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by
Ferenc Forgó

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Ferenc Forgó

Department of Operations Research, Corvinus University of Budapest, Fővám tér 8, 1093 Budapest, Hungary
(e-mail: ferenc.forgo@uni-corvinus.hu)

Abstract Social dilemmas, in particular the prisoners' dilemma, are represented as congestion games, and within this framework soft correlated equilibria as introduced by Forgó F. (2010, A generalization of correlated equilibrium: A new protocol. Mathematical Social Sciences 60:186-190) is used to improve inferior Nash payoffs that are characteristic of social dilemmas. These games can be extended to several players in different ways preserving some important characteristics of the original 2-person game. In one of the most frequently studied models of the n-person prisoners' dilemma game we measure the performance of the soft correlated equilibrium by the mediation and enforcement values. For general prisoners' dilemma games the mediation value is ∞, the enforcement value is 2. This also holds for the class of separable prisoners' dilemma games.

Keywords prisoners' dilemma, congestion games, soft correlated equilibrium, mediation value, enforcement value

JEL Classification Number C72

1 Introduction

The prisoners' dilemma has been one of the most intriguing subjects in game theory ever since its early days in the middle of the last century. It has fascinated many trades as far apart as psychology and physics. The strategic complexity hidden behind its simplicity has made it a favorite subject for analysis, generalizations, experiments and heated discussions. The basic issue is to predict the outcome of a game situation where players can choose between cooperation and defection when cooperation offers the best outcome if done collectively and defection is tempting if done individually (or in small groups). The classic version involves only two players but generalizations to many players have also emerged throughout the years.

Numerous attempts have been made to embed the problem in a more complex scenario (e.g. allowing repetition of the game and find a socially superior equilibrium outcome in the extended game). In general games correlation, the ingenious invention of Aumann (1974), may retain equilibrium in an extended game yet Pareto-improve Nash outcomes. The prisoners' dilemma, however, is resistant against correlation: the only Nash outcome is the only correlated outcome in the 2-person prisoners' dilemma. The generalization of Aumann's correlation introduced by Moulin and Vial (1978) and termed "coarse correlation" or alternatively "weak correlation" also falls short of providing equilibria other than Nash's for the simple reason that the prisoners' dilemma is a binary game, where weak correlation coincides with classical correlation. Another generalization of the classical correlation, "soft correlated equilibrium" introduced by Forgó (2010), however, was shown to be able to Pareto-improve the Nash outcome.

As it will be demonstrated in this paper, another area of game theory, simple congestion games are in close relation with prisoners' dilemma games. Thereby, techniques successfully used in congestion games to measure the effectiveness of different kinds of correlation can be brought to bear on prisoners' dilemma games. In particular, the mediation value and the enforcement value as defined in Ashlagi et al. (2008) can be determined or estimated for some special prisoners's dilemma type games. We will show that the mediation value of the soft correlated equilibrium for the linear n-person prisoners' dilemma game (as defined by Hamburger,1973) is ∞ which means that for any large number K one can find an instance of the n-person prisoners' dilemma game, where the ratio of the social welfare achievable by soft correlation is more than K-times the social welfare of the unique pure Nash equilibrium. The enforcement value of soft correlation is 2, i.e. the absolute maximum of the social welfare cannot be more than 2-times the social welfare that soft correlation can produce. We get the same result for separable n-person games. Social welfare is defined as the sum of the payoffs (utilities) of the players.

The paper is organized as follows. Section 2 includes the necessary preliminaries and establishes the connection between social dilemma games and simple congestion games. In Section 3 we determine the mediation and enforcement value of soft correlated equilibrium for special classes of simple congestion games and prisoners' dilemma games. Section 4 concludes.

