The role of longevity risk in Hungary

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Abstract

In this paper, I analyze the role of longevity risk in Hungary in the public pension system and the life annuity segment of the life insurance market, which are two primary financial sectors of relevance to this special type of actuarial risk, using state-of-the-art econometric methodology. To this end, I present an overview and the mathematical background of several important current mortality forecasting techniques from the Lee–Carter model up to unifying paradigm of the Age–Period–Cohort family of models. After presenting the findings of a case study on the public pension system based on the paper of Bajkó, Maknics, Tóth and Vékas, I conclude that longevity risk jeopardizes the sustainability of the Hungarian public pension system in the long run. In another case study, I present an analysis of the role of longevity risk in the premium of private pension annuities, a relevant topic due to recent changes in a law on Hungarian voluntary pension funds, following an earlier analysis of Májer and Kovács. Based on the criterion on out-of-sample forecasting accuracy, I find that the Cairns–Blake–Dowd mortality forecasting model aimed specifically at modeling old-age mortality outperforms the Lee–Carter model applied by Májer and Kovács . Based on numerical results, I finally conclude that the role of longevity risk in the Hungarian life annuity market has increased significantly in the past decade and is likely to further increase in the future.

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1 Introduction

1.1 The problem of longevity risk

It is an empirical fact that average human lifespans have increased very rapidly during the past centuries, primarily due to advances in medicine and nutrition. The increase is significant even in the short run. Average lifespans of Hungarian males and females nearly doubled between 1900 and 2014. It is a very positive phenomenon in itself, however it is a source of methodological problems in actuarial science, where it makes the traditional assumption of time-invariant mortality rates inadequate. This is especially problematic in the fields of pension systems and life annuities, where the time-invariance assumption may cause gross errors in actuarial planning, as demonstrated by two case studies in my paper. Longevity risk may also be interpreted at an individual level, as increasing human lifespans pose a challenge for old-age individual financial well-being, as well. My paper focuses on the actuarial aspect of the problem, even though it has serious implications in other fields such as demography and quantitative finance, as well.

1.1.1 Longevity risk in the Solvency II framework

As an important recent development of my topic, the Solvency II (EU [2009]) risk modeling and management framework of European insurance companies and pension funds just entered into force in practice on 1st January 2016. The Solvency II Directive treats longevity risk as a submodule of the life underwriting risk module, which itself is an important component of the base solvency capital of the company, and lays down rules for calculating the solvency capital requirement due to longevity risk, which may be modeled by means of the so-called standard formula or an internal model tailored to the individual characteristics of the company in question. The use of mortality forecasting methods is indispensable in any actuarially adequate internal model of longevity risk.

1.2 Literature overview

In this section, I present a brief overview of the relevant international and Hungarian literature of my chosen topic, focusing on recent developments and papers that are relevant to my own research.

1.2.1 International literature

The paper of Lee–Carter [1992] has probably been the most important breakthrough in the history of mortality forecasting. The authors propose a relatively simple parametric, age- and time-dependent log-bilinear structure of mortality rates, which gives a surprisingly accurate description of the evolution of age-specific mortality rates in the United States between 1900 and 1989. As the second step of the procedure, the authors propose the reestimation of the time-dependent parameters (the so-called mortality index) of the model, prescribing the equality of expected and observed death counts. Finally, they forecast the time series of the reestimated mortality index by using autoregressive integrated moving average (ARIMA, see Asteriou–Hall [2015]) processes. They find the simple specification of a random walk with drift sufficiently accurate. In this framework, the mortality index is expected to decrease in a linear pattern, and the forecasted mortality rates are expected to decrease exponentially. The procedure has attained the status of the leading mortality forecasting method in the world by the start of the new millenium (Deaton–Paxson [2001]). Several international applications of the Lee–Carter [1992] model are summarized by Lee [2000] and Tuljapurkar–Li–Boe [2000].

The Lee–Carter [1992] model is an extrapolative statistical method, thus it ignores the underlying causes of mortality decline (advances in healthcare, lifestyle etc.).

Keilman ([1998] and [2008]) argues that official demographic projections, which frequently rely on expert opinion, systematically and significantly underestimate the rate of decrease of mortality rates and thereby the magnitude of longevity risk. Lee–Miller [2001] and Wong-Fupuy–Haberman [2004] conclude that applying the Lee–Carter [1992] model retrospectively results in far more precise forecasts than the corresponding official estimates.

It is an open question how long the past and present rapid decrease of mortality will continue in the future. Based on the inaccuracy of pessimistic official forecasts and the good performance of the Lee–Carter [1992] model, Wong-Fupuy–Haberman [2004] expect the decreasing trend to continue in the future, whereas Carnes–Olshansky [2007] question the applicability of extrapolative methods and expect that average human lifespans in developed countries will attain a peak and may stagnate or decrease afterwards.

Several papers have criticized the assumptions of the Lee–Carter [1992] model and proposed its extensions. As a first extension, Lee–Carter [1992] themselves propose the introduction of binary variables in the equations of mortality rates to capture the mortality shock experienced in the years of the Spanish flu epidemic following the First World War.

Wilmoth [1993] criticizes the homoskedasticity assumption of the model, arguing that the error variance of logarithmic mortality rates is in fact inversely proportional to the observed death counts. To remedy this shortcoming, he proposes the weighted least squares method instead of the Singular Value Decomposition applied by Lee–Carter [1992].

Lee–Miller [2001] propose a new procedure of reestimating the mortality index in the Lee–Carter [1992] model and suggest that the last observed rates should serve as jump-off points in the forecasts.

Brouhns et al. [2002a] assume that death counts follow a Poisson distribution. This variant is known as the Poisson Lee–Carter method, which has several advantages over the original method: it does not assume the homoskedasticity of the error terms, it explicitly takes the exposures and death counts into account, it does not apply the – rather heuristic – reestimation step proposed by Lee–Carter [1992], and it may be embedded more easily into actuarial applications. As an illustration of the latter, Brouhns et al. [2002a] present an analysis of anti-selection in the life annuity market based on their model variant.

