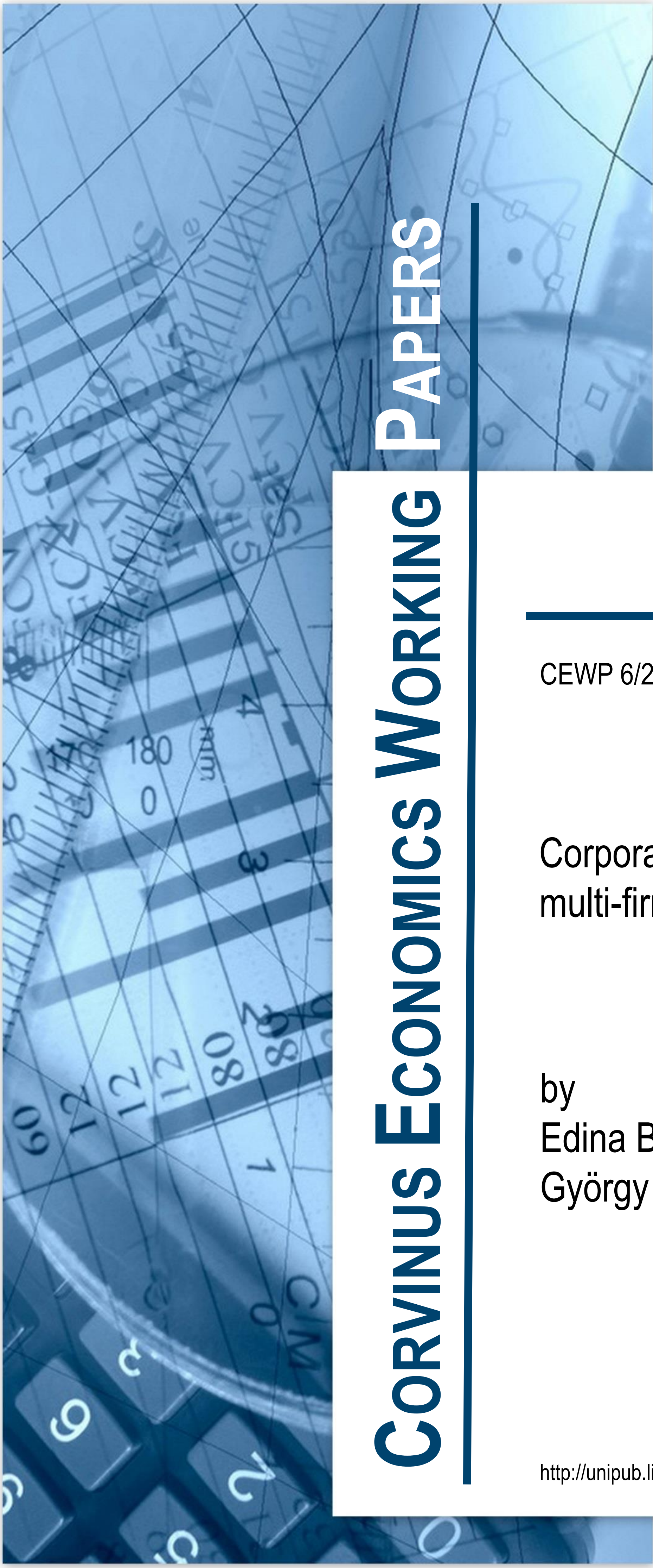

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# Corporate cash-pool valuation in a multi-firm context: a closed formula

by
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# Corporate cash-pool valuation in a multi-firm context: a closed formula

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**Abstract**

Following our earlier paper on the subject, we present a general closed formula to value the interest savings due to a multi-firm cash-pool system. Assuming normal distribution of the accounts the total savings can be expressed as the product of three independent factors representing the interest spread, the number and the correlation of the firms, and the time-dependent distribution of the cash accounts. We derive analytic results for two special processes one characterizing the initial build-up period and the other describing the mature period. The value gained in the stationary system can be thought of as the interest, paid at the net interest spread rate on the standard deviation of the account. We show that pooling has substantial value already in the transient period. In order to increase the practical relevance of our analysis we discuss possible extensions of our model and we show how real option pricing technics can be applied here.