2 Congestion games, social dilemmas and correlation

Simple congestion games are models of situations where non-distinguishable players choose from a finite set of facilities and payoffs depend only on how many of them have chosen a particular facility. We will focus on the case of two facilities. An n-player, 2-facility simple congestion game can be given by the "congestion form": two nonnegative n-vectors $a = (a_1, ..., a_n)$, $b = (b_1, ..., b_n)$ meaning that if j many players choose facility 1 ($F1$), then each one gets utility a_j and if k many players choose facility 2 ($F2$), then each one gets utility b_k. The associated congestion game is defined by the player set N, the strategy set $\{F1, F2\}$ for each player, briefly denoted by $\{1, 2\}$, and the payoffs determined by the utility vectors a and b. A strategy profile of the n players is $(i_1, ..., i_n)$ where $i_j \in \{1, 2\}$, $j \in N$. For example, if $n = 4$, then $(1, 1, 2, 1)$ denotes the situation when players $1, 2, 4$ choose $F1$ and player 3 chooses $F2$.

A special class is when the utilities are determined by linear functions. A 2-facility simple linear congestion game is defined by the utilities $a_j = jx+u, b_j = y + jz$, $j = 1, ..., n$, where x, u, y, z are parameters. An important property of congestion games (not only linear) is that they have at least one pure Nash equilibrium point (PNE), see Rosenthal R W (1973).

Social dilemma games (SD) are symmetric 2-person binary games with some strategic features of "dilemma" type: counterintuitive or problematic Nash equilibria (NE). The most famous SD is the prisoners' dilemma (PD). Other SD's are e.g. the battle of sexes, chicken, stag hunt. For a more detailed discussion see Osborne and Rubinstein (1994). The following two propositions state a close relationship between SD's and 2-facility simple congestion games.,

Proposition 1 Every SD is a (linear) 2-facility simple congestion game.

Proof An SD is determined by four parameters a, b, c, d and is given as a bimatrix game

$$\begin{array}{ccc} & A & B \\ A & a, a & b, d \\ B & d, b & c, c \end{array} .$$

The associated congestion form is

$$\begin{array}{ccc} & F1 & F2 \\ \text{No. of users} & A & B \\ 1 & b & d \\ 2 & a & c \end{array} . \tag{1}$$

$\square$.

Example: A PD is given by the parameters $0 \leq b < c < a < d$. In the associated 2-facility simple congestion game A stands for "cooperate" (C), B stands for "defect" (D). It is a "mixed" congestion game: increasing for $F1$ and decreasing for $F2$.

The reverse of Proposition 1 is also true.

Proposition 2 Every 2-person 2-facility simple congestion game can be represented as a symmetric 2-person binary game.

Proof If the congestion form is given as in (1), then each player has two choices $F1$ and $F2$ and the game is given by the payoff matrices

$$\begin{array}{ccc} & F1 & F2 \\ F1 & a, a & b, d \\ F2 & d, b & c, c \end{array} .$$

$\square$.

The two most important properties of a PD from which many others can be deduced (Hamburger, 1973) are the following:

$P1$ Each player has a dominant strategy (D),

$P2$ (D, D) is the only Nash equilibrium (NE).

There are many ways to generalize the PD to n players, see Carrol (1988). The minimum requirement for the generalization is to preserve $P1$ and $P2$.

Following Hamburger (1973) we define the "cooperators' function" $C(k), k = 1, ..., n$ which is interpreted as the payoff to a C-chooser provided there are k of them and the "defectors' function" $D(k), k = 0, 1, ..., n-1$ which gives the payoff a D-chooser gets provided there are k C-choosers. $C(0)$ and $D(n)$ are undefined. We assume that

$(Q1)$ $C(k) < D(k-1), k = 1, ..., n-1,$

$(Q2)$ $C(n) > D(0)$.

Assumptions $Q1, Q2$ are meant to ensure that $P1$ and $P2$ carry over to the n-person case. $Q1$ means that for a single player it is profitable to leave the set of cooperators no matter how many of them there are. $Q2$ makes collective cooperation preferable to collective defection.

Assuming linearity of the cooperators' and defectors' functions seems to be a reasonable first step in the analysis of n-person PD's. In this case the game can be represented as a mixed simple linear 2-facility congestion game.

