# A characterization of stable sets in assignment games

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**Abstract**

We consider von Neumann-Morgenstern stable sets in assignment games. In the symmetric case Shapley (1959) proved some necessary conditions of vNM stability. In this paper we generalize this result for any assignment game. We show that a $\mathcal{V}$ set of imputation is stable if and only if (i) is internally stable, (ii) is connected, (iii) contains an imputation with 0 payoff to all buyers and an imputation with 0 payoff to all sellers, (iv) contains the core of the semi-imputations in the rectangular set spanned by any two points of $\mathcal{V}$. With this characterization we give a new proof to the existence of stable sets. Moreover using these reult if the core is not stable we can construct infinite many stable set.

## 1 Introduction

Assignment games (Shapley and Shubik, 1972) are models of two-sided matching markets with transferable utilities where the aim of each player on one

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side is to form a profitable coalition with a player on the other side. Since only such bilateral cooperations are worthy, these games are completely defined by the matrix containing the cooperative worths of all possible pairings of players from the two sides.

Shapley and Shubik (1972) showed that the core of an assignment game is precisely the set of dual optimal solutions to the assignment optimization problem on the underlying matrix of mixed-pair profits. This implies that (i) every assignment game has a non-empty core; (ii) the core can be determined without explicitly generating the entire coalitional function of the game; and (iii) there are two special vertices of the core, in each of which every player from one side of the market receives his/her highest core-payoff while every player from the other side of the market receives his/her lowest core-payoff.

Besides the above fundamental results concerning the core, several important contributions dealing with other solution concepts have been published in the last decade. The classical solution concept proposed and studied by von Neumann and Morgenstern (1944) in their monumental work has remained an intriguing exception, although Solymosi and Raghavan (2001) characterized a subclass of assignment games where the core is the unique stable set. The existence question in the general case was settled affirmatively by Núñez and Rafels (2009), who proved that, as conjectured by Shapley (cf. Section 8.4 in (Shubik, 1984)), the union of the cores of certain derived subgames is always a stable set.

In special cases we know much more then the existence of stable sets. Bednay (2014) described every stable set in the one-seller assignment games as a graph of a special monotonic function. Shapley (1959) considered the symmetric market game (glove market). He showed some nice properties of the stable sets, for example every stable set is a monotonic curve end in one endpoint of this curve every buyer gets zero payoff in the other endpoint every seller gets zero payoff. In this paper we generalize the results of Shapley (1959) we show that most of the properties what he showed in the symmetric case also holds in any assignment games (with little changes) for stable sets. We add a new condition and with this we can characterize the stable sets in assignment games. With this characterizaton we can easily prove the result of Solymosi and Raghavan (2001) and the result of Núñez and Rafels (2009). Moreover we can prove that if the core of an assignment game is not stable then the game has infinite many stable sets. We can also prove that the stable set conjectured by Shapley is not only unique in the principal section of the game but it is the unique stable set which contains the buyeroptimal and the selleroptimal elements of the proóincipal section.

# 2 Preliminaries

## 2.1 Basic definitions

A transferable utility cooperative game on the nonempty finite *set* $P$ *of players* is defined by a *coalitional function* $w : 2^P \to \mathbb{R}$ satisfying $w(\varnothing) = 0$. The function $w$ specifies the worth of every *coalition* $S \subseteq P$.

Given a game $(P, w)$, a *payoff allocation* $x \in \mathbb{R}^P$ is called *feasible*, if $x(P) \leq w(P)$; *efficient*, if $x(P) = w(P)$; *individually rational*, if $x_i = x(\{i\}) \geq w(\{i\})$ for all $i \in P$; *coalitionally rational*, if $x(S) \geq w(S)$ for all $S \subseteq P$; where, using the standard notation, $x(S) = \sum_{i \in S} x_i$ if $S \neq \varnothing$, and $x(\varnothing) = 0$. We denote by $\mathcal{I}'(P, w)$ the *semi-imputation set* (i.e., the set of feasible and individually rational payoffs), by $\mathcal{I}(P, w)$ the *imputation set* (i.e., the set of efficient and individually rational payoffs), and by $\mathcal{C}(P, w)$ the *core* (i.e., the set of efficient and coalitionally rational payoffs) of the game $(P, w)$. Semi-imputations which are not efficent are called strict semi-imputations.

The game $(P, w)$ is called *superadditive*, if $S \cap T = \varnothing$ implies $w(S \cup T) \geq w(S) + w(T)$ for all $S, T \subseteq P$; *balanced*, if its core $\mathcal{C}(P, w)$ is not empty.

Given a game $(P, w)$, the *excess* $e(S, x) := w(S) - x(S)$ is the usual measure of gain (or loss if negative) to coalition $S \subseteq P$ if its members depart from allocation $x \in \mathbb{R}^P$ in order to form their own coalition. Note that $e(\varnothing, x) = 0$ for all $x \in \mathbb{R}^P$, and

$$\mathcal{C}(P, w) = \{x \in \mathbb{R}^P : e(P, x) = 0,\ e(S, x) \leq 0\ \ \forall\, S \subset P\},$$

i.e., the core is the set of allocations which yield nonpositive excess for all coalitions.

We say that *allocation* $y$ *dominates allocation* $x$ *via coalition* $S$ (notation: $y \operatorname{dom}_S x$) if $y(S) \leq w(S)$ and $y_k > x_k\ \forall\, k \in S$. We further say that *allocation* $y$ *dominates allocation* $x$ (notation: $y \operatorname{dom} x$) if there is a coalition $S$ such that $y$ dominates $x$ via $S$. We can also define the core of a game with the dominance relation. The core of a game consist the preimputations which are not dominated by any other preimputation. Similarly to this new definition of the core we can define the core of a set $\mathcal{X}$ by the elements of $\mathcal{X}$ which are not dominated by any other element of $\mathcal{X}$. Note that the dominance relation is irreflexive but need not be either asymmetric or transitive. This is the major source of the difficulties encountered when working with the following solution concept advocated by von Neumann and Morgenstern (1944). A (nonempty) set $\mathcal{Z}$ of imputations is called a *stable set* if the following two conditions hold:

- *(internal stability):* there exist no $x, y \in \mathcal{Z}$ such that $y \operatorname{dom} x$
- *(external stability):* for every $x \in \mathcal{I} \setminus \mathcal{Z}$ there exists $y \in \mathcal{Z}$ such that $y \operatorname{dom} x$.

Note that evry stable set is closed and the core is always a set of imputations which satisfies internal stability. It is commonly known that in superadditive games the core is precisely the set of imputations which are not dominated by any other imputation. Consequently, the core is a subset of any stable set.

