand for abatement is modelled in the following way:

- the demand for abatement inputs is allocated to the delivery sectors through fixed coefficients;

- the total delivery for abatement is added to the intermediate demand and these inputs are valued as the other intermediate deliveries.

4.1.3. The consumer's behaviour

The consumer's side modelling is rather similar to the one used for the firm, with one difference regarding the payment of the environmental taxes to the government. While in case of firms, the environmental taxes are paid by the branch causing the pollution, for the households the tax is paid by the branch delivering the product causing pollution to the household. The environmental tax is therefore treated as the other indirect taxes paid by households. This has the following implications for the modelling of the price equations:

- the price of energy in the consumer allocation decision, includes the abatement cost and the tax; it is modelled as a user's cost of energy;
- the price of delivery of energy to the household includes the pollution and/or energy tax;
- the abatement cost-price is defined.

The abatement expenditures of households are modelled as in the case of the production sectors (allocation to branches through fixed coefficient and valued as the other deliveries). They are not added to the private consumption and do not enter directly in the allocation of total consumption by categories, only indirectly through the user’s cost of durables as they are considered as a 'linked' consumption (to energy) and are added directly to the consumption by goods of production (i.e. the deliveries by branches to the households).

4.1.4. End-of-pipe abatement costs

The average abatement cost reflects annualised costs and the value for the parameters in the equation are based on the RAINS database. The cost functions that were derived from this data are represented by the marginal abatement cost function.

$$m\tilde{c}^{ab}_{p,s}(a_{p,s}) = \beta_{p,s} \cdot (1 - a_{p,s})^{\gamma_{p,s}} \text{ ,}$$

where

$a_{p,s}$: degree of abatement of pollutant p of sector s,

$\beta_{p,s}, \gamma_{p,s}$: estimated parameters $\left(\beta_{p,s} \geq 0, \gamma_{p,s} \leq 0\right)$.

By integrating the above formula one obtains the total cost curve per ton unabated emission assuming constant returns to scale:

$$\tilde{c}^{ab}_{p,s}\left(a_{p,s}\right) = \frac{-\beta_{p,s}}{1+\gamma_{p,s}} \cdot \left(1 - a_{p,s}\right)^{\gamma_{p,s}+1} + K_{p,s} \text{ .}$$

The degree of abatement $a_{p,s}$ can be exogenous or determined through the implicit equation, imposing equality between marginal cost of abatement and tax (see bellow).

The abatement cost function of sector s for pollutant p, given the output X_s of this sector and the degree of abatement $a_{p,s}$ is then

$$\tilde{C}^{ab}_{p,s} = \tilde{c}^{ab}_{p,s}\left(a_{p,s}\right)\cdot EM^{pot}_{p,s} = \tilde{c}^{ab}_{p,s}\left(a_{p,s}\right)\cdot \sum_i \left(ef_{p,i,s}\cdot \mu_{i,s}\cdot \alpha_{i,s}\cdot X_s\right),$$

where

$ef_{p,i,s}$: emission factor for pollutant p of input i in sector s,

$\mu_{i,s}$: share of energetic use of demand of input i.

These costs indicate an additional intermediate demand $\widetilde{ABI}_{p,i,s}$; the allocation of these costs to the delivering sectors is based on the assumption of fixed coefficients. The main deliveries for abatement technologies are investment goods, energy (due to a decrease in efficiency) and services (maintenance).

$$\widetilde{ABI}_{p,i,s} = tab_{p,i,s}\cdot \tilde{C}^{ab}_{p,s},$$

where

$tab_{p,i,s}$:share of deliveries of sector *i* for abatement of pollutant p in sector s.

The price index of abatement per unit of energy by branch is determined by the prices of the required intermediate inputs.

$$PC^{ab}_{p,s} = \sum_i tab_{p,i,s}\cdot\left(1+t_{i,s}\right)\cdot PY_i \;,$$

where $t_{i,s}$ is the indirect tax rate.

This allows for calculating the average abatement cost per unit of energy by branch in value, corresponding to the abatement level:

$$c^{ab}_{p,s} = PC^{ab}_{p,s}\cdot \tilde{c}^{ab}_{p,s}$$

This cost will be used to compute the user cost of energy which the firms and households use in their decision process regarding energy inputs. For the variables $\tilde{C}^{ab}_{p,s}$ and $\widetilde{ABI}_{p,i,s}$ the corresponding values $C^{ab}_{p,s}$ and $ABI_{p,i,s}$ are evaluated analogously.

Including the expenditure for abatement in the computation of intermediate demand one obtains the following input-output coefficients $\tilde{\alpha}_{i,s}$:

$$\tilde{\alpha}_{i,s} = \frac{\alpha_{i,s}\cdot X_s + \sum_p \widetilde{ABI}_{p,i,s}}{X_s + \sum_{p,i}\widetilde{ABI}_{p,i,s}},$$

where $\alpha_{i,s}$ is the I-O coefficient without environmental policy impacts.

If an environmental policy is linked to taxes or permits, there are not only costs for emission reduction but for the actual emission caused as well:

$$C^{ef}_{p,s} = c^{ef}_{p,s}\left(a_{p,s}\right)\cdot(1-a_{p,s})\cdot EM^{pot}_{p,s} = c^{ef}_{p,s}\left(a_{p,s}\right)\cdot(1-a_{p,s})\cdot\sum_i\left(ef_{p,i,s}\cdot\mu_{i,s}\cdot\alpha_{i,s}\cdot X_s\right),$$

The unit cost of an actual emitted unit of a pollutant $c^{ef}_{p,s}\left(a_{p,s}\right)$ depends on $a_{p,s}$ and the type of policy instrument imposed. While an emission standard gives no extra cost to the remaining emissions ($c^{ef}_{p,s}\left(a_{p,s}\right) = 0$), emission taxes and permits lead to a $c^{ef}_{p,s}\left(a_{p,s}\right)$ greater than zero.

The total costs of emission of pollutant *p* emitted by sector *s* are thus

$$C^{em}_{p,s} = C^{ab}_{p,s} + C^{ef}_{p,s}$$

The end-of-pipe abatement costs of households are specified in the same way except that for each durable a separate abatement cost function is specified. The cost prices of the inputs are the prices of the deliveries to household (by consumption categories) instead of the input prices of the firms. Using a matrix to transform abatement expenditure of households into consumption demand by goods of production closes the loop induced by the modelling of explicit abatement measures.

4.1.5. User cost of energy

In the model emissions are generated by energy consumption only. Hence, the user cost becomes a function of the price of the energy input and of the additional costs per unit of energy linked to emission or the energy content, i.e. the tax a firm has to pay for its actual emissions (and/or the energy) and the costs of abatement depending on the rate of abatement and the baseline emissions (and/or energy) coefficient. Introducing a new variable for the user cost, $PFU_{i,s}$, its equations is, for each branch s and each energy input i,

$$PFU_{i,s} = \left(1+t_{i,s}\right)\cdot PY_i + c^{en}_s \cdot ec_i \cdot \chi_{i,s} + \sum_p \left(\left[\left(1-a_{p,s}\right)\cdot c^{ef}_{p,s}\left(a_{p,s}\right) + c^{ab}_{p,s}\left(a_{p,s}\right)\right]\cdot ef_{p,i,s}\cdot \mu_{i,s}\right),$$

where

c^{en}_s : tax on energy,

ec_i : coefficient for energy content of energy input i (equal across sectors),

$\chi_{i,s}$: share of energy related use of input i in sector s.

This user cost of the energy product influences the choice between the energy products and between aggregate inputs (as it is used in the price-function of the energy aggregate F_s).

The price of the energy aggregate PF_s is then

$$PF_s = \left[\sum_i \delta^{\sigma_F}_{i,s} \cdot \overline{PFU}_{i,s}{}^{1-\sigma_F}\right]^{\frac{1}{1-\sigma_F}},$$

where

δ_i: distribution parameter of energy component *l*

σ_F: elasticity of substitution

$\overline{PFU_i} = {PFU_i}/{g_i(t)}$: price-diminishing technical progress.

The input price of electricity is affected only because of the energy tax:

$$PEL_s = PELU_s = \left(1+t_{El,s}\right)\cdot PY_{El} + c^{en}_s \cdot ec_{El}\cdot \chi_{El,s}.$$

Electricity and the fuels aggregate are components of the unit cost function PD_s. Hence, a more restrictive environmental policy, which increases $PFU_{i,s}$ and PF_s or PEL_s, will cause an increase in the unit cost and consequently, in the deflator of total demand, PY_s.

The price $\left(1+t_{i,s}\right)\cdot PY_i$ remains the delivery price of energy by the energy branches, for the valuation of $F_{i,s}$. This implies that the emission generating branch pays the environmental tax receipts, if there are any, to the government and not the branch delivering the energy product, as is the case for the other production taxes. The environmental taxes are clearly attributed to the branch generating the pollution.

In the case of the households, in the absence of any environmental constraint, the user cost price of durable good (p_{dur_j}) is specified similarly as follows:

$$p_{dur_j} = p_j\left(r+\delta_j+\cdot t_j^{prop}\cdot(1+r)\right)+\sum_l \vartheta_{l,j}\cdot \tilde{p}_{l,j},$$

where

- r : interest rate,
- δ_j : depreciation rate of durable good j,
- t_j^{prop} : property taxes for durable j,
- $\vartheta_{l,j}$: minimum consumption of non-durable l that is linked to the use of durable j
- $\tilde{p}_{l,j}$: price of linked non-durable good l including value added tax.

If emission costs for households are imposed, the user cost-price of the durable goods is increased by the costs of abatement as well as by the costs for the actual emissions:

$$\tilde{p}_{l,j} = p_l + c^{en}\cdot ec_l + \sum_p\left(\left[\left(1-a_{p,j}\right)\cdot c_{p,j}^{ef}\left(a_{p,j}\right)+a_{p,j}\cdot c_{p,j}^{ab}\left(a_{p,j}\right)\right]\cdot ef_{p,l,j}\cdot \mu_{l,j}\right).$$

4.1.6. Abatement decision

Based on the above specification, the profit maximising firms decide whether to abate or pay taxes, where the profit function takes the following form:

$$G_s = PX_s\cdot X_s - VC_s$$

where VC_s as the variable cost function.

To ease the notation we define an input price PY_i^{act} that includes emission and/or energy-taxes as well as indirect taxes.

$$PY_i^{act} = \left(1+t_i\right)\cdot PY_i + c_s^{en}\cdot ec_i\cdot \chi_{i,s} + \sum_p\left[ef_{p,i,s}\cdot \mu_{i,s}\cdot\left(c_{p,s}^{ab}\left(a_{p,s}\right)+c_{p,s}^{ef}\left(a_{p,s}\right)\cdot\left(1-a_{p,s}\right)\right)\right]$$

The variable cost function VC_s is then given by

$$VC_s = \sum_{i=1}^{n+2} v_i\cdot PY_i^{act},$$

where

$v_{i,s}$: intermediate demand of input i by sector s.

As the indices $n+1$ and $n+2$ denote labour and capital, PY_{n+1}^{act} is equal to PL and PY_{n+2}^{act} is equal to PK_{post}. The notification of intervals in the following equations is suppressed.

The first order conditions of the profit maximizing firm serve to determine supply and the degree of abatement. For the description of the environmental module only the latter is of interest. As the abatement costs are not distinguished by inputs, the formula for the optimal degree of abatement of pollutant p can be reduced to the following expression:

$$\frac{\partial G_s}{\partial a_{p,s}} = -\frac{\partial VC_s}{\partial \vec{P}Y^{act}} \cdot \frac{\partial \vec{P}Y^{act}}{\partial a_{p,s}} = -\sum_i v_{i,s} \cdot \frac{\partial\left(\left[\left(c^{ab}_{p,s}\left(a_{p,s}\right) + c^{ef}_{p,s}\left(a_{p,s}\right)\cdot\left(1-a_{p,s}\right)\right)\right]\cdot \sum_i ef_{p,i,s}\cdot \mu_{i,s}\right)}{\partial a_{p,s}}$$

$$= -\sum_i \left(v_{i,s}\cdot ef_{p,i,s}\cdot \mu_{i,s}\right)\cdot \frac{\partial\left(c^{ab}_{p,s}\left(a_{p,s}\right) + c^{ef}_{p,s}\left(a_{p,s}\right)\cdot\left(1-a_{p,s}\right)\right)}{\partial a_{p,s}} = 0$$

$$\Rightarrow \frac{\partial\left(c^{ab}_{p,s}\left(a_{p,s}\right) + c^{ef}_{p,s}\left(a_{p,s}\right)\cdot\left(1-a_{p,s}\right)\right)}{\partial a_{p,s}} = 0$$

$$\Rightarrow mc^{ab}_{p,s}\left(a_{p,s}\right) + c^{ab}_{p,s}\left(a_{p,s}\right) + mc^{ef}_{p,s}\left(a_{p,s}\right)\cdot\left(1-a_{p,s}\right) - c^{ef}_{p,s}\left(a_{p,s}\right) = 0$$

Hence, given an exogenous emission tax rate of $t^{env}_{p,s}$ ($c^{ef}_{p,s} = t^{env}_{p,s}$ and $mc^{ef}_{p,s} = 0$) the (cost minimising) degree of abatement $a_{p,s}$ can be derived (numerically) by the following implicit equation:

$$\frac{\partial G_s}{\partial a_{p,s}} = mc^{ab}_{p,s}\left(a_{p,s}\right) + c^{ab}_{p,s}\left(a_{p,s}\right) - t^{env}_{p,s} = 0$$

The abatement decision of households can be derived similarly. To reduce the complexity of the analytical solution, it is assumed that only the fixed part of the linked non-durable demand is affected by the end-of-pipe emission reduction measures. Hence, the degree of abatement is independent of the prices and quantities of the linked consumption.

The derivation of the cost minimising degree of abatement can be reduced according to the following expressions:

$$\frac{\partial \tilde{u}}{\partial a_{p,j}} = \frac{\partial \tilde{u}}{\partial z_j}\frac{\partial z_j}{\partial p_{dur_j}}\frac{\partial p_{dur_j}}{\partial a_{p,j}} = 0$$

$$\Rightarrow \frac{\partial p_{dur_j}}{\partial a_{p,j}} = \sum_l \vartheta_{l,j}\cdot ef_{p,l,j}\cdot \mu_{l,j}\cdot \frac{\partial\left(c^{ab}_{p,j}\left(a_{p,j}\right) + c^{ef}_{p,j}\left(a_{p,j}\right)\cdot\left(1-a_{p,j}\right)\right)}{\partial a_{p,j}} = 0$$

$$\Rightarrow mc^{ab}_{p,j}\left(a_{p,j}\right) + c^{ab}_{p,j}\left(a_{p,j}\right) + mc^{ef}_{p,j}\left(a_{p,j}\right)\cdot\left(1-a_{p,j}\right) - c^{ef}_{p,j}\left(a_{p,j}\right) = 0$$

Under an exogenous emission tax $t^{env}_{p,j}$ ($c^{ef}_{p,j} = t^{env}_{p,j}$ and $mc^{ef}_{p,j} = 0$), the optimal degree of abatement $a_{p,j}$ is given by the following implicit equation:

$$mc^{ab}_{p,j}\left(a_{p,j}\right) = t^{env}_{p,j}.$$

4.1.7. The 'State of the Environment' module

The 'state of the environment' module computes the emissions, their transportation over different EU countries and the monetary evaluation of the damages caused by the emissions and depositions. The analysis is conducted on a marginal basis, i.e. it assesses the incremental effects and costs compared to a reference situation. It proceeds in three consecutive steps :

1. the computation of emissions of air pollutants from the different economic activities, through the use of emission factors specific to these activities;
2. the determination of pollutants' transformation and transportation between countries, i.e. the trans-boundary effect of emissions;
3. the assessment of the value of the environmental damages caused by the incremental pollution compared to a reference situation in monetary terms.

ad 1. Emission calculations start from the potential emission $EM_{p,s}^{pot}$ a sector s produces before end-of-pipe measures have been undertaken. These emissions are linked to the endogenous output, the price-dependent (endogenous) input coefficient, the exogenous emission factor and the share of the energetic use of the input demand.

$$EM_{p,s}^{pot} = \sum_i ef_{p,i,s} \cdot \mu_{i,s} \cdot \alpha_{i,s} \cdot X_s \qquad i \in I ,$$

where

$ef_{p,i,s}$: emission factor for pollutant p using input i in the production of sector s ,

$ef_{p,i,s} = 0$ for $i \neq$ emission causing energy input,

$\mu_{i,s}$: share of energetic use of demand of input i in sector s ,

$\alpha_{i,s} \cdot X_s$: intermediate demand of input i for output X_s in sector s ,

I : set of inputs.

For the households we write analogously:

$$EMH_{p,j}^{pot} = \sum_i ef_{p,i,j}^{h} \cdot \mu_{i,j}^{h} \cdot \vartheta_{i,j} \cdot z_j^{fix} \; i \in I_j ,$$

where

$ef_{p,i,j}^{h}$: emission factor for pollutant p using linked non-durable good i to operate durable good j, $ef_{p,i,j}^{h} = 0$ for $i \neq$ emission causing energy input,

$\mu_{i,j}^{h}$: share of energetic use of demand of linked non-durable good i to operate durable good j,

$\vartheta_{i,j} \cdot z_j^{fix}$: fixed part of the demand for linked non-durable good i induced by use of durable good j.

$i \in I_j$: set of non-durable goods linked to the use of durable good j.

Installing abatement technologies reduces total emissions. With respect to the degree of abatement specified above one obtains the abated emissions $EM_{p,s}^{ab}$ or $EMH_{p,j}^{ab}$.

$$EM_{p,s}^{ab} = a_{p,s} \cdot EM_{p,s}^{pot}$$

$$EMH^{ab}_{p,j} = a^{h}_{p,j} \cdot EMH^{pot}_{p,j} \ .$$

The remaining actual emissions ($EM^{ef}_{p,s}$ or $EMH^{ef}_{p,j}$) are then given as residual:

$$EM^{ef}_{p,s} = EM^{pot}_{p,s} - EM^{ab}_{p,s} = \left(1 - a_{p,s}\right) \cdot EM^{pot}_{p,s}$$

$$EMH^{ef}_{p,j} = EMH^{pot}_{p,j} - EMH^{ab}_{p,j} = \left(1 - a_{p,j}\right) \cdot EMH^{pot}_{p,j}$$

The actual emissions of primary pollutants are thus related to the use of energy sources, the rate of abatement, the share of energy use of the demand of input i and the baseline emission coefficient of a pollutant. Hence, for every pollutant, sector and fuel, a reference baseline emission factor is needed, relating the baseline emissions before abatement to the energy use. The emission coefficient must be related to the energy consumption in monetary term. A conversion factor (from energy unit to monetary unit) is derived from the base price of energy. Moreover, at the aggregation level of GEM-E3, energy consumption by branch includes both energy causing and not causing emissions, therefore a parameter reflecting the fraction the energy used in its own production (parameter μ) is also computed in the data calibration.

ad 2. This step establishes the link between a change in emissions and the resulting change in concentration levels of primary and secondary pollutants. The model accounts for the transport of SO_2, NO_x, VOC and particulates emissions between countries (or grids). In the case of tropospheric ozone (a secondary pollutant), besides the trans-boundary aspect, the relation between VOC and NO_x emissions, the two ozone precursors, and the level of ozone concentration has also to be considered.

The concentration/deposition *(IM)* at time *t* of a pollutant *ip* in a grid *g* is, in theory, a function of the total anthropogenic emissions before time t, some background concentration *(BIM)* in every country, and other parameters such as meteorological conditions, as derived in models of atmospheric dispersion and of chemical reactions of pollutants:

$$IM_{ip,g}\ (t) \equiv im_{ip,g}\ (\ EM_{p,c}(t' \leq t),\ BIM_{ip,g}\ (t),..\ \forall\ p,\ c)\ ,$$

For the model, the equations are made static and the problem is made linear by transfer coefficients *TPC.* They reflect the effect the emitted pollutants in the different countries have on the deposition/concentration of a pollutant ip in a specific grid, such as to measure the incremental deposition/concentration, compared to a reference situation:

$$\Delta IM_{ip,g} = \sum_{p}\sum_{c} TPC_{p,ip}\left[g,c\right] \cdot \Delta EM_{p,c}\ ,$$

where TPC[g,c] is an element of the transport matrix TPC with dimension GxC. In the models the grids considered are the countries and deposition/concentration levels are national averages.