One might wonder whether linearity is too strong an assumption and covers only irrelevant trivial cases. To dismiss this fear it is enough to recall that compound games are linear. An n-person PD is said to be compound, if each of n players simultaneously plays the same 2×2 PD game with all the other players and each is required to make the same move in the $n-1$ games she is playing.

n-person PD games have extensively been studied in many contexts. A good sample of references is Hamburger (1973), Carrol (1988), Szidarovszky and Szilagyi (2002) and Szilagyi (2003). Since CE cannot help increase SW, no attention has been paid to correlation as a means of pulling PD out of the trap of the bad NE. The case is different if we consider generalized correlation.

We confine ourselves to describe classical correlated equilibrium (CE) Aumann (1974), weak (coarse) correlated equilibrium (WCE) Moulin and Vial (1978) and soft correlated equilibrium (SCE) Forgó (2010) by the pre-game scenario they are based on. For each of the three kinds of equilibria the scenario begins with a move of an umpire who randomly selects a strategy profile by a commonly known probability distribution. In CE he recommends for each player to use her strategy without telling her or anybody else what his recommendation is for the others. Then each player, without any commitment, chooses one of her strategies. The probability distribution is a CE if no player can get a larger expected payoff by deviating from the recommendation provided everybody else plays the recommendation. WCE demands the players to commit themselves to follow the recommendation blindly. If they refuse to do so, then they can choose any strategy. Equilibrium is defined the same way as in case of CE: no player benefits from not committing if everybody else commits. SCE is almost the same as WCE except that when choosing non-commitment, the player is not allowed to choose the strategy that would have been recommended to her had she committed. Again, in equilibrium, unilateral deviation is unprofitable in expected payoff.

As shown in Forgó (2010), WCE and SCE are generalizations of CE but not of each other.

An important motivation of correlation and its generalizations is the desire to improve social welfare (SW) while creating a situation where the players pursuing their own goals wind up in a socially more preferable state than the one achievable without it. Since we will be concerned with SCE in the multiperson PD and the corresponding mixed 2-facility simple congestion game, it is worth to interpret the scenario of SCE in a different context as is done in Forgó (2014).

We may think of a "club" which has a library and a chess room with no capacity limits. There are premium and regular members. Every week a lottery is done according to a commonly known probability distribution assigning either the library or the chess room to each player. Premium members must take the room assigned to them, regular members must not, i.e. they have to take the one not assigned to them by the lottery. Utilities are such that less people in the library mean higher utility (privacy, less noise) whereas utility increases in the chess room if there are more players (it is easier to find someone of similar level to play with, tournaments can be organized). In an SCE everybody buys premium membership, no player has an incentive to go regular provided that all the others stay premium.

It is proved in Forgó (2010) that for binary games $WCE = CE$. Therefore for SD games only SCE can (if at all) improve SW. Using the idea of Ashlagi et al (2008) we will measure the ability of a correlation scheme, in particular that of SCE, by how much the improvement compared to the SW of the best NE is and how close to the absolute (unconditional) maximum of SW over some class of games (in this case a class of n-person PD's or mixed simple linear 2-facility congestion games) we can get. For measurement we use the mediation value (MV) and the enforcement value (EV).

Let $G \in \Gamma$ be a game from a class of finite games, $P(G)$ the set of probability distributions over the strategy profiles of G, $M(G)$ the set of probability distributions generated by mixed Nash equilibria and $S(G)$ the set of SCE's. Let $SW(p)$ be the social welfare (sum of the expected utilities of the players) of a probability distribution p. Define the mediation value of SCE in G, as

$$MV(G) = \frac{\max_{p \in S(G)} SW(p)}{\max_{p \in M(G)} SW(p)}$$

and the enforcement value as

$$EV(G) = \frac{\max_{p \in P(G)} SW(p)}{\max_{p \in S(G)} SW(p)}.$$

Then the mediation value MV and the enforcement value EV of SCE over the class of games Γ are defined as

$$MV = \sup_{G \in \Gamma} MV(G), \; EV = \sup_{G \in \Gamma} EV(G).$$

It is worth noticing that MV is a "best case", while EV is a "worst case" indicator. For MV the best possible value is ∞, for EV the best possible value is 1.