A common criticism of the Lee–Carter [1992] model is that it only captures the uncertainty arising from the error terms of the time series of the mortality index in the confidence intervals of the forecasts, while taking the estimated parameters of the model for granted. Brouhns et al. [2005] demonstrate that it is possible to adequately model parameter uncertainty in the Poisson Lee–Carter variant proposed by Brouhns et al. [2002a] by means of bootstrapping (Efron [1979]).

Many authors criticize the Lee–Carter [1992] model for failing to take into account any cohort effects beyond age and time effects. The most popular cohort-based extension of the original method is the Renshaw–Haberman [2006] model. As this procedure has proved to be numerically unstable, Haberman–Renshaw [2011] simplify their model by assuming the age-independence of the cohort effect.[1]

Another way of extending the model is to introduce additional time series of mortality indices, resulting in so-called multi-factor mortality forecasting models. Booth–MainDonald–Smith [2002] present a multi-factor extension of the Lee–Carter [1992] model based on retaining further singular vectors in the Singular Value Decomposition proposed by the original authors. The authors conclude that it is rather complicated to include the additional mortality factors in the forecasts, and give recommendations for the reestimation of the mortality indices and the selection of the appropriate estimation base period.

1. As Hunt–Villegas [2015] point out, even the simplified model may pose computational challenges.

Beyond mortality rates, the Lee–Carter [1992] method and its extensions may also be used to forecast age-specific fertility rates, as illustrated by Hyndman–Ullah [2007] and Wiśniowski et al. [2015]. The Age–Period–Cohort (APC) model (Hobcraft et al. [1982] and Carstensen [2007]), which is a special case of the simplified model of Haberman–Renshaw [2011], is another simple and popular mortality forecasting method.
The two-factor Cairns–Blake–Dowd [2006] model, which aims to capture old-age mortality, along with its three-factor generalization by Plat [2009], are widely applied methods in actuarial science. Plat [2009] recommends a simplifed two-factor version of his model to forecast old-age mortality. This variant is a cohort-based extension of the Cairns–Blake–Dowd [2006] model.
Based on an analysis of Finnish and Swedish mortality data, Lovász [2011] proposes the Plat [2009] model for actuarial applications. Cairns et al. [2009] propose the Cairns–Blake–Dowd [2006] model for British and the Renshaw–Haberman [2006] model for American data. The authors suggest the introduction of a quadratic age effect in the Cairns–Blake–Dowd [2006] model.
Researchers and practitioners naturally need a unifying framework of the countless intricate mortality forecasting methods which have evolved based on the criticism of the Lee–Carter [1992] model. An appropriate generalized family of these models has been proposed recently by Hunt–Blake [2014], Currie [2016] and Villegas et al. [2016]. Among other possible names, this framework is known as the Generalized Age–Period–Cohort (GAPC) model, and is motivated by the Generalized Linear Model (GLM, McCullagh–Nelder [1989]), which is widespread in actuarial applications. The GAPC family unifies age- and time-dependent, log-bilinear and logit-bilinear, one- and multi-factor, as well as cohort-free and cohort-based mortality forecasting models. The resulting wide family includes the Poisson Lee–Carter (Brouhns et al. [2002a]), Renshaw–Haberman [2006], Age–Period–Cohort (Carstensen [2007]), Cairns–Blake–Dowd [2006] and Plat [2009] models besides several other procedures. Additionally, the GAPC framework also unifies several individually tailored parameter estimation, model selection and forecasting procedures.
A comprehensive report of the International Monetary Fund (IMF [2012]) on the financial impact of longevity risk concludes among others that every one-year increase of the life expectancy at age 63 increases the value of pension liabilities by 3 per cent in the United States.
Brouhns et al. [2002b] examines the role of longevity risk in the premium calculation of life annuities based on the Poisson Lee–Carter model variant of Brouhns et al. [2002a]. The authors simulate the model parameters from the multivariate normal distribution (Deák [1990] and Gassmann–Deák–Szántai [2002]) based

on their maximum likelihood estimator and Fisher information matrix, and compute the net premium of a life annuity for each replication to approximate its probability distribution. The paper of Hári et al. [2008] applies the two-factor Lee–Carter model (Booth–MainDonald–Smith [2002]) for this purpose.
Some prominent sources of the quantitative financial aspect of longevity risk, which itself is not examined in my paper: Krutov [2006] and Cairns–Blake–Dowd [2008] examine the securitization of longevity risk, a phenomenon which emerged in the financial markets in the new millennium, Blake et al. [2006] and Bauer et al. [2010] present pricing methods for longevity bonds, and Dowd et al. [2006] analyze the associated longevity swaps from a financial point of view.

1.2.2 Hungarian literature

Baran et al. [2007] fit a three-factor generalized Lee–Carter [1992] model to Hungarian mortality data, and conclude that the estimation period of the years 1949–2003 is inappropriate due to structural breaks in the mortality patterns, which are absent if the period 1989–2003 is used instead. The authors warn practitioners that forecasts should be interpreted with caution due to the past variability of Hungarian mortality.
Since the mortality of annuitants may differ substantially from the general population, and at the same time, the short history and small size of the Hungarian life annuity market mostly rule out the possibility of the creation of company- and product-specific life tables, Arató et al. [2009] recommend the use of past life tables of other countries which are sufficiently similar to the experience of the company in question. They propose three possible metrics to assess the similarity of life tables along with a simulation procedure which may be used to compute the critical values of the associated test statistics, and recommend a forecasting procedure based on a simple parametric mortality law.
Májer–Kovács [2011] apply the Lee–Carter [1992] model on mortality data of Hungarian people aged between 65 and 100 years in the period 1970–2006, and compute the life expectancy at the current retirement age of 65 years as well as the net single premium of a life annuity based on two assumptions: the static life table of 2006 and the dynamic life table obtained by their forecasts. According to their results, the static model underestimates the life expectancy at retirement by 6.33 per cent and the net single premium of the life annuity by 4.51 per cent compared to the dynamic approach. The authors present confidence intervals based on two approaches: by considering only the uncertainty arising from the error terms of the mortality index, as proposed by Lee–Carter [1992], and by treating the stochastic trend parameter of the mortality index as a random variable, thereby capturing some

of the parameter uncertainty inherent in the model. Additionally, they demonstrate that life annuities do not become risk-free for the annuity provider even in an infinite portfolio of policies if longevity risk is present.

