Which rationing rule does a single consumer follow?*

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Abstract

We will investigate the amount of residual demand in a market consisting of only one consumer and two producers. Since there is only one consumer, we cannot really speak about a rationing rule, but we can ask ourselves whether a known rationing rule reflects the consumer's utility maximizing behavior. We will show that, if the consumer has a Cobb-Douglas utility function, then the amount purchased by the consumer from the high-price firm lies between the values determined according to the efficient rationing rule and the random rationing rule. We will show further, that if the consumer has a quasilinear utility function, then in the economically interesting case his residual demand function will be equal to the residual demand function under efficient rationing.

1 Introduction

In Bertrand-Edgeworth duopolies quantities and prices are both decision variables. At first sight the simultaneous admittance of these two control variables leads to an underspecified model. Particularly, in the context of partial equilibrium analysis, where the consumers' side of the duopoly market is given by the aggregate demand curve, we can not determine the quantity demanded from the high-price firm. The missing item in the model is called a rationing rule. The aggregate demand function and the rationing rule together contain enough information to determine the sales of both duopolists. The only case in which the knowledge of the aggregate demand curve suffices is when the low-price firm covers the entire market. In Bertrand-Edgeworth type duopolies the low-price firm typically is not able or not interested in covering the entire market at the low-price. The cause for this behavior can be either capacity constraints or a U-shaped marginal cost functions.

There are many applicable rationing rules, but the two most frequently used rationing rules are the so called random rationing rule and the efficient rationing rule.

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2 Rationing rules

First we give a formal definition of a rationing rule. Let us denote the set of the admissible demand curves with $\mathcal{D} \subset \mathbb{R}_+^{\mathbb{R}_+}$.

Definition 1. A function is called a **rationing rule** if it assigns to every admissible demand function and to the duopolists' every quantity and price choices the saleable amount of products. Formally a rationing rule is a $h : \mathcal{D} \times \mathbb{R}_+^2 \times \mathbb{R}_+^2 \to \mathbb{R}_+^2$ function.

It is of main interest to find reasonable rationing rules. We will only discuss the two main rationing rules, namely the random and the efficient one.

In case of the random rationing rule the ratio of the satisfied demand at the low-price to the entire demand remains constant for all price levels above the low-price. In fact from the definition below the ratio is $1 - q_i/D(p_i)$.

Definition 2. A rationing rule $h : \mathcal{D} \times \mathbb{R}_+^2 \times \mathbb{R}_+^2 \to \mathbb{R}_+^2$ is called **random**, if $\forall j \in [1..2]$:

$$h_j(D, p_1, p_2, q_1, q_2) := \begin{cases} D(p_j) & \text{if} \quad p_j < p_i, \ i \neq j; \\ \frac{q_j}{q_1+q_2} D(p_j) & \text{if} \quad p_j = p_i, \ i \neq j; \\ \max\left((1 - \frac{q_i}{D(p_i)}) D(p_j), 0\right) & \text{if} \quad p_j > p_i, \ i \neq j. \end{cases}$$

By the efficient rationing rule the consumer with a higher reservation price is served before a consumer with a lower reservation price. Therefore, if we shift the demand curve leftward by the amount of sales at the low-price, then we will obtain the residual demand curve. This rationing rule is called efficient because at given prices and quantities it maximizes consumer surplus (see Tirole (1988)). Let us give also a formal definition for the efficient rationing rule.

Definition 3. A rationing rule $h : \mathcal{D} \times \mathbb{R}_+^2 \times \mathbb{R}_+^2 \to \mathbb{R}_+^2$ is called **efficient**, if $\forall j \in [1..2]$:

$$h_j(D, p_1, p_2, q_1, q_2) := \begin{cases} D(p_j) & \text{if} \quad p_j < p_i, \ i \neq j; \\ \frac{q_j}{q_1+q_2} D(p_j) & \text{if} \quad p_j = p_i, \ i \neq j; \\ \max(D(p_j) - q_i, 0) & \text{if} \quad p_j > p_i, \ i \neq j. \end{cases}$$

For market situations in which the application of the efficient or the random rationing rule is reasonable see, for example, Allen and Hellwig (1986), Gelman and Salop (1983), Tirole (1988) and Wolfstetter (1993).