In Forgó (2014) MV and EV of SCE for n-player, 2-facility simple non-increasing linear congestion games was determined or estimated for certain cases. They are summarized in the next table

Number of players	MV	MVP	EV
2	∞	∞	1
3	$\frac{4}{3}$	$\frac{4}{3}$	1
4	?	$\geq \frac{4}{3}$	1.007478...
...	...	...	...
n	?	$\geq \frac{n^2}{4(n-1)}$	$\leq \frac{4}{3}$

The column of MVP stands for the case when in the definition of MV only pure NE's are considered. The research task naturally comes to mind: determine as many values as possible in the above table for n-player, 2-facility mixed simple linear congestion games, in particular for PD games.

3 Mediation and enforcement values of soft correlated equilibrium for prisoners' dilemma games

Consider the congestion form for an n-player, 2-facility simple mixed linear congestion game

$$\begin{array}{cc} F1 = C & F2 = D \\ a_1 = 0 & b_1 = y + (n-1)z \\ a_2 = x & b_2 = y + (n-2)z \\ a_3 = 2x & b_3 = y + (n-3)z \\ ... & ... \\ a_n = (n-1)x & b_n = y \end{array}.$$

with parameters $x, y, z > 0$. In terms of the cooperators' and defectors' functions the above table takes the form

$$\begin{array}{cc} F1 = C & F2 = D \\ C(1) = 0 & D(n-1) = y + (n-1)z \\ C(2) = x & D(n-2) = y + (n-2)z \\ C(3) = 2x & D(n-3) = y + (n-3)z \\ ... & ... \\ C(n) = (n-1)x & D(0) = y \end{array}.$$

Since we will concentrate on PD games, the facilities have been given other names C and D (cooperate and defect) as well. Cooperation is paying off more and more as the number of cooperators increases and defection gets more and more profitable as the number of defectors decreases. It is no infringement on generality that the lowest utility is fixed at 0. Most of the ensuing analysis, however, is valid not only for PD's but also for 2-facility simple mixed linear congestion games with the above congestion form. It should be remarked that there are simple mixed linear congestion games where $b_n < a_1$ (e.g. the chicken game for $n = 2$) but we will not consider those in this paper.

Let t denote the number of players playing D, $t = 0, 1, ..., n$. Then, after some rearrangement, the incentive constraint for SCE is (see Forgó 2014)

$$(-y + (n-1)(x-z))q_0 + \frac{1}{n}\sum_{t=1}^{n-1}(ty - t(n-t)(x-z) +$$

$$(n-t)((n-t-1)(x-z) - y))q_t + yq_n \geq 0. \qquad (2)$$

This constraint is meant to make unprofitable, in terms expected payoffs, to deny premium membership (in the interpretation given in the previous section). Probabilities must be nonnegative and add up to 1

$$\begin{aligned} \sum_{t=0}^{n} q_t &= 1, \qquad (3) \\ q_t &\geq 0,\ t = 0, 1, ..., n. \end{aligned}$$

We want to maximize SW which is the sum of the players' expected payoff

$$SW = n(n-1)xq_0 + \sum_{t=1}^{n-1}(t(y+(n-t)z) + (n-t-1)(n-t)x)q_t + nyq_n \ . \quad (4)$$

In order for this game to represent an n-person PD assumptions $Q1$ and $Q2$ must be satisfied. If t players play D, then $n - t$ play C. By $Q1$, $a_{t+1} \leq b_{n-t}$ for $t = 1, ..., n-1$ and thus the parameters should satisfy

$$y + tz > tx,\ t = 1, ..., n \ . \quad (5)$$

All these inequalities are implied by the single inequality

$$y + (n-1)z > (n-1)x. \qquad (6)$$

Taking assumption $Q2$ in account, we get $C(n) = (n-1)x > D(0) = y$. So, for a 2-facility simple mixed linear congestion games to represent an n-person PD it is necessary that the positive parameters x, y, z satisfy

$$0 < \frac{1}{n-1}y < x < \frac{1}{n-1}y + z \text{ if } n \geq 2. \qquad (7)$$

Determining MV is quiet simple for any $n \geq 2$. It turns out that it is the best possible.