Observe that for $x, y \in \mathcal{I}$, if $y \operatorname{dom}_S x$ then (i) $x(S) < w(S)$, i.e. an imputation can be dominated only via coalitions having positive excess at that imputation; and (ii) $2 \leq |S| \leq |P| - 1$, i.e. among imputations domination can occur only via a proper coalition containing at least two players. Another useful observation is that inessential coalitions are redundant for the domination relation. We call coalition $S$ inessential in a game $w$, if $w(S) \leq \sum_{1 \leq j \leq r} w(S_j)$ for a partition $S = \bigcup_{1 \leq j \leq r} S_j$, and call $S$ essential if it is not inessential. Suppose now that $y \operatorname{dom}_S x$ for some $S$ that is inessential because $w(S) \leq \sum_{1 \leq j \leq r} w(S_j)$. Then we must have $y \operatorname{dom}_{S_j} x$ for some $1 \leq j \leq r$. Consequently, if $\mathcal{E}(P, w)$ denotes the set of all essential coalitions in game $(P, w)$ then $\operatorname{dom} = \bigcup_{S \in \mathcal{E}(P,w)} \operatorname{dom}_S$.

We say a set $\mathcal{Z}$ is $\mathcal{X}$-stable if $\mathcal{Z} \subseteq \mathcal{X}$ and

- *(internal stability):* there exist no $x, y \in \mathcal{Z}$ such that $y \operatorname{dom} x$
- *(external stability):* for every $x \in \mathcal{X} \setminus \mathcal{Z}$ there exists $y \in \mathcal{Z}$ such that $y \operatorname{dom} x$.

This is a generalization of the stable set concept. The „normal” stable sets are the $\mathcal{I}$-stable sets (or $\mathcal{I}'$-stable sets).

## 2.2 Assignment games

In this paper we consider a special type of cooperative games. The player set is $P = M \cup N$ with $M \cap N = \emptyset$, players $i \in M = \{1, \dots, m\}$ are called sellers, and players $j \in N = \{1', \ldots, n'\}$ are called buyers. The coalitional function $w = w_A$ is generated from the $m \times n$ nonnegative matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{bmatrix}$$

consisting of the profits that pairs of a seller and a buyer can make. We define

$$w_A(S) = \max_{\sigma \in \Pi(S \cap M, S \cap N)} \sum_{i=1}^{m} a_{i\sigma(i)}$$

Where $\Pi(X, Y)$ denotes the value of the maximal matching between sets $X$ and $Y$. Notice that $w_A(S) = 0$ if $S \subseteq M$ or $S \subseteq N$. In particular, $w_A(\{k\}) = 0$ for all $k \in P$.

Assignment games are obviously superadditive. To simplify notation, we drop reference to $w_A$ or $A$ whenever this causes no confusion.

To emphasize the special role of the sellers and buyers, we shall write the payoff allocations as $(\boldsymbol{u}; \boldsymbol{v}) \in \mathbb{R}^m \times \mathbb{R}^n$.

In assignment game $w_A$ if domination occurs among semi-imputations it also occurs via coalitions $\{i, j'\}$ with $a_{ij} > 0$. We shall simply write $(\boldsymbol{u}; \boldsymbol{v}) \operatorname{dom}_{ij} (\boldsymbol{u}'; \boldsymbol{v}')$ if $u_i + v_j \leq a_{ij}$ and $u_i > u'_i$, $v_j > v'_j$. Since the set of essential coalitions consists of mixed-pair coalitions with positive value and the single-player coalitions, but domination between imputations is not possible via the 0-value single-player coalitions, we clearly have

$$\operatorname{dom} = \bigcup_{i \in M, j' \in N : a_{ij} > 0} \operatorname{dom}_{ij}.$$

We say that the mixed-pair $\{i, j'\}$ is active at imputation $(\boldsymbol{u}; \boldsymbol{v})$ if $0 < a_{ij} - (u_i + v_j)$, since $(\boldsymbol{u}; \boldsymbol{v})$ could be dominated by another imputation via the mixed-pair coalition $\{i, j'\}$.

# 3 The characterization

In this section we show that

**Theorem 3.1** *A set $\mathcal{V} \subseteq \mathcal{I}$ is stable in an assignment game if and only if it*

1. *is internally stable,*

2. *is connected,*

3. *contains an imputation with* 0 *payoff to all buyers and an imputation with* 0 *payoff to all sellers,*

4. *contains the core of the semi-imputations in the rectangular set spanned by any two points of* $\mathcal{V}$.

The necessity of these properties was proved by Shapley (1959) for glove markets (assignment games with $a_{ij} = 1$ for all $i \in M$ and $j \in N$). The proof of the necessity in the general case is similar to the proof of Shapley. Before the proof we need some preparation. Suppose that $\mathcal{V}$ is a subset of the set of imutations which satisfies the four conditions in 3.1 Theorem. We denote the coordinatewise maximum of the vectors $\boldsymbol{x}$ and $\boldsymbol{y}$ by $\vee$ and the minimum by $\wedge$. Observe that if $(\boldsymbol{x};\boldsymbol{y})$ dominates $(\boldsymbol{u}^1 \vee \boldsymbol{u}^2;\boldsymbol{v}^1 \wedge \boldsymbol{v}^2)$, then it also dominates $(\boldsymbol{u}^1;\boldsymbol{v}^1)$ or $(\boldsymbol{u}^2;\boldsymbol{v}^2)$. The set $\mathcal{X} \subseteq \mathcal{I}'$ is said to be a lattice if for every $(\boldsymbol{u}^1;\boldsymbol{v}^1),(\boldsymbol{u}^2;\boldsymbol{v}^2) \in \mathcal{X}$ the payoff vectors $(\boldsymbol{u}^1 \vee \boldsymbol{u}^2;\boldsymbol{v}^1 \wedge \boldsymbol{v}^2),(\boldsymbol{u}^1 \wedge \boldsymbol{u}^2;\boldsymbol{v}^1 \vee \boldsymbol{v}^2)$ are also in $\mathcal{X}$. Shapley and Shubik (1972) showed that the core of an assignment game is a lattice and Shapley (1959) showed that this also holds for stable sets in glove markets. This property is also true in assignment games. To see this suppose that for some stable set $\mathcal{V}$ the vector $(\boldsymbol{u}^1 \vee \boldsymbol{u}^2;\boldsymbol{v}^1 \wedge \boldsymbol{v}^2)$ is not in $\mathcal{V}$. If it is a semi-imputation it is dominated by an element of $\mathcal{V}$. In this case this vector also dominates $(\boldsymbol{u}^1;\boldsymbol{v}^1)$ or $(\boldsymbol{u}^2;\boldsymbol{v}^2)$ in contradiction with the internal stability of $\mathcal{V}$. If it is not a semi-imputation then $(\boldsymbol{u}^1 \wedge \boldsymbol{u}^2;\boldsymbol{v}^1 \vee \boldsymbol{v}^2)$ is a strict semi-imputation and since $\mathcal{V} \subseteq \mathcal{I}$ we have $(\boldsymbol{u}^1 \wedge \boldsymbol{u}^2;\boldsymbol{v}^1 \vee \boldsymbol{v}^2) \notin \mathcal{V}$ which leads to the same contradiction. See also in Núñez and Rafels (2013).