3 The behavior of a single consumer market

Now we turn to the case, where the demand side of the market contains only one consumer. We have now two possibilities to determine the residual demand of the single consumer. First, given the microeconomic theory of consumer behaviour, we can formulate and solve the adequate consumer's utility maximizing problem explicitly for a given type of utility function. Second, we can determine the consumer's residual demand from the consumer's individual demand with the help of an explicitly chosen rationing rule. Of course the first method gives the right solution for the residual demand. But it is an interesting task to compare the results of the two methods. Of course, we cannot really speak about a rationing rule in a single consumer market, but we ask ourselves, which rationing rule is applicable.

Now we formulate of our problem. Following the main oligopolistic literature, our analysis will be of partial nature. Our consumer's utility function is $U(x,m)$, where x is the amount consumed from the duopolists' product and m is his consumption from a composite commodity, which we call from now on simply money. Furthermore, we assume that U is twice continuously differentiable, $U_x > 0$, $U_m > 0$. We denote with $\overline{m}$ our single consumer's amount of money and assume that this value is strictly positive. Assuming the first firm to be the low-price firm ($p_1 < p_2$), our consumer's utility maximizing problem takes the form as below:

$$\begin{array}{rcl} U(x_1 + x_2, \overline{m} - p_1 x_1 - p_2 x_2) & \rightarrow & \max \\ x_1 & \leq & q_1 \\ p_1 x_1 + p_2 x_2 & \leq & \overline{m} \\ x_1, x_2 & \geq & 0 \end{array} \tag{1}$$

The purchased amount of products from firm 1 and 2 are denoted by x_1 and x_2.

From our consumer's utility function we can derive his demand function. So we can determine the residual demand function belonging to a given rationing rule. To compare the residual demand function obtained by the second method with the solution of problem (1) is quite demanding, perhaps even impossible, because for general utility functions we can not solve problem (1) explicitly. But we can get positive results for special types of utility functions.

3.1 Cobb-Douglas utility function

In the Cobb-Douglas case we can relate the solution to the two main rationing rules. The results are summarized in the next proposition.

Proposition 1. *Assume that there is only one consumer on a duopoly market with utility function $u(x,m) = Ax^{\alpha}m^{\beta}$, where $0 < \alpha$, $0 < \beta$ and $\alpha + \beta \leq 1$. His money stock is positive and denoted by $\overline{m}$. The duopolists' prices are given, and let $0 < p_1 < p_2$. The low-price firm is offering $q_1 > 0$. Then the there exists a unique solution to the consumer's utility maximizing problem. Furthermore*

1. *if*
$$\overline{m} > p_1 q_1 + \frac{\beta}{\alpha} p_2 q_1, \tag{2}$$
then the optimal solution will be $x_1^* = q_1$,
$$x_2^* = \frac{\alpha \overline{m} - q_1(\alpha p_1 + \beta p_2)}{(\alpha + \beta) p_2} \tag{3}$$
and x_2^* is lying between the values suggested by the efficient and the random rationing rule;

2. if $\overline{m} \leq p_1 q_1 + \frac{\beta}{\alpha} p_2 q_1$*, then* $x_2^* = 0$.

Proof: Our utility maximizing consumer has to solve the following problem:

$$\begin{array}{rcl} A(x_1+x_2)^\alpha(\overline{m}-p_1x_1-p_2x_2)^\beta & \rightarrow & \max \\ x_1 & \leq & q_1 \\ p_1x_1+p_2x_2 & \leq & \overline{m} \\ x_1, x_2 & \geq & 0 \end{array} \tag{4}$$

We can check that the object function is strictly concave because of our restrictions imposed on the parameters α and β. Hence, uniqueness is guaranteed. The Lagrangian belonging to problem (4) is

$$\mathcal{L}(x_1,x_2,\lambda_1,\lambda_2) = A(x_1+x_2)^\alpha(\overline{m}-p_1x_1-p_2x_2)^\beta - \lambda_1(x_1-q_1) - \lambda_2(p_1x_1+p_2x_2-\overline{m})$$

and the appropriate Kuhn-Tucker conditions (5) are as follows.