Proposition 3 For the n-person PD the mediation value $MV = \infty$.

Proof Take the parameter values $x = 1, y = \varepsilon, z = 1$, $\varepsilon > 0$. Then $q_0 = q_n = \frac{1}{2}, q_i = 0, i = 1, ..., n-1$ is an SCE that can easily be verified by substituting into (2). Also, x, y, z satisfy (7) for all $n \geq 2$. The SW of the n-person PD with these parameters is obtained by substituting into (4) and is found to be $\frac{1}{2}n(n-1) + \frac{1}{2}n\varepsilon$. The only NE of the n-person PD is when all players defect and its SW is $n\varepsilon$. The MV ratio is then

$$\frac{\frac{1}{2}n(n-1) + \frac{1}{2}n\varepsilon}{n\varepsilon}$$

which goes to ∞ if $\varepsilon \to 0$. $\square$

Since every PD is a 2-facility simple mixed linear congestion game, it follows from Proposition 5 that $MV = \infty$ for this wider class of games as well. Now we turn to estimating the EV for the class of 2-facility simple mixed linear congestion games.

Proposition 4 For 2-facility simple mixed linear congestion games the enforcement value $EV \leq 2$.

Proof. Define for any real number $t \in [0, n]$ the quadratic function W by

$$W(t) = \sum_{t=0}^{n} (t(y + (n-t)z) + (n-t-1)(n-t)x). \tag{8}$$

The maximum $W^*$of $W(t)$ over $[0, n]$ is an upper bound of the absolute maximum of SW which is attained at some integer point in $[0, n]$. The coefficient of the quadratic term in (8) is $x - z$. Depending on the sign of $x - z$, we distinguish two cases.

Case 1. $x \geq z$. In this case
$W^* = \max\{W(0), W(n)\} = \begin{matrix} ny & \text{if } x \leq \frac{1}{n-1}y \\ n(n-1)x & \text{if } \frac{1}{n-1}y < x \end{matrix}$.

The set of probabilities $q_t = 0, t = 0, 1, ..., n-1, q_n = 1$ is always an SCE since $y > 0$, therefore we trivially have $EV = 1$ if $W^* = ny$. Consider now the subcase when $W^* = n(n-1)x$. The set of probabilities $q_0 = q_n = \frac{1}{2}, q_t = 0, t \neq 0, n$ is easily seen to be an SCE by substituting into (2) and getting

$$\frac{1}{2}(-y + (n-1)(x-z)) + \frac{1}{2}y = (n-1)(x-z) \geq 0.$$

The SW of this SCE is

$$\frac{1}{2}n(n-1)x + \frac{1}{2}ny.$$

Thus we get the estimation

$$EV \leq \frac{n(n-1)x}{\frac{1}{2}n(n-1)x + \frac{1}{2}ny} < 2.$$

Case 2. $x < z$. Assume first that n is even. An SCE can be obtained by setting the probabilities $q_{\frac{n}{2}} = 1, q_i = 0, i \neq \frac{n}{2}$. This satisfies (2) because

$$\frac{n}{2}y - \frac{n}{2}(n-\frac{n}{2})(x-z) + (n-\frac{n}{2})((n-\frac{n}{2}-1)(x-z)-y) = \frac{n}{2}(z-x) > 0. \tag{9}$$