Banyár [2012] presents a comprehensive treatise of the practical and theoretical questions of the modeling of life annuities provided to pension fund members.

Bajkó–Maknics–Tóth–Vékás [2015] apply the Lee–Carter [1992] model to forecast age-specific Hungarian mortality and fertility rates and analyze the sustainability of the Hungarian public pension system based on their forecasts and a cohort-based pension model.

2 The applied methodology

2.1 Statistical indicators of mortality

The most fundamental statistical indicator of mortality is the mortality rate, which may be obtained as the ratio of the death count and the corresponding exposure in a given population and period:

$$m = \frac{D}{E}, \tag{1}$$

where m is the mortality rate, $D \in \mathbb{N}$ is the number of deaths in the given population and period and $E > 0$ is the corresponding exposure, which may be defined as the number of people in the given population at the beginning of the period (the so-called initial exposed to risk, denoted by E^0) or the average number of people alive in the given population during the same period (the so-called central exposed to risk, denoted by E^c).

Among several important types of mortality rates, age-specific mortality rates are of cardinal importance. In this case, the given population is the group of people belonging to the age group $x \in \{1, 2, \ldots, X\}$ at the beginning of the period. The age group is usually denoted in the lower index of mortality rates, death counts and exposures. Thus age-specific initial and central mortality rates are computed as follows:

$$\begin{aligned} m_x^0 &= \frac{D_x}{E_x^0} \quad (x = 1, 2, \ldots, X), \\ m_x^c &= \frac{D_x}{E_x^c} \quad (x = 1, 2, \ldots, X). \end{aligned} \tag{2}$$

These two quantities may be related to each other based on approximations of the average lifetime of the deceased.

2.2 The Lee–Carter model

In this section, I assume that m_{xt} denotes the central death rates m^c associated with the age group $x \in \{1, 2, \ldots, X\}$ and calendar year $t \in \{1, 2, \ldots, T\}$, suppressing the upper index ^{c} for simplicity. The calendar years are assumed to be consecutive.
The Lee–Carter [1992] model assumes that the central death rates $m_{xt} > 0$ are known for all age groups $x \in \{1, 2, \ldots, X\}$ and calendar years $t \in \{1, 2, \ldots, T\}$, and are described by the following equation:

$$\ln m_{xt} = a_x + b_x k_t + \varepsilon_{xt} \quad (x = 1, 2, \ldots, X, \quad t = 1, 2, \ldots, T), \tag{3}$$

where $X \geq 2$ and $T \geq 2$ are integers.
In Equation (3), a_x, b_x and k_t are age- and time-dependent parameters and ε_{xt} are error terms, which are assumed to be independent and normally distributed with mean 0 and equal variance $\sigma^2 > 0$:[2]

$$\varepsilon_{xt} \sim \mathcal{N}(0, \sigma^2) \quad (x = 1, 2, \ldots, X, \quad t = 1, 2, \ldots, T).$$

In this case, the age- and time-specific mortality rates are independent with the following distributions:

$$\ln m_{xt} \sim \mathcal{N}(a_x + b_x k_t, \sigma^2) \quad (x = 1, 2, \ldots, X, \quad t = 1, 2, \ldots, T).$$

Introducing the following parameter vectors facilitates the remainder of the presentation of the model:

$$\mathbf{a} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_X \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_X \end{bmatrix}, \quad \mathbf{k} = \begin{bmatrix} k_1 \\ k_2 \\ \vdots \\ k_T \end{bmatrix}.$$

The model has $2X + T + 1$ parameters, which are meant to give a parsimonious description of the original XT mortality rates.
It causes an identification problem that the parameters $(\mathbf{a}, \mathbf{b}, \mathbf{k})$ may be transformed as follows, while remaining valid in Equation 3:

$$(\tilde{\mathbf{a}}, \tilde{\mathbf{b}}, \tilde{\mathbf{k}}) = \left(\mathbf{a} + \alpha\mathbf{b}, \frac{1}{\beta}\mathbf{b}, \beta(\mathbf{k} - \alpha\mathbf{1})\right) \quad (\alpha, \beta \in \mathbb{R}, \beta \neq 0)$$

2. Lee–Carter [1992] do not name a specific distribution beyond the mean and variance, but inserting the additional normality assumption leads to their original parameter estimates in a rigorous maximum likelihood setting.

To eliminate any degrees of freedom, Lee–Carter [1992] propose the following additional constraints:[3]

$$\begin{aligned} \sum_{x=1}^{X} b_x &= \mathbf{1}^T \mathbf{b} = 1, \\ \sum_{t=1}^{T} k_t &= \mathbf{1}^T \mathbf{k} = 0. \end{aligned} \tag{4}$$

The logarithmic transformation of the mortality rates on the left-hand side of Equation (3) is motivated by two circumstances: the variance-stabilizing transformation increases the validity of the homoskedasticity assumption, and additionally, the negativity of the untransformed mortality rates is ruled out this way.
The maximum likelihood estimates of the parameters a_x $(x = 1, 2, \ldots, X)$ are equal to the average log-mortality rates by age group:

$$\hat{a}_x = \frac{1}{T} \sum_{t=1}^{T} \ln m_{xt} \quad (x = 1, 2, \ldots, X).$$

For notational simplicity, centralized logarithmic mortality rates $\tilde{m}_{xt}$ and their matrix may be defined as follows:

$$\tilde{m}_{xt} = \ln m_{xt} - \frac{1}{T} \sum_{t=1}^{T} \ln m_{xt} \quad (x = 1, 2, \ldots, X, \quad t = 1, 2, \ldots, T),$$

$$\mathbf{M} = \begin{pmatrix} \tilde{m}_{11} & \tilde{m}_{12} & \ldots & \tilde{m}_{1T} \\ \tilde{m}_{21} & \tilde{m}_{22} & \ldots & \tilde{m}_{2T} \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{m}_{X1} & \tilde{m}_{X2} & \ldots & \tilde{m}_{XT} \end{pmatrix}.$$

Using this notation, the maximum likelihood estimators of the vectors **b** and **k** are equal to the dominant left and right singular vectors of the matrix **M** after proper normalization according to the identification constraints (4), and the maximum likelihood estimator of the error variance is equal to the root mean squared error of the model:

$$\hat{\sigma}^2 = \frac{1}{XT} \sum_{x=1}^{X} \sum_{t=1}^{T} (\ln m_{xt} - \hat{a}_x - \hat{b}_x \hat{k}_t)^2.$$

The parameter vector **k** is called the mortality index, which represents the evolution of overall mortality by calendar year in the model, while the elements of the parameter vector **b** represent the age-specific sensitivities of the log-mortality rates with respect to changes in the mortality index.

