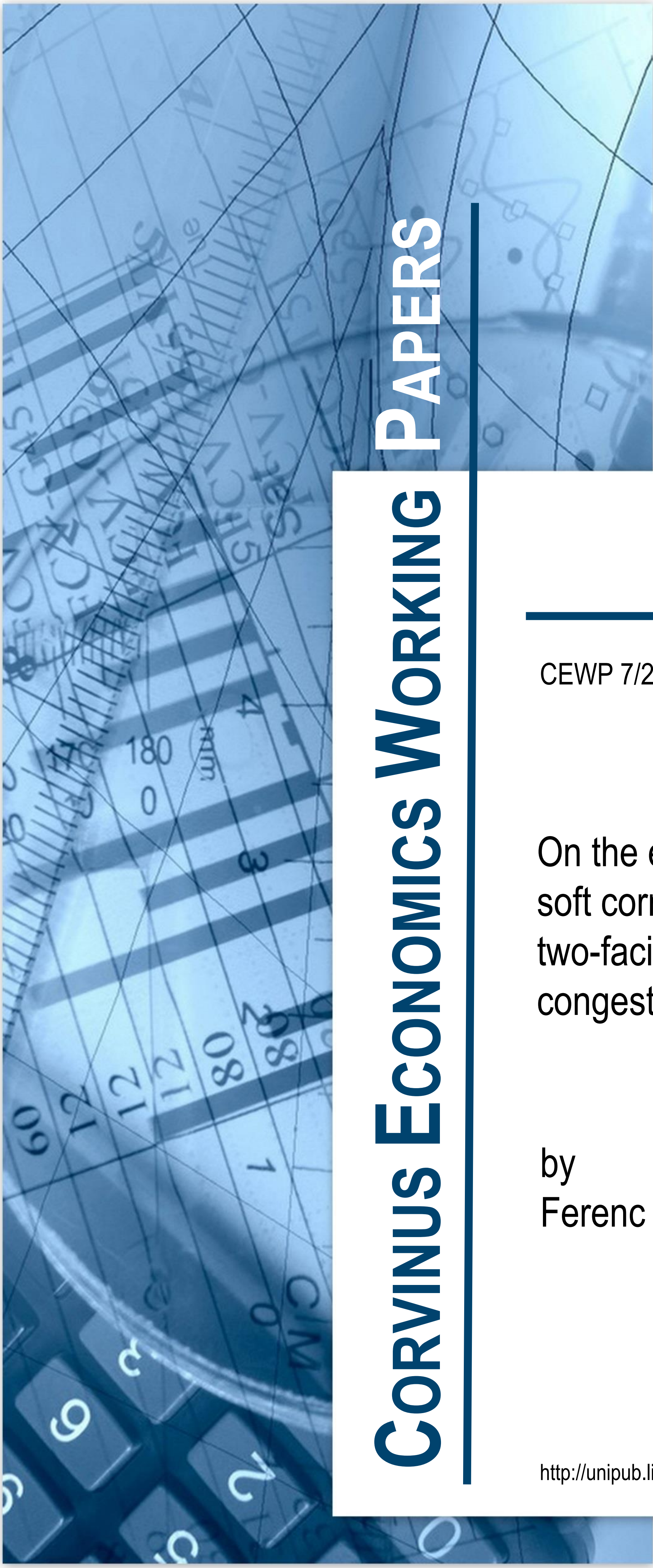

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On the enforcement value of soft correlated equilibrium for two-facility simple linear congestion games

by
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Abstract Exact enforcement values (Ashlagi I, Monderer D and Tennenholz M (2008) Journal of Artificial Intelligence 33:575-613) of soft correlated equilibrium (Forgó F (2010) Mathematical Social Sciences 60:186-190) for non-decreasing and mixed two-facility simple linear congestion games (including n-person chicken and prisoners' dilemma games) are determined and found to be 1 and 2, respectively. For non-inreasing two-facility simple linear congestion games lower and upper bounds are given for the enforcement value. The upper bound $1,265625$ is significantly better than the previously known $1,333$.

Keywords Soft correlated equilibrium, congestion games, chicken game, prisoners' dilemma, enforcement value

JEL Classification Number C72

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1 Introduction

Correlated equilibrium (CE) was introduced by Aumann (1974) as a generalization of Nash equilibrium (NE), Nash (1950, 1951). By adding a pre-game phase to a normal-form game it is defined as an NE of the extended game. Originally the pre-game phase presupposes a mediator who does a lottery according to a commonly known distribution over the strategy profiles and then, without letting the others know it, recommends each player to play her strategy in the selected strategy profile. Then she either accepts the proposal and implements it or chooses some other strategy. The probability distribution is said to be a CE if following collectively the recommendations is an NE of the extended game i.e. each player's expected utility (payoff) cannot be improved by deviating from the recommendation provided the rest of the players do accept the mediator's advice. By agreeing to participate in the extended game the social welfare (SW) as measured e.g. by the sum of the players's utility (or average utility) can be more than the SW in any NE. There are, however, games where CE is of no help in improving NE outcomes.