The SW belonging to this SCE is

$$W(\frac{n}{2}) = \frac{n}{2}(y + \frac{n}{2}z) + \frac{n}{2}(\frac{n}{2} - 1)x) = \frac{n}{2}(y + \frac{n}{2}z + \frac{n}{2}x - x).$$

In this case the quadratic function $W(t)$ attains its continuous absolute maximum at $r = \frac{y+nz-(2n-1)x}{2(z-x)}$. If $r > n$, then just as in Case 1, we have $EV = 1$. If $r < 0$, then $W(0) > W(t)$ for all $t \in [0, n]$ and we have the estimation

$$EV \leq \frac{W(0)}{W(\frac{n}{2})} = \frac{n(n-1)x}{\frac{n}{2}(y + \frac{n}{2}z + \frac{n}{2}x - x)} = 4\frac{(n-1)x}{2y + nz + (n-2)x} =$$

$$4\frac{(n-1)x}{2y + nz + (n-2)x} < 2.$$

which truly holds since $z > x$.

Consider now $r \in (0, n)$. We claim that the coefficient of every q_t in (2) is positive if $\frac{n}{2} \leq t \leq n$. To see this, observe that if t is considered a continuous variable, then the coefficient of q_t is a concave quadratic function of t since the coefficient $x - z$ of the quadratic term is negative. The coefficient y of q_n is positive by assumption, so is the coefficient of $q_{\frac{n}{2}}$ by (9). The positivity at the endpoints of an interval implies positivity at all points by concavity establishing our claim.

Thus if the continuous maximumpoint r of the function W falls in the interval $[\frac{n}{2}, n]$,so does the integer maximumpoint t^*(being one of the neighboring integers of r). The coefficient of q_{t^*} being positive, $q_{t^*} = 1, q_t = 0, t \neq t^*$ is an SCE and $EV = 1$. Then we have to consider only the case when $0 \leq r < \frac{n}{2}$. The continuous maximum can be bounded from above as

$$W^* = W(r) = n(n-1)x + r^2(z-x) < n(n-1)x + \frac{n^2}{4}(z-x)$$

because $r < \frac{n}{2}$ and $z - x > 0$. For the EV we have

$$EV < \frac{W^*}{W(\frac{n}{2})} = \frac{n(n-1)x + \frac{n^2}{4}(z-x)}{\frac{n}{2}(y + \frac{n}{2}z + \frac{n}{2}x - x)} = \frac{4(n-1)x + n(z-x)}{2y + nz + (n-2)x} < 2.$$

Now we turn to the case when n is odd, $n \geq 3$. We have already seen that the coefficients of q_t in inequality (2) are positive for $\frac{n+1}{2} \leq t \leq n$ and if the integer maximumpoint t^* of W falls in this interval, then $EV = 1$. Therefore it is enough to deal with the case when $0 \leq r < \frac{n+1}{2}$. Consider the $SCE = q_{\frac{n-1}{2}} = \frac{1}{2}, q_{\frac{n+1}{2}} = \frac{1}{2}, q_i = 0, i \neq \frac{n-1}{2}, \frac{n+1}{2}$. The SW of this SCE is

$$\begin{aligned} W &= \frac{1}{2}W(\frac{n-1}{2}) + \frac{1}{2}W(\frac{n+1}{2}) = \\ & \frac{1}{2}(\frac{n-1}{2}(y + \frac{n+1}{2}z) + \frac{n-1}{2}\frac{n+1}{2}x + \frac{n+1}{2}(y + \frac{n-1}{2}z) + \frac{n-1}{2}\frac{n-3}{2}x). \end{aligned}$$

Then we have the estimation

$$\begin{aligned} EV &\leq \frac{n(n-1)x + r^2(z-x)}{\frac{1}{2}W(\frac{n-1}{2}) + \frac{1}{2}W(\frac{n+1}{2})} \leq \\ & \frac{n(n-1)x + \frac{(n+1)^2}{4}(z-x)}{\frac{1}{2}(\frac{n-1}{2}(y + \frac{n+1}{2}z) + \frac{n-1}{2}\frac{n+1}{2}x + \frac{n+1}{2}(y + \frac{n-1}{2}z) + \frac{n-1}{2}\frac{n-3}{2}x)}. \end{aligned}$$