**JEL-Codes**: G15, G21, G32

**Keywords**: cash management, analytical solution, Brown motion, mean reversion, real option pricing

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## Introduction

Cash-pool is a centralized cash management service offered by almost all commercial banks to corporate groups. In a cash-pool system accounts of different companies (even of different legal entities) are introduced into a single bank account structure settled in a mutual cash-pool agreement. It centralizes all balances of the subaccounts into a central master account. Amounts are consolidated and deposit and credit interest rates are automatically calculated and charged. The literature focuses on the technical solutions, the main characteristics and the corresponding risks, (Turtle et al. 1994; Dolfe and Koritz 1999; Hillman, 2011; Jansen and de Gruyter, 2011; Walter and Kenesei, 2015); and summarizes the costs and benefits of cash pooling systems in detail (Rebel, 2007). However, the explicit modeling and valuation of such products are still missing.

In our previous paper (Berlinger et al. 2016) we valued a cash-pool of two firms with the help of a Monte Carlo simulation from the point of view of the firms. In this paper we derive a general formula for a cash-pool of $n$ uniform firms and we investigate the special cases when the accounts are stationary or are following a Brown motion. Contrary to our previous paper, here the time is continuous and the benefit of the cash-pool comes exclusively from the savings on the interest rate spread. Otherwise, all other characteristics of the model remain the same.

The paper is structured as follows. In Section 2 a general formula is derived, in Section 3 two special cases are analyzed, in Section 4 some possible extensions are discussed and finally in Section 5 conclusions are summarized.

### 1. General formula

The following notation will be used:

| | |
|---|---|
| $t$ | time, in years |
| $d$ | yearly deposit rate |
| $c$ | yearly credit rate |
| $s=c-d$ | net interest spread |
| $\sigma$ | standard deviation of the net account position of the individual firms |
| $\rho$ | correlation between the net account positions of any two firms |
| $A_i(t)$ | cash account position of firm $i$ at time $t$ |
| $R_i(t)$ | incremental return on the account of firm $i$ at time $t$ |
| $A(t)$ | cash account position of the pool at time $t$ |
| $R(t)$ | incremental return on the account of the pool at time $t$ |
| $\Delta A(t)$ | reduction in the account position due to the cash-pool at time $t$ |
| $\Delta R(t)$ | excess incremental return on the account due to the cash-pool at time $t$ |

*TES* total expected saving per firm

Let the cash account position of firm $i$ on day $t$ be $A_i(t)$. If the position is positive, the firm receives interest with yearly rate $d$ from a deposit account. If the position is negative, the firm pays interest with yearly rate $c$ to a credit account. Typically, $c > d$, both are fixed and are calculated on an annual basis. As the investigated time horizons are only a few years at maximum, and the cash account fluctuates widely, it makes sense to consider simple linear interest. For the same reasons, we will neglect discounting. We define the (positive) net interest spread as the difference between credit and deposit rates $s = c - d$. The $R_i(t)$ incremental return on the account during the small time interval $dt$ can be written as

$$R_i(t) = \left(cA_i(t) - s(A_i(t))^+\right)dt \tag{1}$$

where $(x)^+ = \max(0, x)$ is the positive part function. Now consider two firms that pool their cash accounts. The pool position will evolve as $A(t) = A_1(t) + A_2(t)$ and the incremental return on the pool account becomes

$$R(t) = \left(c(A_1(t) + A_2(t)) - s(A_1(t) + A_2(t))^+\right)dt \tag{2}$$

We can express the excess return due to pooling $\Delta R(t) = R(t) - R_1(t) - R_2(t)$, as

$$\Delta R(t) = s\left((A_1(t))^+ + (A_2(t))^+ - (A_1(t) + A_2(t))^+\right)dt = s\Delta A(t)dt \tag{3}$$

where $\Delta A(t)$ is the decrease in the aggregate position due to cash pooling which is also the decrease in the aggregate credit or in the aggregate deposit, by definition. It can be easily seen that both $\Delta A(t)$ and $\Delta R(t)$ are nonnegative.

In all our subsequent analysis, the positions $A_i(t)$ will have a joined normal distribution, with zero mean, uniform time-dependent variance $\Sigma^2(t)$, and uniform time-independent correlation $\rho$. We derive the expression for the expectation $E[\Delta A(t)]$ using these assumptions, as follows:

$$E\left[(A_1(t))^+\right] = E\left[(A_2(t))^+\right] = \frac{\Sigma(t)}{\sqrt{2\pi}} \tag{4}$$

$$E\left[(A_1(t) + A_2(t))^+\right] = \frac{\Sigma(t)}{\sqrt{2\pi}}\sqrt{2(1+\rho)} \tag{5}$$

$$E[\Delta A(t)] = \frac{\Sigma(t)}{\sqrt{2\pi}}\left(2 - \sqrt{2(1+\rho)}\right) \tag{6}$$