$$\begin{array}{rcl} \frac{\partial \mathcal{L}}{\partial x_1} & = & A\alpha(x_1+x_2)^{\alpha-1}(\overline{m}-p_1x_1-p_2x_2)^\beta - \\ & & A\beta p_1(x_1+x_2)^\alpha(\overline{m}-p_1x_1-p_2x_2)^{\beta-1} - \lambda_1 - \lambda_2 p_1 \leq 0 \\ \frac{\partial \mathcal{L}}{\partial x_2} & = & A\alpha(x_1+x_2)^{\alpha-1}(\overline{m}-p_1x_1-p_2x_2)^\beta - \\ & & A\beta p_2(x_1+x_2)^\alpha(\overline{m}-p_1x_1-p_2x_2)^{\beta-1} - \lambda_2 p_2 \leq 0 \\ \frac{\partial \mathcal{L}}{\partial \lambda_1} & = & q_1 - x_1 \geq 0 \\ \frac{\partial \mathcal{L}}{\partial \lambda_2} & = & \overline{m} - p_1x_1 - p_2x_2 \geq 0 \quad \text{and} \\ & & x_1 \geq 0, \quad x_2 \geq 0, \quad \lambda_1 \geq 0, \quad \lambda_2 \geq 0 \quad \text{and} \\ & & x_1\frac{\partial \mathcal{L}}{\partial x_1} = 0, \quad x_2\frac{\partial \mathcal{L}}{\partial x_2} = 0, \quad \lambda_1\frac{\partial \mathcal{L}}{\partial \lambda_1} = 0, \quad \lambda_2\frac{\partial \mathcal{L}}{\partial \lambda_2} = 0. \end{array} \tag{5}$$

Observe that the Kuhn-Tucker conditions are not defined on set

$$S := \{(x_1,x_2) \in I\!R^2_+ | p_1q_1 + p_2q_2 = \overline{m}\}. \cup \{(0,0)\} \tag{6}$$

The values in S cannot be optimal, since their associated utility level is zero, but positive utility levels are obviously attainable. Therefore, $\lambda_2^* = 0$.

1. First let us assume that the optimal solution x_2^* is positive. We will show that $x_2^* > 0$ implies $x_1 > 0$ and $\lambda_1 > 0$. Let us assume that $\lambda_1 = 0$. Hence, if the first condition in (5) is satisfied, then the second condition will hold as strict inequality. Thus, $x_2 = 0$ would follow; a contradiction. Therefore, we conclude $\lambda_1 > 0$. Now $\lambda_1 > 0$ implies $x_1 > 0$ because $x_1 = q_1$ holds by the third complementary condition. Therefore, we have equalities in the first three condition of (5). We now have to look for nonnegative λ_1 fulfilling the first two equalities. From the second equality in (5) we can get

$$\frac{A\alpha}{p_2}(q_1+x_2^*)^{\alpha-1}(\overline{m}-p_1q_1-p_2x_2^*)^\beta - A\beta(q_1+x_2^*)^\alpha(\overline{m}-p_1q_1-p_2x_2^*)^{\beta-1} = 0$$

From the first equality in (5) the existence of nonnegative λ_1 follows because of $p_1 < p_2$. We can express x_2^* from the second equality in (5) and we will get (3). We can check that condition (2) is equivalent to $x_2^* > 0$ given in (3). It can be verified that x_2^* satisfies the budget constraint.