As far as global warming is concerned, the global atmospheric concentration matters only, which is only a function of the total anthropogenic emission of greenhouse gases:

$$\Delta CC_{ip,g}\ (t) = \Delta CC_{ip}\ (t) = cc_{ip}\ (\ \Delta TAEM_{p}\ (t' \leq t)..\forall p)\ .$$

and then, the concentration of GHG's (greenhouse gases) must be translated into radiative forcing R and global temperature increase ΔT,

$$R(t) = f_1(\Delta CC_{ip}\ (t)\ \ \forall ip)\ ,$$

$$\Delta T(t) = f_2(R(t))\ .$$

ad 3. The approach followed in damage evaluation is entirely based on the framework and data derived in the *ExternE project*, though at a much more aggregated level. Following the ‘damage or dose-response function approach’, the incremental physical damage DAM per country is given as a function of the change in deposition/concentration,

$$\Delta DAM^{c}_{ACID,d}(t) = dam^{c}_{ACID,d}(\Delta IM_{ip,c}\ (t),..\ \forall\ ip)\ ,$$

In the case of global warming, damage is a function of the temperature rise,

$$\Delta DAM^{c}_{GLOBWAR,d}(t) = dam^{c}_{GLOBWAR,d}(\Delta T(t),..)\ \ .$$

The damages categories considered in the model are

- damage to public health (acute morbidity and mortality, chronic morbidity, but no occupational health effect)
- global warming
- damage to the territorial ecosystem (agriculture and forests) and to materials.

For the monetary valuation of the physical damage, a valuation function VAL is used:

$$VAL^{c}_{o}(t) = val^{c}_{o}(\Delta DAM^{c}_{o,d}(t),..\ \forall d)\ .$$

The economic valuation of the damage should be based on the willingness-to-pay or willingness to accept concept. For market-goods, the valuation can be performed using the market price. When impacts occur in non-market goods, three broad approaches have been developed to value the damages. The first one, the contingent valuation approach, involves asking people open- or closed-ended questions for their willingness-to-pay in response to hypothetical scenarios. The second one, the hedonic price method, is an indirect approach, which seeks to uncover values for the non-marketed goods by examining market or other types of behaviour that are related to the environment as substitutes or complements. The last one, the travel cost method, particularly useful for valuing recreational impacts, determine the WTP through the expenditure on e.g. the recreational impacts.

4.1.8. Instruments and policy design

While standards can be imposed on emissions or energy use, taxes and permits can be based on emissions, energy content, depositions, and damage. There is a wide range of tax proposals concerning the tax base to be used. This includes pure energy taxes, pure emission taxes, and mixtures of both with varying weights of the two.

The permit designs suggested by environmental economists reach from undifferentiated emission permits to regionally differentiated emission permits. GEM-E3 is able to analyse undifferentiated emission permits and ambient discharge permits. The latter consider the source and the sink of an emission geographically (on a country level). It is also intended to analyse permit systems that are based on the damage caused in the countries. This application could give a rough estimation of the economic effects and the financial transfers due to an EU-wide implementation of the polluter-pays-principle.

Using market-based policy instrument requires a range of information about how these instruments should be implemented in practice. This includes e.g. the purpose of tax receipts or the principle how permits should be allocated initially.

Principally there are many ways of refunding the receipts raised by emission and/or energy taxes. Not all of them are reasonable and only a few are in the current political discussion. In GEM-E3 the following refunding mechanisms were introduced and tested:

- government keeps the tax receipts and reduces public deficit (e.g. in order to fulfil the Maastricht criteria)
- government uses receipts for public consumption and investment (e.g. expecting to reduce unemployment)
- lump sum transfer to households
- tax reform 1: social security rates are reduced according to the tax receipts (double dividend analysis)
- tax reform 2: taxes on firms and capital income are lowered according to the tax receipts.

For the permit markets a variety of designs is possible. The initial allocation of permits can be based on the grandfathering principle or on auctioneering. In a grandfathered allocation permits are given free of charge to the polluters according to their actual emissions or a comparable rule. If permits are sold by auction, an additional income for the government is raised, for which again an appropriate refunding mechanism has to be chosen (not interesting at this level of analysis because it is equivalent to a tax). Other aspects, like the duration of permits (limited or unlimited) etc., are supported by the current version of the model.

GEM-E3 supports the simulation of market-based instruments (permits and taxes) on a national and/or on the multinational level. The multinational level can cover a selection of some countries only or the entire EU-15. This feature can be used for permit markets as well. They can be installed on a national and/or a multinational level where permits are traded between sectors and countries. All kinds of exemptions can be simulated, i.e. taxes and permits can be introduced for some sectors or the households only.

The GEM-E3 model's environmental module has been implemented in a single country model developed for Hungary (see Revesz, T.- Zalai, E. – Pataki, A. [1999]). In this model (called HUGE i.e. HUngarian General Equilibrium model) we developed a generalized (nested-CES-type) welfare function was used, in which environmental quality enters also as a third component, besides the consumption and leisure. In this model it is also possible to select from among several closure options depending on the possible environmental policy concern, e.g. endogenous emission tax rates, several mechanisms for recycling environmental tax revenues.

4.2. Multiple households

Rising inequality, deprivation, and poverty are direct concern of the economic policy. In addition, policy goals are influenced by indirect effects of differential development of the individual social groups. Since the savings rate and consumption pattern of the strata are quite different, demand for the products and hence employment of the individual industries depend very much on the within-household sector income distribution. Similarly, shifts within the household sector influence other macroeconomic aggregates too, like import or investment.

Representing multiple households and their relationships with the labour market and income distribution is a common practice in CGE models. Several applied modelling experiences can be found in the literature, in particular within the stream of CGE models used for policy analysis in the World Bank (see, for example, Harrison et al [2002], Kalb [2000], Shoven, J. B. and J. Whalley [1998]). Among the project participants, P. Capros, D. Van Regemorter and Zalai and Révész have also conducted, separately and collectively, several successful CGE modelling exercises with multiple households in the past.

However, including more than one representative household into a CGE-model arises various conceptual and technical problems.

In the models of multiple households each class of them receive labour and capital income from each sector according to specific distribution schemes. Household classes also receive government transfers, pay income taxes. Their savings can be represented either as a fixed proportion of their after-tax income, or determined together with their consumption of goods and services based on the usual assumption of utility maximization subject to their budget constraints, each of the household classes having its own preferences.

The construction of the household accounts usually proceeds as follows. First a wage and salary distribution matrix is created by combining industry-occupation data with average salaries, and then mapping the earnings by individuals according to their occupations to households. Capital-related income by industry is aggregated into an economy-wide enterprise account and then mapped to household classes according to data derived from individual income tax returns and/or household budget surveys (HBS)[11]. The statistical data on individual income tax returns forms the basis also for calibrating household related income taxes and government transfers. Personal consumption is disaggregated according to various household classes using data from the Consumer Expenditure Survey. Household savings can be determined as the residual income (income-expenditure) or its estimate can be based on the data of specific sociological surveys on savings behaviour and its amount.[12]

The consumption behaviour of each household class can be modelled by using the same type of expenditure system as in the standard single household GEM-E3 models, but their coefficients will of course differ from one household class to another, reflecting differences in tastes and habits. As a matter of fact, empirical research shows that preferences differ across

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[11] In some countries this aggregation method can be rather problematic when certain household types (e.g. farmers) capital endowments concentrate on specific sectors (agriculture)

[12] The training document MultHh-Hu05.doc provides an example how HBS data can be used to disgregate the household related data of the GEM-E3 model.

household classes according to differences in income levels, occupational, educational positions and urban/rural residence of the classes. The time series of the EU Consumer Expenditure Surveys provide adequate information basis for the disaggregated representation of consumer expenditure systems in the model.

The choice of the household classes to represent in the model is a practical issue that depends first of all on available statistical data. The choice can be based on the cross section between three sets of statistical information: the consumer expenditure surveys, the industry-occupation data and the income tax return statistics. Pure income classes are not desirable, because empirical research shows that income is not the only, not even the dominant factor in explaining the differences in the consumer expenditure systems of various household groups. A more common factor that explains these differences is the occupational-educational position of the family head. On that basis, cross mappings with labour skills and tax income categories is also easier to define. The occupational-educational distribution of households is strongly correlated with income distribution. Unfortunately, in some countries - because of the poor availability of statistical data - it is not possible to differentiate classes according to urban-rural criteria.

An advantage of using the occupational-educational dimension for definition of various household classes is that it makes it possible to represent societal and demographic evolutions in the long term scenarios prepared for the model. As a matter of fact, a multitude of factors that are not necessarily resulting from changes in the economic indicators simulated with the model may explain the evolution in the future of the number of households per class as defined according to the occupational-educational dimension. There have been attempts in the literature to link such an evolution with the projections of economic growth, income and the labour market, however, there is poor evidence about their direct causal (functional) relationships that would allow the modellers to include such a mechanism in the model.

Therefore, it is advisable that in the forecast of the number of households per class be exogenous in the model, and be quantified as part of the construction of the exogenous assumptions of scenarios. Evidently, to build such a scenario, one should implicitly relate the evolution of classes with other exogenous assumptions, such as population growth and regional welfare convergence trends. Of course, one may vary these assumptions across scenarios.

While the number of households per class is exogenously determined in the model, all other changes related to their behaviour shall be endogenously determined. For example, linkages between consumer choices per class, income per class in relation to the labour markets, tax-income policy and industry-occupational levels per class will all be endogenous variables in the model.

Expenditures and labour supply in each class of households should be modelled separately, using the type of nested linear expenditure system typically applied in the GEM-E3 models. Labour per class and capital can be assumed to be perfectly mobile across sectors, but imperfectly mobile across countries. Within each country the labour market must clear according to an imperfect competition mechanism (see for example the labour market clearing in the Worldscan model which is inspired from the research work of G. Pissarides), through a nested scheme involving clearing for labour demand, supply and salaries at each class level and globally at each country's economy.

4.3. Illustrative programs

The special programs accompanying this training material contain illustrations for household disaggregation possibilities and socio-economic group-formation criteria. The MultHhMod-MENG.doc file describes the MULTHH.GMS program, a CGE model calibrated for Hungarian data for 1998 distinguishing 3 sectors and 10 household groups. The accumulation capacity of the various household groups differ much are more than their consumption levels and patterns, therefore only dynamic models can illustrate properly the different impacts of economic policy measures on the prospects of the individual social groups. To render this possible we introduced into our model group specific human capital (which also can be accumulated by the “productive” part of the personal consumption), capital incomes (imputed rent income and net interest) and group specific financial wealth. Consumption of foreign tourists is also treated separately in our model, since it does not belong to any resident household groups. To make the model more flexible and realistic we also introduced a CET-labour supply function, and alternative closures rules.

5. Statistical background of the GEM-E3 model

5.1. The primary data requirements of the model

To calibrate the parameters[13] of the GEM-E3 model, one has to get benchmark year data on the production technology (incl. emission of air pollutants), consumer preferences (patterns), prices, income distribution, savings and final demands. Concretely, these data can be derived from the following datasets, which list shows the primary data requirements of the model:

1. ‘Symmetric’ *I-O table* at basic prices and supplementary tables (import matrix if in the original table import and domestic flows are not added together, matrix of indirect taxes and subsidies, e.g. of the import duties)

2. *Import duty matrix* (import duties by commodity groups and supplier countries). It is not always necessary (the vector break-down by commodity groups suffices), since it is assumed that all duties are imposed on the imports from the ROW (within the EU there are no such duties), but when some countries joined the EU since the benchmark year (see the accession countries in 2004 and 2007) it can be very useful to separate out those duties which were applied to the trade with them (and then it has to be decided whether the model is to be run so as if these countries had been already members in the benchmark year).

3. *Foreign trade matrices* (exports and imports by commodity groups and partner countries). Usually the rows represent the commodity groups and columns the countries (or group of countries). For the model each of the 27 EU-countries (although Belgium and Luxembourg need not be split) have to be treated separately, while the other countries need not be broken down, so they will form the "Rest of the World" in the model. The bilateral import matrix is similar but by definition it refers to the import flows.

4. *Consumption transformation matrix* (consumption by categories and branch of origin, the last row and column show the VAT and consumption taxes by category and branch respectively). Therefore the consumption matrix translates the demand per consumer categories into deliveries by branch.

5. *Investment transformation matrices* by institutional sectors (for each institutional sector the matrix shows the investment by branch of origin and by investing /destination/ branch). Investment Matrix. Thefore the investment matrix translates the demand of investment goods by branches into deliveries by branches. The matrix, which has been constructed to portray the investment transactions between sectors of the United Kingdom Economy, is showed in Appendix 5.

6. *Income distribution* (national accounts) data, more specifically:

6/a: by branches: output, value added and its primary distribution /wages, social security, production taxes, production subsidies, operating surlus (preferably the amortization too)

6/b: by institutional sectors: income-expenditure balance sheets (incl. saving and investment).

[13] The list and meaning of these parameters can be found in the Para-dyn.prn file.

7. *Environmental data*. The environmental data needed for GEM-E3 cover four types of data:

7/a: emission coefficient per type of activity for the pollutants in the model, CO2, SO2, NOx, VOC and PM (suspended particulates)

7/b: marginal abatement cost functions (including the parameters for the input structures) for the pollutants SO2, NOx, VOC and PM

7/c: pollutants' transformation and transportation between countries coefficients to arrive at air concentration and deposition

7/d: damage per pollutant and its monetary valuation

Since there are no end-of-pipe-technologies for reducing greenhouse gases at reasonable costs, the end-of-pipe abatement technologies considered in GEM-E3 are limited to the primary pollutants SO_2, NO_x, VOC and particulates.

In GEM-E3 we distinguish twenty emission relevant sectors (firms) or uses (households): the 18 branches, the heating systems of households and private traffic. A distinction has been made between emissions linked to production and emission linked to energy consumption with the same specification of the cost function.

8. *Auxiliary data* (factor endowment data, interest rates, inflation rate in the base year, demographic data, foreign tourists' domestic consumption expenditure by supplier branches and the related VAT and consumption tax, energy balance sheets, energy taxes, stocks of energy consuming durable goods, share of gasoline and gas-oil within motor-fuel demand, share of non-energetic use of the energy carriers, etc.)

9. Data for special extensions of the model (multiple households, financial module, imperfect competition, etc.)

The GEM-E3 model needs the above data in the following break-downs:

The GEM-E3 model identifies the following <u>products/sectors</u>:

NO.	Sector Name	NACE-CLIO R25 aggregation (see the names in Appendix 1)
1	Agriculture	010 Agriculture, forestry and fishery products
2	Coal	of 060: 031 Coal and coal briquettes 033 Lignite and lignite briquettes 050 Products of coking
3	Crude oil and refined oil products	of 060: 071 Crude petroleum 073 Refined petroleum products
4	Natural gas	of 060: 075 Natural gas 098 Manufactured gases

5	Electric Power	of 060: 097 Electric Power 110 Nuclear fuels 099 Steam, hot water, compressed air
6	Ferrous and non-ferrous ore and metals	130
7	Chemical products	170
8	Other energy intensive industries	150 Non-metallic mineral products 190 Metal products except machinery & transp 470 Paper and printing products
9	Electrical goods	250
10	Transport equipment	280
11	Other equipment goods	210 Agricultural and industrial machinery 230 Office and data processing machines, etc.
12	Consumer goods industries	360 Food, beverages, tobacco 420 Textile & clothing, leather, footwear 480 Other manufacturing products 490 Rubber & plastic products
13	Building & construction	530
14	Telecommunication services	670
15	Transport	610 Inland transport services 630 Maritime & air transport services 650 Auxiliary transport services
16	Credit & insurance	690
17	Other market services	560 Recovery, repair services, wholesale & retail trade 590 Lodging and catering services 740 Other market services
18	Non-market services	860

The classification of the consumption of households by purpose (‘wants’ or categories) is listed in the table bellow (ND stands for non durables and D for durables):

No	**Purpose Name**	**Status**	**EUROSTAT code**
1	Food, Beverages and Tobacco	ND	1
2	Clothing and Footwear	ND	2
3	Housing and Water	ND	31
4	Fuels and Power	ND	32
5	Housing Furniture and Operation	ND	41+42+44+45+46
6	Heating and Cooking Appliances	D	43
7	Medical Care and Health Expenses	ND	5
8	Transport Equipment	D	61
9	Operation of Transport Equipment	ND	62
10	Purchased Transport	ND	63
11	Telecommunication services	ND	64
12	Recreation, Entertainment, Culture, etc.	ND	7
13	Other Services	ND	8

It should be added that the above list shows the ideal situation. When some of the above datasets or details are missing, one has to apply certain (proportionality, etc.) assumptions. In case of inconsistencies certain balancing methods (e.g. RAS) can be applied. In case of different classifications one has to apply certain transformation techniques.

5.2. Sources of the primary data

The sources of the above listed primary data categories are the following:

• I-O tables (usually at basic prices) and in many countries the import matrices too (at c.i.f. prices) are published by the CSOs (Central Statistical Office of the given country) or by the Eurostat (for the EU15).

• Indirect tax and subsidy matrices are either supplied (as supplementary tables) by the CSO or have to be estimated (e.g. by RAS) from margins obtained from the CSO and the Budget Reports (see the Hungarian example later). If an import duty matrix is to be estimated, one can use the www.wto.com web-site's useful material for the the effective duty rates by countries and commodities.

• Trade Data of bilateral trade flows: for the EU15 they can be found in the COMEXT. For other countries, reference data for estimating trade flows can be found in the GTAP data base (which we obtained for 1995 and 1997). For example, the GTAP project's published (simplified) export and import matrices for 1997 show the trade flows by 56 commodity-groups and for 57 countries or regions.

Alternatively, one can use the annual Yearbook for Foreign Trade Statistics publications or electronic trade databases of the individual countries. In the foreign balance of payments several trade related data can be found (including for the service trade) buti n many cases they are somewhat different (due to methodological and practical reasons) from the NA data.

• Consumption transformation matrix is estimated by using available information on the structures in the national accounts and consumption statistics. When consumption statistics is not available or its break-down by categories is not suitable for the margin of the consumption matrix, data (in fact for 2000) can be found in the OECD/Eurostat's Purchasing Power Parity project's background worksheets.

• Tourist expenditures: There are country specific surveys for some years, or NA consistent "satellite accounts". However, their commodity break-down is seldom satisfactory or reliable. Since indirect tax content (especially by branch break-down) is usually not published, one has to estimate them. The commodity structure of the tourist consumption has to be estimated only if the national consumption pattern has to be estimated, or when only HBS data (which correspond to the national consumption concept) are available for the estimation of the domestic consumption structure. However, in the GEM-E3 model, for the time being the domestic consumptions are dealt with, so it is not a present problem.

- Investment transformation matrix is estimated by using available information on the structures in the investment statistics, and the national accounts and I-O tables.
- National accounts by sector and by branch (for the EU15 from Cronos of Eurostat), that - after aggregation and supplemented by the Income distribution data (State Budget Reports, National Accounts, Foreign Balance of Payments) - can be used to complete the income distribution part of the Social Accounting Matrix by country.

- Emission statistics for the atmospheric pollutants represented in the model can be found in the RIVM (EDGAR)[14], IPCC, UN[15] and other sources. The Greenhouse gas emission data can be found in the http://cdr.eionet.europa.eu/ro/eu/ghgmm/envrayfoa web-pages.
- The estimation of the abatement cost functions is based on the RAINS data computed for the CAFE project.
- Auxiliary macroeconomic data:
- Factor endowment data (capital, labour, employment, stocks, etc.): these are usually found in the CSO (employment) and ILO publications (e.g. labour costs including those which are regarded to be production taxes or intermediate costs).
- Demographic data: Total and active population data were collected from the ENERDATA database but they also can be found in the national statistical yearbooks and labour surveys.
- Interest rates, inflation rate in the base year, etc.: CSO, NB (National Banks)
- Energy balance sheets (by Eurostat methods): IEA/OECD or Eurostat (if not available for they are not public, one can estimate it from national publications as in the case of Hungary).
- Energy taxes and energy consuming durable goods: Data can be found in the national energy authorities' publications or in the OECD databases. Many useful (including special) data (e.g. the share of the gasoline within the motor fuel use) can be found on the World Resource Institue's[16] http://earthtrends.wri.org/searchable_db web-site.
- Data for extension modules can be found in specific sources. Of these we just mention the case of the multiple household version of the model, which is under development for the GEM-E3 model, but which can be found in many CGE models:
- Data for household groups: The main sources are the Household Budget Surveys (or in some cases separate Income and Expenditure Surveys) the census (for the stock of durable goods) and Tax Returns (mainly for individuals). For the other household group specific data sources it is worth mentioning for example, the Hungarian Central Statistical Office conducted and published surveys for the *Inter-household transfers* (see KSH [2004]).

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[14] The Netherlands National Institute for Public Health and the Environment/The Netherlands Environmental Assessment Agency (RIVM/MNP) and the Netherlands Organization for Applied Scientific Research (TNO). 2005 and 2001. The Emission Database for Global Atmospheric Research (EDGAR) 3.2 Fast Track 2000 and 3.2. Acidifying gases: SO2 (Sulfur Dioxide): Extended Emissions 2000 and Aggregated Emissions 1990/1995. The Netherlands: RIVM. Electronic database available online at: http://www.mnp.nl/edgar/.