3. $\mathbf{1}^T = (1, 1, \ldots, 1)$ is the summation vector of the appropriate dimension.

The maximum likelihood estimators uniquely exist under a set of sufficient regularity conditions.

If age- and period-specific central exposures $E^c_{xt} > 0$ and death counts $D_{xt} \in \mathbb{N}$ are known for all $x \in \{1, 2, \ldots, X\}$ and $t \in \{1, 2, \ldots, T\}$, then Lee–Carter [1992] propose the reestimation of the mortality index so that the equation

$$\hat{\mathbf{k}}^{(adj)} = \{\mathbf{k} \in \mathbb{R}^T : \quad D_t = \sum_{x=1}^{X} D_{xt} = \sum_{x=1}^{X} E^c_{xt} e^{a_x + b_x k_t} \quad (t = 1, 2, \ldots, T)\}$$

holds, which ensures that estimated and observed death counts are equal in all calendar years. This step is motivated by the fact that the original estimates of the mortality index do not take the sizes of age groups into consideration, thereby disproportionately tilting the fit of the model towards otherwise less relevant younger ages.

Lee–Carter [1992] model the reestimated mortality index as an ARIMA process (see e.g. Asteriou–Hall [2015]). They argue that the ARIMA$(0, 1, 0)$ specification of a random walk with drift (RWD) typically provides a satistfactory fit, and this specification is used in the majority of applications:

$$\hat{k}_t^{(adj)} = \hat{k}_{t-1}^{(adj)} + s + \phi_t \quad (t = 2, 3, \ldots, T), \tag{5}$$

where $\hat{k}_1^{(adj)} \in \mathbb{R}$ is a given initial value, $s \in \mathbb{R}$ is the drift parameter and the terms ϕ_t denote the error terms, which are assumed to be independent normal random variables with mean 0 and identical variance $\sigma^2_{RWD} > 0$. Additionally, the error terms are independent from those of the log-mortality rates. The maximum likelihood estimates of the parameters of Equation (5) are as follows:

$$\hat{s} = \frac{1}{T-1} \sum_{t=2}^{T} (\hat{k}_t^{(adj)} - \hat{k}_{t-1}^{(adj)}) = \frac{\hat{k}_T^{(adj)} - \hat{k}_1^{(adj)}}{T-1}, \tag{6}$$

$$\hat{\sigma}^2_{RWD} = \frac{1}{T-1} \sum_{t=2}^{T} (\hat{k}_t^{(adj)} - \hat{k}_{t-1}^{(adj)} - \hat{s})^2.$$

Future log-mortality rates may be estimated by extrapolating Equation (5) and applying Equations (3) and (6):

$$\ln \hat{m}_{x,T+j} = \hat{a}_x + \hat{b}_x (\hat{k}_T^{(adj)} + j\hat{s}) \quad (x = 1, 2, \ldots, X, \quad j = 1, 2, \ldots).$$

Forecasting uncertainty may be modeled by the Monte Carlo simulation (Deák [1990]) of the error terms of Equation (5).

2.3 The Generalized Age–Period–Cohort (GAPC) family of models

The Generalized Age–Period–Cohort (GAPC) family of models (Hunt–Blake [2014], Currie [2016] and Villegas et al. [2016]) is a

unifying framework of several mortality forecasting methods. In this section, m_{xt} denotes either the inital m^0 or central m^c mortality rates associated with the age group $x \in \{1, 2, \ldots, X\}$ and calendar year $t \in \{1, 2, \ldots, T\}$.
In order to apply this methodology, it is necessary to know the death counts $D_{xt} \in \mathbb{N}$ and the set of central or initial exposures $E^c_{xt} > 0$ or $E^0_{xt} \in \mathbb{N}_{>0}$ for all $x \in \{1, 2, \ldots, X\}$ and $t \in \{1, 2, \ldots, T\}$. The model considers the death counts D_{xt} to be realizations of random variables $\tilde{D}_{xt}$, which are either Poisson on binomially distributed, depending on the type of exposures:

$$\begin{aligned} &\tilde{D}_{xt} \sim Poisson(E^c_{xt} m^c_{xt}) \quad (x = 1, 2, \ldots, X, \quad t = 1, 2, \ldots, T) \\ \text{vagy} \quad &\tilde{D}_{xt} \sim Bin(E^0_{xt}, m^0_{xt}) \quad (x = 1, 2, \ldots, X, \quad t = 1, 2, \ldots, T). \end{aligned} \tag{7}$$