With the lattice property of the set $\mathcal{V}$ we can easily see the necessity of the third condition: since $\mathcal{V}$ is a closed lattice, there is a vector $(\underline{\boldsymbol{u}};\overline{\boldsymbol{v}}) \in \mathcal{V}$ which gives the minimal payoffs to the sellers and the maximal payoffs to the buyers. If $\underline{\boldsymbol{u}} \neq \boldsymbol{0}$, then $(\boldsymbol{0};\overline{\boldsymbol{v}})$ is a strict semi-imputation which is not dominated by $\mathcal{V}$ because no buyers can get more in $\mathcal{V}$ which contradicticts the external stability of $\mathcal{V}$.

Since $\operatorname{med}(x,y,z) = (x \vee y) \wedge (y \vee z) \wedge (z \vee x) = (x \wedge y) \vee (y \wedge z) \vee (z \wedge x)$ where $\operatorname{med}(x,y,z)$ denotes the median of $x, y$ and $z$, we have that the median of every three elements of $\mathcal{V}$ is also in $\mathcal{V}$. Observe that if $(\boldsymbol{x};\boldsymbol{y})$ is between $(\boldsymbol{u}^1;\boldsymbol{v}^1)$ and $(\boldsymbol{u}^2;\boldsymbol{v}^2)$ (which means $(\boldsymbol{x};\boldsymbol{y}) = \operatorname{med}((\boldsymbol{u}^1;\boldsymbol{v}^1),(\boldsymbol{x};\boldsymbol{y}),(\boldsymbol{u}^2;\boldsymbol{v}^2))$, $(\boldsymbol{u}^3;\boldsymbol{v}^3)\operatorname{dom}_{ij}(\boldsymbol{x};\boldsymbol{y})$ and $(\boldsymbol{u}^1;\boldsymbol{v}^1),(\boldsymbol{u}^2;\boldsymbol{v}^2),(\boldsymbol{u}^3;\boldsymbol{v}^3)$ don't dominate each other, then

$\operatorname{med}((\boldsymbol{u}^1;\boldsymbol{v}^1),(\boldsymbol{u}^2;\boldsymbol{v}^2),(\boldsymbol{u}^3;\boldsymbol{v}^3))\operatorname{dom}_{ij}(\boldsymbol{x};\boldsymbol{y})$.

If we use this observation for a vector $(\boldsymbol{x};\boldsymbol{y}) \notin \mathcal{V}$ which is between two elements $(\boldsymbol{u}^1;\boldsymbol{v}^1)$ and $(\boldsymbol{u}^2;\boldsymbol{v}^2)$ of $\mathcal{V}$, we have more than the external stability of $\mathcal{V}$: we get an element of $\mathcal{V}$ which dominates $(\boldsymbol{x};\boldsymbol{y})$ and this vector is between $(\boldsymbol{u}^1;\boldsymbol{v}^1)$ and $(\boldsymbol{u}^2;\boldsymbol{v}^2)$. From this property we get immediately the necessity of the fourth condition. We can also get the second one: we show that between every two points of $\mathcal{V}$ there is also a third point. Let $(\boldsymbol{u}^1;\boldsymbol{v}^1)$ and $(\boldsymbol{u}^2;\boldsymbol{v}^2)$ be two elements of $\mathcal{V}$. If the average of these two points is in $\mathcal{V}$ then we have a third point between $(\boldsymbol{u}^1;\boldsymbol{v}^1)$ and $(\boldsymbol{u}^2;\boldsymbol{v}^2)$. If the average is not in $\mathcal{V}$ then there is a vector $(\boldsymbol{u}^3;\boldsymbol{v}^3) \in \mathcal{V}$ which is between $(\boldsymbol{u}^1;\boldsymbol{v}^1)$ and $(\boldsymbol{u}^2;\boldsymbol{v}^2)$ and this vector dominates $(\boldsymbol{x};\boldsymbol{y})$. With the closedness of $\mathcal{V}$ we can prove following Shapley (1959) that every stable set is connected. To prove the sufficiency of these properties we need a couple of lemmas:

**Lemma 3.1** *Every set $\mathcal{V}$ satisfying the four properties in theorem 3.1 is a lattice.*

PROOF.
Let $(\boldsymbol{u}^1;\boldsymbol{v}^1), (\boldsymbol{u}^2;\boldsymbol{v}^2)$ be two elements of $\mathcal{V}$. Observe that the vectors $(\boldsymbol{u}^1 \vee \boldsymbol{u}^2;\boldsymbol{v}^1 \wedge \boldsymbol{v}^2)$ and $(\boldsymbol{u}^1 \wedge \boldsymbol{u}^2;\boldsymbol{v}^1 \vee \boldsymbol{v}^2)$ are not dominated by any vectors between $(\boldsymbol{u}^1;\boldsymbol{v}^1)$ and $(\boldsymbol{u}^2;\boldsymbol{v}^2)$. Because of the fourth condition if $(\boldsymbol{u}^1 \vee \boldsymbol{u}^2;\boldsymbol{v}^1 \wedge \boldsymbol{v}^2)$ or $(\boldsymbol{u}^1 \wedge \boldsymbol{u}^2;\boldsymbol{v}^1 \vee \boldsymbol{v}^2)$ is an imputation then it is also an element of $\mathcal{V}$. If $(\boldsymbol{u}^1 \vee \boldsymbol{u}^2;\boldsymbol{v}^1 \wedge \boldsymbol{v}^2)$ or $(\boldsymbol{u}^1 \vee \boldsymbol{u}^2;\boldsymbol{v}^1 \wedge \boldsymbol{v}^2)$ is a strict semi-imputation, then by the fourth condition it is an element of $\mathcal{V}$ which contradicts the condition $\mathcal{V} \subseteq \mathcal{I}$. If $(\boldsymbol{u}^1 \vee \boldsymbol{u}^2;\boldsymbol{v}^1 \wedge \boldsymbol{v}^2)$ or $(\boldsymbol{u}^1 \wedge \boldsymbol{u}^2;\boldsymbol{v}^1 \vee \boldsymbol{v}^2)$ is not a semi-imputation, then the other one is a strict semi-imputation which leads to a contradiction.
□

**Lemma 3.2** *Every two points of $\mathcal{V}$ is connected with a coordinatewise monotonic curve in $\mathcal{V}$.*

PROOF.