[15] Mainly the country specific National Inventory Report prepared for submission in accordance with the UN Framework Convention on Climate Change (UNFCCC) [including electronic Excel spreadsheet files containing Common Reporting Format (CRF) data]

[16] Climate Analysis Indicators Tool (CAIT) version 3.0. (Washington, DC: World Resources Institute, 2005). Available at http://cait.wri.org. WRI calculates carbon dioxide emissions from 3 sources:

EIA. 2004. International Energy Annual 2002. Available online at: http://www.eia.doe.gov/iea/carbon.html.

IEA. 2004. CO2 Emissions from Fuel Combustion (2004 edition). Available online at: http://data.iea.org/ieastore/co2_main.asp.

Marland, G., T.A. Boden, and R. J. Andres. 2005. Global, Regional, and National Fossil Fuel CO2 Emissions. in Trends: A Compendium of Data on Global Change. Carbon Dioxide Information Analysis Center, Oak Ridge National Laboratory, U.S. Department of Energy, Oak Ridge, Tenn., U.S.A. Available online at: http://cdiac.esd.ornl.gov/trends/emis/meth_reg.htm.

5.3. Data availability and problems

- I-O tables: For some countries the I-O tables were not available for the selected base year (1995). In this case the I-O table had to be compiled from the MAKE and USE matrices (see the example of Greece in Revesz-Zalai (2003)).

The first usual problem is that the break-down of the I-O tables (and the nace 2-digit level break-down of the national accounts data for branches) are not sufficiently detailed in the case of the energy sectors. It can be seen from the branch classification table in section 5.1.

The second frequent problem of the I-O tables is that they are not consistent with the National Accounts. The NA data figures differ from the I-O table data mostly in the case of the foreign trade flows, but to a less extent it is so in the case of the gross outputs, wages, and stock accumulation. In the case of the foreign trade, the national accounts contain certain double accounting of the value of the materials to be processed, or of the goods to be repaired or reexported. Although in the I-O table (and output indicator) only the processing fee is accounted as export, in some of the I-O tables such reexport-like figures can be found, which is rather difficult to explain within the standard CGE modelling theoretical framework, so it is difficult to decide what to do with them (e.g. netting out, which however, for certain years may result in negative net export values due to the time lag between the import and export of such goods). This problem will be further discussed below, when we present the other problems of foreign trade data too.

The NA also showed some phenomena, which is difficult to take into account in CGE models. For example, in some of the national accounts the foreign sector had wage income (Hungary, Austria) (from which industry?) and SSC income, while it paid indirect taxes and received indirect subsidies (estimated tax content of the expenditures of the inbound tourists?).

Different I-O tables may account the imputed output of bank services (FISIM) differently. Ideally they have to be allocated to actual users (estimating the interest margins by clients using reference interest rates for the domestic and foreign currencies), but this can not be done accurately using just the figures of the I-O table. Note, that in some countries (e.g. in Bulgaria) the Statistical Office just eliminated the FISIM by treating it as the own-consumption of the financial sector. However, it has certain problems, mainly the resulting apparent negative value added and the implicit assumption that the user-structure of the indirect services (interest margin) is the same as the users of the directly charged banking services.

In the GEM-E3 model the NPISHs are aggregated with the household sector (while in the I-O table they have separate columns in the final consumption). So in the model we had to treat the NPISHs as part of the household sector too. However, several coutries do not provide enough data for the NPISHs (Jellema et al [2004]) so the missing data have to be estimated.

- Foreign Trade Data: Apart from the above mentioned problem of value of the double-accounting of materials to be *processed* (or repaired or the like), the most common problem is the apparent inconsistency of the bilateral foreign trade data of the partner countries, which arises partly from the different valuation of the trade flows (the import is at c.i.f. parity while the exports are valuated at f.o.b. parity, so a so-called c.i.f.-f.o.b. correction is needed), but also from different accounting methodology, classification and missing or false reporting. To see the resulting differences usually mirror-checks (i.e. one country's reported import from an other country is the same as the other country's reported export to the first country) are made by commodity groups (branches) and partner countries.

An other notable problem of these trade statistics, is that they usually contain the merchandise trade, but they do not include the services (for which data are rather unreliable and may be based on the foreign balance of payment statistics or specific surveys). A further general problem is that exports and imports with '*custom-free-zones*' are not allocated to countries of origin or destination.

In the case of the printed publication of foreign trade statistics an additional special fundamental problem is that it contains only the *large* trade flows, which means that it is almost impossible to reconstruct the structure of the export and the import with small countries like Danemark or Portugal. The national account data for exports and imports by branch (in the new Cronos database) are too bad to be used (Eurostat is currently modifying them).

➢ Consumption related data: The macrostatistical data for personal consumption by categories show the domestic consumption, while HBS data by definition contain the national consumption. Therefore, one has to estimate the tourists' consumption by categories too.

➢ Investment data: From the New Cronos database it was feasible to extract information regarding to gross fixed capital formation by branch, changes in inventories and acquisitions less disposals of valuables. In addition information on investments by product was available through the main aggregates of EUROSTAT. Combining this information with the investment structure derived from the projected tables the final investment by product transaction could be obtained.

The distinction of investments between the institutional sectors (Households, Firms, Government) was made by incorporating the information realised from the investment matrices as well as from the respective structure of Greece and United Kingdom.

➢ Environmental data: Emission of air pollutants can be estimated quite accurately from the energy consumption data. However, in some cases uniform emission coefficients – derived from the EU15 practice - may result in rather unrealistic emission estimates. For example, in Hungary the extracted 'coal' is barely more than lignite, so its (per Joule or per ton) emission coefficients are far higher than those of any coal products within the EU15. Similarly the modellers have to bear in mind that in the Baltic countries fossil energy extraction practically means only pet and oil shale which are not typical product neither of the coal nor the oil industry.

➢ Auxiliary data: When missing, the capital stocks have to be estimated from the capital incomes or amortization data (e.g in the case of Austria using exogenous type or industry-specific amortization rates)

➢ Data for household groups: *Inter-household transfers* do not appear in the national accounts since by definition their aggregate value is zero. Therefore we can not adjust proportionately the HBS data for the macroeconomic total because it would set each inter-household transfer figures to zero. Hence one has to find another way how to assess the representativity of the HBS's figures for the inter-household transfers.

The usual problems in the HBSs are the need for reclassifying the special categories to scientific (national accounts) categories, the uneven representativity of the various income and expenditure categories, the underrepresentation of the poor, the rich, the overloaded, and the mobile (moving, commuting, immigrant, etc.) households, the annualization of the data (covering usually a period of one month or less), the matching of household incomes (and consumption) with members, the imputation of missing data (in-kind benefits, various capital incomes).

Apart these general data problems in chapter 8 we present the Hungarian example, showing how to cope with the missing data.

5.4. Data processing methods

Most of the primary data of the GEM-E3 model can be filled into a SAM (Social Accounting Matrix) scheme by which one can check the consistency of the data and their plausibility (by computing proportions and comparing them with data of other countries. Therefore, the construction of the SAM is the starting point of the model building work.

Figure 5.1: GEM-E3 SAM according to ESA 95 methodology.

<table>
<tr><th></th><th>Industries</th><th>Labour</th><th>Capital</th><th colspan="2">Consumption</th><th>Firms</th><th colspan="3">Investment.</th><th>Ch. in Stocks</th><th>Exports (F.O.B)</th><th>Total</th></tr>
<tr><td></td><td>1………..18</td><td></td><td></td><td>House-holds</td><td>Govern-ment.</td><td></td><td>.</td><td>.</td><td>.</td><td></td><td></td><td></td></tr>
<tr><td>Products 1 . . 18</td><td>Intermediate consumption at producer’s prices.</td><td>-</td><td>-</td><td colspan="2">Demand of household and government. consumption. (incl. NPIHS)</td><td>-</td><td colspan="3">Demand for investment goods</td><td></td><td>Demand for exports</td><td>Total demand for goods</td></tr>
<tr><td>Wages and Salaries</td><td rowspan="3">Primary factors’ income.</td><td colspan="2" rowspan="3">-</td><td colspan="3" rowspan="3">-</td><td colspan="3" rowspan="3">-</td><td rowspan="3">-</td><td rowspan="3">Income transfers from foreign.</td><td rowspan="3">Total factor revenues</td></tr>
<tr><td>Social Security Contribution</td></tr>
<tr><td>Operating Surplus</td></tr>
<tr><td>Households</td><td>-</td><td colspan="2" rowspan="3">Factor payments to agents according ownership.</td><td colspan="3" rowspan="10">Domestic income transfers between agents.</td><td colspan="3" rowspan="10">-</td><td rowspan="10">-</td><td rowspan="10">Income transfers from/to abroad.</td><td rowspan="10">Total income of agents.</td></tr>
<tr><td>Firms</td><td>-</td></tr>
<tr><td rowspan="8">Government</td><td>Gov. Firms</td></tr>
<tr><td>Gov. Foreign</td><td colspan="2" rowspan="7">-</td></tr>
<tr><td>Direct Tax</td></tr>
<tr><td>Subsidy</td></tr>
<tr><td>VAT</td></tr>
<tr><td>Duty</td></tr>
<tr><td>Social Security</td></tr>
<tr><td>Indirect Tax</td></tr>
<tr><td>G. Transfers</td><td>-</td><td colspan="2" rowspan="3"></td><td colspan="3" rowspan="3"></td><td colspan="3" rowspan="3">-</td><td rowspan="3">-</td><td rowspan="3"></td><td rowspan="3"></td></tr>
<tr><td>Imports</td><td></td></tr>
<tr><td>Savings</td><td>-</td></tr>
<tr><td>Total</td><td>Total supply of goods</td><td colspan="2">Total payments of factors</td><td colspan="3">Total spending of agents</td><td colspan="6"></td></tr>
</table>

A SAM is a square matrix of monetary flows that describes all transactions taking place between the economic agents of an economy for a determined year. The number of actors constitutes the dimension of the square matrix. By convention, columns represent expenditures while rows represent receipts. A schematic representation of the GEM-E3 SAM is shown in Figure 5.1.

The balance is conceived as the tautological equality (guarranteed by displaying the savings as a component of the 'expenditures') between the sum by row and the sum by column. In addition, a SAM ensures the fulfillment of the Walras law in the base year, since by construction the algebraic sum of surplus or deficits of agents is equal to zero. The GEM-E3 SAM represents flows between production sectors, production factors and economic agents. The production sectors produce an equal number of distinct goods (or services), as in the Input-Output table.

Production factors include, in the SAM, only primary factors, namely labor and capital. The economic agents, namely households, firms, government and the foreign sector, are owners of primary factors, so they receive income from labor and capital rewarding.

In addition, there exist transactions between the agents, in the form of taxes, subsidies and transfers. The agents distribute their income between consumption and investment, and form final domestic demand. The foreign sector also makes transactions separately with each sector. These transactions represent imports (as a row) and exports (as a column) of goods and services. The difference between income and spending (on consumption and investment) by an economic agent determines his surplus or deficit.

5.4.1. Processing the Input-Output table

As can be seen the I-O table occupies the upper and left section of the SAM table. So the first task is to obtain the I-O table data. However, for many countries such table is not available for the selected benchmark year, or only the Supply and Use tables are available. The method of the compilation of the I-O tables from these tables is described in the SNA 1968 volume[17]. A schematic method (using Hungarian data) of this in Excel-worksheet can be found in the **MakeUse.XLS** file. This shows how the branch by branch (square) matrix can be obtained and the import matrix separated of the matrix of domestic flows. However, the method does not deal with the conversion of the basic prices to users prices (as it is sometimes needed, when the use matrix is available only at users' prices). An other shortcoming of the method used in the MakeUse.Xls file is that, as one can observe, that although the method takes into account that exports and government consumption do not contain any imports, the inevitable assumption of uniform import shares across the rest of the users resulted in too low (machine) import for investment while too high imports for the other uses. Naturally, the schematic method can be refined, as needed by the user. Note that for many countries the import matrix is published (e.g. Austria, Hungary), so the reclassification can be done separately for the domestic and import 'Use' matrices.

[17] A shortened version of this can be found in the Appendix.

For many other countries, which publish I-O and National Accounts data only in the 2-digit nace code break-down, nace sectors 11 (Crude petroleum and natural gas; services incidental to oil and gas extraction excluding surveying), 23 (Coke, refined petroleum products and nuclear fuels) and 40 (Electrical energy, gas, steam and hot water) have to be split first, before aggregating.

In order to do that, one has to obtain data for the output, value added, etc. their components (industry statistics, more detailed SUT, etc.). Splitting can be done then by using proportionality assumptions (like the 'industry-technology assumption' in the construction of symmetric I-O tables from the Supply and Use tables). In some countries, however, some of the components are just missing, and 'row-wise' the crude oil can be separated from the natural gas quite easily, since it is used exclusively by the refinery industry, so the rest of the elements of the row must be just natural gas. Also, 'gas' in sector 40 means only manufactured gas (which has ceased in most of the countries) and the gas distribution. The question is how the natural gas is accounted for in the I-O tables: either directly at the users or routed through sector 40. Usually in the case of those big gas users which get the natural gas via their specific pipelines from their foreign supplier, this gas consumption is accounted as direct import, while in the case of small users and the retail trade the indirect accounting is the common treatment.

When the I-O table is not available for the benchmark year, stock accumulation data can be estimated from the data of the CSOs’ specific inventory survey, or in the case of energy carriers, from the energy balance sheets.

The *aggregation* of the data to 18 sectors can be done by the following procedure:

The aggregation starts from a G aggregator matrix (N rows, n columns, where N is the number of the original sectors, while n is the number of the aggregated sectors - in our case 25). This G matrix has to be filled so that <u>if</u> P proportion of the i-th original sector (output, consumption, investment, export, whatever category is just to be aggregated) will go to the j-th aggregate <u>then</u> it should be P (in most cases in each row there will be a 1 while the other elements of G will be 0). If the aggregation is pure aggregation (i.e. P = 1 or P = 0) then the filling procedure can be simplified by designing a V column vector in which the i-th element's value is just j. Then G is computed from V the automatically so that G(i,j) = 1 if V(i) = j, otherwise G(i,j) = 0.

Then the aggregate of the A (NxN) matrix can be obtained by the G^*AG matrix product, where G^* is the transpose of G (so multiply A from the left by G^* while by G from the right). Matrices which has to be aggregated only from one side has to be multiplied by G or G^*. More precisely if the rows have to be aggregated, then G^*A has to be used, and if the columns, then the AG matrix product gives the result.

The advantage of this approach is that it can be done easily ina n Excel worksheet and if you change your mind of the aggregation scheme (or if some original sectors even has to be split or actually disagregated) you have just to modify V, and then the Excel formulas do all the necessary changes in an instant.

For example for the Excel file which contains the aggregation scheme from the 21 sectors of the Hungarian I-O table for 1995 to the model's 18 sectors is in the 'dom' worksheet of the **akm95d.xls** file. In this sheet V is in the E7:E27 cells (yellow), in which the elements corresponding to the sectors to be split are zeros. The G matrix is in the E7:W27 range. The rows of the sectors which are not to be split in matrix G are computed automatically (although in a little bit tricky way by using the ROUND function and a big exponent to replace the time consuming IF-s by a proxy characteristic function).

Note that for those sectors which have to be split, then ideally separate matrices have to be used for the domestic and the import, in which the respective shares are used for the splitting of the domestic and import flows. However, for the time being we use only one aggregation scheme in which for a given commodity the same shares are assumed for the domestic and import sources.

5.4.2. The compilation of the other blocks of the GEM-E3 SAM.

Combining the Input-Output data, adapted to market prices and to the national product concept (instead of the domestic product concept), and the data of the National Accounts by sector allows building the Social Accounting Matrix for each country.

TREATMENT OF THE SEPARATE ACCOUNTS.

The allocation of the adapted Input-Output totals to the different sectors, household, government, firms and rest of the world is rather straightforward using National Accounts data, and can be summarized as follows:

- The total labor value added is allocated to the households except for the part going to the Rest of the World
- The capital income is distributed between household, firms and government as in the National Accounts
- The social security contributions are paid by households to the government and to the firms
- Households and firms pay the direct taxes to the government.

In the SAM it is assumed by construction that all subsidies are paid by the government to the branches (firms). In fact a part of the subsidies is paid by the foreign sector. In order to take into account this issue an imputed flow was created in the SAM representing the difference between the subsidies received by the branches and the actual subsidies paid by the government (this difference is attributed to the foreign sector).

Since the government does not receive the sum of the taxes on product paid by the branches (a part goes to the foreign sector) a similar treatment to the one applied on subsidies has been established.

(i) Social security contribution:

The employers' contribution had to be displayed in a branch break-down in the intersection of row 'Social Security' and columns of activities (branches). In the case of Hungary, this component of the labour income is transferred to the households' account, who in turn pay it to the government along with the employees' contribution (in the intersection of row

’Government: Social Security Revenue’ and column ’Households’). As an alternative solution, this component can be displayed in the intersection of ‘Social Security’ and ‘Labour’, in order to arrive at the value of the total compensation of employees.

(ii) Interest payments:

In the GEM-E3 model interest does not appear explicitly as a separate item. Therefore, in the SAM they were included in the row of (‘before-interest’-) ’Savings’, which also included the balance of the net interest payments (as if these payments were automatically used as savings).

(iii) Capital and investment transfers:

They were treated in a manner similar to interest payments (i.e. as if they belonged to the use side of the savings).

(iv) Dividends:

Dividends paid on foreign direct investment had to be displayed in the row of ’Expenditures abroad’.

(v) Retained profits:

With the exception of government institutions retained profits have not been explicitly taken into account. Savings statistics, as a rule, does not include retained profits, so they do not affect the row of ’Savings’. The foreign balance of payments, for example, shows only the dividends (distributed profits) and the resulting balance is regarded to be the saving of the foreign sector. Similarly, the state budget deficit is computed only by taking into account the dividends of the state-owned companies, and the government saving is the net sum of the deficit (-), government investment and interest payments, and the government investment and capital transfers.

The retained earning of government institutions is displayed in the intersection of row ’Gov. firms’ and column ’Capital’.

(vi) In-kind benefits:

The government financed part of them must be included in the ‘government consumption’ while the NPISHs provided part must be aggregated with the households consumption expenditures. This standard procedure may be modified only if in the analysis the personal-social (collective) consumption is preferred to the private-government consumption distinction. In this case the government provided in-kind benefits (education, health care, etc.) should be included in the cell ’Households income from government’ along with other (cash) benefits. The use of this income item shows up in the corresponding elements of the column of the ‘Household consumption’. This allows one to separate the in-kind benefits out of the total benefits.

(vii) Other transfers:

Social (cash and in-kind) benefits, non-labour income of the households received from the firms, foreign aid, gambling income, penalties, rents (that were not accounted as imputed output and capital income) belong to this category.

Statistical information, in most cases, do not reveal who pays to whom these transfers (only the net amount is known), therefore one is free to select an agent (e.g. ’firms’), who is assumed to collect and distribute these transfers.

(viii) FISIM (financial services)

In the data set of the GEM-E3 model FISIM is distributed among branches in proportions to their ex ante operating surplus (i.e. one calculated without the cost paid for financial services). By subtracting the cost of financial services, as calculated above, one arrives at the value of the estimated (ex post) operating surplus that represents capital income. This amount should be, in principle, positive. If it is not, special subsidies are introduced to correct for the negative number.

(ix) Direct taxes:

Apart from personal and corporate income taxes, one has to account here for the property taxes, domestic (stamp, etc.) duties, local taxes and so on. Although they are not directly proportionate with income, they correlate to income to a considerable degree, especially at the aggregate level.

(x) Indirect taxes and Subsidies:

For the GEM-E3 model the ‘Subsidies’ row of the SAM contains all subsidies, independently from the fact that they are related to products or to domestic production alone. In the case of Hungary, for example, we accounted here the production taxes (e.g. rents on mineral extraction) as well, since they change in proportion with the level of domestic production.

Similarly, ‘Indirect taxes’ account for all (non-VAT) taxes on products (e.g. fuel excise tax).

The row ‘VAT’ contains all the VAT paid, irrespectively from the area of use (intermediate, household and tourist consumption, investment, etc.). As a consequence, the intermediate consumption has to be computed at user prices in the SAM too.

The foreign trade statistical data have to be reclassified to Input-Output table sectors and adjusted to them so that the I-O export column should be equal with the sum of the countries (columns) of the bilateral export matrix. This adjustment is needed partly for the mentioned methodological reasons.

As long as no investment matrix exists in the national statistics, its column totals (demand of investment goods by branch) are taken from the National Accounts by branches (Gross fixed capital formation, by ownership branch), while the row-totals (total deliveries by branch) are taken from the corresponding column of the Input-Output table. Finally appropiate assumptions have to be made for every country individually to fill the matrix. We usually use the RAS-method, for which (initial) reference investment matrices can be found in the OECD Input Output Database. These data were available for the following regions: Australia, Japan and United States. For the compilation of investment matrices for non-EU countries, other countries’ already compiled investment matrices can be used as reference.