The model family assumes that the random variables $\tilde{D}_{xt}$ $(x = 1, 2, \ldots, X,\ t = 1, 2, \ldots, T)$ are independent.
The mortality rates on the right-hand side of Equation (7) are given by the following equation:

$$g(m_{xt}) = \eta_{xt} \quad (x = 1, 2, \ldots, X, \quad t = 1, 2, \ldots, T), \tag{8}$$

where η_{xt} is called the systematic component of the model and $g: \mathbb{R}_{>0} \to \mathbb{R}$ is a continously differentiable, strictly increasing function (the so-called link function). Hunt–Blake [2014] propose the logarithmic link function $g(y) = \ln y$ $(y > 0)$ for central death rates and exposures and the logit link function $g(y) = \ln\left(\frac{y}{1-y}\right)$ $(0 < y < 1)$ for initial death rates and exposures. I follow this convention in my paper.
The systematic component (8) is the following function of ages, periods and cohorts:

$$\eta_{xt} = a_x + \sum_{i=1}^{N} b^{(i)}_x k^{(i)}_t + b^{(0)}_x c_{t-x} \quad (x = 1, 2, \ldots, X, \quad t = 1, 2, \ldots, T), \tag{9}$$

where $N \in \mathbb{N}$ is the number of age-period interactions in the model, a_x and $\{b^{(i)}_x\}_{i=0}^N$ are age-dependent, $\{k^{(i)}_t\}_{i=1}^N$ are period-dependent and c_{t-x} are cohort-dependent, real-valued parameters. The model has $(N+1)(X+T)+2X-1$ parameters, which capture the original XT mortality rates.
Typically, Equation (9) must be accompanied by identification constraints, which depend on the model specification in question. At the end of this section, I shall present in more detail five popular models which are members of the GAPC family.

The parameter vector of the GAPC family may be written as

$$\begin{aligned}\boldsymbol{\zeta}^T = (&a_1, a_2 \dots, a_X,\\ &b_1^{(0)}, b_2^{(0)}, \dots, b_X^{(0)},\\ &b_1^{(1)}, b_2^{(1)}, \dots, b_X^{(1)},\\ &b_1^{(2)}, b_2^{(2)}, \dots, b_X^{(2)},\\ &\vdots\\ &b_1^{(N)}, b_2^{(N)}, \dots, b_X^{(N)},\\ &k_1^{(1)}, k_2^{(1)}, \dots, k_T^{(1)},\\ &k_1^{(2)}, k_2^{(2)}, \dots, k_T^{(2)},\\ &\vdots\\ &k_1^{(N)}, k_2^{(N)}, \dots, k_T^{(N)},\\ &c_{1-X}, c_{1-X+1}, \dots, c_{T-1}) \in \mathbb{R}^{(N+1)(X+T)+2X-1},\end{aligned}$$

which may be estimated by means of the maximum likelihood method. In the case of central exposures, the log-likelihood of the model is

$$\ell(\boldsymbol{\zeta}) = \sum_{x=1}^{X}\sum_{t=1}^{T} \chi_{xt}\Big(- E_{xt}^c m_{xt}^c + D_{xt}(\ln E_{xt}^c + \ln m_{xt}^c) - \ln(D_{xt}!)\Big),$$

where

$$m_{xt}^c = g^{-1}(\eta_{xt}) = e^{\eta_{xt}} \quad (x = 1, 2, \dots, X, \quad t = 1, 2, \dots, T),$$

and in the case of initial exposures, the log-likelihood function is

$$\ell(\boldsymbol{\zeta}) = \sum_{x=1}^{X}\sum_{t=1}^{T} \chi_{xt}\Big(\ln \binom{E_{xt}^0}{D_{xt}} + D_{xt} \ln m_{xt}^0 + (E_{xt}^0 - D_{xt}) \ln(1 - m_{xt}^0)\Big),$$

where

$$m_{xt}^0 = g^{-1}(\eta_{xt}) = \frac{1}{1 + e^{-\eta_{xt}}} \quad (x = 1, 2, \dots, X, \quad t = 1, 2, \dots, T).$$

The log-likelihood function has to be maximized with respect to $\boldsymbol{\zeta}$, subject to the identification constraints of the model variant in question. Brouhns et al. [2002a] propose the Newton method for the numerical optimization of the log-likelihood function.
Model selection may be perfomed using the likelihood ratio test, the Akaike and Bayesian information criteria or by estimating models on a training period and minimizing the out-of-sample forecasting error of the model in a subsequent test period.
The mortality indices $k_t^{(i)}$ $(i = 1, 2, \dots, N,\ t = 1, 2, \dots, T)$ are usually modeled and forecasted jointly as a multivariate random

walk with drift, and the cohort effects c_j ($j = 1-X, 2-X, \ldots, T-1$) are normally modeled and forecasted as an ARIMA process.[4] Point estimates of the systematic component of the model may be obtained by substituting the forecasted values of the mortality indices and the cohort effect into Equation (9) as follows:

$$\hat{\eta}_{x,T+j} = \hat{a}_x + \sum_{i=1}^{N} \hat{b}_x^{(i)} \hat{\mathbb{E}}(\hat{k}_{T+j}^{(i)}) + \hat{b}_x^{(0)} \hat{\mathbb{E}}(\hat{c}_{T+j-x}) \quad (x = 1, 2, \ldots, X, \quad j = 1, 2, \ldots). \tag{10}$$

Subsequently, point estimates of future mortality rates may be obtained by applying Equations (10) and (8) as follows:

$$\hat{m}_{x,T+j} = g^{-1}(\hat{\eta}_{x,T+j}) \quad (x = 1, 2, \ldots, X, \quad j = 1, 2, \ldots).$$

Forecasting uncertainty due to the error terms of the time series of mortality indices and cohort effects may be modeled using Monte Carlo simulation (Deák [1990]). However, this procedure underestimates the actual forecasting uncertainty as it fails to take any parameter uncertainty into account by treating the estimated parameters of the model for granted. In order to include parameter uncertainty in the estimates, Brouhns et al. [2005] propose semiparametric bootstrapping (Efron [1979]), while Koissi et al. [2006] proposes residual bootstrapping.
In the semiparametric bootstrap procedure proposed by Brouhns et al. [2005], $B \in \mathbb{N}_{>0}$ bootstrap samples of death counts D_{xt}^b ($b = 1, 2, \ldots, B$, $x = 1, 2, \ldots, X$, $t = 1, 2, \ldots, T$) are generated from Poisson distributions with means equal to the observed death counts, and the parameters of the selected GAPC and time series specifications are reestimated in each of the bootstrap samples. The probability distribution of the analyzed indicators (e.g. death rates or life expectancies) may be approximated by their empirical bootstrap distributions, and the approximation is asymptotically consistent as the number of bootstrap samples converges to infinity.
I now present the five particular members of the GAPC family which I applied in my analysis.