Now we show that the value in (3) lies really between the values suggested by the efficient ($x_2^e := D(p_2) - q_1$) and by the random ($x_2^r := D(p_2) - q_1 \frac{D(p_2)}{D(p_1)}$) rationing rules. We need the demand function of the Cobb-Douglas utility function

$$D(p) = \frac{\alpha \overline{m}}{p(\alpha+\beta)}, \tag{7}$$

which is well known (see for example Varian (1992)). Hence,

$$x_2^* = D(p_2) - q_1 \left(\frac{\alpha}{\alpha+\beta} \frac{D(p_2)}{D(p_1)} + \frac{\beta}{\alpha+\beta} \right) \tag{8}$$

Now using the fact that $D(p_2) < D(p_1)$ because of $p_1 < p_2$, we can verify that $x_2^e < x_2^* < x_2^r$ regarding the equalities below.

$$1 > \frac{\alpha}{\alpha+\beta} \frac{D(p_2)}{D(p_1)} + \frac{\beta}{\alpha+\beta} > \frac{D(p_2)}{D(p_1)} \tag{9}$$

To complete the proof of the first part of the proposition we still have to show that if $x_2 = 0$ is a solution of (4), then (2) cannot hold. We have to consider three cases: $x_1 = 0$, $0 < x_1 < q_1$ and $x_1 = q_1$.

(i) $x_1 = x_2 = 0$ cannot be a solution to (4) because $u(0, \overline{m}) = 0$ and a positive utility level is attainable.

(ii) If $0 < x_1 < q_1$, then from (5) $\lambda_1 = 0$ follows immediately. We already know that $\lambda_2 = 0$. Solving now for x_1, we will get $x_1 = \frac{\alpha \overline{m}}{(\alpha+\beta)p_1}$. But substituting this into the third inequality in (5), we will get a contradiction with (2).

(iii) If $x_1 = q_1$, then from the budget constraint we will obtain $p_1 q_1 \leq \overline{m}$. This is in contradiction with (2).

2. Controversially, let us assume that $x_2 > 0$ is a solution and (2) does not hold. We already saw in the first part that if $x_2 > 0$ is a solution, then x_2 must take the value given by (3). But by our assumption this has to be positive. Hence, (2) must hold. Therefore, we obtained a contradiction. □

Remark 1. Considering equation (8), we can see that if β is close to zero, than our consumer will act approximately according to the random rationing rule, while if α is close to zero, than our consumer will act approximately according to the efficient rationing rule.

3.2 Quasilinear utility function

Now, we assume that the single consumer has a quasilinear utility function. In particular, his utility function is $U(x, m) = u(x) + m$. Furthermore, we assume that u is twice continuously differentiable, $u' > 0$ and $u'' < 0$. Assuming that the first firm is the low-price firm ($p_1 < p_2$), the consumer's utility maximizing problem is

$$\begin{array}{rcl} u(x_1 + x_2) + \overline{m} - p_1 x_1 - p_2 x_2 & \rightarrow & \max \\ x_1 & \leq & q_1 \\ p_1 x_1 + p_2 x_2 & \leq & \overline{m} \\ x_1, x_2 & \geq & 0 \end{array} \tag{10}$$

Let us write down the Lagrangian belonging to problem (10):

$$\mathcal{L}(x_1, x_2, \lambda_1, \lambda_2) = u(x_1 + x_2) + \overline{m} - p_1x_1 - p_2x_2 - \lambda_1(x_1 - q_1) - \lambda_2(p_1x_1 + p_2x_2 - \overline{m})$$

The object function is twice continuously differentiable, strictly concave and the constraint functions are convex. Furthermore, Slater's condition is satisfied because of the assumptions $q_1 > 0$ and $\overline{m} > 0$. Therefore, the Kuhn-Tucker conditions (11) are equivalent to our problem (10).

$$\begin{array}{lll} \frac{\partial \mathcal{L}}{\partial x_1} = u'(x_1 + x_2) - p_1 - \lambda_1 - \lambda_2 p_1 \leq 0 & \text{and } \frac{\partial \mathcal{L}}{\partial x_1} = 0, & \text{if } x_1 > 0; \\ \frac{\partial \mathcal{L}}{\partial x_2} = u'(x_1 + x_2) - p_2 - \lambda_2 p_2 \leq 0 & \text{and } \frac{\partial \mathcal{L}}{\partial x_2} = 0, & \text{if } x_2 > 0; \\ \frac{\partial \mathcal{L}}{\partial \lambda_1} = q_1 - x_1 \geq 0 & \text{and } \frac{\partial \mathcal{L}}{\partial \lambda_1} = 0, & \text{if } \lambda_1 > 0; \\ \frac{\partial \mathcal{L}}{\partial \lambda_2} = \overline{m} - p_1x_1 - p_2x_2 \geq 0 & \text{and } \frac{\partial \mathcal{L}}{\partial \lambda_2} = 0, & \text{if } \lambda_2 > 0. \end{array} \tag{11}$$