Formula (6) was obtained for two firms pooling their cash accounts. It can readily be generalized for the case of $n$ firms (still assuming variances and correlations be uniform across all the firms):

$$E[\Delta A(t)] = \frac{\Sigma(t)}{\sqrt{2\pi}}\left(n - \sqrt{n(1+(n-1)\rho)}\right) \tag{7}$$

To estimate the magnitude of the savings per firm due to cash pooling for a given time horizon $T$, we calculate the total expected saving per firm (TES):

$$TES = \frac{1}{n}\int_0^T \mathrm{E}[\Delta R(t)] = \frac{s}{n}\int_0^T \mathrm{E}[\Delta A(t)]dt \tag{8}$$

Substituting our result for $E[\Delta A(t)]$, we obtain the general result:

$$TES = s\frac{n - \sqrt{n(1+(n-1)\rho)}}{\sqrt{2\pi n}}\int_0^T \Sigma(t)dt = s \cdot m \cdot A \tag{9}$$

According to (9) $TES$ is the product of three factors. The first factor $s$ will be referred as the spread factor and is assumed to be constant. The second factor $m = \frac{n - \sqrt{n(1+(n-1)\rho)}}{\sqrt{2\pi n}}$ will be called the multi-firm factor, since it shows the dependence on the number of firms $n$ and on the correlation $\rho$ And finally, the third factor $A = \int_0^T \Sigma(t)dt$ depends on the accounts' stochastic process which is uniform for each firm, so we will call it the account factor.

As expected, the multi-firm factor $m$ is a decreasing function of correlation, indeed it vanishes at $\rho = 1$, showing there is no pooling benefit for firms with perfectly correlated cash accounts. In the limit of many firms pooling their accounts, the factor converges to the finite value of $\frac{1-\sqrt{\rho}}{\sqrt{2\pi}}$. This shows that although the overall pooling benefit grows with the number of participants, the benefit *per firm* saturates.

The account factor $A$ is the time-integrated standard deviation of a single firm's position, therefore it depends on the stochastic processes the accounts follow. In the next section, we consider two important special cases for the account process and derive factor $A$ and hence $TES$ accordingly.

## 2. Special cases

Two processes are examined in detail. In both cases the expected value of the accounts are assumed to be zero and we concentrate on the behavior of the standard deviation over time. First, in the stationary model the standard deviation is constant, whereas in the subsequent Brownian model it is an increasing function of time.

The real world cash account of a firm can be best modeled by a mean reversion process, (e.g. Ornstein-Uhlenbeck) which can be interpreted as an "interpolation" of these two extremal models. At short time scales, in the initial stage it coincides with the diffuse Brownian motion, while in the long run it saturates to the stationary case.

### 3.1 Stationary model

In this model, we assume the individual firms' accounts have reached stationary normal distributions, all with the same time-independent standard deviation $\Sigma$, and with uniform pair-wise correlations $\rho$. Using the general result from the previous section, calculation of *TES* is straightforward, and we obtain that $A = \Sigma \cdot T$, therefore

$$TES_{stationary} = s\frac{n - \sqrt{n(1+(n-1)\rho)}}{\sqrt{2\pi n}}\Sigma \cdot T = s \cdot m \cdot \Sigma \cdot T \tag{10}$$

As expected, within this stationary model, the savings due to pooling for each firm aggregate linearly throughout the period up to the time horizon. The value gained can be thought of as the interest, paid at rate $s$ on an account with a fixed size of $m \cdot \Sigma$.

### 3.2 Brownian motion

The previous model offers estimation on the benefit of pooling in a stationary situation, when the participants have already established the pooling of their accounts. In this analysis we consider firms whose cash accounts are zero at the beginning. For such firms, pooling does not offer any immediate benefit, yet it may still be advantageous in the future.

We now model the individual account processes $A_i(t)$ as scaled Brownian Motions:

$$A_i(t) = \sigma W_i(t) \tag{11}$$

Once again we assume a homogeneous model; the volatility of all account processes are $\sigma$, and the pair-wise correlation between any two Brownian motions is $\rho$. Subsequently, the accounts at any given time $t$ follow a joined normal distribution with uniform correlation $\rho$ and with the same standard deviation

$$\Sigma(t) = \sigma\sqrt{t} \tag{12}$$

Using the general formula we get $A = \frac{2}{3}\sigma \cdot T^{\frac{3}{2}}$, therefore

$$TES_{Brownian} = s \cdot \frac{n - \sqrt{n(1+(n-1)\rho)}}{\sqrt{2\pi n}} \cdot \frac{2}{3}\sigma \cdot T^{\frac{3}{2}} = \frac{2}{3} \cdot s \cdot m \cdot \Sigma(T) \cdot T \tag{13}$$

The result is very similar to (10) obtained in the stationary case. In particular, the spread factor $s$ and the multiform factor $m$ are the same as before. However, in this case when a diffuse stochastic process, a Brownian motion was assumed, time-dependence became super-linear ($T^{\frac{3}{2}}$).