Let $(\boldsymbol{u}^0;\boldsymbol{v}^0)$ and $(\boldsymbol{u}^1;\boldsymbol{v}^1)$ be two elements of $\mathcal{V}$. We can assume that $\boldsymbol{u}^0 \leq \boldsymbol{u}^1$ and $\boldsymbol{v}^0 \geq \boldsymbol{v}^1$ because we showed in lemma 3.1 that $(\boldsymbol{u}^0 \vee \boldsymbol{u}^1;\boldsymbol{v}^0 \wedge \boldsymbol{v}^1) \in \mathcal{V}$ and if there is a monotone curve between $(\boldsymbol{u}^0;\boldsymbol{v}^0)$ and $(\boldsymbol{u}^0 \vee \boldsymbol{u}^1;\boldsymbol{v}^0 \wedge \boldsymbol{v}^1)$ and another one between $(\boldsymbol{u}^0 \vee \boldsymbol{u}^1;\boldsymbol{v}^0 \wedge \boldsymbol{v}^1)$ and $(\boldsymbol{u}^1;\boldsymbol{v}^1)$ and we connect these two curves together we get a monotone curve between $(\boldsymbol{u}^0;\boldsymbol{v}^0)$ and $(\boldsymbol{u}^1;\boldsymbol{v}^1)$. Since $\mathcal{V}$ is connected there is a continuous curve $(\boldsymbol{u}^t;\boldsymbol{v}^t)_{t\in[0;1]} \subseteq \mathcal{V}$ between $(\boldsymbol{u}^0;\boldsymbol{v}^0)$ and $(\boldsymbol{u}^1;\boldsymbol{v}^1)$. Let $\boldsymbol{u}'^t = \mathrm{med}(\boldsymbol{u}^0, \boldsymbol{u}^t, \boldsymbol{u}^1)$ and $\boldsymbol{v}'^t = \mathrm{med}(\boldsymbol{v}^0, \boldsymbol{v}^t, \boldsymbol{v}^1)$.

Since $\mathcal{V}$ is a lattice $(\boldsymbol{u}'^t;\boldsymbol{v}'^t)_{t\in[0;1]} \subseteq \mathcal{V}$. Let $\boldsymbol{u}''^t = \min_{s\le t}\boldsymbol{u}'^s$ and $\boldsymbol{v}''^t = \max_{s\le t}\boldsymbol{v}'^s$. Obviously the curve $(\boldsymbol{u}''^t;\boldsymbol{v}''^t)_{t\in[0;1]}$ is monotone, $(\boldsymbol{u}''^0;\boldsymbol{v}''^0) = (\boldsymbol{u}^0;\boldsymbol{v}^0)$, $(\boldsymbol{u}''^1;\boldsymbol{v}''^1) = (\boldsymbol{u}^1;\boldsymbol{v}^1)$ and since $\mathcal{V}$ is a lattice $(\boldsymbol{u}''^t;\boldsymbol{v}''^t)_{t\in[0;1]} \subseteq \mathcal{V}$. $\square$

With this lemma we can prove a condition which is stronger then the internal stability.

**Corollary 3.1** *Let $(\boldsymbol{x};\boldsymbol{y}),(\boldsymbol{u};\boldsymbol{v}) \in \mathcal{V}$ such that $x_i > u_i$ and $y_j > v_j$ for some $i \in M$ and $j' \in N$ then $u_i + v_j \ge a_{ij}$ (the internal stability states only $x_i + y_j > a_{ij}$*

PROOF.
Suppose that $u_i + v_j < a_{ij}$. Let $s,t \in \mathbb{R}$ such that $s+t \le a_{ij}, u_i < s < x_i$ and $v_j < t < y_j$. $(\boldsymbol{u}\vee\boldsymbol{x},\boldsymbol{v}\wedge\boldsymbol{y}),(\boldsymbol{u}\wedge\boldsymbol{x},\boldsymbol{v}\vee\boldsymbol{y}) \in \mathcal{V}$ because $\mathcal{V}$ is a lattice. There is a vector $(\boldsymbol{x}^1,\boldsymbol{y}^1) \in \mathcal{V}$ in the monotonic curve connecting $(\boldsymbol{u};\boldsymbol{v})$ and $(\boldsymbol{u}\vee\boldsymbol{x},\boldsymbol{v}\wedge\boldsymbol{y})$, and a point $(\boldsymbol{x}^2,\boldsymbol{y}^2) \in \mathcal{V}$ in the monotonic curve connecting $(\boldsymbol{x};\boldsymbol{y})$ and $(\boldsymbol{u}\wedge\boldsymbol{x},\boldsymbol{v}\vee\boldsymbol{y})$ such that $x_i^1 = s = x_i^2$. Note that $y_j^1 = v_j$ and $y_j^2 = y_j$. There is a vector $(\boldsymbol{x}^3,\boldsymbol{y}^3) \in \mathcal{V}$ in the monotonic curve connecting $(\boldsymbol{x}^1,\boldsymbol{y}^1)$ and $(\boldsymbol{x}^2,\boldsymbol{y}^2)$ such that $y_j^3 = t$. For this vector $x_i^3 = s$ means that $(\boldsymbol{x}^3;\boldsymbol{y}^3)\,\mathrm{dom}_{ij}(\boldsymbol{u};\boldsymbol{v})$ which contracicts the internal stability. $\square$