In Appendix 5 the UK matrix is given as example. This example shows that the deliveries were basically made by the branches of Other Energy Intensive Industries (number 08), Electrical Goods (number 09), the Transport Equipment (number 10), Other Equipment Goods Industries (number 11), Building & Construction (number 13) among industrial sectors and to a lesser extent the branch which represents the market related services (number 17).

Investment matrices were available only for Greece and UK. For the computation of the rest of the matrices the information available was the investment by branch and by product. Since there was insufficient information on the transformation matrix a RAS procedure was adopted. The initial tables for the RAS procedure were based on the Greek and UK investment matrix modified appropriately in order to serve the specific investment structure of each country.

For countries where a consumption transformation matrix exists, they are usually expressed in consumer’s prices, i.e. they include the VAT and the margins are included in the price of the delivery and not considered as a separate delivery by a service branch. For the use in the model, the matrix has to be in producer’s prices and with explicit delivery by service branches. Therefore, the following corrections have to be made:

• given VAT rates for the different consumer categories, a consumption matrix without VAT is computed,

• the margins included in the deliveries by branch are evaluated as the difference between the consumption matrix deliveries (without VAT) and the IO deliveries.

• these margins are allocated between the services branches based on exogenous parameter compatible with the Input-Output deliveries of these branches.

For the countries, where such matrix was not available, the matrix was computed using the following procedure:

• the consumption per consumer category is taken from the National Accounts (final consumption of households on the economic territory, by purpose) and corrected for the consumption by tourist.

• given VAT rates for the different consumer categories, the total per category without VAT is computed

• the total deliveries are taken from the Input-Output tables and appropriate assumptions were made to allocate the total per categories to the delivery branch.

If the the row margin (column totals) of the consumption transformation matrix shows the consumption by the usual 12 COICOP categories break-down, then 3 such categories have to be broken further down (to be able to compute the consumption by the 13 GEM-E3 consumption categories) according to the following:

category 4 (Housing, water, electricity, gas and other fuels) has to be split to water and energy

category 5 (Furnising, household equipment and routine maintenance of the house) has to be split to heating and cooking appliances and the rest

category 7 (Transport) has to be broken down to its trasport equipment, operation of transport equipment and purchased transport components

For the GEM-E3 model, several assumptions have been made to allocate the EUROSTAT energy balance sheet values to the branches and products of the IO table:

1. energy consumption by energy branches : combustion of solid fuels are allocated to branch '2'. Combustion of liquid fuels is allocated to branch '3'. Combustion of natural gas is allocated to branch '4'.

2. energy consumption by tertiary sector : the total energy inputs are allocated to the tertiary branches on the basis of ratios derived from the IO tables.

3. transportation : only LPG, gasoline and diesel oil are used for road transport. The total road transportation input figures are allocated to the different branches and households on the basis of ratios observed or estimated for Belgium[18]. For the computation of the emission coefficient, product '3' is explicitly split into a fraction used for road transport purposes and a fraction used for other purposes. The energy inputs for non-road transport are allocated to the branch transportation services.

4. manufactured gases: since the GEM-E3 IO-table do not include transfers between branches, the manufactured gases have to be handled as a delivery of solid fuels '2'. Total blast furnace gas consumption is allocated as a delivery of product '2' to branch iron and steel. A correction is made for the demand of electricity of this branch (efficiency = 0.4). Coke oven gas used for 'power generation' and 'own consumption' is allocated as a delivery of product '2' to branch '2'. A correction is made for the demand for electricity '5' of the branch '2' (efficiency = 0.4). Coke oven gas used by 'I&S' is allocated as a delivery of product '2' to branch iron and steel.

5. non-energy use: the bunkers are allocated as a delivery of product '3' to branch transportation services. The transformation input are allocated to their respective branch, with the exception of blast furnace gas which is already handled as relevant energy use. For the non-energetic final input, the chem goes to branch chemical and the other is allocated to the other branches on the basis of 1980 ratios which are extrapolated to 1985.

With these computations, one obtains, for GEM-E3, one sheet with the relevant deliveries of the energy branches to all branches and one sheet with total deliveries. This allows for the computation of the relevant fraction of total input, i.e. the m parameter in GEM-E3, which is needed to compute emissions in GEM-E3.

[18] Note that for example the Hungarian Energy Statistical Yearbook contains in the Appendix the so-called EU-conform balance sheets which shows the motor-fuel consumption by industries too.

5.4.3. The baseline emission coefficients

- CO_2 emission coefficients

The CO_2 emission coefficients are those used by EUROSTAT, in ‘Environment Statistics’, if no country specific information available.

- NO_x and SO_2 emission coefficients

These emission coefficients have been computed based on the RAINS baseline scenario. The SO_2 emission coefficients are computed taking into account the sulphur content in the fuel, the fraction of sulphur retained and the net heating value of the fuel. The NO_x emission coefficients are those computed by Coherence in the HECTOR model (1993). The NO_x coefficients are fixed on the basis of technological assumptions. Some corrections were made when more complete information was available.

- VOC emission coefficients

Emission coefficients for VOC were added to the database and were considered equal across countries but specific to the fuel used. Their source is the following: Part 3, Default Emission Factors Handbook, CORINAIR Inventory, 1992

- PM_{10} emission coefficients

These emission coefficients have been computed based on the RAINS baseline scenario. Information about the contribution of each sector to the emission of PM_{10} is coming from ExternE project (stationary sources), VIA, RWTH (1995) and TRENEN project (mobile sources). ExternE distinguishes the main sources of PM_{10} for each activity within the fuel cycle (mining, transport, electricity generation, etc.), however given the structure of GEM-E3 and PRIMES models, only the data for electricity generation (coal, lignite, oil and gas) was considered. We assumed the same emission coefficient for the industry sector. For the conversion from g/MWh to ton/PJ (stationary sources), we used the efficiency of the power plant considered and the conversion factor 3600KJ/kWh. As for VOC, the emission coefficients are assumed to be equal across countries.

The parameters of the *abatement functions* have been estimated from the RAINS database.

Based on some further information, the following deliveries of abatement expenditures are used throughout all sectors and pollutants.

Table 5.1: Break down of deliveries of abatement technologies (in % of total costs)

cost type	%	assignment to GEM-E3 classification
investment costs	77	equipment goods industries
labour costs	3	labour
waste costs and other variable costs	12	services
fuel costs	8	main energetic input of the sector

Matrices of Pollutant Transport and Transformation Coefficients (Units: % of total emissions, Columns: Emitters, Rows: Receivers) for NOx and SO2 are derived from the EMEP model.

Auxuliary data: when missing, the capital stocks have to be estimated from the capital incomes or amortization data (e.g in the case of Austria using exogenous type or industry-specific amortization rates).

5.5. Techniques for estimating the missing data

5.5.1. Using proxies:

For example, temporarily lacking better information, we assumed that all stock accumulation takes place in the sectors of origin (i.e. the producers has to store them).

5.5.2. Computing as residual:

This can be done either from stock-flow and source-use balances. One has to be carefult not to select a relatively small item as residual, since even the relative small statistical errors in the larger items may result in extremely large relative estimating errors in this small residual item.

5.5.3. Routing through

When we can not tell who pays to whom we can use collecting accounts through which these payments are routed through. This collecting account can be even the account of a standard agent of the model (e.g. the transfer payments are routed through the government).

5.5.4. 'Rooking'

The „rooking" method is well-known in the operational research. It is applied when certain elements of the matrix have to be modified so that the row-totals and column totals remain unchanged. Usually a rectangle is selected and when we modify one of its corners by shifting a certain amount to (or from) an adjacent corner, simultaneously we shift the same amount but in the opposite direction between the other two corners of the rectangle. An application of this method is described in section 6 describing the estimation of the Hungarian consumption transformation matrix.

5.5.5. Miniature programming methods

In some cases during the estimating process a small-scale or partial programming model is solved. An example for this also can be found in section 6 when describing the estimation of the Hungarian custom duty matrix, where in the first round commodity group specific standard duty rates were estimated by minimizing the squared error of the actual and estimated duty costs of the individual users.

5.6. Techniques for the reconciliation (adjustment techniques) of inconsistent data

5.6.1. The RAS-method

The RAS-method is generally applied for the adjustment of a reference matrix to given margins (see Lecomber [1975], Polenske [1977], Robinson et al [2001]). The RAS-method can be regarded as a bi-proportional adjustment, since it tries to maintain the original proportions of the column and row structures. An Excel-representation of the RAS-method can be found in the **RAS72x61.XLS** file, which can adjust an initial matrix up to 109 rows and 68 columns (shown in the C5:BR113 range). The procedure is programmed so that first the row-wise proportional adjustment takes place in the BW5:EL113 range, then the column-wise adjustment is done in the C118:BR226 range. Finally, by an Excel macro (which can be run by pressing the Ctr-i buttons) the values of this range are copied back to the initial C5:BR113 range, so that the BW5:EL113 and C118:BR226 ranges now show the adjustments of this new reference matrix. This iterative procedure can be repeated infinitely (by pressing the Ctr-r buttons 10 such iteration steps are done) and usually converges to a matrix which (as it can be proven) is equivalent to the solution of a hyperbolic programming problem (where the objective function is the sum of certain 'relative errors'). However, in certain circumstances the algorithm does not work properly: for example, when the (subset) of row totals are not consistent with the (subset) of column totals, the iterations lead nowhere (usually the some elements of the matrix oscillate). This happens mainly in 'rare' matrices like the investment transformation matrix where only several rows and the main diagonal elements are different form zeros[19].

Negative element may also cause problems of the sonvergence of the RAS-iterations. In some cases (mainly when in the column of the I-O table's stock accumulation there are both large negative and positive elements while the total is a small number) they turn the sign of the whole vector causing further inversions of the sign of the row- and column totals.

An advanced version of the RAS-method is the so-called bounded RAS-method[20].

Of course, there are many more sophisticated matrix balancing methods, which are generally called entropy method (see a comparison of several such methods in Schneider, M. H. – Zenios, S. A. [1990]). However, most of these methods (incl. the RAS itself) retains the zero values even if we have information on its positive value (e.g. technology development leads to the use of new materials and services). An ad-hoc treatment of this problem is either the exogenous setting of this element or (if we do not know its magnitude precisely) imputing an initial positive value into the corresponding (hitherto zero) element of the reference matrix.

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[19] The most extreme case is when in a selected row and column there is only one non-zero element, i.e. their intersection. In this case, if the prescribed totals of this row and column differ, this single element tries to satisfy both, resulting in oscillating between them.

[20] RAS-problems with exogenous elements can be traced back to standard RAS-problems by setting to zero the corresponding element of from the reference matrix and by subtracting its value from the corresponding row- and column totals (see e.g. the RAS2004.XLS file which updates the 2000 Hungarian I-O table for 2004).

5.6.2. The ‘additive’ RAS-method

When some of the elements of the row- or column margins (totals) are zero, the RAS method is not applicable since it would turn all elements of the corresponding row or column to zero even if these data should be different from zero. For example, when the Household Budget Survey based income and expenditure data (where the savings are treated as part of the ‘expenditures’) are arranged in such a format that each columns represent the incomes and the expenditures of a given group so that the incomes are displayed by positive numbers while the expenditures by negative ones, the column totals should be zeros by definition. To solve this problem (without setting exogenously the total income or expenditure of the individual groups) I developed a so called „additive” RAS method. Since expenditures are displayed as negative numbers in the matrix of the household budgets (where a rows represent the items, and columns represent the groups) the column-totals - by definition - add up to zero. This makes the RAS method inappropriate. Instead, the modified iteration method is the following:

$$H_{i,j} = H0_{i,j} + \left(HTOT_j - \sum_k H0_{k,j} \right) - SH_{i,j}$$

where $H_{i,j}$ is the new cell-value, $HO_{i,j}$ is the previous cell-value, $HTOT_j$ is the desired column-total (in our case 0), and $SH_{i,j}$ is the original absolute value share. This method adjusts the individual cells proportionately to their absolute value, but the direction of the change depends not on the sign of the cell, but rather on the sign of the total discrepancy, which has to be eliminated. By this method one can estimate the change in the income inequality (shares of the individual groups within the total) too. For example, when we applied this additive RAS-adjustment to the 1991-1994 period in Hungary we found that relative income position of the rural and larger households (i.e. those with children) deteriorated.

A GAMS version of the additive RAS-algorithm (applied to the updating of the income and expenditure data of the household groups) can be found in the Rasadd01.gms file, which uses the RAS98.PRN as input (reference matrix, prescribed new margins) and which puts the results into the RESULTS.RAS file.

The source code of a (Borland) C^{++} version of the additive RAS-method can be found in the **addras.cpp** file, while its executable version is the **addras.exe** file. The DOS-command line for the running of this program is the following:

addras.exe <input_txt_file_name> [/it <number_of_iterations >]

where the typing of the parameters in the [],<> brackets are optional. The output is generated on the output.txt file. An example is the following:

Original matrix	Column 1	Column 2	Desired row-totals
Row 1	11	34	67
Row 2	13	14	38
Row 3	-24	-48	-85
Desired column-totals	20	0	20

The results of the additive-RAS procedure is shown by the following table:

Resulting matrix	Column 1	Column 2	Desired row-totals
Row 1	19.778	47.222	67
Row 2	20.746	17.254	38
Row 3	-20.523	-64.476	-85
Desired column-totals	20	0	20

It can be seen clearly, that the zero column total for Column 2 is achieved so that the structural information of the column is retained.

5.7. Summary

First the 8 main statistical datasets needed by the GEM-E3 model were listed and their meaning was explained. Then conversion tables were introduced to show how the original classification of these datasets can be converted to the break-downs needed by the GEM-E3 model. After discussing the usual data sources and availability problems a general guideline was given how to process the rough data and how to arrange the processed data into the standard data table forms of the GEM-E3 model. Since in newly accessed EU-countries have few modellers experienced in such data processing methods, we devoted a section for the discussion of techniques for estimating the missing data and for the reconciliation (adjustment techniques) of inconsistent data. Here several methods and softwares developed by ourselves were presented, e.g. the “additive-RAS” technique for the bi-proportional adjustment (balancing) of matrices with zero margins or the method for flexible and automatic aggregation and disaggregation (implemented in Excel programs).

6. Compilation of the database for the GEM-E3 model: the example of Hungary

To help the prospective modellers of the newly accessed EU-countries this section contains a detailed case study which we present the whole process for compiling the Hungarian data for the GEM-E3 model. Here we highlight several country and year (in our case 1995) specific statistical problems and the original way how we solved them. This is intended to show that although in the newly accessed countries there are many similar novelties, but still there are many common approaches and techniques in their solution.

6.1. Domestic output and imports

We started the work by transforming the 21 sector I-O data for 1995 to the desired 18 sector breakdown of the project. We transformed the import matrix and the matrix of the domestic flows separately. The following re-classification scheme shows that to which new branch (represented by their serial number) the original branches were put. More than one values in one line indicate that the original branch had to be split, and the individual parts of it were put to different new branches.

Name of Original Branch	**NEW BRANCH NO.**
Agriculture, hunting and fishing	1
Forestry	1
Mining and quarrying	2,4,6
Food and tobacco industry	12
Light industry	8,12
Manufacture of chemicals	3,7,12
Manufacture of other non-metallic mineral products	8
Manufacture of basic metals and metal products	8
Machine industry	9,10,11
Manufacture of products not mentioned above	12,17
Electricity, gas, steam and water supply	4,5
Construction	13
Wholesale and retail trade and repair	17

Hotels and restaurants	17
Transport, storage	15
Post and telecommunications	14
Financial intermediation	16
Real estate, renting and business services	17
Public administration and other service activities	17,18
Education	18
Health and social work	18

The name of the new GEM-E3 branches and their content was already presented earlier. The splitting of the rows and columns of the I-O table belonging to the above branches was done proportionately to the output (or in the case of the import matrix proportionately to the total import) share of its components. In doing this all but one output figures could be found in the national accounts. In the case of the gas production and distribution sector (which is rather important for the model) most of the output of this sector is not published separately in the national accounts. Instead, it is dispersed in the ’oil&gas extraction’ and ’refinery’ sectors (which is the main activity of the large MOL company which, however, deals with gas-extraction too). Therefore, we only could estimate the output of the gas sector as the sum of the outputs of the gas distribution and oil&gas extraction industries. In this way one part of the gas-extraction output is overestimated, while the other part of it is underestimated. Although these errors more or less cancel out each other, the future operation of the model may indicate that the estimate has to be revised (refined).

Of course, in general this proportionality assumption may have to be refined in the future. In the case of the import, for example, it may prove to be necessary when by linking the country-models, differences between the data of ’EU import from Hungary’ and ’Hungarian export to EU’ will be observed.

As far as the latter is concerned, from the import vector obtained by the outlined reclassification procedure, we separated out the EU trade by using the merchandise trade statistics. Note, that for the service trade we did not have EU and non-EU breakdown, thus we had to use plausible assumptions about the share of the EU in our import of services.

6.2. Indirect taxes

In the next step, by a mathematical programming procedure (by minimizing the squared errors of the estimate) we determined the import duty matrix from the standard duty rates and the duties paid by users. The resulting duty matrix can be found in rows 81-100 of the ‘IMP’ worksheet of the akm95d.xls file (table 7 of the Data95pap.doc paper). Since the I-O tables account the import matrix only at c.i.f. prices, the duty matrix was necessary to convert the imports to domestic basic prices which is the unit of measurement for quantities (volumes) in the model.

However, this principle was not upheld in the case of the domestic indirect taxes. Although they are rather branch-specific (mainly due to the lack of VAT-refund apparatus for the budgetary institutions, households housing investment, small enterprises and agriculture) the guideline required the accounting of these taxes also in the rows of the branches. Since the Hungarian I-O table is compiled at net (basic) prices, the indirect taxes are accounted by users in a separate row of the table. Therefore the total indirect taxes had to be split to related inputs (by branch affiliation) and added to the matrix of the intermediate demand. Since the category of net indirect taxes is rather mixed (e.g. the local sales taxes are paid not after the inputs but after the output), we separated its main components and distributed them among branches according to their own characteristics (e.g. the fuel tax was added obviously to the row of the „OIL" branch). The resulting intermediate indirect tax an subsidy matrices can be found in rows 116-161 of the 'DOM' worksheet of the AKM95D.XLS file.

6.3. Investment transformation matrix

In the next step we compiled the simplified investment matrix, i.e. investment by 18 sectors of origin and by institutional sectors or agents. The procedure can be seen in the 'IMP' worksheet of the file AKM95D.XLS, in rows 102-153. The two margins were available in the I-O table and the NA respectively. During the process the main problems were the identification of the own-account investment and design costs, and the transformation of the 14 investment destination branches to the 3 agents. Here we also had to use certain homogeneity and proportionality assumptions. Concretely, we assumed that investments in the same industry has the same investment good structure whatever agent is concerned. Note, that the households investment is published together with the non profit institutions (NPIS) investment, but fortunately the latter is not significant (less than 1 percent of the sum of the two).

6.4. Consumption transformation matrix

The compilation of the consumption transformation matrix was based on a similar table for 1994 which we obtained from the HCSO. We had to overcome several difficulties arising from the different classifications, and different treatments of indirect taxes. The procedure can be seen in details (incl. the formulas) in the file **LF95.XLS**. In the following concise description we refer to the corresponding rows of the last worksheet of this file in parenthesis.

First of all, we had to modify the original data for 1994 so that it be a proper reference matrix for the RAS updating procedure. First, the trade margins had to be separated out of the expenditures (see the 'tax94separ' sheet of the file). In the next step, we aggregated the 34 categories (wants) to the required 13 categories break-down. Here the only serious problem was with the last of the 34 original consumption category. The problem is that this „Correction for the welfare services provided by the branches" is an artificial category which is difficult to incorporate into the standard aggregation scheme. Fortunately, this item is not significant or important in the model either, so we simply omitted this column and left for the RAS method to do the necessary adjustments. The resulting consumption transformation matrix at consumer's prices can be found in rows 47-67 of the last ('basic') sheet of the file LF95.XLS.

By the RAS algorithm we updated the 1994 matrix to 1995 using the consumption statistics and I-O data as prescribed margins. However, to get consistent margins we had to convert the I-O figures to consumers prices by adding the estimated net indirect taxes by sectors (first in the original 21 sector break-down, then converted to the 18 branches of the model). These indirect taxes were estimated in the following way: First we computed the rates for 1994 and then we proportionately adjusted them to the 1995 total as given by the 1995 I-O table. Because of the proportional adjustment, the tax on telecommunication services was exempted, being the only service for which the tax could be computed directly. This could be done by taking advantage of the fact, that this is the only sector which has a match among the 'wants', so the tax could be computed as the difference between the figures in the consumption statistics (at consumer prices) and the I-O table (at basic prices).