After multiplying the numerator and the denominator by 4 and deleting positive terms from the denominator we get

$$EV \leq \frac{4n(n-1)x + (n+1)^2(z-x)}{(n-1)(n+1)z + \frac{(n-1)(n+1)}{2}x + \frac{(n-1)(n-3)}{2}x}.$$

We would like to prove that this ratio is no more than 2. This means that the following inequality must hold

$$4n(n-1)x + (n+1)^2 z - (n+1)^2 x \leq 2(n^2-1)z + 2(n-1)^2 x.$$

By rearranging we get

$$(n-3)(n+1)x \leq (n-3)(n+1)z$$

which obviously holds by the assumption $x < z$. Thus we have that for 2-facility simple mixed linear congestion games $EV \leq 2$.□

One might hope that for some important subclasses of PD games better bounds can be gained for the EV. An important subclass is the class of PD's where $x = z$. For these games the reward of a player's joining an already existing group of cooperators increases at the same constant speed as the speed at which the benefit of defection decreases. These games are called "separable" and exhibit many interesting properties, see for example Hamburger (1973).

Proposition 5 For the n-person separable PD the enforcement value $EV = 2$.

Proof In Proposition 4 we have already seen that for 2-facility simple mixed linear congestion games $EV \leq 2$ if $x \geq z$ (Case 1), in particular for $x = z$.

There is only one question to answer: is this bound tight? Consider the n-person separable PD with parameters $x = 1, y = \varepsilon, z = 1$. These parameters satisfy (7) if ε is small enough. By substituting into (2) and (3) we get the following LP whose solution provides the SW maximizing SCE :

$$\max n(n-1)q_0 + \sum_{t=1}^{n-1}(t\varepsilon + (n-1)(n-t))q_t + n\varepsilon q_n.$$

$$\text{subject to} \qquad \begin{aligned} nq_0 - \sum_{t=1}^{n-1}(n-2t)q_t - nq_n &\leq 0, \\ q_0, q_1, ..., q_n &\geq 0, q_0 + q_1 +, ..., +q_n = 1. \end{aligned}$$

Assume that n is even. Then $q_{\frac{n}{2}} = 1, q_j = 0, j \neq \frac{n}{2}$ is a feasible solution, and $u = \frac{1}{2}(n-1) - \frac{\varepsilon}{2}, v = \frac{n}{2}(n-1) + \frac{n}{2}\varepsilon$ is a feasible solution to its dual with the common objective value $\frac{n}{2}(n-1) + \frac{n}{2}\varepsilon$. The absolute maximum of SW is $n(n-1)$. Therefore the EV ratio is

$$\frac{n(n-1)}{\frac{n}{2}(n-1) + \frac{n}{2}\varepsilon} = 2\frac{n-1}{n-1+\varepsilon}$$

which goes to 2 if $\varepsilon \to 0$.

The analysis for odd n is similar. In this case we have to consider the feasible solution $q_{\frac{n-1}{2}} = \frac{1}{2}, q_{\frac{n+1}{2}} = \frac{1}{2}, q_j = 0, j \neq \frac{n-1}{2}, \frac{n+1}{2}$.□

Corollary 1 For 2-facility simple mixed linear congestion games the enforcement value $EV = 2$.

Corollary 2 For the n-person PD the enforcement value $EV = 2$.

Conclusion

It was shown how n-person prisoners' dilemma games can be represented as 2-facility simple mixed linear congestion games. This representation is used to measure the performance of soft correlated equilibrium in Pareto-improving the Nash outcome. The mediation value was found ∞ , and the enforcement value was proved to be 2 for any n. The enforcement value is also 2 for separable n-person prisoners' dilemma games. Other social dilemma games and their generalizations for n players can also be studied by the methods used in this analysis.

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