**Lemma 3.3** *Every set $\mathcal{V}$ satisfying the four properties in Theorem 3.1 is closed.*

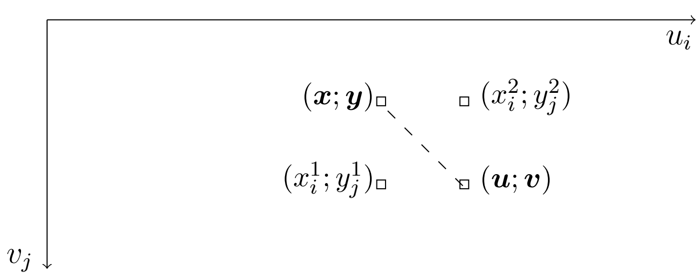
PROOF.
Let $(\boldsymbol{u}^i;\boldsymbol{v}^i)_{i\in\mathbb{N}} \subseteq \mathcal{V}$ and let $(\boldsymbol{u};\boldsymbol{v})$ be the limit of this sequence. Since each $(\boldsymbol{u}^i;\boldsymbol{v}^i)$ is in $\mathcal{V} \subseteq \mathcal{I}$ we get $(\boldsymbol{u};\boldsymbol{v}) \in \mathcal{I}$. By the second condition, there are elements $(\overline{\boldsymbol{u}};\boldsymbol{0})$ and $(\boldsymbol{0};\overline{\boldsymbol{v}})$ in $\mathcal{V}$. As $\mathcal{V}$ is a lattice, every element of $\mathcal{V}$ is between these two vectors. Since each $(\boldsymbol{u}^i;\boldsymbol{v}^i)$ is between $(\overline{\boldsymbol{u}};\boldsymbol{0})$ and $(\boldsymbol{0};\overline{\boldsymbol{v}})$ we get that $(\boldsymbol{u};\boldsymbol{v})$ is also between them.
Now suppose that $(\boldsymbol{u};\boldsymbol{v}) \notin \mathcal{V}$. Then $(\boldsymbol{u};\boldsymbol{v})$ is between $(\overline{\boldsymbol{u}};\boldsymbol{0})$ and $(\boldsymbol{0};\overline{\boldsymbol{v}})$, thus there is a mixed pair $\{i;j'\}$ which can dominate $(\boldsymbol{u};\boldsymbol{v})$ with a vector between $(\overline{\boldsymbol{u}};\boldsymbol{0})$ and $(\boldsymbol{0};\overline{\boldsymbol{v}})$. Because of lemma 3.2, there is a vector $(\boldsymbol{x};\boldsymbol{y}) \in \mathcal{V}$ between $(\overline{\boldsymbol{u}};\boldsymbol{0})$ and $(\boldsymbol{0};\overline{\boldsymbol{v}})$ such that $x_i - y_j = u_i - v_j$. If $x_i > u_i$ and $y_j > v_j$ then $\exists k : x_i > u_i^k, y_j > v_j^k$ and $u_i^k + v_j^k < a_{ij}$ in contradiction with 3.1 corollary.
Now we can assume that $x_i \le u_i$ and $y_j \le v_j$. Let $(\boldsymbol{x}^1;\boldsymbol{y}^1) = (\boldsymbol{u}\wedge\boldsymbol{x};\boldsymbol{v}\vee\boldsymbol{y})$, $(\boldsymbol{x}^2;\boldsymbol{y}^2) = (\boldsymbol{u}\vee\boldsymbol{x};\boldsymbol{v}\wedge\boldsymbol{y})$. There are two cases:

- $(\boldsymbol{x}^1;\boldsymbol{y}^1)$ or $(\boldsymbol{x}^2;\boldsymbol{y}^2)$ is a semi-imputation but is not in $\mathcal{V}$: assume that $(\boldsymbol{x}^1;\boldsymbol{y}^1)$ is this vector. By lemma 3.1, $(\boldsymbol{u}'^i;\boldsymbol{v}'^i) = (\boldsymbol{u}^i\wedge\boldsymbol{x};\boldsymbol{v}^i\vee\boldsymbol{y}) \in$

$\mathcal{V}$ $\forall i \in \mathbb{N}$ and $\lim(\boldsymbol{u}'^i; \boldsymbol{v}'^i) = (\boldsymbol{x}^1; \boldsymbol{y}^1) = (\boldsymbol{u}'; \boldsymbol{v}')$. Thus, $(\boldsymbol{x}^1; \boldsymbol{y}^1)$ is between $(\mathbf{0}; \overline{\boldsymbol{v}})$ and $(\boldsymbol{x}; \boldsymbol{y})$ but between these points there is no vector which dominates $(\boldsymbol{x}^1; \boldsymbol{y}^1)$ via the mixed-pair $\{i; j'\}$.

- both $(\boldsymbol{x}^1; \boldsymbol{y}^1), (\boldsymbol{x}^2; \boldsymbol{y}^2) \in \mathcal{V}$: since lemma 3.1, $(\boldsymbol{u}'^i; \boldsymbol{v}'^i) = \operatorname{med}((\boldsymbol{x}^1; \boldsymbol{y}^1); (\boldsymbol{u}^i; \boldsymbol{v}^i); (\boldsymbol{x}^2; \boldsymbol{y}^2)) \in \mathcal{V}$ $\forall i \in \mathbb{N}$ and $\lim(\boldsymbol{u}'^i; \boldsymbol{v}'^i) = (\boldsymbol{u}; \boldsymbol{v}) = (\boldsymbol{u}'; \boldsymbol{v}')$. Thus, $(\boldsymbol{u}; \boldsymbol{v})$ is between $(\boldsymbol{x}^1; \boldsymbol{y}^1)$ and $(\boldsymbol{x}^2; \boldsymbol{y}^2)$ but between these points there is no vector which dominates $(\boldsymbol{u}; \boldsymbol{v})$ via the mixed-pair $\{i; j'\}$.

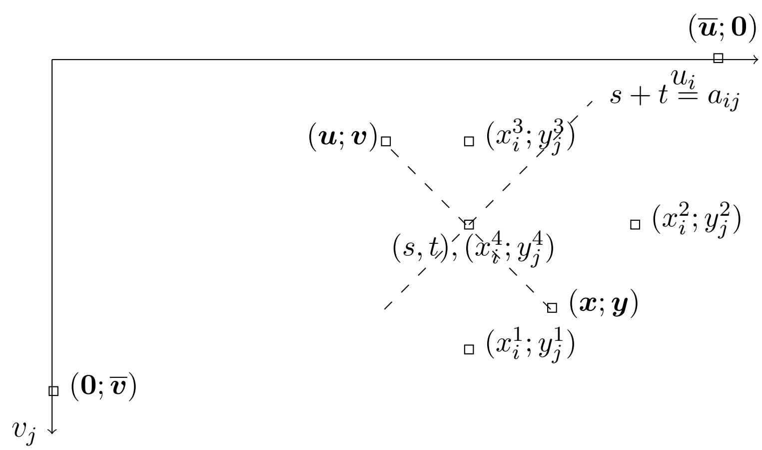
In both cases we got two points from $\mathcal{V}$ and a sequence $(\boldsymbol{u}'^i; \boldsymbol{v}'^i) \subseteq \mathcal{V}$ between them such that the limit of this sequence is outside of the set $\mathcal{V}$ and this limit is not dominated by any vector in the rectangular set spanned by the two points of $\mathcal{V}$. Now change $(\overline{\boldsymbol{u}}; \mathbf{0})$ and $(\mathbf{0}; \overline{\boldsymbol{v}})$ to these two points, $(\boldsymbol{u}; \boldsymbol{v})$ to $(\boldsymbol{u}'; \boldsymbol{v}')$ and the sequence $(\boldsymbol{u}^i; \boldsymbol{v}^i)$ to $(\boldsymbol{u}'^i; \boldsymbol{v}'^i)$. If we do this step again we can exclude another possible dominating mixed-pair. After a finite number of steps we exclude all mixed-pairs and we get two points of $\mathcal{V}$ and a third outside of $\mathcal{V}$ between them which is not dominated by any vector of the rectangular set spanned by the two vectors of $\mathcal{V}$ in contradiction with the fourth property. $\square$




Now we can prove the sufficiency of the four conditions. The proof will be very similar to the proof of lemma 3.3.