From the resulting domestic consumption matrix (updated to 1995 by the RAS method, see rows 73-93.) we separated out the domestic and import components and reclassified the matrix from 21 branches to 18 using similar proportionality assumptions as before (see rows 99-116. and 122-139.). Then, by computing branch specific custom duty rates for the consumption (based on the duty matrix), we could determine the consumption-duty matrix and the consumption matrix at 'uniform' prices, i.e. consumer prices net of the domestic indirect taxes ('VAT'). By subtracting the margins of this matrix from the total indirect taxes one can determine the domestic indirect taxes by category and by branch as well (ranges C165:O165 and Q145:Q162). In the consumption matrix at 'uniform' prices we had to do some corrections since the proportionality assumption (applied to the chemical industry) allocated too much fuel to the „medical expenses" category and on the other hand too much medicines too the „operation of transport equipment" category. We made the correction by the „rooking" method well-known from the operational research (i.e. the delivery of the 'oil' sector was shifted from the housing and medical services to the want of transportation and equal large delivery of the 'chemical' products from the want of transportation to the housing and medical services). Although the resulting matrix (see rows 145-162, where the corrected cells are in bold characters) is not perfect yet for statisticians, for the model in question it can be used without any problem.

Finally from the so computed domestic consumption matrix we had to estimate the matrix of national consumption (rows 169-186.). This we did by the proportionately adjustment of each rows to the new row-totals, which we took from the consumption column of the SAM, but which in turn was determined by modifying the vector of the domestic consumption by the balance of the inbound and outbound tourist's consumption (by the 18 branches). The next section tells how the tourist consumptions were estimated.

6.5. Accounting for tourism

Tourist exports and imports had to be added to the enterprise export and import. This was to be done in the SAM and the model does not distinguish between the two types of foreign trade. The estimated VAT content of the tourist export was allocated to the row of ’indirect taxes’.

Branch breakdown of tourism expenditures and revenues does not exist in the official statistics. We could, however, rely on various surveys and studies (HCSO Survey on Inbound Tourists 1994, Horvath (1999), WTTC (2000)). In this respect the problem is not only the registration of the expenditure pattern of the inbound and outbound tourists, but also we have to get to know where they got the currency from, and whether the goods they bought were registered in the already existing (production, consumption, etc.) statistics. Only by this one could account the tourism consistently. For the time being we could only distribute the spending of officially exchanged amounts (as shown by the I-O table or the balance of payment statistics) to branches.

6.6. The Social Accounting Matrix (SAM)

Finally, we filled the SAM form partly by the above mentioned data, partly by the income distribution data of the national accounts (NA), balance of payments statistics (BP) and the Government Budget Reports (BR). Here some minor methodological problems about the necessary content of certain cells and the placement of several unusual categories still remain. For example, the export subsidies could not be put into the column of the branches since the original (I-O table) figures of the exports already contain them. The related statistical problem is that the Budget Reports (BR) and the NA show different figures for the aggregate export subsidy (perhaps it is only due to the fact that the BR usually publish cash-flows, while the NA follows the „due payment” or accrual approach). Neither source publish any break-down of this subsidy (from the 1998 BR we know that in that year 77 % went to the food industry and 23 % to the agriculture). As a solution, we accounted the export subsidies as a direct transfer from the government to the foreign sector (as a consequence its direct relationship with the export is not visible). Also there are further minor inconsistencies between the data. For example, it is worth mentioning the inconsistency between the ’dividend to the foreign sector’ figures in the National Accounts and the Balance of Payments statistics. Also for the government net interest expenditure and (theoretically meaningful) deficit we could derive somewhat different figures from the BR and the NA.

The greatest problem was the unclear content of various budget transfer payments and their relationship with the NA categories. It was very difficult to make sure that double accounting is not made. Fortunately, we did not have to deal seriously with the capital and investment transfers, since the model will not deal with these explicitly. Interest payments are not explicitly taken into account either. Therefore, the official figures for savings had to be modified only by them to get the current (or primary or operational) savings.

Table 6.1: The content of the income distribution of the 1995 Hungarian SAM (SAM18HU3.XLS file's SAM18 worksheet)

Cell name	Category	Content (in Million HUF) and source (NA=National Accounts, BR=Budget Report)
AA31	Households' other transfer payments	361326 (NA) other transfer outlays (routed through firms)
AA33	Households tax and duty payments	411965 (NA) (of which 383744 was income tax)
AA34	Households social security contribution	894523 (NA)=698879+31007 employer (incl. imputed) +140478 employee+24159 other
AA47	Saving of households net of interest income	NA, 598640 saving-138790 net interest income
AB30	Households' cash and in-kind benefits (routed through gov.!)	NA: 910459 cash benefits+763186 in-kind benefits
AB31	Non-export subsidies not accounted as production subsidies	=BR (1500+7302 rehabil.+433 defunct+5580 guarrantee +20384 extrabudg.funds) -29629 (NA) **+42428 balancing**
AB45	Net foreign transfers of the government	BoP: 13*125,69
AB47	Primary saving of government (incl. inv.transf. by LG,EBF too)	=-361100 Flow of funds data by NBH+AJ23 (investment) +(200100+138790-170450+AC23) net interest (residually) +(79247-37546)+12066+(28679-14141) inv. transfers -2500 coal correction-**17649 balancing**
AD30	Households transfer and property income without interest	NA:485354 transfers+(196726-138790) property income
AD33	Firms' income tax, dividend to state	NA: 92301+22206 income tax of enterprises & banks +10912 dividend
AD45	Dividends,other transfers +discrepancy between the BoP and SAM for. trade acc.	24200+(6700-3500+1500)-(143300+218500-130000 -301400-23300+25400)+(AG24+AG36-AB46-V46)
AD47	Saving of firms & banks, net of interest (incl.reinvested profit)	370196+97547+(99572+26900) reinvested-(AC23-170450) +2500 correction of coal sector's saving
AE47	Saving of the foreign sector net of interest income	=296500 current account deficit -200100 net interest
AF36	Zero (by GEM-E3 model's assumption. Alternatively: Tourist export's VAT content)	may be estimated only
Y30	Households non-labor income of production	279533+720149 (NA), operating surplus+mixed income
Y31	Firms' capital income (enterprises, NPIS, banks respectively)	908765+2015+153609 (NA) op.surplus-16921 net other taxes (rent on mining) (BR)+2500 corr. in coal sector
Y39	Government capital income	248182 (NA), operating surplus of government institutions

The following comments have to be made for the SAM cells presented in Table 6.1:

AA31: An unpublished background worksheet of the HCSO showed that of the NA accounted 361326 million HUF transfers paid by the household 83 per cent (299942 Million HUF) were ‘transfer expenditures in foreign currency’. The rest were insurance (43685) and gambling (8186). Similarly, in

AD30 of the 485354 received transfers of the households 90 per cent (436627) was ‘transfer income in foreign currency’. The rest were insurance (40541) and gambling (8186, apparently accounted as income received from other households).

AA34: the ‘other’ SSC consists of SSC of the self-employed, the voluntary SSC and SSC after people on unemployment benefit, child care leave and sick leave.

AB47: Non-households investment transfers were not published in the NA before 2002. Hence the data are estimated from the Budget Reports.

AD30: Household other property income consisted of dividend (35432), insurance (15132) and (land) rent income (7372).

AD45: Data for the net dividend income of the RoW are rather different in the NA (58535-3665=54870) and in the BoP (24200=‘direct investment income’, where reinvested profits were not included here yet at that time).

AD47: Since in the NA savings are accounted after subtracting reinvested profits, here we had to add them back (since in the transfer cells of the SAM we did not account them for). For the coal correction see Y31.

AE47: In contrary, in the BoP the current account deficit (i.e. the mirror image of the savings of the RoW) was accounted without computing the reinvested profits. Therefore we did not have to add it back to the BoP figure for the savings of the RoW.

AF36: If the SAM displays the domestic consumption, then no VAT should appear here. Even if the SAM displays the residents consumption, and hence the exports include the consumption of inbound tourists, the GEM-E3 model assumes does not distinguishes between the 2 types of the exports (enterprise and tourist), so it assumes that all VAT is accounted for in the column of household consumption. If the SAM’s export data are at user’s prices (as in the NA), VAT on foreign tourists’ consumption remain unseparated by definition. However, if I-O table data are used, which usually at basic prices, the difference between users process and basic prices (incl. export taxes and subsidies) have to be accounted for somewhere in the accounts of the RoW. In any case, in our SAM these differences (6504 tax-54712 subsidy=-48208 net tax, i.e. negative net tax) are added to the basic prices. It however means that exports are not really at uniform (users) prices, since foreign consumers will have to pay the foreign VAT and other consumer taxes when purchasing the products exported by Hungary.

Y31 and AB47 and AD47: The capital income of the coal sector seemed negative in the Hungarian statistics. However, it is impossible that the real cost of capital was negative. Therefore we had to find in the Budget Report that yet unaccounted government transfer which helped the survival of the coal sector. However, note that these bailing out transfers usually take rather hidden forms, ranging from write-offs and preferential interest rates to equity lift-offs.

After filling the above block of the SAM without the residual corrections (indicated by the bold letters in Table 6.1), total savings and total accumulation did not match exactly, but the difference was surprisingly small (at least relative to what one would expect after learning the above mentioned methodological inconsistencies and coverage problems). We concluded that the remaining error certainly must be in the accounts of the government. It is not only because of the above mentioned uncertainty in the interest expenditures, capital transfers and savings, but also because of the incomplete or not guarranteedly correct accounting of certain transfers between the firms and the government. For example, the (although small amount of) transfers of the „non-profit institutions" are not necessarily properly accounted for, since their account in the SNA is also incomplete and the government budget data do not separate them out.

However, by determining some cells as residual (see the residual corrections in Table 6.1 in bold letters), the SAM could be balanced easily. Table 6.2 shows the aggregate balanced SAM. Table 6.3 shows the aggregate balanced SAM converted to Euros and after eliminating the FISIM (as mentioned in section 5.3. and 5.4.1.) We think the resulting SAM can be used by the model well, but if any further corrections should prove to be necessary, the elaborated framework will make it easy to do.

6.7. Bilateral trade matrix

As we have seen in section 6.1 for Hungary import matrices in 21 sectors break-down were officially published for 1995 as a supplement for the I-O table.

Bilateral (country by country) trade matrices had to be compiled for the export and import turnover to and from the European Union (EU 15), the other ECE countries and the 'rest of the world' (inclusive tourist expenditures).

In Hungary, the main problem with the available mentioned published data (Foreign Trade Statistical Yearbook) was that in the case of bilateral trade it contained only the largest shipments, so in many cases the structure of trade with smaller EU- and accession countries could only be estimated. In addition, the published data by country was available only for the 20 main categories (sections) of the HS nomenclature (for custom free zones the country break-down was not available at all). Therefore, we had to split Section 2, 5, 15 and 16 to get the break-down by our 18 industries. A minor problem is that within Section 1-3 there are intermediate goods too, while we assign them to "Consumer goods" exclusively. Similar problems can be mentioned in connection with other sections too.

Trade matrices for merchandise exports and imports have been compiled by converting the foreign trade data given in HS classification into NACE classification, and aggregated into the sector format of the GEM-E3 model.

Table 6.2: The aggregate balanced Hungarian SAM for 1995

(in million Forints, 1 Euro=125,69 HUF)

USERS--> / Branches	Branches TOTAL	*Labour*	*Capital*	*Total*	*Consumption Households*	*Govern.*	*Banks*	*Firms*	*Exports*	*Tourist export*	*Total Export*	*Investments Househ.+ NPIS*	*Private*	*Govern.*	*Total*	*Change in Stocks*	**Total**
TOTAL of branches	5748257	0	0	0	3723955	617700	219248	0	1849552	223495	2073047	281848	678717	164824	1125389	218346	13725942
SSC	729886			0							0						729886
Wages (withoutSSC)	1905116			0					0		0						1905116
Capital	2297832			0							0						2297832
Total Value Added	4932834	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	4932834
Actual Output-Subs	10681091			0							0						10681091
HHS	0	2635002	999682	3634684		1673645	0	543290	0	0	0						5851619
FIRMS	0		1049968	1049968	361326	47998	0				0						1459292
Indirect Taxes (net)	134653			0							0						134653
Direct Taxes	0			0	411965		0	125419	0		0						537384
Social Security	0			0	894523				0		0						894523
Subsidies&mine rent	-63883			0					0		0						-63883
VAT taxes	580257			0	0				*0*	0	0						580257
Duties	249430			0							0						249430
Gov. Foreign	0			0							0						0
Gov. firms	0		248182	248182							0						248182
Total Taxes	900457	0	248182	248182	1306488	0	0	125419	0	0	0	0	0	0	0	0	2580546
Distr. Output	0			0							0						0
EC	1249273			0							0						1249273
NON-EC	762810			0							0						762810
Total Imports	2012083	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	2012083
Expend.abroad	132312			0		*1634*	0	23418			0						157364
Total Imports	2144395	0		0	0	1634	0	23418	0	0	0	0	0	0	0	0	2169447
Savings	0			0	459850	239568	0	547917	96400		96400						1343735
Total Resources	13725943	2635002	2297832	4932834	5851619	2580545	219248	1240044	1945952	223495	2169447	281848	678717	164824	1125389	218346	32063416

Table 6.3: The aggregate balanced Hungarian SAM for 1995 (in Millions of Euro)

				Total	***Consumption***				***Total Exports***	***Investments***			***Total***	*Change in Stocks*	***Total***
	Total	*Labour*	*Capital*	***Total***	*Househ.*	*Govern.*	*FIRMS*	*Exports*		*Househ.*	*Private*	*Govern.*	***Investm.***		
Total	**36 701**	**0**	**0**	**0**	**22 896**	**3 798**	**0**	**12 745**	**12 745**	**1 733**	**4 173**	**1 013**	**6 919**	**1 342**	**84 402**
Wages	**11 713**	0	0	0	0	0	0	0		0	0	0	**0**	0	**11 713**
SSC	**4 487**	0	0	0	0	0	0	0		0	0	0	**0**	0	**4 487**
Capital	**12 779**	0	0	0	0	0	0	0		0	0	0	**0**	0	**12 779**
Total Value Added	**28 980**														**28 980**
Actual Output	**65 681**														**65 681**
HHS	**0**	16 200	6 146	22 347	0	10 290	3 340	0		0	0	0	**0**	0	**35 977**
FIRMS	**0**	0	6 455	6 455	2 222	295	0	0		0	0	0	**0**	0	**8 972**
Indirect Taxes	**828**	0	0	0	0	0	0	0		0	0	0	**0**	0	**828**
Direct Taxes	**0**	0	0	0	2 533	0	771	0		0	0	0	**0**	0	**3 304**
Social Security	**0**	0	0	0	5 510	0	0	0		0	0	0	**0**	0	**5 510**
Subsidies	**-393**	0	0	0	0	0	0	0		0	0	0	**0**	0	**-393**
VAT taxes	**3 568**	0	0	0	0	0	0	0		0	0	0	**0**	0	**3 568**
Duties	**1 535**	0	0	0	0	0	0	0		0	0	0	**0**	0	**1 535**
Gov. Foreign	**0**	0	0	0	0	0	0	0		0	0	0	**0**	0	**0**
Gov. firms	**0**	0	1 526	1 526	0	0	0	0		0	0	0	**0**	0	**1 526**
Total Taxes	**5 537**	**0**	**1 526**	**1 526**	**8 043**	**0**	**771**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**15 877**
Distr. Output	**65 681**			0											**65 681**
Imports	**13 184**	0	0	0	0	10	144	0							**13 338**
Total Imports	**13 184**	**0**	**0**	**0**	**0**	**10**	**144**	**0**	**0**						**13 338**
SAVINGS	**0**	0	0	0	2 827	1 473	3 369	593		0	0	0	**0**	0	**7 669**
Total Resources	**84 403**	**16 200**	**14 127**	**30 328**	**35 987**	**15 866**	**7 624**	**13 338**	**12 745**	**1 733**	**4 173**	**1 013**	**6 919**	**1 342**	

It is worth mentioning, that in spite of these problems, the resulting data (see in the file **Trade95.xls**) are fairly consistent with what we could have got, provided we had access to the more detailed (electronic) trade statistics (like the Polish partners had) in time. Later obtained data (in 2 and 4 digit code breakdown of the HS nomenclature and in branch and sub-sector breakdown of the origin of the products) converted to the GEM-E3 model's 18 sectors break-down (see the **TradAgAc.Xls, TradSzag.Xls** files for the exports to the accession countries, the **TrSzagEU.Xls** file for Hungary's trade with the EU15, and the EU15's exports to Hungary as registered in the Comext database and shown in the Tradenis.xls file) show a satisfactory coincidence with what we estimated from incomplete (in the case of small countries fragmentary) data. The exceptions are Hungary's export to Austria, Greece, Great Britain and Spain, and Hungary's import from Austria, Denmark, Finland, and Sweden, where in at least 3 branches the ratio of the two corresponding data (coming from the different estimations) is close to or above 2. On the aggregate level (which means that in quite a few countries) we observe that trade data in branch of origin break-down resulted in significantly lower estimates for the GEM-E3 sectors of Agriculture, Coal, Metal products, and Electrical goods than estimates based on the HS-code commodity break-down dataset. On the other hand, Chemical industry, Energy intensive industries and Equipments the former method's estimates were lower than those of the latter.

In any case, in the following step we had to adjust the trade data (already in the 18 sector break-down) to the I-O table's (also already aggregated to the 18 sector break-down) corresponding (export or import) trade figures. However, since the value of exports in the national accounts and the input-output tables include *export subsidies* as well (they are at basic prices), while the trade statistics data are essentially at users prices (contract prices, usually on fob parity), they had to be separated from the I/O table figures and displayed among the subsidies. Export is measured at *f.o.b. prices*, whereas import at *c.i.f. prices*[21].

The adjustment eliminated not only the above mentioned (mainly aggregation) errors, but also eliminated those discrepancies which were due to the different methodology of the I-O table and the trade statistics (as discussed earlier, in relation with the processed materials and the like). However, in the SAM tourist exports are also added to the total exports, although we do not have any information of its country break-down. In this case we can assume that the country-structure of the tourist spending is the same as that of the company conducted foreign trade. If this again causes assymmetricity in the estimated trade flows between countries, it can be eliminated either pair-wise or only at the aggregate level of the EU or the world market.

Data for foreign trade of *services* (as found in the I-O table) is usually not available in country break-down either, and even its industry break-down information is not very reliable (usually they are rather different from the Foreign Balance of Payments statistics, which, by the way, has a different break-down of the service trade). In any case, the GEM-E3 model has a subroutine which deals with the balancing of this service trade on the aggregate EU- and world market level.

[21] Note the calibration module of the GEM-E3 estimates the difference between the export and import prices as linked export of transportation and other services.

To estimate the *custom duty* data, one could use various sources on foreign trade data and official tariff rates. Based on the above information duty matrices have been estimated, making use of proportionality assumptions (within the blocks of the EU and ECE countries, as well as commodity groups). For Slovenia there was more detailed information available so that it produced more differentiated duty rates than the other countries. It is interesting to note that the Hungarian custom policy treated Poland and Slovenia almost the same way as the EU countries.

6.8. Energy balance sheets

The GEM-E3 model requires rather detailed and EU-comfort data on emission. For the EU countries these are computed from appropriately designed (based on the Eurostat methodology) energy-balance-sheets. Some sorts of energy balance sheets are available in the accession candidate countries, as well, their format is, however, usually different from both the OECD and the EUROSTAT energy balance formats. The estimation of EU-comfort energy balance sheets required reclassifications (energy types, types of plants, activities or purposes of energy final demand, etc.), imputations (blast furnace gas, coke-oven), standardizations (statistical and distributional losses, auxiliary use of transformation, etc.), and proportional adjustments, which meant much time and effort.

When the energy matrix is compiled directly from national energy statistics, the following questions have to be asked:

Brown and hard coal are separated?

What types of power plants are distinguished?

Heat and electricity generation are separated (input and outputs)?

How the coke-oven and blast furnace inputs and outputs treated (conversion!)?

Within the uses: technology, transportation, material (feedstock), conversion (heat generation) uses are separated?

How the own consumption, auxiliary consumption, distribution losses are treated?

Are there any fictive (statistical) losses accounted in the electricity data?

Is the industry classification in conformity with the GEM-E3?

Although we managed to get a copy of the official 1995 Energy balance Sheet, its format was different from both the OECD and The Eurostat energy balance formats. The necessary reclassifications, standardizations and proportional adjustments required much time and effort. A notable problem was the aggregate accounting of lignite and (brown and hard) coals, although their separation would have been necessary for a reasonable estimate of environmental (air) pollution. The resulting energy balance sheet can be found in the **Enbal_hu.xls** file. Further details can be seen in the cell-comments of this file.

6.9. Emission data

We could not identify any estimates related to emission in the format and detail required by the model. Therefore we had to use an indirect method (using energy consumption specific emission coefficients) to estimate them based on the energy balances. The method (which is executed by a specific GAMS program which was developed for that purpose) is described in detail in the GEM-E3 manual (Capros et al.(1997)).