2.3.1 The Poisson Lee–Carter (LC) model

Brouhns et al. [2002a] suggest the Poisson Lee–Carter model with systematic component

$$\eta_{xt} = a_x + b_x k_t \quad (x = 1, 2, \ldots, X, \quad t = 1, 2, \ldots, T). \tag{11}$$

4. Renshaw–Haberman [2006] propose an ARIMA$(1, 1, 0)$ process with drift, while Plat [2009] suggest an ARIMA(2,2,0) process with drift.

Brouhns et al. [2002a] assume central exposures and propose the logarithmic link function. In contrast to Equation (3), Equation (11) contains no error terms, as the error variance is contained by the Poisson distributions of the hypothetical death counts in this model variant. The identification constraints of this model variant are identical to the ones proposed by Lee–Carter [1992]:

$$\sum_{x=1}^{X} b_x = 1,$$
$$\sum_{t=1}^{T} k_t = 0$$

The reestimation of the mortality index is not necessary in this model. It is worth noting that the original Lee–Carter [1992] model itself is not a member of the GAPC family.

2.3.2 The Renshaw–Haberman (RH) model

Renshaw–Haberman [2006] extend Equation (11) with a cohort effect:

$$\eta_{xt} = a_x + b_x^{(1)} k_t^{(1)} + b_x^{(0)} c_{t-x} \quad (x = 1, 2, \ldots, X, \quad t = 1, 2, \ldots, T).$$

They assume central mortality rates and the logarithmic link function.
Due to the numerical instability of the model, Haberman–Renshaw [2011] propose a simplified version of their model with the systematic component

$$\eta_{xt} = a_x + b_x k_t + c_{t-x} \quad (x = 1, 2, \ldots, X, \quad t = 1, 2, \ldots, T). \quad (12)$$

The authors propose the following identification constraints:

$$\sum_{x=1}^{X} b_x = 1,$$
$$\sum_{t=1}^{T} k_t = 0,$$
$$\sum_{i=1-X}^{T-1} c_i = 0.$$

I apply the simplified variant of the model according to Equation (12) in my analysis.

2.3.3 The Age–Period–Cohort (APC) model

The Age–Period–Cohort model (Carstensen [2007]) is a special case of the RH model. Its systematic component is

$$\eta_{xt} = a_x + k_t + c_{t-x} \quad (x = 1, 2, \ldots, X, \quad t = 1, 2, \ldots, T),$$

This model is typically applied in conjunction with central exposures, the logarithmic link function and the following identification constraints:

$$\sum_{t=1}^{T} k_t = 0,$$

$$\sum_{i=1-X}^{T-1} c_i = 0,$$
$$\sum_{i=1-X}^{T-1} i c_i = 0.$$

2.3.4 The Cairns–Blake–Dowd (CBD) model

The Cairns–Blake–Dowd [2006] model of old-age mortality has the systematic component

$$\eta_{xt} = k_t^{(1)} + (x - \bar{x})k_t^{(2)} \quad (x = 1, 2, \ldots, X, \quad t = 1, 2, \ldots, T), \tag{13}$$

where $\bar{x} = \frac{1+X}{2}$ is the arithmetic mean of the age indices in the model. The authors propose the application of the model in ages above $x_0 = 60$ years. η_{xt} in Equation (13) describes the mortality of people aged $x_0 + x$. Cairns–Blake–Dowd [2006] assume initial exposures and the logit link function. No identification constraints are necessary in this model.

2.3.5 The Plat model

The systematic component of the Plat [2009] model is

$$\begin{aligned}\eta_{xt} = a_x + k_t^{(1)} + (x - \bar{x})k_t^{(2)} + (\bar{x} - x)^+ k_t^{(3)} + c_{t-x} \\ (x = 1, 2, \ldots, X, \quad t = 1, 2, \ldots, T),\end{aligned}$$

where $\bar{x} = \frac{1+X}{2}$ is the arithmetic mean of age indices in the model and $(\bar{x} - x)^+ = \max\{\bar{x} - x; 0\}$.

In the special case of modeling old-age mortality, the author recommends the omission of the third mortality index $k_t^{(3)}$, resulting in the systematic component

$$\eta_{xt} = a_x + k_t^{(1)} + (x - \bar{x})k_t^{(2)} + c_{t-x} \quad (x = 1, 2, \ldots, X, \quad t = 1, 2, \ldots, T). \tag{14}$$

η_{xt} in Equation (14) describes the mortality of those aged $x_0 + x$, where x_0 is a base age (e.g. $x_0 = 60$).

I use the old-age variant of the Plat model according to Equation (14) in my analysis. Plat [2009] assumes central exposures and the logarithmic link function, and proposes the following identification constraints:

$$\sum_{t=1}^{T} k_t^{(1)} = 0,$$
$$\sum_{t=1}^{T} k_t^{(2)} = 0,$$
$$\sum_{i=1-X}^{T-1} c_i = 0,$$
$$\sum_{i=1-X}^{T-1} ic_i = 0.$$

3 Results

I present two case studies in this paper. The first case study analyzes the Hungarian public pension system, while the second one examines the role of longevity risk in the premium calculation of pension annuities.

3.1 Case study I: The sustainability of the Hungarian pension system

My first case study is built on the paper of Bajkó–Maknics–Tóth–Vékás [2015], which uses the Lee–Carter [1992] model to forecast some main demographic indicators of Hungary until 2035, and presents a cohort-based pension model, which the authors use to analyze the impact of the present demographic and macroeconomic trends as well as some hypothetical parametric pension policy measures on the sustainability of the Hungarian public pension system.
The authors use mortality data of the Hungarian general population from the period 1950–2012, and select the estimation base period of the years 1980–2012 based on the criterion of out-of-sample forecasting accuracy in the test period of the years 2001–2012, where accuracy is measured by the χ^2 test statistic for life tables (Benjamin–Pollard [1993]). They forecast the gender-specific mortality indices using an ARIMA$(1,1,1)$ specification, which they select by means of the Box–Jenkins methodology (Asteriou–Hall [2015]). They forecast life expentancies at birth to increase up to 82.12 years for females and 75.95 years for males by 2035 from the current values of 78.91 years for females and 72.13 years for males.
Following Hyndman [2007], Bajkó–Maknics–Tóth–Vékás [2015]

forecast age-grouped fertility rates using the estimation base period of 2000–2012 and official fertility data, and demonstrate the rapid increase of the mean childbearing age, which they expect to continue in the future. Their estimates indicate a slighlty increasing total fertility rate, which is nevertheless expected to remain well below the approximate critical replacement value of 2.1 (Espenshade et al. [2003]) throughout the forecasting period.