The internal stability of $\mathcal{V}$ is our first condition thus we only need to prove the external stability of $\mathcal{V}$. Let $(\boldsymbol{u}; \boldsymbol{v})$ be a semi-imputation outside of $\mathcal{V}$. We can assume that $(\boldsymbol{u}; \boldsymbol{v})$ is between $(\mathbf{0}; \overline{\boldsymbol{v}})$ and $(\overline{\boldsymbol{u}}; \mathbf{0})$. To see this suppose that this claim does not hold and let $(\boldsymbol{u}'; \boldsymbol{v}') = \operatorname{med}((\mathbf{0}; \overline{\boldsymbol{v}}); (\boldsymbol{u}; \boldsymbol{v}); (\overline{\boldsymbol{u}}; \mathbf{0}))$. This vector is also a semi-imputation outside of $\mathcal{V}$ and if this is dominated by a vector from $\mathcal{V}$, this vector also dominates $(\boldsymbol{u}; \boldsymbol{v})$.
By the fourth condition, there is at least one mixed pair which can dominate

$(\boldsymbol{u};\boldsymbol{v})$ between $(\mathbf{0};\overline{\boldsymbol{v}})$ and $(\overline{\boldsymbol{u}};\mathbf{0})$. The proof is similar to the proof of the closedness of $\mathcal{V}$. There are two cases:
1. There exists a mixed pair $\{i;j'\}$ such that $(\boldsymbol{u};\boldsymbol{v})$ can be dominated via this coalition between $(\mathbf{0};\overline{\boldsymbol{v}})$ and $(\overline{\boldsymbol{u}};\mathbf{0})$ and there is a vector $(\boldsymbol{x};\boldsymbol{y})\in\mathcal{V}$ between $(\mathbf{0};\overline{\boldsymbol{v}})$ and $(\overline{\boldsymbol{u}};\mathbf{0})$ such that $x_i>u_i$ and $y_i>v_j$
2. For each mixed pair $\{i;j'\}$ such that $(\boldsymbol{u};\boldsymbol{v})$ can be dominated via this coalition between $(\mathbf{0};\overline{\boldsymbol{v}})$ and $(\overline{\boldsymbol{u}};\mathbf{0})$ there is no vector $(\boldsymbol{x};\boldsymbol{y})\in\mathcal{V}$ between $(\mathbf{0};\overline{\boldsymbol{v}})$ and $(\overline{\boldsymbol{u}};\mathbf{0})$ with $x_i>u_i$ and $y_i>v_j$.
In the second case we can do the same as in the proof of the closedness of $\mathcal{V}$ because by the internal stability of $\mathcal{V}$ if $(\boldsymbol{u}';\boldsymbol{v}')$ is dominated by a vector from $\mathcal{V}$ the vector $(\boldsymbol{u};\boldsymbol{v})$ is also dominated via the same coalition.




In the first case if $x_i+y_j\le a_{ij}$ then $(\boldsymbol{x};\boldsymbol{y})$ dominates $(\boldsymbol{u};\boldsymbol{v})$. Let $x_i+y_j>a_{ij}$. Because of the connectedness of $\mathcal{V}$ we can assume that $u_i-v_j=x_i-y_j$. Let $s,t\in\mathbb{R}$ such that $s+t=a_{ij}$ and $s-t=u_i-v_j=x_i-y_j$. By lemma 3.2, there are two vectors $(\boldsymbol{x}^1;\boldsymbol{y}^1),(\boldsymbol{x}^2;\boldsymbol{y}^2)\in\mathcal{V}$ such that $(\boldsymbol{x}^1;\boldsymbol{y}^1)$ is between $(\mathbf{0};\overline{\boldsymbol{v}})$ and $(\boldsymbol{x};\boldsymbol{y})$, $(\boldsymbol{x}^2;\boldsymbol{y}^2)$ is between $(\boldsymbol{x};\boldsymbol{y})$ and $(\overline{\boldsymbol{u}};\mathbf{0})$, $x_i^1=s$ and $y_j^2=t$. Let $(\boldsymbol{x}^3;\boldsymbol{y}^3)=(\boldsymbol{x}^1\vee\boldsymbol{u};\boldsymbol{y}^1\wedge\boldsymbol{v})$ and $(\boldsymbol{x}^4;\boldsymbol{y}^4)=(\boldsymbol{x}^2\wedge\boldsymbol{x}^3;\boldsymbol{y}^2\vee\boldsymbol{y}^3)=\operatorname{med}((\boldsymbol{x}^1;\boldsymbol{y}^1),(\boldsymbol{u};\boldsymbol{v}),(\boldsymbol{x}^2;\boldsymbol{y}^2))$. Since $x_i^4=s$ and $y_j^4=t$, the vector $(\boldsymbol{x}^4;\boldsymbol{y}^4)$ dominates $(\boldsymbol{u};\boldsymbol{v})$. If it is in $\mathcal{V}$, we have proved that $\mathcal{V}$ dominates $(\boldsymbol{u};\boldsymbol{v})$. If $(\boldsymbol{x}^4;\boldsymbol{y}^4)\notin\mathcal{V}$ then there are two cases:
1. If $(\boldsymbol{x}^4;\boldsymbol{y}^4)$ is a semi-imputation, then it is enough to show that $\mathcal{V}$ dominates $(\boldsymbol{x}^4;\boldsymbol{y}^4)$ because if a vector from $\mathcal{V}$ dominates $\operatorname{med}((\boldsymbol{x}^1;\boldsymbol{y}^1),(\boldsymbol{u};\boldsymbol{v}),(\boldsymbol{x}^2;\boldsymbol{y}^2))$, then it also dominates one of $(\boldsymbol{x}^1;\boldsymbol{y}^1)$, $(\boldsymbol{u};\boldsymbol{v})$, $(\boldsymbol{x}^2;\boldsymbol{y}^2)$. Because of the internal