In the case of Hungary we could obtain an emission matrix prepared for 1994, which had a 25 branches break-down. To transform the data into the required 18 branches break-down one had to split some of the branches. This turned out to be a rather sensitive operation, which resulted in some places in odd coefficients (see the results in the **hunemi95.xls** file). Nevertheless that made it possible to estimate emission coefficients in two ways for Hungary. That made it possible to test the results of the above mentioned GAMS estimates.

It was found that the emission coefficients estimated by the two different methods are, in many respects, sufficiently close to each other. (Although, apparently mainly due to the aggregated accounting of different types of coal, the GAMS estimate – which was based on average EU coefficients for SO2 – resulted in only one third of the expected figure.) Comparing the results obtained for the three countries and Austria (as a reference EU country) gave us enough confidence in the reliability of the applied method.

For many countries direct estimate of the emission matrix can be based on the the Greenhouse Gas (GHG) inventory information published by the so-called National Communications. These documents are regularly published by countries that joined the UN FCCC (Unites Nations' Framework Convention on Climate Change). The Communications contains the GHG inventories in a unified format, and the emissions are calculated uniformly by the methodology developed by the IPCC (Intergovernmental Panel on Climate Change). All the EU countries has already joined the UN FCCC, therefore our spreadsheet model can be used in each EU-member country. This seems to be more expedient than to use different data sources and/or different methods to develop the emission module of the GEM-E3.

The 'SAM18HUN.xls' file illustrates this process for 2005 for Hungary (see also in the **Emission-Hu05.doc** report). Its 'Emission' sheet contains a spreadsheet model, which in turn is based on another spreadsheet model, which was developed for gathering all the necessary energy data (see the 'Energy' sheet and the **Energy-Hu05.doc** report). Both the energy and the emission data are gathered and processed in the respective worksheets of the '**SAM18HUN.xls**' file.

7. Implementation of the GEM-E3 model

Since the GAMS software is not well-known in Central-Eastern Europe and since the GEM-E3 model uses only part of it, in this section we summarized the structure of the GAMS program of such CGE models, the main characteristics of their solver modules and a list of the most important syntactic rules. To make all this information more practical an example program is given with explanatory comments. Finally the GAMS-Excel interface is presented, which shows how one can read in data from Excel format and how one can put the results into an Excel sheet. This is also illustrated by an example, which is elaborated for a price-model for Austria.

7.1. The GAMS software

The www.gams.com website contains the documentation and the system files of the GAMS package. The GAMS is a rather efficient and model-builder friendly software to handle and solve large nonlinear models with 'well-behaving' (twice differentiable, etc.) functions in its equations. The latest version of the GEM-E3 model is involves a system of about 60,000 non-linear equations per time period. The GEM-E3 model has been successfully transformed as a mixed complementarity model and solved in GAMS using the PATH solver. Previous attempts to solve the model in other solution algorithms (as with MINOS and CONOPT) have been unsuccessful mainly due to the model's large size and complexity.

The PATH solver on the other hand, has been successful in solving very large scale models and through the complemenetarity approach that it uses, enables the expansion of GEM-E3 to include inequalities and a separate optimisation energy sub-module.

The general structure of the GAMS programs are the following (following the logic of mathematitians):

Inputs	Outputs
- `Sets (SET)` - Data (`Parameters, Tables, Scalar`) - `Variables` - Assignment of bounds and/or initial values (optional) - `Equations` - `Model` and `Solve` statements - Algorithm selection (for NLP: MINOS, CONOPT, for system of equations: PATH) - `Display` statement (optional)	- Echo Print - Reference Maps - Equation Listings - Status Reports - Results

An example for GAMS programs (transportation problem):

```
Sets
  i canning plants / seattle, san-diego /
  j markets / new-york, chicago, topeka / ;
Parameters
  a(i) capacity of plant i in cases
```

```
    / seattle 350
      san-diego 600 /
  b(j) demand at market j in cases
    / new-york 325
      chicago 300
      topeka 275 / ;
Table d(i,j) distance in thousands of miles
       new-york chicago topeka
  seattle       2.5     1.7    1.8
  san-diego     2.5     1.8    1.4 ;
Scalar f freight in dollars per case per thousand miles /90/ ;
Parameter c(i,j) transport cost in thousands of dollars per case ;
     c(i,j) = f * d(i,j) / 1000 ;
Variables
  x(i,j)         shipment quantities in cases
  z    total transportation costs in thousands of dollars ;
Equations
  cost             define objective function
  supply(i)        observe supply limit at plant i
  demand(j)        satisfy demand at market j ;
cost ..            z =e= sum((i,j), c(i,j)*x(i,j)) ;
supply(i) ..       sum(j, x(i,j)) =l= a(i) ;
demand(j) ..       sum(i, x(i,j)) =g= b(j) ;
Model transport /all/ ;
Solve transport using lp minimizing z ;
Display x.l, x.m ;
```

The most important syntactic rules of the GAMS are the following:

Notation	Meaning
$include filename	The text of ’filename’ file is inserted there
$(….)	Conditional statement, executed only if the statement in the bracket is true (or if the value of the expression is positive)
LOOP	Beginning a loop cycle
..	Declaration of an equation (instead of the ’:’ notation)
sum(i	Summation over the set i
ord(i)	(serial) Index of the elemenst of the set i
=e=	Required equality (in equation definitions)
=l=	Required less than equal relationship
=g=	Required greater than equal relationship
Display x.l	Writes the value of variable x to the list (*.lst) file

`Solve transport using lp minimizing z`	Solves the model (set of equations listed within / / brackets in the declaration of the model) called 'transport' by minimizing the value of the variable z using the selected (or default) LP solver

The next section and the referred files contain further comments on specific features (syntax) of the GAMS language.

7.2. Reading in the data from CSV files and Excel tables

7.2.1. Import data from Excel to GAMS

Overview

Usually databases are given in Microsoft Excel spreadsheet. A tool what convert spreadsheet data from Excel to GAMS could be very useful. The GAMS software has a small add-on for this job, it is called XLS2GMS.

When running the executable XLS2GMS.EXE without command line parameters the tool will run interactively with a built-in GUI interface. Alternatively XLS2GMS can be run in batch mode which is useful when calling it directly from a GAMS model using the $call command.

The process is very simple. First of all, XLS2GMS reads data from the spreadsheet as a text. Than the text is exported to a GAMS include file (*.inc). Which can be read by GAMS using the $include command.

Pros:

- Easily Import data from Excel to GAMS. GUI and batch mode.
- When *some* changes occur in the database, it's not necessary to rewrite the GMAS code; it's enough to reload the Excel file.

Cons:

- Sometime it is needed to edit the Excel file before using XLS2GMS.
- Row vectors can't be imported directly, it's necessary to make some modifications. (See below).
-

Interactive use

When the tool is called without command line parameters, it will startup interactively. Using it this way, one can specify the spreadsheet file (.XLS file), the range and the final destination file (a GAMS include file) using the built-in interactive environment.

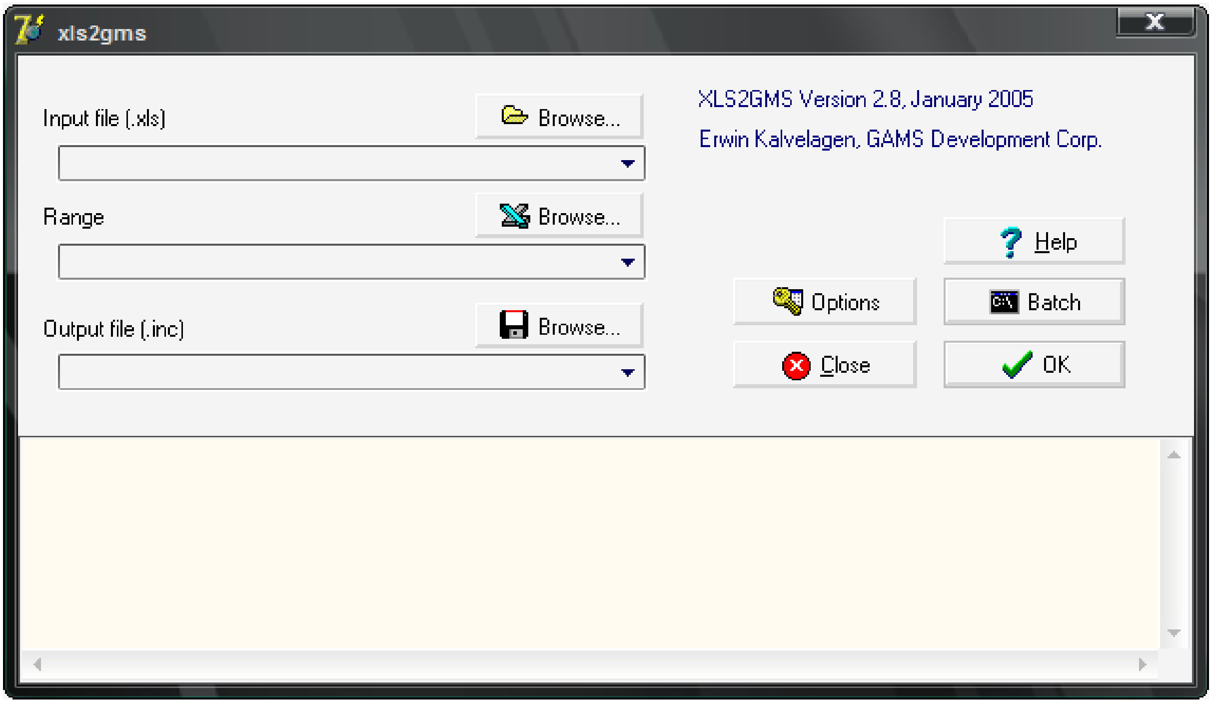
Input file (*.XLS). This is the combo box to specify the input file.

Range. The range can be a single cell (e.g. A1), a block (e.g. B2:J23), or a region within a sheet (e.g. Sheet1!A1:C10). The range can also be a name if the spreadsheet contains named ranges.

Output GAMS Include file (*.INC). This is the combo box to specify the location of the input file.

If the OK button is pressed the query will be executed and an include file will be generated.

Pressing the batch button will give information on how the current extract command can be executed directly from GAMS in a batch environment. The batch call will be displayed and can be copied onto the clipboard.



Command line

When calling XLS2GMS directly from GAMS we want to specify all command and options directly from the command line or from a command file.

This is the general batch call for xls2gmas:

```
$call =xls2gms r=<range> s=”,” i=<input_file_neve>
```

s

that have blanks in them will be quoted.

Let’s see an example. Our task is to import this table above from an Excel spreadsheet to a GAMS model.

	A	B	C	D
1		Household	Public	
2	Industry	1026	582	
3	Other	506	790	
4				
5				

The location of the input file is: C:\excel\test.xls

According to the GAMS programming structure, we need to declare a SET for the Users (Household and Public) and one for the Sectors (Industry and Other). The table name will be Table1.

This process is too slow if we want to import more than a few data set. Every time xls2gms will be executed and the excel file will be opened and closed. We can use a command file to make this faster.

In the first part, we create a simple text file using the echo command. After it we can use a command line to run xls2gms with specified parameters.

Comments and tips

XLS2GMS has some limits. If the table - to which we want to import - has a label in the upper-left corner, GAMS will not recognize it, so we need to delete it or create another table with clear upper-left corner.

	A	B	C	D
1	~~Table 1~~	Household	Public	
2	Industry	1026	582	
3	Other	506	790	
4				
5				

When we want to import a row vector data set (as a Parameter) which is stored like this:

	A	B	C	D	E	F
1	Basic industry	Processing indrustry	Agriculture and forestry	Other material industries	Services	
2	9714	3715	4246	20939	27064	
3						
4						

We cannot do it by one simple command. We need to insert a column to our spreadsheet and supplement the data set to a table.

	A	B	C	D	E	F
1		Basic industry	Processing indrustry	Agriculture and forestry	Other material industries	Services
2	Temp	9714	3715	4246	20939	27064
3						
4						

Than we can import it to GAMS as a table, and convert it into parameters, like this:

After a GAMS model was successfully solved, sometimes it is necessary to put the results (back) to the Excel spreadsheet. *Exporting data to an Excel file could be also useful if there is a better option in Excel to reach our goals than we have it in GAMS.*

For example, during the GAMS process we need to invert a matrix. We can simply export the matrix to a spreadsheet (which, of course has the necessary functions). And at the next line, we can import the results from the Excel file by using xls2gms.

Manual

The first step is to unload data from GAMS to an exchange file (*.gdx: Gams Data eXchange file). After that, we can put them into the Excel spreadsheet:

Where execute_unload and execute are GAMS commands. Price is the name of the variable (L=Level, see GAMS section for details). O=Output file name, var means variable (if we want to export a parameter or a table, we should write par=… or table=…). Rng is for the Excel range.

Exercise 1 - Austria

Based on a 5-sectors 'B-type' (i.e. in which the import is displayed separately as the n+1. commodity) Input-Output table for Austria, such price system and rate of return to capital has to be determined, which satisfies the following macroeconomic criteria:

- The price system is a so-called 'production'-price-system , i.e. in which the surplus is generated proportionately to the capital.
- The domestic currency is devaluated by 2 per cent in real terms (the basket of the foreign currencies appreciate 2 per cent relative to the consumer price index).
- Real-wages fall by 5 %.
- The revaluation rate for the fixed capital and the amortization lags behind the investment price index by 10 percentage points.
- The consumer price index is unchanged ($CPI = 1$).

The basic price model

First of all we need to build a formal comprehensive price model:

Sectoral basic price index: $\mathbf{p}^{\mathrm{h}} = \mathbf{p}^{\mathrm{h}}\mathbf{A} + p_{\mathrm{m}}\cdot\mathbf{a}^{\mathrm{m}} + p_{\mathrm{c}}\cdot\mathbf{c}^{\mathrm{w}} + p_{\mathrm{k}}\cdot\mathbf{c}^{\mathrm{a}} + \mathbf{c}^{\pi}$

Consumption price index: $p_{\mathrm{c}} = \mathbf{p}^{\mathrm{h}}\mathbf{c}^{\mathrm{h}} + p_{\mathrm{m}}\cdot c_{\mathrm{m}}$

Import price index: $p_{\mathrm{m}} = \mathbf{p}^{\mathrm{h}}\mathbf{s}^{\mathrm{z}} + p_{\mathrm{m}}\cdot c_{\mathrm{re}}$

Investment price index: $p_{\mathrm{b}} = \mathbf{p}^{\mathrm{h}}\mathbf{b}^{\mathrm{h}} + p_{\mathrm{m}}\cdot b_{\mathrm{m}}$

Capital price index: $p_{\mathrm{k}} = \mathbf{pb}$

Return to Capital: $\mathbf{c}^{\pi} = \pi\cdot p_{\mathrm{k}}\cdot\mathbf{k}$

Price normalization rule: $p_{\mathrm{c}} = 1$

where : $\mathbf{p}^{\mathrm{h}}$, $\mathbf{c}^{\pi}$ are vector variables, p_{c}, p_{m}, p_{b}, p_{k} and π are scalar variables, and the rest of the letter notations refer to various parameters (e.g., $\mathbf{k}$ is the vector of sectoral capital/output ratios).

The final price model

To solve the given problem we need to expand our previous model.

- The cost of the import depends on the real exchange rate, which will be denoted by α (In the base $\alpha = 1$). It is defined as the ratio of the import price index and the domestic price index of the export basket (assuming the foreign trade prices in foreign currency do not change)
- The cost of the sectoral wages depends on the real wages. We denote the real wage index by β. (In the base $\beta = 1$)
- The price index of the capital was pegged to the price index of the investments, but there can be a difference between these two indexes. Their ratio will be denoted by γ. (In the base $\gamma = 1$)

So, our final model will be this:

Sectoral basic price index: $\mathbf{p}^{\mathrm{h}} = \mathbf{p}^{\mathrm{h}}\mathbf{A} + \alpha\cdot p_{\mathrm{m}}\cdot\mathbf{a}^{\mathrm{m}} + \beta\cdot p_{\mathrm{c}}\cdot\mathbf{c}^{\mathrm{w}} + p_{\mathrm{k}}\cdot\mathbf{c}^{\mathrm{a}} + \mathbf{c}^{\pi}$

Domestic price index of the export basket: $p_{\mathrm{m}} = \mathbf{p}^{\mathrm{h}}\mathbf{s}^{\mathrm{z}} + p_{\mathrm{m}}\cdot c_{\mathrm{re}}$

Consumption price index:	$p_c = \mathbf{p}^h\mathbf{c}^h + \alpha \cdot p_m \cdot c_m$
Investment price index:	$p_b = \mathbf{p}^h\mathbf{b}^h + \alpha \cdot p_m \cdot b_m$
Capital price index:	$p_k = \gamma \cdot p_b$
Unit Return to Capital (per unit of output):	$c^\pi = \pi \cdot p_k \cdot \mathbf{k}$
Price normalization rule ('numeraire'):	$p_c = 1$

According to the macroeconomic scenario:

- The real exchange rate of the foreign currency increases by 2 % => $\alpha = 1{,}02$
- The real wages were decreased by 5 % => $\beta = 0{,}95$
- Revaluation rate of the capital lags behind the investment price index by 10 per cent => $\gamma = 0{,}9$

Model Summary for GAMS programing

Parameters (Exogenous Variables):

A:	I/O Coefficients	Matrix
a^m:	Import Structure	Row Vector
c^w:	Wage Coefficients	Row Vector
c^a:	Amortization Coefficients	Row Vector
s^z:	Export Structure	Column Vector
c_{re}:	Re-export	Scalar
c^h:	Structure of the Household Consumption	Column Vector
c_m:	Import intensity of Household demand	Scalar
b^h:	Domestic Coefficients of Investment	Column Vector
b_m:	Import intensity of Investment	Scalar
k:	Capital/Output Ratios	Row Vector

Endogenous Variables:

p^h:	Sectoral basic price index	Row Vector	5
π:	Rate of return	Scalar	1
p_k:	Capital price index	Scalar	1
p_b:	Investment price index	Scalar	1
p_c:	Consumption price index	Scalar	1
p_m:	Import (Export) price index	Scalar	1
c^π:	Return to capital	Row Vector	5
		Total:	15 variables

Equations:

Equation for Sectoral basic price index	5
Equation for Export price index	1
Equation for Consumption price index	1
Equation for Capital price index	1
Equation for Investment price index:	1
Equation for Unit Return to capital	5
Equation for Price normalization	1
Total:	15 equations

Solve the Model – GAMS

By using XLS2GMS we import data from the Excel spreadsheet (data.xls). After the necessary declarations, we include the imported·.inc files to the GAMS program. According to the Model Summary section, we can build our model in GAMS. It is not a typical Non-Linear Problem (NLP), but we can simply convert it to an NLP. We need to add a fictious variable and a fictious equation to minimize:

Fictious equation: $\text{diff} = (\mathbf{p}^{h} - (\mathbf{p}^{h}\mathbf{A} + \alpha \cdot p_{m} \cdot \mathbf{a}^{m} + \beta \cdot p_{c} \cdot \mathbf{c}^{w} + p_{k} \cdot \mathbf{c}^{a} + \mathbf{c}^{\pi}))^{2}$

The minimum value for *diff* must be 0. Now we can solve our model by using GAMS' NLP solver.

The GAMS results:

```
----    130 VARIABLE Ph.L

            Basic industry            1.037

            Processing industry       0.947

            Agriculture and forestry  0.759

            Other material industries 1.466

            Services                  0.906

----    130 VARIABLE pm.L      =      1.026

            VARIABLE pb.L      =      1.015

            VARIABLE pc.L      =      1.000

            VARIABLE pi.L      =      0.283

            VARIABLE diff.L    =      0.000
```

One can see that the resulting rate of return (pi.L) is 28.3 %. It is reasonable, since for the labour we did not prescribe any rate of returns, which could cover the employers' social security contribution (not accounted in the gross wages). For other countries, in the next section the results are presented also in Excel format.

Exercise 2 – All EU Countries

Our task is the same as it was in Exercise 1. The only difference is that now, we need to answer the question for all countries. The Capital Coefficients, the changes of the real exchange rate (α) and the real wages (β), and the revaluation rate (γ) can be modified by editing the Excel file.

For the solution see data2.xls! It is an example Excel file, that runs the GAMS program by VBA macro and after the GAMS program executed successfully, it exports the results to the Excel file. The country can be chosen from the combo box.