Generalizations of CE's aim at improving SW beyond the levels CE's can reach. This is done by changing the protocol of the pre-game phase. The price to pay is a stricter protocol, more commitment required of the players. The protocol of coarse correlated equilibrium (CCE) introduced by Moulin and Vial (1978) requires the players to commit to blindly follow the recommendation of the mediator whatever it may be. Each player is allowed to deny commitment and play freely any strategy. A CCE is a probability distribution for which no player can improve her payoff by denying commitment provided everybody else commits. There are examples and entire classes of games Moulin and Varet (1978), Moulin et al (2014a), (2014b) where CCE outperforms CE (and consequently NE). Soft correlated equilibrium (SCE) Forgó (2010) is another generalization of CE. The protocol of CE is "slightly" different from that of CCE: a player who does not want to commit can choose freely any other strategy except the one selected by the lottery for her. CCE and SCE are both generalizations of CE but not of each other as shown in Forgó (2010). There are games, however, where SCE is a generalization of CCE. An important class where this is true are binary games, i.e. games where every player has only two choices.

In this paper we are concerned with measuring the performance of SCE over a class of games C by how close it can approach the absolute maximum of SW that can be achieved if players obey a benevolent dictator. We will use the enforcement value (EV) as defined by Ashlagi et al. (2005). For a game $G \in C$ the enforcement value $EV(G)$ is the ratio of the absolute maximum of SW and the maximum an SCE is able to realize. The EV of the class C is then defined as $EV = \sup_{G \in C} EV(G)$. EV is a typical worst-case indicator commonly used in computer science. It is a close relative to "price of stability", where the social cost of the best NE (or CE) is related to the absolute minimum of the social cost (see Anshelevich et al (2004) and Christodoulou and Koutsoupias (2005)). Results about the price of stability in cost models cannot be carried over to utility models by simple means as demostrated by Ashlagi et al (2005).

The class of games considered in this paper are two-facility simple linear congestion games. In these games players can choose between two facilities and the utility they get linearly depends on the number players using the particular facility chosen. We will determine the exact value of EV for two subclasses: non-decreasing and mixed games. These are in turn 1 and 2. Certain social dilemma games (SD) such as the prisoners' dilemma and chicken games (see Osborne and Rubinstein (1996), Hamburger (1973),Bornstein et al (1997), Szilagyi and Somogyi (2010)) are subclasses of mixed games. We will determine the EV for these games as well. It will turn out that in the general case the EV does not change, however, for the 2 and 3-person chicken game $EV = 1,5$. For non-increasing games we determine a lower bound $1,125$ and an upper bound $1,265625$. The latter is better than the previously known $1,333$.

One might wonder whether linearity is too strong an assumption and covers only irrelevant trivial cases? This is not by far the case. It is straightforward to show that semi-compound games are linear. An n-person SD is said to be semi-compound, if each of the n players simultaneously plays the same 2×2 SD game with a fixed number k of all the other players and each is required to make the same move in the k games she is playing. If $k = n - 1$, then the game is compound, as defined for prisoners' dilemma games by Hamburger (1973)

The paper is organized as follows. Section 2 contains the necessary preliminaries and definitions. Section 3 deals with the class of non-increasing and non-decreasing simple linear congestion games. Section 4 addresses the class of "chicken-like games" whereas Section 5 is devoted to "prisoners' dilemma-like games". Section 6 concludes.

2 Preliminaries, notation and definitions

We begin with the definition of SCE. To this end we need some notation and definitions of basic game theory. Let $G = \{S_1, ..., S_n; f_1, ..., f_n\}$ be an n-person game in normal (strategic) form with finite strategy sets $S_1, ..., S_n$ and payoff functions $f_1, ..., f_n$. The basic ingredients in the definition of various kinds of correlated equilibria are the "incentive" constraints which compare the expected utility when following the advice of the mediator to that of turning it down. We will formulate the incentive constraints for a particular player i and suppress index i if it does not cause any confusion. Introduce the following notation:

$N = \{1, ..., n\}$: set of players.

$I = \{1, ..., m\}$: strategy set of player i represented by the indices of strategies.

S_-: Cartesian product of strategy sets of all players but i.

$s_- \in S_-$: strategy profile of all players but i.

(j, s_-), $j \in I, s_- \in S_-$: strategy profile of all players.

$S = \{(j, s_-) : j \in I, s_- \in S_-\}$: set of strategy profiles.