stability of $\mathcal{V}$, we get that this vector dominates $(\boldsymbol{u};\boldsymbol{v})$. Thus $(\boldsymbol{x}^4;\boldsymbol{y}^4)$ is between $(\boldsymbol{x}^1;\boldsymbol{y}^1)$ and $(\boldsymbol{x}^2;\boldsymbol{y}^2)$ and between these vectors the coalition $\{i;j'\}$ can't dominate anything. Thus we excluded one coalition.
2. If $(\boldsymbol{x}^4;\boldsymbol{y}^4)$ is not a semi-imputation, then $(\boldsymbol{u}\wedge\boldsymbol{x}^1;\boldsymbol{v}\vee\boldsymbol{y}^1)$ or $(\boldsymbol{u}\vee\boldsymbol{x}^2;\boldsymbol{v}\wedge\boldsymbol{y}^2)$ is a strict semi-imputation (because if $(\boldsymbol{x}^3;\boldsymbol{y}^3)$ is a semi-imutation, then $(\boldsymbol{x}^3\vee\boldsymbol{x}^2;\boldsymbol{y}^3\wedge\boldsymbol{y}^2)=(\boldsymbol{u}\vee\boldsymbol{x}^2;\boldsymbol{v}\wedge\boldsymbol{y}^2)$ or $(\boldsymbol{x}^4;\boldsymbol{y}^4)$ is a semi-imputation and if $(\boldsymbol{x}^3;\boldsymbol{y}^3)$ is not a semi-imputation, then $(\boldsymbol{u}\wedge\boldsymbol{x}^1;\boldsymbol{v}\vee\boldsymbol{y}^1)$ is a semi-imputation). Let this vector be $(\boldsymbol{x}^5;\boldsymbol{y}^5)$. If $(\boldsymbol{x}^5;\boldsymbol{y}^5)$ is dominated by $\mathcal{V}$ then $(\boldsymbol{u};\boldsymbol{v})$ is also dominated thus it is enough to show that $\mathcal{V}$ dominates $(\boldsymbol{x}^5;\boldsymbol{y}^5)$.

Now we can do the same, once again with $(\boldsymbol{x}^5;\boldsymbol{y}^5)$ instead of $(\boldsymbol{u};\boldsymbol{v})$, and $(\boldsymbol{x}^1;\boldsymbol{y}^1)$ instead of $(\overline{\boldsymbol{u}};\boldsymbol{0})$ or $(\boldsymbol{x}^2;\boldsymbol{y}^2)$ instead of $(\boldsymbol{0};\overline{\boldsymbol{v}})$. But now $(\boldsymbol{x}^5;\boldsymbol{y}^5)$ is a strict semi imputation, and because of the closedness of $\mathcal{V}$ there exists $\epsilon>0$ for all $(\boldsymbol{x};\boldsymbol{y})\in\mathcal{V}$ such that $x_i>x_i^5$ and $y_j>y_j^5$ satisfying $x_i+y_j>a_{ij}+\epsilon$ If we do the same the coalition $\{i;j'\}$ get more than in $(\boldsymbol{x}^5;\boldsymbol{y}^5)$ with at least $\epsilon/2$ thus after a finite number of repetition we get a vector $(\boldsymbol{x}^k;\boldsymbol{y}^k)\notin\mathcal{V}$ such that $x_i^k+y_j^k\geq a_{ij}$. If $(\boldsymbol{x}^k;\boldsymbol{y}^k)$ is dominated by $\mathcal{V}$ then $(\boldsymbol{u};\boldsymbol{v})$ is also dominated via the same coalition. Thus after a finite number of steps we can exclude one coalition.

□

Based on the above characterization we can give a simpler proof than in Núñez and Rafels (2013) to the conjecture by Shapley is stable. We can assume that in the main diagonal of $A$ there is an optimal assignment. We call principal section the subset of imputations in which all mixed pair in the main diagonal (which is a maximal matching) gets exactly their value. Shapley stated but have not prooved that the core of the principal section is stable. We will denote this set by $\mathcal{CB}=\{(\boldsymbol{x};\boldsymbol{y})\in\mathcal{I}:\ \forall\, i,j: x_i+y_j\geq a_{ij}$ or $x_i=a_{ii}$ or $y_j=a_{jj}\}$ Núñez and Rafels (2013) proved that this set is stable and it is the unique stable set in the principal section. Using the characterization we can get a stronger result for uniqueness. If we denote by $\boldsymbol{d}$ the main diagonal of $A$, we can easily see that this set is the core of the semi-imputations between $(\boldsymbol{0};\boldsymbol{d})$ and $(\boldsymbol{d};\boldsymbol{0})$. Observe that if this set is stable it is the unique stable set containing the vectors $(\boldsymbol{0};\boldsymbol{d})$ and $(\boldsymbol{d};\boldsymbol{0})$. To prove the stability of $\mathcal{CB}$ we will check the four conditions:

1. The internal stability is obvious from the definition of $\mathcal{CB}$
2. Similarly to the proof of the lattice property of the core or the stable sets it can be shown that $\mathcal{CB}$ is a lattice. It is known Shapley, Shubik (1972)

that the core of the game is nonempty. Let $(\boldsymbol{u};\boldsymbol{v}) \in \mathcal{CB}$. Obviously from the definition of $\mathcal{CB}$ for each $x \in \mathbb{R}$ the vector $\operatorname{med}((\mathbf{0};\boldsymbol{d}),(\boldsymbol{u}+\mathbf{1}x;\boldsymbol{v}-\mathbf{1}x),(\boldsymbol{d};\mathbf{0})) \in \mathcal{CB}$. Thus there exists a curve $(\boldsymbol{u}^t;\boldsymbol{v}^t)$ between $(\mathbf{0};\boldsymbol{d})$ and $(\boldsymbol{d};\mathbf{0})$. Let $(\boldsymbol{x}^1;\boldsymbol{y}^1)$ and $(\boldsymbol{x}^2;\boldsymbol{y}^2)$ be two elements of $\mathcal{CB}$. Since $\mathcal{CB}$ is a lattice the curve $\operatorname{med}((\boldsymbol{x}^1;\boldsymbol{y}^1);(\boldsymbol{u}^t;\boldsymbol{v}^t);(\boldsymbol{x}^2;\boldsymbol{y}^2))$ is a curve in $\mathcal{CB}$ connecting $(\boldsymbol{x}^1;\boldsymbol{y}^1)$ and $(\boldsymbol{x}^2;\boldsymbol{y}^2)$.