Data folder: C:\Documents and Settings\Markell\My Documents\gamsdir\
New location: ...
Change data folder
Clear results table
Execute GAMS file
Language: English
Country: Austria

Austria

Changes	New value
Real exc. rate	**102%**
Real wages	**95%**
Capital - investment	**90%**

	Basic industry	Processing industry	Agriculture and forestry	Other material industries	Services
Capital coefficient	**0.8753**	**0.4200**	**0.3768**	**2.2669**	**0.9738**
Rate of amortization	5%	6%	5%	4%	3%

Results	Basic industry price index:	Processing industry price index:	Agriculture and forestry price index:	Other material industries price index:	Services price index:	Private consumption price index:	Investments price index:	Import (Export) price index:	Rate of return:
Austria	1.0368	0.9466	0.7591	1.4661	0.9063	1	1.0149	1.0258	28.310%

	Basic industry	Processing industry	Agriculture and forestry	Other material industries	Services	Private consumption	Public consumption	Investments	Export	Change in stocks	Total
Basic industry	14563	3512	1534	4066	5772	5312	0	2940	10004	-686	47018
Processing industry	993	4333	791	1278	2017	7774	0	1376	13022	759	32344
Agriculture and fore	320	1893	1989	157	1155	6156	0	873	4794	-191	17144
Other material indust	1077	607	291	4277	6434	9898	0	5908	8130	1750	38371
Services	3612	3558	1539	4484	23194	36048	36512	22802	11558	-382	142925
Import	5222	5985	2480	3655	7020	18134	250	6008	20865	378	69997
Wages	10138	7158	2930	11453	45426						77105
Amortization	2058	815	323	3479	4176						10851
Operational Surplus	9039	4480	5266	5519	47731						72036
Total	47022	32341	17143	38368	142924	83321	36762	39908	68373	1628	507790

Yo u can also see the results for all countries in the data.xls file:

Data folder: C:\Documents and Settings\Markell\My Documents\gamsdir\
New location: ...
Reload data file
English
Clear results table
Run GAMS

Changes	New value
Real exc. rate	**102%**
Real wages	**95%**
Capital - investment	**90%**

	Basic industry	Processing industry	Agriculture and forestry	Other material industries	Services
Capital coefficient	0.8753	0.4200	0.3768	2.2669	0.9738
Rate of amortizatior	5%	6%	5%	4%	3%

	Results	Basic industry price index:	Processing industry price index:	Agriculture and forestry price index:	Other material industries price index:	Services price index:	Private consumption price index:	Investments price index:	Import (Export) price index:	Rate of return:
HU	Hungary	1.1459	0.9735	0.8855	1.5067	0.9561	1	1.2218	1.0200	26.326%
PA	Poland	1.2821	1.0660	0.9194	1.6346	0.8214	1	1.3542	1.1923	25.054%
SL	Slovenia	1.1198	1.0113	0.8922	1.4319	0.9601	1	1.2114	1.0537	19.040%
AU	Austria	1.0368	0.9466	0.7591	1.4661	0.9063	1	1.0149	1.0258	28.310%
BE	Belgium	0.8596	0.8379	0.7191	1.4136	0.9411	1	1.0024	0.8698	25.520%
DE	Germany	1.0459	0.9536	0.8793	1.4192	0.8399	1	0.9687	0.9854	28.222%
FR	France	1.1032	0.9868	0.9180	1.4709	0.9868	1	1.2536	1.0097	22.873%
FI	Finland	1.0125	0.8889	0.6432	1.4177	0.9582	1	1.0109	0.9646	28.546%
IR	Ireland	0.9065	0.7558	0.8194	1.5862	1.1165	1	1.0482	0.8686	38.015%
IT	Italy	1.0601	0.9777	0.7967	1.6571	0.9040	1	1.0490	1.0152	35.464%
NL	Netherlands	0.9705	0.9438	0.8125	1.4987	0.9381	1	1.0593	0.9744	25.772%
PL	Portugal	0.9417	0.8995	0.7299	1.4416	0.9770	1	1.0648	0.9395	28.716%
SP	Spain	1.0421	0.9847	0.8491	1.6020	0.9682	1	1.3345	1.0636	25.617%
SW	Sweden	0.9345	0.9778	0.7528	1.4590	0.9438	1	1.0389	0.9760	27.373%
UK	United Kingdom	1.0130	0.9320	0.8463	1.4824	0.9885	1	1.1978	0.9896	20.831%
DK	Denmark	1.0500	0.9681	0.8359	1.4686	0.9719	1	1.2106	1.0486	22.675%
GR	Greece	1.1918	0.9663	0.7731	1.7346	0.9722	1	1.4758	1.0707	31.907%

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APPENDIX 1: THE I-O TABLE IN GEM-E3 NOMENCLATURE

The aggregation of Nace-Clio R59 has been used for the initial Input-Output table. For some of the countries where this aggregation was not available, the Nace-Clio R25 aggregation was used instead. The procedure that has been followed to convert the table of 59 branches into the specific one of 18 branches for the GEM-E3 model, is showed in the following structure:

A1. Table: Codes and names of the Nace-Clio R59 branches

010 Agricultural, forestry & fishery products

031 Coal & coal Briquettes

033 Lignite & lignite briquettes

050 Products of coking

071 Crude petroleum

073 Refined petroleum products

075 Natural gas

095 Water (collection, purification, distribution)

097 Electric power

098 Manufactured gases

099 Steam, hot water, compressed air

110 Nuclear fuels

135 Iron ore & ECSC iron & steel products

136 Non-ECSC iron & steel products

137 Non-ferrous metal ores,non-ferrous metals

151 Cement, lime, plaster

153 Glass

155 Earthenware & ceramic products

157 Other minerals & derived products

170 Chemical products

190 Metal products

210 Agricultural & industrial machinery

230 Office machines

250 Electrical goods

270 Motor vehicles & engines

290 Other transport equipment

310 Meat & meat products

330 Milk & dairy products

350 Other food products

370 Beverages

390 Tobacco products

410 Textiles & clothing

430 Leather & footwear

450 Timber & wooden furniture

471 Pulp, paper, board

473 Paper goods, products of printing

490 Rubber & plastic products

510 Other manufacturing products

530 Building & civil engineering works

550 Recovery & repair services

570 Wholesale & retail trade

590 Lodging & catering services

611 Railway transport services

613 Road & transport services

617 Inland waterway services

631 Maritime & coastal transport services

633 Air transport services

650 Auxiliary transport services

670 Communications

690 Credit & insurance

710 Business services provided to enterprises

730 Renting of immovable goods

750 Market services of education & research

770 Market services of health

790 Market services n.e.c.

810 General public services

850 Non-market services of education & research

890 Non-market services of health

930 Non-market services n.e.c.

APPENDIX 2: CORRESPONDENCE BETWEEN NACE AND EXTERNAL TRADE PRODUCT CODE

GEM code	NACE code	External Trade code
1	01 Agriculture, forestry and fishery products	Section 1: life animals & animal products Section 2: Vegetable products, chapters: 6+7+8+9+10
	06 Fuel and Power Products	Section 5: Mineral products, chapter 27
2	Lignite, coal & coke	2701+2702+2704+2708
3	Oil products	2709+2710+2712+2714+?271119
4	Gas	2711-?271119
5	Electricity	
6	13 Ferrous & non ferrous ores & metals	Section 5, chapter 26 Section15, Chapters 72+73+74+75+76+78+79+80+81
7	17 Chemical products	Section 6
8	15 Non metallic mineral products	Section 5, chapter 25 Section 13
	47 Paper and printing products	Section 10
	19 Metal products exc. machines & transp	Section15, chapters 82+83
9	25 Electrical goods	Section 16, chapter 85
10	28 Transport equipment	Section 17
11	21 Agricultural & industrial machinery	Section 16, chapter 84 (minus the one in23)
	23 Office & data processing machines	Chapter 84 (8451+8452+8453+8454+8455)
12	36 Food, beverages, tobacco	Section 2, chapters 11+12+13+14 Section 3, Section 4
	42 Textile & clothing	Section 8, Section 11, Section 12 chapter 64+65
	48 Other manufacturing prod	Section 9, Section 12 chapter 66+67, section 14 Section 18, Section 19, Section 20, Section 21
	49 Rubber & plastics	Section 7

APPENDIX 3: THE DERIVATION OF SYMMETRIC I-O TABLES

For GEM-E3 the construction of a symmetric input-output table is required. The symmetric IO table is a product-by-product or industry-by-industry matrix describing the domestic production processes and the transactions in products of the national economy in detail. In the following, we follow the methodology from Holub and Schnabl (1994) to construct a symmetric input-output table.

The conversion of the make and use matrices into the square input-output matrix hinges on two types of technology assumptions:

1. industry technology, assuming that all products in a product group produced in a branch are produced with the same input structure;
2. product technology, assuming that all products in a product group have the same input structure, whichever industry produces them.

Given the IO matrix:

	commodities	activities	final demand	sum
commodities		use matrix U	final demand matrix Y	use (q)
Activities	make matrix V			production profits (g)
imports	imports M			
value added		value added matrix W		
Sum	costs (q´)	costs of the production (g´)		

We define three matrices :

$B = U * < g >^{-1}$ Matrix of the IO coefficients with dimension commodities x industries. The use matrix is divided by the column sum.

$C = V' * < g >^{-1}$ Product-mix matrix with dimension goods x industries. It shows the shares of the particular commodity on the overall output for each industries. The column sum is accordingly equal to 1.

$D = V * < (q - m) >^{-1}$ Market-shares-matrix with dimension industries x commodities. It shows the production share (market share) for each good of the particular industry from domestic production. The sum of the column items is equal to 1.

Activity technology: All commodities (characteristic as well as non-characteristic) produced from one industry are being produced with the same input structure i.e. each activity uses a particular technology independent from which commodities are being produced. IO coefficients are given as weighted averages of all intermediate structures, which produce the respective commodities. The market shares are used as weights.

The square input output matrices in the commodity *x* commodity or industry *x* industry dimension are given by the following formulas.

$$A_{I,CxC} = B * D = U * < g >^{-1} * V * < (q - m) >^{-1}$$

$$A_{I,AxA} = D * B = V * < (q - m) >^{-1} * U * < g >^{-1}$$

The process is illustrated by the MakeUse5.xls file.

APPENDIX 4: THE CONSUMPTION MATRIX OF GREECE (IN M. ECU 1995)

Purpose[22]. / Product	Beverages and	Housing and Water	Fuels and Power	Housing Furniture and	Heating and Cooking Appliances	Medical Care and Health	Transport Equipment	on of Transport	Purchased Transport	Telec. services	Recreation, Ent, Culture etc	Other Services	Housing and Water	Total
Agriculture	2194	0	0	60	0	0	0	0	0	0	0	0	0	*2254*
Coal	0	0	0	0	0	0	0	0	0	0	0	0	0	*0*
Oil	0	0	0	627	0	0	0	0	1349	0	0	0	0	*1976*
Natural Gas	0	0	0	60	0	0	0	0	0	0	0	0	0	*60*
Electricity	0	0	211	952	0	0	0	0	0	0	0	0	0	*1162*
Ferrous & Non-Ferrous Metals	0	0	0	0	0	0	0	0	0	0	0	0	0	*0*
Chemical Products	0	0	42	0	379	0	376	0	0	0	0	192	680	*1669*
Other Energy-Intensive Industries	0	0	0	0	585	35	0	0	0	0	0	577	259	*1457*
Electrical Goods	0	0	3	0	10	78	0	0	0	0	0	196	0	*287*
Transport Equipment	0	0	0	0	0	0	3	877	0	0	0	18	0	*898*
Other Equipment Goods Industries	0	0	0	0	2	301	34	0	0	0	0	94	0	*431*
Consumer Goods Industries	9657	4499	44	0	1488	0	0	0	40	0	0	166	341	*16235*
Building and Construction	0	0	387	0	0	0	0	0	0	0	0	0	0	*387*
Telecommunication Services	0	0	0	0	0	0	0	0	0	0	980	0	0	*980*
Transports	0	0	0	0	0	0	0	0	0	1847	0	0	114	*1961*
Services Of Credit And Insurance	0	0	0	0	0	0	0	0	189	0	0	0	521	*710*
Other Market Services	0	0	9976	0	351	19	364	0	114	100	0	307	17150	*35638*

Source: GEM-E3 Database.

						6	5		5			5		
Non-Market Services	0	0	0	0	0	0	0	0	0	0	0	0	335	335
Total	11851	4499	10663	1699	2816	610	4057	877	2723	1947	980	4318	19401	66441

APPENDIX 5: THE UNITED KINGDOM INVESTMENT MATRIX (IN M. ECU 1995)

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	Total
Agriculture	498	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	498
Coal	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Oil	0	0	1179	131	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1310
Natural Gas	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Electricity	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Ferrous & Non-Ferrous Metals	50	54	1029	122	463	43	78	239	67	469	75	112	519	116	263	607	530	511	5346
Chemical Products	0	0	0	0	0	0	12	0	0	0	0	0	4	0	0	0	0	0	16
Other Energy-Intensive Industries	0	0	0	0	0	0	0	227	0	0	0	0	381	0	0	0	5	0	613
Electrical Goods	26	13	56	226	1581	239	770	441	997	512	401	312	52	4475	659	3461	1849	3104	19174
Transport Equipment	388	60	158	112	80	123	119	102	56	26	109	142	399	351	2989	4403	987	324	10928
Other Equipment Goods Industries	706	186	339	219	836	714	795	1296	632	939	1307	1491	899	212	776	1747	3058	1598	17751
Consumer Goods Industries	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Building and Construction	795	125	2286	702	1594	188	392	329	503	275	545	565	29710	466	2886	4218	4771	9760	60111
Telecommunication Services	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	165	165
Transports	21	1	661	73	0	4	2	6	2	14	4	5	8	13	10	53	69	33	979
Services Of Credit And Insurance	151	48	727	104	104	138	256	549	229	159	404	567	4304	215	256	1965	661	1282	12119
Other Market Services	263	56	233	116	552	222	346	452	343	372	390	423	372	926	555	1450	1184	1035	9289
Non-Market Services	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	483	0	1617	2099

Total	2897	542	6669	1805	5209	1670	2771	3642	2830	2766	3234	3617	36648	6774	8394	18388	13114	19429	140398

Source: GEM-E3 Database.

APPENDIX 6: Models of Optimal Resource Allocation

Introduction

The following models are illustrated by the MULTHH-OPT-SCEN.GMS program with 1998 data, in 3 sectors and 10 household groups. A further unusual generalization of the resource allocation problem is in these models that gross fixed capital formation is also broken down to 3 sectors.

Formal description of the models

We deal with the static version (1 period). Simpler models are formulated in a more general framework, so that the GAMS can treat them as special cases. Simpler formulas can be obtained as special cases of the more sophisticated ones. Parameters and non-parametrized functions are denoted by small letters, while endogenous variables are denoted by capital letters. Those categories which may appear in certain models as endogenous variables are declared variables. (the GAMS does not allow the redefinition of PARAMETERS as VARIABLES and vice versa). Exogenous variables are not distinguished from. Since the GAMS does not allow the optimization of an objective function formula, a separate variable is defined to measure the value of the objective function.

The NLP2 (primal optimal resource allocation) model

Sets:

G : household groups (general element is referred by index g , dimension: N_g),

I: branches (general element is referred by index i or j, dimension: N_i)(the set is also denoted by J)

Functions:

$e_i (Z_i)$ Export price-volume function,

$f_i (M_i , XD_i)$ Import-domestic substitution function,

$g_i (RL_i , RK_i)$ Capital-Labour substitution function,

$h_g (CV_{g,i})$ Consumer welfare function by groups,

t_i (Z_i , XD_i) Export-domestic transformation function

Variables:

CES_i , $CESLK_i$, $INVS_i$, X_i , XD_i , M_i , Z_i , RL_i , RK_i , PE_i , $CVg_{,i}$, CL_g , BTR, IL, IK, IG, OBJ

(altogether $10 \cdot N_i + N_g \cdot N_i + N_g + 5$ variables, which in the case of $N_g = 10$ and $N_i = 3$ yields 75)

Conditions:

(precisely inequalities, in the form of inputs ≥outputs, source ≥ use and use ≥ presribed value):

Name	**Shadow price**	**Formula**
Definitions:		
$ECESLK_i$	(μ_i)	$g_i (RL_i, RK_i) \geq CESLK_i$
$ECES_i$	(λ_i)	$f_i (M_i, XD_i) \geq CES_i$
$EPA2_i$	(PA_i)	$X_i \geq t_i (Z_i , XD_i)$
EV2	(V)	$\sum_i (PE_i \cdot Z_i - pwm_i \cdot M_i) \geq BTR$
EW2	(W)	$IL \cdot tl \geq \sum_i (RL_i \cdot X_i)$
ER2	(R)	$IK \cdot tk \geq \sum_i (RK_i \cdot X_i)$
EOBJ	(φ)	$BTR \geq OBJ$
Behavioral and technnological equations, balance identities:		
EPE_i	(ε_i)	$e_i (Z_i) \geq PE_i$
ECL_g	(γ_g)	$h_g (CV_{g,i}) \geq CL_g$
$EPR2_i$	(PR_i)	$CESLK_i \geq 1$
$EPHM2_i$	(PHM_i)	$CES_i \geq \sum_j (ahm_{i,j} \cdot X_j + bhm_{i,j} \cdot INVS_j) + \sum_g (cf_{g,i} + CV_{g,i}) + gb_i \cdot IG + stacc_i + texp_i$
EIL	(α)	$ile \geq IL$
EIK	(δ)	$ike \geq IK$
EIG	(PG)	$IG \geq ige$

$ECPIS_g$	$CPIS_g$	$CL_g \geq tc_g$
$EINVS_i$	$PINVS_i$	$INVS_i \geq ibe_i \cdot invs0_i$

(altogether $7 \cdot N_i + 2 \cdot N_g + 7$ conditions, which in the case of $N_g = 10$ and $N_i = 3$ yields 48, so if the constraints are binding then the degree fo freedom is $3 \cdot N_i + N_g \cdot (N_i - 1) - 2$, which in the case of $N_g = 10$ and $N_i = 3$ yields 27)

Task:

OBJ -> max

Note, that introducing OBJ as a separate variable is due to the fact that the GAMS program can maximize only the value of a single variable, so a formula can not be maximized directly.

The NLP3 model (first order conditions of the NLP2 model)

If the above inequalities hold in the form of strict equalities (which can be proved to happen in the optimum in the case of continously substitutable, differentiable functions), then the optimal solution can be derived by using the Lagrange-multiplier method.

The **Lagrangian** (with some reordering to get common summations) is the following:

$£ := OBJ + \sum_i \{ \mu_i \cdot (g_i (RL_i, RK_i) - CESLK_i) + \lambda_i \cdot (f_i (M_i, XD_i) - CES_i) + PA_i \cdot (X_i - t_i (Z_i, XD_i)) + PHM_i \cdot (CES_i - [\sum_j (ahm_{i,j} \cdot X_j + bhm_{i,j} \cdot INVS_j) + \sum_g (cf_{g,i} + CV_{g,i}) + gb_i \cdot IG + stacc_i + texp_i]) + PINVS_i \cdot (INVS_i - ibe_i \cdot invs0_i) + \varepsilon_i \cdot (e_i (Z_i) - PE_i) + PR_i \cdot (CESLK_i - 1) \} + V \cdot (\sum_i (PE_i \cdot Z_i - pwm_i \cdot M_i) - BTR) + W \cdot (IL \cdot tl - \sum_i (RL_i \cdot X_i)) + R \cdot (IK \cdot tk - \sum_i (RK_i \cdot X_i)) + \varphi \cdot (BTR - OBJ) + \sum_g \{ \gamma_g \cdot (h_g (CV_{g,i}) - CL_g) + CPIS_g \cdot (CL_g - tc_g) \} + \alpha \cdot (ile - IL) + \delta \cdot (ike - IK) + PG \cdot (IG - ige)$

The first order conditions of the optimum are obtained by setting the partial derivatives of the Lagrangian to zero. Concretely, setting the partial derivatives of the Lagrangian according to the shadow prices to zero bring back the original conditions (in the form of equations), while setting the partial derivatives of the Lagrangian according to the primary (quantity) variables to zero we get the well-known marginality conditions. These are the following:

$$\delta £ / \delta X_i \equiv PA_i - W \cdot RL_i - R \cdot RK_i - \sum_j PHM_j \cdot ahm_{j,i} = 0 \qquad (1)$$

$$\delta £ / \delta Z_i \equiv -PA_i \cdot \delta t_i / \delta Z_i + V \cdot PE_i + \varepsilon_i \cdot \delta e_i / \delta Z_i = 0 \qquad (2)$$

$$\delta £ / \delta XD_i \equiv -PA_i \cdot \delta t_i / \delta XD_i + \lambda_i \cdot \delta f_i / \delta XD_i = 0 \qquad (3)$$

$$\delta £ / \delta M_i \equiv \lambda_i \cdot \delta f_i / \delta M_i - V \cdot pwm_i = 0 \qquad (4)$$

$$\delta £ / \delta RL_i \equiv \mu_i \cdot \delta g_i / \delta RL_i - W \cdot X_i = 0 \qquad (5)$$

$$\delta £ / \delta RK_i \equiv \mu_i \cdot \delta g_i / \delta RK_i - R \cdot X_i = 0 \qquad (6)$$

$$\delta£ /\delta CV_{g,i} \equiv - PHM_i + \gamma_g \cdot \delta h_g /\delta CV_{g,i} = 0 \qquad (7)$$

$$\delta£ /\delta BTR \equiv -V + \varphi = 0 \qquad (8)$$

$$\delta£ /\delta CES_i \equiv -\lambda_i + PHM_i = 0 \qquad (9)$$

$$\delta£ /\delta CESLK_i \equiv - \mu_i + PR_i = 0 \qquad (10)$$

$$\delta£ /\delta PE_i \equiv V \cdot Z_i - \varepsilon_i = 0 \qquad (11)$$

$$\delta£ /\delta CL_g \equiv - \gamma_g + CPIS_g = 0 \qquad (12)$$

$$\delta£ /\delta IG \equiv -\sum_i PHM_i \cdot gb_i + PG = 0 \qquad (13)$$

$$\delta£ /\delta INVS_j \equiv -\sum_i PHM_i \cdot bhm_{i,j} + PINVS_j = 0 \qquad (14)$$

$$\delta£ /\delta IL \equiv -\alpha + W\cdot tl = 0 \qquad (15)$$

$$\delta£ /\delta IK \equiv -\delta + R\cdot tk = 0 \qquad (16)$$

$$\delta£ /\delta OBJ \equiv 1 - \varphi = 0 \qquad (17)$$

By counting the number of equations and variables one can see that the equation system is regular. Majority of the variables are irrelevant or can be computed recursively after solving the simultaneous block. Therefore only the simultaneous block has to be dealt with.