Based on their estimates of mortality and fertility, they forecast the population of the country by means of a simple recursion, and estimate the population size to decrease to 8 647 505 people by 2035, 51.5 per cent of whom are expected to be female. It is important to note that their results do not take the effect of migration into account. Additionally, they forecast a rapidly and progressively increasing old-age dependency ratio, which seriously jeopardizes the sustainability of the system.

They build a cohort-based model of the incomes and expenditures of the system using their own demographic forecasts and simple additional forecasts of future employment rates and real wages. Besides the mean estimates, they construct optimistic and pessimistic scenarios of the latter two indicators, which they apply for scenario-based sensitivity analysis. They forecast the incomes of the system based on their forecasts of age- and gender-specific population counts, employment rates and real wages as well as the legislation on pension contributions (Parliament [2014]). They do not take some minor sources of income (such as fines and government contributions) into account, which they consider impossible to model due to the extent of their past variability. They forecast gender-specific initial pensions at the retirement age using the pension formula defined by law, determine the percentages of pension-aged people who are entitled to government pensions based on projections and index already existing pensions by the assumed inflation rate. Beyond regular old-age pensions, they take other types of pensions into account by projecting current trends.

They validate their model by forecasting the main indicators of the model for the years 2012 to 2014 and comparing them to their actual realizations, and find the short-term forecasting accuracy of their model surpisingly good.

Numerical results of the model and the sensitivity analyses are discussed in Section 3.3. Overall the authors find the sustainability of the system highly problematic in the medium term, and beyond the short term measures of increasing pension contributions and the retirement age and keeping the indexation of pensions in line with the inflation rate, they propose two important long-term solution strategies: providing more incentives for childbearing as well as for middle-aged employment, where there is still room for improvement despite the otherwise increasing overall employment rate.

3.2 Case study II: Longevity risk in the premium calculation of pension annuities

The motivation of the second case study is the methodological upgrade of the analysis presented by Májer–Kovács [2011] and the assessment of the change in the impact of longevity risk on the premium calculation of pension annuities in the period between 2006 and 2014. I apply the Generalized Age–Period–Cohort family of models (Hunt–Blake [2014], Currie [2016] and Villegas et al. [2016]) in conjunction with the bootstrapping procedure proposed by Brouhns et al. [2005] to forecast old-age mortality rates and quantify the impact of longevity risk as well as the associated forecasting uncertainty.

I base my analysis on unisex mortality data (as mandated by EU [2004]) of the age group of 65 to 99 years of the Hungarian general population[5] from the period 1975–2014, and select the Cairns–Blake–Dowd [2006] model based on the estimated out-of-sample forecasting error of the five GAPC models presented in Section 2.3.[6] I assessed the out-of-sample forecasting accuracy on the period 2005–2014 by applying the χ^2 test statistc for life tables (Benjamin–Pollard [1993]) in the age group of 65 to 84 years.

The importance of the analysis is increased by the new legislation on pension annuities in relation to voluntary pension funds (Parliament [2015]), which mandates these funds to offer its members the option to purchase life annuities at retirement, complementing the option to purchase closed-term financial annuities, which have been available for a long time in this context, albeit being less appropriate for the desired purpose of providing old-age financial security to fund members.

I shall present further numerical results and their interpretation in Section 3.3.

3.3 Summary

I now summarize my main research questions and the answers to them in the following list:

- What are the expected trajectories of Hungarian male and female age-specific mortality rates, fertility rates, the popu-

5. Banyár [2012] elaborates on the question of the selection of the appropriate life table for pension annuities. As no company-specific mortality data were available, I had no option but to work with data for the general population.
6. I performed the necessary calculations in the statistical environment R (R [2008] and Villegas et al. [2016]).

lation of the country, the life expectancies at birth and the old-age dependency ratio until the year 2035?
Answer: The forecasts presented in the paper of Bajkó–Maknics–Tóth–Vékás [2015], which rely on the Lee–Carter [1992] model, indicate that both male and female age-specific mortality rates are expected to decrease significantly until 2035, besides a slightly increasing fertility rate, which is nevertheless expected to remain well below the critical value of 2.1. As a result of these processes, the population of Hungary is expected to drop below 8 648 000 people by 2035, and the age distribution of the population is expected to change significantly. Life expectancies at birth are expected to rise up to 76 years for men and 82 years for women, while the old-age dependency ratio is expected to increase above 40 per cent by 2035, which is one and a half times its current value, posing a serious challenge to the sustainability of the Hungarian public pension system.

- How long is the current balance of the incomes and expenditures of the Hungarian public pension system sustainable given the present trends in employment, mortality, fertility, real wage improvement and the current retirement age? What is the expected trajectory of the balance of the system? Which parametric changes may prolong the equilibrium of the system and how long are they effective?
Answer: The cohort-based pension model presented in the paper of Bajkó–Maknics–Tóth–Vékás [2015] predicts the balance of the system to drop into the negative range starting in 2026, and the deficit is expected to reach the size of 8 per cent of all pension-related taxes and contributions by 2035. According to the model, the problem may be resolved by increasing pension-related contributions as a percentage of gross wages by 4 per cent by 2035, or by increasing the share of the social contribution tax allocated to the public Pension Insurance Fund from the value of 85.46 per cent in 2015 back to the value of 96.3 per cent in 2014. However, this would be detrimental to the Health Insurance Fund, which is the other recipient of the same tax. If real wages evolve in the future according to the assumed optimistic and pessimistic scenarios then a deficit will first emerge in 2035 and 2022, respectively, assuming that all other parameters are unchanged. Similary, if employment evolves in the future according to the assumed optimistic and pessimistic scenarios then the first deficit will appear in the system in 2034 and 2023, respectively, given that all other parameters behave according to the base scenario. However, if the retirement age is continuously increased according to the expected increase of the life expectancy at the retirement age, as in Denmark, then the model indicates no deficit throughout

the forecasting period up to 2035.