3. It is obvious from $(\mathbf{0};\boldsymbol{d}),(\boldsymbol{d};\mathbf{0}) \in \mathcal{B}$

4. Suppose not and there exist vectors $(\boldsymbol{u}^1;\boldsymbol{v}^1),(\boldsymbol{u}^2;\boldsymbol{v}^2) \in \mathcal{CB}$ and a vector $(\boldsymbol{x};\boldsymbol{y})$ between them and for every mixed pair at least one of the following condition holds: $x_i + y_j \geq a_{ij}$ or $x_i = u_i^1 \vee u_i^2$ or $y_j = v_j^1 \vee v_j^2$. In this case $(\boldsymbol{x};\boldsymbol{y})$ is also an element of $\mathcal{B}$. We can assume that $\boldsymbol{u}^1 \leq \boldsymbol{u}^2$ and $\boldsymbol{v}^1 \geq \boldsymbol{v}^2$. If $x_i = u_i^2$ then since $(\boldsymbol{u}^2;\boldsymbol{v}^2) \in \mathcal{B}$ at least one of the following must hold: $a_{ij} \leq u_i^2 + v_j^2 (\leq x_i + y_j)$ or $a_{ii} = u_i^2 (= x_i^2)$ or $a_{jj} = v_j^2 (\leq x_j^2 \leq v_j^1 \leq a_{jj})$. Similarly, we can prove that if $y_j = v_j^1$ then $a_{ij} \leq x_i + y_j$ or $a_{ii} = x_i$ or $a_{jj} = y_j$ must hold.

Since this set $\mathcal{CB}$ always contains the core and it is the core if and only if the matrix $A$ has a dominant diagonal we proved that the core of an assignment game is stable if and only if the matrix of the game has a dominant diagonal.

**Remark 3.1** *We can get a similar characterization of $\mathcal{X}$-stable sets if $\mathcal{X}$ is a connected lattice and it is a subset of the semi-imputation set. A set $\mathcal{V} \subseteq \mathcal{X}$ is $\mathcal{X}$-stable if and only if it*

1. *is internally stable,*
2. *is connected,*
3. *contains an buyeroptimal and the selleroptimal imputation of $\mathcal{X}$,*
4. *contains the core of the elements of $\mathcal{X}$ in the rectangular set spanned by any two points of $\mathcal{V}$.*

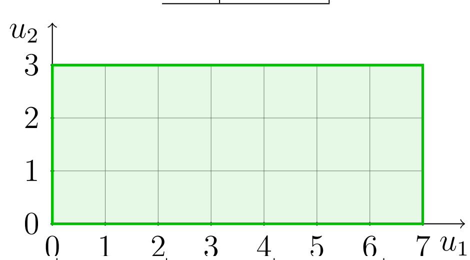
**Corollary 3.2** *Let $A, A' \in \mathcal{R}^{m\times n}$ such that $A \leq A'$ and $w_A(P) = w_{A'}(P)$. If $\mathcal{V}$ is stable in the assignment game belonging to the matrix $A$ and $\mathcal{V}'$ is $\mathcal{V}$-stable in the assignment game belongs to the matrix $A'$, then $\mathcal{V}'$ is stable (not only $\mathcal{V}$-stable) in the game belonging to $A'$.*

It can be easily checked that if $A$ and $A'$ differ in only one element the core of $V$ in the game belonging to $A'$ is always $V$-stable (and also stable) in the game belonging to $A'$.

With these corollaries we can construct stable sets, and give an other proof to the theorem of Núñez and Rafels (2009): if $A$ is a diagonal matrix then the principal section is obviously stable. In the first step we increase one element of the matrix $A$ and take the core of the original stable set in the new game. This set is stable in the new game. Then we inrease another element of the matrix and so on.
We can see how it works in the following example. In the first game the matrix is diagonal so the principal section $\{(u_1, u_2;\ v_1, v_2) \in \mathbb{R}^4_+ \text{such that} u_1 + v_1 = 7, u_2 + v_2 = 3\}$ is stable. Than we increase the $a_{12}$ element of the matrix to 4 and we take the core of the principal section in the new game. This is the trapezoid with vertices $(1; 0), (4; 3), (7; 3)$ and $(7; 0)$ (the payoffs of the sellers) and the line segment between $(0; 0)$ and $(1; 0)$. This set is setable in the new game. In the last step we increase the $a_{21}$ element of the matrix and we take the core (in the new game) of the stable set of the privious game. This is the union of the parallelogram and the two horizontal line segment in the third figure. This set is stable in the last game.

**Example 3.1**

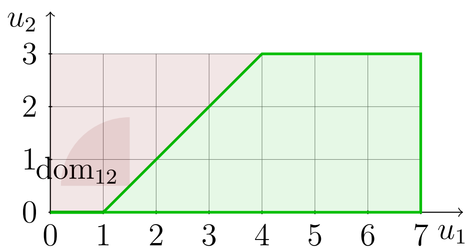
| | $v_1$ | $v_2$ |
|---|---|---|
| $u_1$ | 7 | 0 |
| $u_2$ | 0 | 3 |




| | $v_1$ | $v_2$ |
|---|---|---|
| $u_1$ | 7 | 0 |
| $u_2$ | 0 | 3 |

$\rightarrow$

| | $v_1$ | $v_2$ |
|---|---|---|
| $u_1$ | 7 | 4 |
| $u_2$ | 0 | 3 |



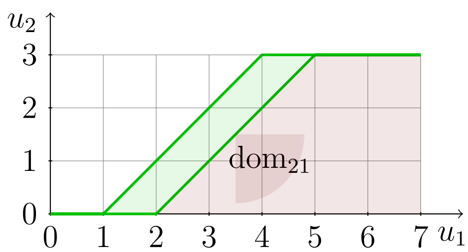

| | $v_1$ | $v_2$ |
|---|---|---|
| $u_1$ | 7 | 0 |
| $u_2$ | 0 | 3 |

$\rightarrow$

| | $v_1$ | $v_2$ |
|---|---|---|
| $u_1$ | 7 | 4 |
| $u_2$ | 0 | 3 |

$\rightarrow$

| | $v_1$ | $v_2$ |
|---|---|---|
| $u_1$ | 7 | 4 |
| $u_2$ | 5 | 3 |




If the core of an assignment game is not stable, then the game has infinite many stable sets.

It can be easily prove that in the 2-buyers, 2-sellers case if the core is not stable then the union of the core and at most 2 monotonic curve is a stable set. onen monotonic curve connects the buyer-optimal point of the core with vector such that every seller gets zero payoff, and the other monotonic curve connects the seller-optimal point of the core with vector such that every buyer gets zero payoff. In the example we can replace the line segment connecting the vectors $(5, 3;\ 2, 0)$ and $(7, 3;\ 0, 0)$ to any monotonic curve connecting the vectors $(5, 3;\ 2, 0)$ and $(a, b;\ 0, 0)$ such that $a \geq 5, b \geq 3$ and $a + b = 10$. Using this result we can easily construct ininite many stable sets in assignmet games with diagonal matrix except one element which is not dominant diagonal. And starting the construction above from these matrix and stable set we get infinite many different stable sets.

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