We can obtain the demand function for the quasilinear utility function easily. The individual demand function of a consumer with a quasilinear utility function is

$$d(p) = \begin{cases} (u')^{-1}(p), & \text{if } u'(\overline{m}/p) < p, \\ \overline{m}/p, & \text{if } u'(\overline{m}/p) \geq p. \end{cases} \tag{12}$$

Now let us turn back to our original problem (11). As a first step let us regard the following proposition:

Proposition 2. *Assume that $0 < p_1 < p_2$, $q_1 > 0$, there is only one consumer with money stock $\overline{m} > 0$ and utility function $U(x, m) = u(x) + m$, where $u \in \mathcal{C}^2(\mathbb{R}_+)$, $u' > 0$ and $u'' < 0$ in the market. Then there exists a unique solution to problem (10) denoted by x_1^*, x_2^*. Moreover,*

1. *if $x_2^* > 0$, $u'(q_1 + x_2^*) > p_2$*
   (a) *and $u'(\frac{\overline{m}}{p_1}) \geq p_1$, then the consumer behaves according to the random rationing rule;*
   (b) *and $u'(\frac{\overline{m}}{p_1}) < p_1$, then the consumer demands even more than determined by the random rule;*
2. *if $x_2^* > 0$ and $u'(q_1 + x_2^*) = p_2$, then the consumer behaves according to the efficient rationing rule.*

Proof: Let x_1^*, x_2^*, λ_1^* and λ_2^* be a solution of problem (11). Our assumptions about u assures the existence and the uniqueness of solution x_1^*, x_2^* of problem (10), since the constraint set is nonempty, compact and convex and the object function is strictly concave.

First, we will show that $x_2^* > 1$ implies $x_1 > 0$ and $\lambda_1 > 0$. Let us assume that $\lambda_1 = 0$. Hence, if the first condition in (11) is satisfied, then the second condition will hold as strict inequality. Thus, $x_2 = 0$ would follow; a contradiction. Therefore, we conclude that $\lambda_1 > 0$. Now $\lambda_1 > 0$ implies $x_1 > 0$, since $x_1 = q_1$ holds by the third complementary condition. We only have to consider two cases in function of the last inequality of (11).

We have to examine the conditions under which there exist $\lambda_1^* > 0$, $\lambda_2^* \geq 0$, such that together with x_1^*, x_2^* we obtain a solution of (11). The first two equalities are

$$\begin{bmatrix} u'(q_1 + x_2) - p_1 \\ u'(q_1 + x_2) - p_2 \end{bmatrix} = \begin{bmatrix} 1 & p_1 \\ 0 & p_2 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} \tag{13}$$

This equality system has to be solvable for positive λ_1 and nonnegative λ_2. The matrix of (13) is invertible and we can obtain the next equivalent system.

$$\begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} = \frac{1}{p_2} \begin{bmatrix} p_2 & -p_1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u'(q_1 + x_2) - p_1 \\ u'(q_1 + x_2) - p_2 \end{bmatrix} \tag{14}$$

$\lambda_1 > 0$ results from $p_2 > p_1$ and $u' > 0$. For λ_2 we have to consider two cases.

1. In the first point of the proposition we made the following assumption.

$$u'(q_1 + x_2^*) > p_2 \tag{15}$$

$\lambda_2 > 0$ is equivalent to assumption (15) by (14). Furthermore, $\lambda_2 > 0$ implies that equality holds in the last inequality of (11) by the complementary conditions. Therefore, our consumer spends his whole money. Solving the last two equalities, we get $x_1^* = q_1$ and $x_2^* = \frac{\overline{m} - p_1 q_1}{p_2}$.