The process of the solution can be e.g. the following:

From (17) $\varphi = 1$, which substituted into (8) can be omitted together with its equation.

Similarly from (10) $\mu_i = PR_i$, which substituted into (5) can be omitted together with (10).

Similarly from (9) $\lambda_i = PHM_i$, which substituted into (3) and (4) can be omitted together with (9).

Similarly from (12) $\gamma_g = CPIS_g$, which substituted into (7) can be omitted together with (12).

Similarly from (11) $\varepsilon_i = V \cdot Z_i$, which substituted into (2) can be omitted together with (11).

Finally, since from equations (13)-(16) one can express $PG = \sum_i PHM_i \cdot gb_i$, $PINVS_j = \sum_i PHM_i \cdot bhm_{i,j}$, $\alpha = W\cdot tl$ and $\delta = R\cdot tk$ variables, which do not appear anywhere else, these can be regarded to be part of the epilogue. Therefore the simultaneous block just consists of equations (1)-(8) and the primal conditions[23]. Naturally, the variables with Greek letters are substituted by their latin letter equivalents.

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[23] Although the number of the primary equations and variables also can be reduced, since IL, IK, IG, CLg, $INVS_i$, $CESLK_i$ are exogenously given and OBJ can be substituted by BTR.

This block consists of $6 \cdot N_i + N_i \cdot N_g + 1$ equation. By adding the number of the primary equations, the total number of equations is $13 \cdot N_i + 2 \cdot N_g + N_i \cdot N_g + 8$. The number of the effective variables is the same, since to the $10 \cdot N_i + N_g + N_i \cdot N_g + 5$ primary variables only $3 \cdot N_i + N_g + 3$ dual variables have to be added (PA_i, PR_i, PHM_i, $CPISG_g$, V, W és R). Therefore this reduced equation system is regular, which is in practice the sufficient condition of the existence and unicity of the solution.

The NLP3 model in the GAMS program only differs from this in that respect, that (for the sake of future generalization) instead of R (through RS_i and $PINVS_i$) and W stand PK_i and PL_i respectively, although not differentiated by sectors if we set the parameter values appropriately (see in the GAMS program[24]). The NLP3 model has an additional 'epilogue' block of 5 equations, which determine the investment by products (investment goods), the investment price indices and the sectoral breakdown of the capital and labour[25].

Naturally, the e_i , f_i , g_i , h_g and t_i functions appear in the GAMS program in concrete a parametrized functional form. Concretely, , f_i , g_i and h_g by CES-functions, t_i –s are represented by CET-function and the $PE_i = e_i(Z_i)$ export price-volume functions (see the definitional equation EPEi among the primary conditions) are specified as

$$PE_i = e_i(Z_i) = \tau_i \cdot Z_i^{\omega(i)}$$

i.e. as an 'isoelastic' function. If we denote $\delta e_i / \delta Z_i$ by $e'(Z_i)$, then we can get the

$$e'(Z_i) = \tau_i \cdot \omega(i) \cdot Z_i^{\omega(i)-1} = \omega(i) \cdot e_i(Z_i)/Z_i$$

relationship.

If we substitute this into (2) (or into that form of (2) in which ε_i is already replaced by $V \cdot Z_i$ – as seen in (11)) and if we define PZ_i (the unit revenue of export at local currency) as

$$PZ_i = V \cdot (1+\omega(i)) \cdot PE_i$$

then we get

$$PA_i \cdot t'(Z_i) = V \cdot PE_i + \varepsilon_i \cdot e'(Z_i) = V \cdot PE_i + V \cdot \omega(i) \cdot e_i(Z_i) = V \cdot (1+\omega(i)) \cdot PE_i = PZ_i \qquad (2')$$

where $\delta t_i / \delta Z_i$ was also denoted by $t'(Z_i)$.

Further NLP model variants (NLPKT, NLPGE) are a somewhat modified and further generalized form of this system of equations, by introducing further auxiliary variables and by using the Euler-theorem and other relationships (see the alternative equivalent conditions of the optimum).

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[24] For example, when computing the optimum, the GAMS program sets the BHM(I,J) parameter of the PINVS(J,Y) =E= SUM(I, PHM(I,Y)*BHM(I,J,Y)) equation so, that it be uniform accross investing sectors:
BHM(I,J,Y)=SUM(JJ, BHM(I,JJ,Y)*INVS0(JJ))/SUM((II,JJ), BHM(II,JJ,Y)*INVS0(JJ))

[25] See equations EB,EPINV,EK, EKS,EL in the GAMS-program (note, that for capital both demand and supply in determined at sectoral level, although in these models this is irrelevant, capital is assumed to be perfectly mobile).

The NLPKT model (modified version of the first order conditions)

Of the „further NLP model variants" the first one is the so-called NLPKT model, which is obtained so, that we introduce the

EPM(I,Y).. PM(I,Y) =E= TXM(I,Y)*V(Y)*PWM(I,Y);

EPZ(I,Y).. PZ(I,Y)=E=TXZ(I,Y)*V(Y)*PE(I,Y);

EPD(I,Y).. PD(I,Y)=E=(PA(I,Y)*X(I,Y)-PZ(I,Y)*Z(I,Y)) / XD(I,Y);

EC(G,I,Y).. C(G,I,Y)=E=CF(G,I,Y)+CV(G,I,Y); {here it is only epilogue or statistics}

definitional equations, and in which (3) and (4) (which are called DXD and DM in the GAMS program) are replaced by EPHMKT(I,Y) and EMKT(I,Y), while (5) and (6) (which are called DRL and DRK in the GAMS program) are replaced by ERLKT(J,Y) and ERKKT(J,Y).

The above equations are defined as follows:

EPHMKT(I,Y).. PHM(I,Y)=E=(XD(I,Y)*PD(I,Y)+PM(I,Y)*M(I,Y)) / CES(I,Y);

EMKT(I,Y).. M(I,Y)=E=MH(I,Y)*(PD(I,Y)/PM(I,Y))**MEL(I,Y)*XD(I,Y);

ERLKT(J,Y).. RL(J,Y)=E=RL0(J,Y)**(1-REL(J,Y))*(AL(J,Y)*PR(J,Y)/PL(J,Y))**REL(J,Y);

ERKKT(J,Y).. RK(J,Y)=E=RK0(J,Y)**(1.-
REL(J,Y))*(AK(J,Y)*PR(J,Y)/PK(J,Y))**REL(J,Y);

where CESLK(J,Y) was replaced by 1 (see equation EPR2).

It is not straitforward to prove that these replacements are equivalent. Here only we can outline how to prove this statement.

For example, to prove that **(3),(4) => EPHMKT,EMKT** we have to multiply (2'),(3) and (4) by Z_i, XD_i and M_i respectively, then we have to add them pairwise (i.e. form the (2')+(3) and (3)+(4) equations) to get equations (5) and (6) respectively. Then in equation (6) we have to apply the $CES_i = f\,'(XD_i)\cdot XD_i + f\,'(M_i)\cdot M_i$ Euler-theorem. Then we have to derive equation (11) by subtracting equation (5) from (6). By observing that due to (3) the $PA_i\cdot t'(XD_i)$- $PHM_i\cdot f\,'(XD_i)$ component in (11) is 0, and by using the definition of PD_i, and by dividing the equation by CES_i we get the EPHMKT equation.

Then by subtracting $PZ_i\cdot Z_i$ from equation (5) and by using the definition of PDi again and finally by dividing by XDi we get the

$$PD_i = PHM_i \cdot f\,'(XD_i) \quad (8)$$

relationship.

Then le tus define mh_i as

$mh_i = (am_i / ad_i)^{1/mel(i)}$

and let us start from the

$mh_i \cdot (PD_i / PM_i)^{1/mel(i)}$ formula. Here by using (8),(4), the defintion of PM_i and the fact that since f is a CES-function so $f\,'(XD_i) = (CES_i/XD_i)^{1/mel(i)}$, we can arrive at the M_i/XD_i ratio. Note that by this we proved the relationship in EMKT, which was our goal.

To prove the statement from the opposite direction, i.e. that **EPHMKT,EMKT => (3),(4)** one may do the following steps:

We start from EPHMKT, into which we replace PD_i by its definition, and by using the Euler-theorem twice again we replace X_i by $t'(XD_i)\cdot XD_i + t'(Z_i)\cdot Z_i$ and CES_i by $f\,'(XD_i)\cdot XD_i + f\,'(M_i)\cdot M_i$. Then we observe that due to (2') the $PA_i \cdot t'(Z_i) \cdot Z_i - PZ_i \cdot Z_i$ component is 0, we get an equation (11) which is just the sum of (3) and (4).

Then from EMKT (using again that $f\,'(XD_i) = (CES_i/XD_i)^{1/mel(i)}$ and $f\,'(M_i) = (CES_i/M_i)^{1/mel(i)}$) we get that

$$f\,'(M_i) = f\,'(XD_i)\cdot PM_i / PD_i \tag{12}$$

Substituting this into (11) for $f\,'(M_i)$ and (9) for PD_i w eget the following relationship:

$$PHM_i \cdot f\,'(XD_i) \cdot (XD_i + M_i \cdot PM_i / PD_i) = PD_i \cdot (XD_i + M_i \cdot PM_i / PD_i) \tag{13}$$

By dividing the equation by the formula in the parenthesis we directly get (8).

Then, multiplying the $X_i = t'(XD_i)\cdot XD_i + t'(Z_i)\cdot Z_i$ Euler-theorem by PA_i and by using the definitions of PZ_i and PD_i and dividing the equation by XD_i we get the

$$PD_i = PA_i \cdot t'(XD_i) \tag{9}$$

relationship.

Now, if we compare the 2 different formulas for PD_i (i.e. (8) and (9)), we can conclude that the other (right hand) sides should also be equal, i.e.

$$PHM_i \cdot f\,'(XD_i) = PA_i \cdot t'(XD_i)$$

which is just equation (3) which we had to prove. Finally, if we subtract this from (11) (i.e from (3)+(4)) we get equation (4) too.

To prove the **(5),(6) ⇔ ERLKT,ERKKT** equivalency is much simpler, so we will leave it for the reader.

The NLPGE model (slightly modified NLPKT)

It differs from NLPKT only in that respect, that the EPR2 composite factor-utility setting (to unity) is replaced by the EPR explicit definition of the shadow-price formula. Naturally, the two formulas are equivalent.

The CGE and CGECLO models (toward a general equilibrium model)

First, in these models the ECL composite consumption-utility definition is replaced by the ECPIS equation, which computes the shadow-price at the optimum (as the ratio of the total expenditure and utility). Naturally, as in the case of the production function, the two formulas are equivalent.

The ‘CGE’ model contains a simplified income-distribution block (including transfers and savings), which, however, is irrelevant given the NLP-type of closure. If we set the parameter values appropriately, then its solution is identical to that of the above NLP-models.

The MultHH-opt-scen.GMS program therefore runs this model twice: first before doing these optimal parameter-settings, then after the NLP-model runs, when the parameters are already set so. The latter run demonstrates that this model is also equivalent to the NLP-models.

This sequence of runs is explained by the fact that it is easy to make omelette from eggs but not vice versa: i.e. after setting the parameters optimally it is difficult to recalibrate them so that they fit the benchmark data 8as usual in the static CGE models).

The ‘CGECLO’ model demonstrates that by changing the macroeconomic closure of the model, one can get a solution different from that of the above NLP-models. Concretely, the fixing the group-specific consumption levels (equation ECPIS2) are replaced by the fixing of the consumption/wage ratio (equation ESAVRAT), which is a proxy for the savings rate (since the model’s income distribution block is not sufficiently elaborated the real savings rate can not be determined straitforwardly).

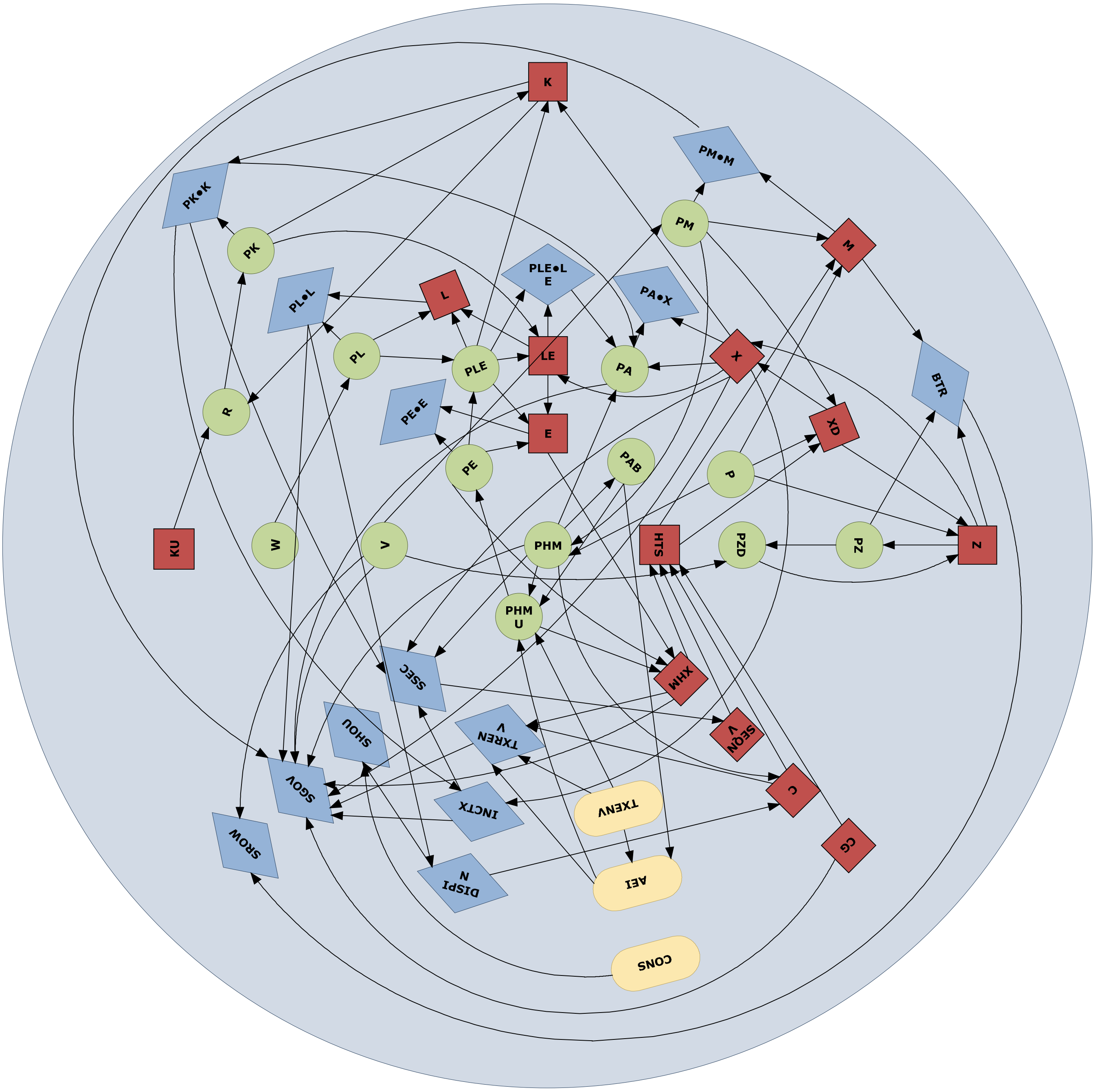
Of course, if both (CGE and CGECLO) models are calibrated to the benchmark data, they reproduce the benchmark both. This is shown in columns E an F of the ‘Results’ sheet of the MultiHHOutput.xls file. However, when running a counterfactual simulation (in our case the government consumption was reduced by 20%, while the consumption tax rates wre increased by 10 percentage points uniformly across goods), the two models yield different results. This is demonstrated in columns G and H. As can be seen, in the CGECLO model (i.e. in which the savings rates are fixed) consumption levels decrease (due to the higher taxes) and the difference is pushed to the export markets, which, however, can be reached by the incentive of real devaluation of the local exchange rate (note that since the nominal exchange rate is kept fixed, this implies the drop of the domestic prices).

Comparing these columns with columns E or F one can see, that in the optimal solution the balance of trade (see in row 66) is higher than in the CGE-models with non-uniform factor prices, markups, indirect taxes and similar ‘distortions’. Not surprisingly, in the counterfactual scenario the NLP models retain their trade balance edge over the CGE-models. In fact, it is more revealing to compare the changes caused by the the same ‘shocks’ in different models. Therefore, the comparison of the difference of H-G may be compared with that of I-J, to see how sensitive the results are to the model specification. This comparison we leave to the reader.

In columns J,K and L the equivalence of the NLP2, NLPKT and CGE models are empirically demonstrated.

Finally, note that in the GAMS version of the model the categories have a time dimension (Y) (and some intertemporal factor accumulation relationships too) which makes it easier to develop them further into a dynamic model. Such a dynamic model is presented in section 4.2.1.

APPENDIX 7: FLOW CHART OF THE HUNGARIAN CGE MODEL



Notes:

1. CPI=1 is the numeraire, so it is not displayed
2. Several insignificant effects are not displayed (e.g. transfers depending on V, income tax depending on L,K)
3. Several intermediate variables (e.g. RS,PR,PHMUC,CL,B) are not displayed (as if we had substituted them out)
4. Irrelevant variables (mostly belonging to the environmen-leisure-consumption optimization) are not displayed

5. If behaviour is optimal shadow-prices depend only on components
6. Radiuses (axis crossing the origo) represent a type of category or a step in the income distribution (e.g. we can start from the left horizontal – i.e. 180^o radius, which represent the factor returns or supply)

Legend:

Circle: Price category

Rectangle: Quantity category

Romboid: Value category (cost, income)

--→ : Direction of direct effect

Notations: *(Y refers to the year, I and J for the sector, G for the household group)*

VARIABLES:

B(I,Y)	total investment by goods
BTR(Y)	balance of trade
C(G,I,Y)	total personal consumptions of commodity i
CG(Y)	government consumption level index
CL(G,Y)	level of the variable consumption
CV(G,I,Y)	variable consumption
INVS(J,Y)	investment level index
K(J,Y)	capital stock
KS(J,Y)	sectoral capital
L(J,Y)	employment
M(I,Y)	import
NTRF(Y)	net transfers of the foreign sector
NTRG(Y)	net transfers of the government
NTRH(G,Y)	net transfers of the household groups
NTRS(J,Y)	net transfers of the sectors
OBJ	objective function average: average total consumption
PA(I,Y)	average sales price
PD(I,Y)	price for domestic sales

PE(I,Y)	export fob price in base years exhange rate
PHM(I,Y)	composite good price
PINV(Y)	investment price index
PINVS(J,Y)	price of investment
PK(I,Y)	calculative capital costs
PL(I,Y)	calculative labour costs
PM(I,Y)	import user price
PR(I,Y)	composite cost of labour and capital per unit output in sector i
PZ(I,Y)	unit revenue of export sales
R(Y)	adjustment factor for the net rate of return on capital
RK(I,Y)	unit capital
RL(I,Y)	unit employment
RS(J,Y)	rate of return to capital by sectors
SF(Y)	saving of the foreign sector
SG(Y)	saving of the government
SH(G,Y)	saving of the household groups
SS(J,Y)	saving of the sectors
V(Y)	foreign exchange rate
W(Y)	standard wage index
X(I,Y)	output
XD(I,Y)	domestic sold output
Z(I,Y)	exports