When comparing our results for the stationary and the Brownian cases, formulas (10) and (13), we get the following relationship

$$TES_{stationary} \cdot \frac{2}{3} = TES_{Brownian} \tag{14}$$

In the diffuse model standard deviation gradually grows from zero to $\Sigma(T)$. The benefit over this transitional period is comparable (two-thirds) of the benefit over the same period in a saturated system with constant $\Sigma(T)$. This shows that pooling can be highly beneficial for firms even when they all start with zero cash accounts and the key factor is the standard deviation.

## 4. Possible extensions

The two models were selected in the previous section because we could obtain simple, yet revealing analytic results. In practical applications, with semi-analytic solutions, many generalizations are possible. In this section we survey some of these possible extensions.

(i) The time value of money, i.e. discounting of the savings, can easily be incorporated by integrating the present value of the expected savings.

If we generalize formula (3) to multiple firms, we get that the reduction in the exposure due to the cash-pool at time $t$, $\Delta A_t$ can be expressed as

$$\Delta A_t = \sum_{i=1}^{n} \left(A_t^i\right)^+ - \left(\sum_{i=1}^{n} A_t^i\right)^+ \tag{15}$$

It follows that $\Delta A_t$ can be considered as a combination of European call options with an exercise date of $t$ and an exercise price $K$=0. The positive terms in (15) are simple European type options, while the negative element is a basket option, because the underlying asset is the sum of the $n$ processes. Hence, the present values can be calculated with the help of technics used for pricing real options according to the formula:

$$PV(\Delta A_t) = n \cdot call_t^{European} - call_t^{Basket} \tag{16}$$

where $call_t^{European}$ and $call_t^{Basket}$ refer to the price (present value) of a European call option and a basket option, respectively. Thus, the value of the discounted savings can be calculated by time-integrating the corresponding option prices, hence (8) changes into

$$TES_{discounted} = \frac{s}{n}\int_0^T PV(\Delta A_t)dt = s\int_0^T \left( call_t^{European} - \frac{1}{n} call_t^{Basket} \right)dt \tag{17}$$

This approach is based on option pricing formulas which are available also for arithmetical Brown motion analyzed in the previous section; see for example Liu (2007) and Kolmar (2013). In most cases (17) can be calculated only in a semi-analytical way as even if closed formulas for the option prices are known, time- integration can only be done numerically.

(ii) It may be interesting and useful to consider firms with different, nonzero initial cash account positions. Such treatment would allow distinguishing between cases when the initial positions are of different directions from cases when they are in the same direction. In the former case, pooling offers benefits right from the start, while in the latter case benefits are much reduced. Mathematically, the main complication arises from the fact that the joined normal distributions will have nonzero mean, therefore $\mathrm{E}\left[(A_i(t))^+\right]$ and $\mathrm{E}\left[\left(\sum_{i=1}^{n} A_i(t)\right)^+\right]$ take more complicated forms. Once again, real option pricing formulae automatically handle these cases.

(iii) Calculations in this paper were only focusing on the benefits of interest rate savings. Models can be easily complemented with other types of benefits due to the cash-pool that are related to the reduction of the firms' exposure, for example the reduction in the counterparty risk the firms run when making a deposit in a bank, see (Berlinger et al. 2016).

## 5. Conclusions

In this paper, we investigated the benefits of a multi-firm cash-pool within a theoretical framework. We concentrated on the netting advantage arising from a spread between credit rates and deposit rates.

According to our general formula (9), the value of a cash-pool is the product of three independent factors representing the interest spread, the number and the correlation of the firms, and the time-dependent account variances. We derived analytic results for two special account process models, one that describes an initial Brownian diffusion period, and another representing mature pools with stationary account distributions.

We find that in the stationary model the value of the cash-pool is linear in $T$ while in the diffuse Brownian model it is superlinear (scales as $T^{\frac{3}{2}}$). Our results show that cash-pool benefits emerge fast even in the case when the initial cash accounts have zero balance. In the long run, the value gained by the participants can be thought of as the interest, paid at the net interest spread rate on the standard deviation of their cash account.

In order to increase the practical relevance of our analysis we discussed three possible extensions of our model (discounting, non-zero initial accounts, and counterparty risk) and we show how real-option pricing technics can be applied here.

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