(a) Now we will show that if (15) and $u'(\frac{\overline{m}}{p_1}) \geq p_1$ holds, our consumer acts according to the random rationing rule. Because $\overline{m}/p_2 < q_1 + x_2$ we conclude $u'(\overline{m}/p_2) > p_2$ from condition (15). Now using (12), we obtain $d(p_2) = \overline{m}/p_2$. If we use (12) again, we will get $d(p_1) = \overline{m}/p_1$ because of our $u'(\overline{m}/p_1) \geq p_1$ assumption. Therefore, the

$$x_2^* = \frac{\overline{m} - p_1 q_1}{p_2} = \frac{\overline{m}}{p_2}(1 - \frac{q_1}{d(p_1)}) = d(p_2)(1 - \frac{q_1}{d(p_1)}) = x_2^r \tag{16}$$

equalities hold true. We can verify, that $q_1 \leq D(p_1)$, because otherwise, we would get $x_2^* = 0$. We recognize the random rationing rule in (16).

(b) Now we assume that (15) and $u'(\overline{m}/p_1) < p_1$ holds true. This second assumption is equivalent to $\overline{m}/p_1 > (u')^{-1}(p_1)$. We apply (12) again. Hence, $d(p_1) = (u')^{-1}(p_1)$. Thus, on the one hand we get inequality

$$x_2^* = \frac{\overline{m} - p_1 q_1}{p_2} = \frac{\overline{m}}{p_2}(1 - \frac{q_1}{\overline{m}/p_1}) > \frac{\overline{m}}{p_2}(1 - \frac{q_1}{d(p_1)}) = d(p_2)(1 - \frac{q_1}{d(p_1)}), \tag{17}$$

which is what we wanted to prove. As a limiting case we will get that if the amount of money spent at the low-price firm is almost negligible in relation to the money stock, then the residual demand would almost be equal to the demand.

Economically condition (15) means, that our consumer's marginal utility is greater than the high-price. So our consumer is willing to spend his entire money stock on the product.

2. We have to consider the case of $\lambda_2 = 0$. In the second point of the proposition we assumed that the optimal solution satisfies the following equality.

$$u'(q_1 + x_2^*) = p_2 \tag{18}$$

This is exactly equivalent to $\lambda_2 = 0$ by (14). Hence, $x_2^* = (u')^{-1}(p_2) - q_1$. Therefore, our consumer acts according to the efficient rationing rule. Economically, condition (18) means that our consumer's marginal utility is equal to the price of the high-price firm. We saw that the utility of holding money could hinder our consumer to spend his entire money stock on the product, since the equality in the forth condition in (11) is not assured. $\square$

At first sight the condition for the efficient rationing rule is more plausible. If we accepted that money means in this context a composite commodity, then it would be quite unrealistic to assume that our consumer consumes only the product offered by our duopolists. In defense for the random rationing rule we could bring forward the extreme case in which the product sold by the duopolists is the only basic good for survival and that our consumer is too poor to spend money on other goods.

Without calculating too much, one would surely suggest the efficient rationing rule to be applied for the following reason. The individual demand function tells our consumer how many products he will buy at the high-price, particularly $d(p_2)$. At the low-price he bought q_1. Now he obviously wants to buy $\max\{d(p_2) - q_1, 0\}$ products from the high-price firm. The only pitfall in this argument is, that we neglect the income effect, which results from the fact that he bought the first q_1 products cheaper and so we must not use directly $d(p_2)$ to calculate his extra demand at price level p_2. In fact, in case of a quasilinear utility function, there will be only an income effect, if the consumer's budget constraint is binding.

The next proposition summarizes the entire solution of problem (10).