$f(j, s_-)$: payoff (utility) to player i if she plays strategy j and the rest of the players play s_-.

p: probability distribution on S.

$p(j, s_-)$: probability assigned by p to profile (j, s_-).

Definition 1 A CE is a probability distribution p satisfying the following incentive constraints for player i, $(i \in N)$

$$\sum_{s_- \in S_-} f(j, s_-)p(j, s_-) \geq \sum_{s_- \in S_-} f(k, s_-)p(j, s_-) \qquad \text{for all } j, k \in I \ .$$

Definition 2 A CCE is a probability distribution p satisfying the following incentive constraints for player i, $(i \in N)$

$$\sum_{j \in I} \sum_{s_- \in S_-} f(j, s_-)p(j, s_-) \geq \sum_{j \in I} \sum_{s_- \in S_-} f(k, s_-)p(j, s_-) \qquad \text{for all } k \in I \ .$$

For the definition of SCE we need the notion of "admissible" sets. For a fixed $j \in I$, consider the constraints

$$\sum_{s_- \in S_-} f(j, s_-)p(j, s_-) \geq \sum_{s_- \in S_-} f(l, s_-)p(j, s_-) \qquad \text{for all } l \in I \ .$$

and call them a j-set (of constraints). Consider the set

$$K = \prod_{j=1}^{m} (I \setminus \{j\}) \ .$$

Elements of K are called admissible (index)sets.

Definition 3 An SCE is a probability distribution p satisfying the following incentive constraints for player i, $(i \in N)$

$$\sum_{j \in I} \sum_{s_- \in S_-} f(j, s_-)p(j, s_-) \geq \sum_{j \in I} \sum_{s_- \in S_-} f(k_j, s_-)p(j, s_-)$$

for all admissible sets $(k_1, ..., k_m) \in K$.

Now we turn to n-player two-facility simple congestion games. This is going to be a brief account, for more details consult Forgó (2014). An n-player, two-facility simple congestion game can be given by the "congestion form": two non-negative n-vectors $a = (a_1, ..., a_n)$, $b = (b_1, ..., b_n)$ meaning that if j many players choose facility 1 ($F1$), then each one gets utility a_j and if k many players choose facility 2 ($F2$), then each one gets utility b_k. The associated congestion game is defined by the player set N, the strategy set $\{F1, F2\}$ for each player, briefly denoted by $\{1, 2\}$, and the payoffs determined by the utility vectors a and b. A strategy profile of the n players is $(i_1, ..., i_n)$ where $i_j \in \{1, 2\}$, $j \in N$. Let $p_{i_1,...,i_n}$ be the probability of the mediator selecting strategy profile $(i_1, ..., i_n)$. Let t denote the number of players using facility $F2$, $t = 0, 1, ..., n$. Let furthermore $S_t = \{(i_1, ..., i_n) \in S : \text{number of players choosing } F2 = t\}$. Taking into account the inherent symmetry of the game we assume that all probabilities $p_{i_1,...,i_n}$, $(i_1, ..., i_n) \in S_t$ are equal and denote this by p_t.

Using this notation the incentive constraint of each player becomes

$$(a_n - b_1)p_0 + \sum_{t=1}^{n-1} (\binom{n-1}{t-1} (b_t - a_{n-t+1}) +$$

$$\binom{n-1}{t} (a_{n-t} - b_{t+1}))p_t + (b_n - a_1)p_n \geq 0. \quad (1)$$

The normalizing and the non-negativity constraints are

$$\sum_{t=0}^{n} \binom{n}{t} p_t = 1 \ , \ \ p_t \geq 0, \quad t = 0, 1, ..., n. \quad (2)$$

and the SW (defined as the sum of the utilities of the players) is

$$SW = \sum_{t=0}^{n} \binom{n}{t} (b_t t + a_{n-t}(n-t)) p_t. \tag{3}$$

Introducing the notation $q_t = \binom{n}{t} p_t, t = 0, 1, ..., n$ (1), (2), (3) can be brought to the more simple form

$$\sum_{t=0}^{n} (t(b_t - a_{n-t+1}) + (n-t)(a_{n-t} - b_{t+1}))q_t \geq 0 \tag{4}$$

$$\sum_{t=0}^{n} q_t = 1, \quad q_t \geq 0, \quad t = 0, 1, ..., n$$

$$SW = \sum_{t=0}^{n} (b_t t + a_{n-t}(n-t))q_t. \tag{5}$$

The maximum SW achievable through SCE can be determined by the solution of the following LP

$$P : \max \sum_{t=0}^{n} (b_t t + a_{n-t}(n-t))q_t$$

$$\sum_{t=0}^{n} (t(b_t - a_{n-t+1}) + (n-t)(a_{n-t} - b_{t+1}))q_t \geq 0$$

$$\sum_{t=0}^{n} q_t = 1, \quad , q_t \geq 0, \quad t = 0, 1, ..., n.$$

The proof of this claim is relegated to the appendix.