- Which widely applied mortality forecasting method provides the most accurate description of Hungarian old-age mortality rates based on the criterion of out-of-sample estimation accuracy?
  Answer: Out of five popular mortality forecasting techniques, the Cairns–Blake–Dowd [2006] model, which is specifically recommended for the modeling of old-age mortality, has the best forecasting accuracy for the ages of 65 to 84 years on the testing period 2005–2014. This model produces the lowest growth rate of the estimation error as the time horizon increases, and a significantly smaller error than the otherwise second most precise Poisson Lee–Carter in the ages between 65 and 70 years, which are of paramount importance in the premium calculation of life annuities. The use of the overparametrized Plat [2009] and Renshaw–Haberman [2006] models leads to overfitting, which may be inferred from their excellent fit on the training period and their weak forecasting performance on the testing period, which deteriorates rapidly as the forecasting horizon increases.
- What is the size of the error and the associated financial loss if the annuity provider calculates the life expectancy at retirement and the net single premium of the life annuity starting at retirement based on the classical actuarial methodology, assuming static – instead of dynamic – mortality?
  Answer: Based on the dynamic unisex life table generated by the Cairns–Blake–Dowd [2006] model, the life expectancy at retirement is approximately two years higher than the corresponding value obtained under the assumption of static mortality. The static calculation underestimates the net single premium of the life annuity starting at retirement by 6.43 per cent, which causes an immediate deficit of 1 million 60 thousand HUF in the premium reserve and a financial loss of the same size in the case of a hypothetical life annuity with a yearly payout of 1 million HUF. This loss is highly significant in actuarial practice.
- Has the size of the premium calculation error resulting from ignoring longevity risk increased significantly in the past eight years?
  Answer: Yes. In the period between 2006 and 2014, the underpricing error caused by static premium calculation has increased from 4.51 per cent, as presented by Májer–Kovács [2011], to 6.43 per cent. This increase is highly significant in actuarial practice, and it is statistically significant beyond doubt due to the enormous sample size of the Hungarian general population, irrespectively of the type of statistical test applied.

Based on the answers to my research questions, I establish the following conclusions in relation to the research hypotheses presented in the introduction of my paper:

- Hypothesis 1: The improvement of Hungarian mortality along with the current trends in fertility and employment are jointly expected to lead to the excess of expenditures versus incomes of the Hungarian public pension system, which implies the unsustainability of the system in the medium term. *Conclusion: Yes, the balance of the Hungarian public pension system is expected to incur a progressively increasing deficit starting in 2026 under the standard model assumptions and the current parameters of the system, as demonstrated by the cohort model of Bajkó–Maknics–Tóth–Vékás [2015].*

- Hypothesis 2: One of the newer mortality forecasting methods that have evolved in the new millennium provides a better description of Hungarian old-age mortality rates than the classical Lee–Carter [1992] model. *Conclusion: Yes, the Cairns–Blake–Dowd [2006] two-factor mortality forecasting method provides the most accurate description of Hungarian old-age mortality rates based on the criterion of out-of-sample forecasting accuracy in the chosen test period.*

- Hypothesis 3: The importance of longevity risk in the premium calculation of life annuities starting at retirement increased in Hungary between 2006 and 2014. *Conclusion: Yes. By relying on a static life table, annuity providers would have incurred a significantly larger financial loss in 2014 than eight years earlier, based on the criteria of both practical and statistical significance, which implies that the significance of longevity risk in the premium calculation of life annuities increased in the same period. Additionally, since the insured population is typically subject to lower mortality rates than the general population, and the insured population who purchase annuity products typically have even lower mortality rates than the whole of the insured population, which may party be attributed to opportunistic anti-selection (Banyár [2003]), even larger errors may be expected in actual portfolios of life annuities. The importance of the question is further increased by the new government regulation on the provision of life annuities to members of voluntary private pension funds (Parliament [2015]), which is likely to lead to an expansion of the Hungarian market of life annuities.*

The results of this paper may and should be extended in several directions in the future. I mention three specific possible lines of future research. The analysis of the Hungarian public pension system should be amended by a similar model of the incomes and expenditures of the public healthcare system, which is also seriously affected by the problem of longevity risk. This is a highly challenging topic, as the methodology of the analysis of morbidity rates and health expenditures is very different from the modeling of pension systems, and it requires much more detailed datasets. Another promising direction is the application of pension microsimulation models. This methodology enables the modeler to take distributional effects into account, thereby enabling more accurate forecasts. Finally, I consider it important to create an integrated statistical and microeconomic model of anti-selection in the life annuity market, as even otherwise very accurate forecasts may be highly misleading if the hypothetical insurance portfolio assumed for premium calculation differs significantly from the actual composition of the portfolio of policies. A thoroughly validated anti-selection model based on a wide dataset would provide a useful tool for actuaries and researchers analyzing the Hungarian market of life annuities.

Beyond the answers to my research questions and the conclusions of my research hypotheses, the main methodological contributions of my analysis are the cohort-based pension model presented by Bajkó–Maknics–Tóth–Vékás [2015], which is tailored to the characteristics of the Hungarian public pension system, the application of the Lee–Carter [1992] model to Hungarian fertility rates and the application of the newest mortality forecasting methods and their unifying framework of Generalized Age–Period–Cohort models to Hungarian mortality data.

The main practical contributions of my analysis are the actuarial forecasting and sensitivity analysis of the indicators of the Hungarian pension system, the analysis of the role of longevity risk in the premium calculation of life annuities in Hungary and the detailed comparison of the most popular mortality forecasting methods on Hungarian mortality data in the GAPC framework.

I hope that researchers as well as practising pension and life insurance actuaries will benefit from my results in the future and successfully use them for the construction of models which take longevity risk into account in a methodologically proper way.

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