Proposition 3. *Under the assumptions of proposition 2 the explicit solution of problem (10) is:*

1. *if* $u'(0) \leq p_1$, *then* $x_1^* = 0$ *and* $x_2^* = 0$;
2. *if* $u'(0) > p_1$ *and* $\overline{m} \leq p_1 q_1$, *then* $x_1^* = \min\{(u')^{-1}(p_1),\ \overline{m}/p_1\}$ *and* $x_2^* = 0$;
3. *if* $u'(0) > p_1$, $\overline{m} > p_1 q_1$ *and* $u'(q_1 + \frac{\overline{m} - p_1 q_1}{p_2}) > p_2$, *then* $x_1^* = q_1$ *and* $x_2^* = \frac{\overline{m} - p_1 q_1}{p_2}$;
4. *if* $u'(0) > p_1$, $\overline{m} > p_1 q_1$, $u'(q_1) > p_1$ *and* $u'(q_1 + \frac{\overline{m} - p_1 q_1}{p_2}) \leq p_2$, *then* $x_1^* = q_1$ *and* $x_2^* = \max\{(u')^{-1}(p_2) - q_1, 0\}$.
5. *if* $u'(0) > p_1$, $\overline{m} > p_1 q_1$, $u'(q_1) \leq p_1$ *and* $u'(q_1 + \frac{\overline{m} - p_1 q_1}{p_2}) \leq p_2$, *then* $x_1^* = (u')^{-1}(p_1)$ *and* $x_2^* = 0$.

Proof: 1. Let us assume that in spite of our assumption x_1^* or x_2^* is positive. Hence, $p_2 > p_1 \geq u'(0) > u'(x_1^* + x_2^*)$. Thus, the first two conditions in (11) could not been satisfied; a contradiction.

2. Obviously, $\overline{m} \leq p_1 q_1$ implies $x_1^* \leq q_1$. If we suppose that $(u')^{-1}(p_1) \geq \frac{\overline{m}}{p_1}$, then we can verify that $x_1^* = \overline{m}/p_1$, $x_2^* = 0$, $\lambda_1^* = 0$ and $\lambda_2^* = \frac{u'(\overline{m}/p_1) - p_1}{p_1}$ is a solution of problem (11).

Otherwise, $x_1^* = (u')^{-1}(p_1)$, $x_2^* = 0$, $\lambda_1^* = 0$ and $\lambda_2^* = 0$ is a solution of problem (11).

3. We have to verify that if $u'(0) > p_1$, $\overline{m} > p_1 q_1$ and $u'(q_1 + \frac{\overline{m} - p_1 q_1}{p_2}) > p_2$, then $x_1^* = q_1$ and $x_2^* = \frac{\overline{m} - p_1 q_1}{p_2}$ will be a solution of (11). We immediately see that the last two conditions in (11) hold as equalities for x_1^* and x_2^*. Therefore, we have to show that there exist nonnegative λ_1^* and λ_2^*, which are together with x_1^* and x_2^* a solution of (11). But regarding our assumptions, we have already shown this in the proof of the previous proposition.

4. We have to consider two cases. In the first case let us assume that $u'(q_1) > p_2$. This implies that there exists a value $\hat{x}_2 \in (0, \frac{\overline{m} - p_1 q_1}{p_2})$ such that $u'(q_1 + \hat{x}_2) = p_2$. Now applying the second part of proposition 2 we obtain what has to be proved.

In the second case we now assume the opposite. In particular, we assume that $u'(q_1) \leq p_2$. We now show that $x_1^* = q_1$ and $x_2^* = 0$ is the solution. Obviously, the last two conditions in (11) are satisfied. From the last one we further get $\lambda_2 = 0$. The second condition can now be written as $u'(q_1) \leq p_2$ which is now fulfilled by our assumption. The first condition takes the form $u'(q_1) = p_1 + \lambda_1$ because of $x_1^* > 0$. This equation is solvable for nonnegative λ_1 because $p_1 < p_2$ and in point 4 we already assumed that $u'(q_1) > p_1$.

5. We have only to verify that $x_1^* = (u')^{-1}(p_1)$, $x_2^* = 0$, $\lambda_1^* = 0$ and $\lambda_2^* = 0$ is a solution to problem (11). But this is obvious. $\square$

4 Summary

We have investigated the residual demand of a single consumer in a duopoly market. We have deduced the behavior of a single consumer in a duopolistic market in cases of quasilinear and Cobb-Douglas utility functions. Finally we have compared the obtained results with the values suggested by the two most frequently used rationing rules.

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