3 The EV for two-facility non-increasing and non-decreasing simple linear congestion games

The performance of SCE for non-increasing simple linear congestion games was the subject of an earlier paper by Forgó (2014). These games were also analyzed by Ashlagi et al (2008) for CE. In these games utility of a player does not increase for either facility as congestion grows. Traffic situations are typical examples. In Forgó (2014) an upper bound $\frac{4}{3}$ was determined for the EV and it was conjectured that this bound can significantly be decreased. As it will turn out, this is the case. Exact values of EV were obtained up to $n = 4$. $EV = 1$ for $n = 2, 3$ and $EV = 1,007478$ for $n = 4$. So we may assume $n \geq 5$.

Here, and throughout the whole paper we will minimally infringe on generality by fixing the level of the lowest utility at 0. This is fairly typical in microeconomics. The purpose is to make the complicated analysis much easier since we have only to consider three parameter instead of four. The congestion form of a non-increasing simple linear congestion game is given in the following table

$F1$			$F2$
$a_1 =$	$(n-1)x$	$b_1 =$	$y + (n-1)z$
$a_2 =$	$(n-2)x$	$b_2 =$	$y + (n-2)z$
	...		...
$a_t =$	$(n-t)x$	$b_t =$	$y + (n-t)z$
	...		....
$a_{n-1} =$	x	$b_{n-1} =$	$y + z$
$a_n =$	0	$b_n =$	y

We assume that x, y, z are all nonnegative, $x > 0$, and at least one of y and z is positive. This will also be assumed for all other simple linear congestion games considered in this paper. Substituting the congestion form into (4) and (5) we get

$$\sum_{t=0}^{n}(t(n+1-2t)x + (2t-n)y + (n-t)(2t-n+1)z)q_t \geq 0$$

$$SW = \sum_{t=0}^{n}(t(n-t)(x+z) + ty)q_t.$$

In order to make the dependence on parameters clear, introduce the notation

$$C(n,x,y,z,t) = -[t(n+1-2t)x + (2t-n)y + (n-t)(2t-n+1)z] \tag{6}$$

$$W(n,x,y,z,t) = t(n-t)(x+z) + ty \tag{7}$$

for any $0 \leq t \leq n, (n \geq 5)$. As seen earlier the maximum SW achievable in an SCE for fixed n, x, y, z,is the optimal objective function value of the following LP

$$P: \ \max \sum_{t=0}^{t=n} W(n,x,y,z,t)q_t$$

$$\sum_{t=0}^{t=n} C(n,x,y,z,t)q_t \leq 0$$

$$\sum_{t=0}^{t=n} q_t = 1, \ q_t \geq 0, \quad t = 0, 1, ..., n.$$

Obviously $\max_{0\leq t\leq n} W(n,x,y,z,t)$ is an upper bound to the highest achievable SW without any mediation whatsoever which happens to be exact if the maximumpoint is an integer. Then for any feasible point $q = (q_0, q_1, ..., q_n)$ of P we have

$$EV \leq \sup_{n,x,y,z,t} \frac{\max_{0\leq t\leq n} W(n,x,y,z,t)}{\sum_{t=0}^{t=n} W(n,x,y,z,t)q_t}. \tag{8}$$

On the other hand, for any set of parameters n, x, y, z

$$EV \geq \frac{\max_{t=0,1,...,n} W(n,x,y,z,t)}{\max_{q\in L_P} \sum_{t=0}^{t=n} W(n,x,y,z,t)q_t} \tag{9}$$

where L_P denotes the feasible set of P.

We state a simple lemma and two corollaries.

Lemma 1 For any n, x, y, z, t and $\lambda > 0$, $W(n, \lambda x, \lambda y, \lambda z, t) = \lambda W(n, x, y, z, t)$ and $\sum_{t=0}^{t=n} C(n, \lambda x, \lambda y, \lambda z, t)q_t = \lambda \sum_{t=0}^{t=n} C(n, x, y, z, t)q_t$.

Proof By substituting into (6) and (7).□

Corollary 1 EV is not affected by scaling with a factor $\lambda > 0$.

Corollary 2 Without loss of generality we may take $y = 1$ if $y > 0$, or when it is more convenient $z = 1$ if $z > 0$.

Theorem 1 For the class of two-facility non-increasing simple linear congestion games $EV \leq \left(\frac{9}{8}\right)^2 = 1,265625$.

Proof It can easily be seen that the absolute (continuous) maximum of $W(n, x, y, z, t)$ with respect to t is attained at

$$t^* = \frac{n}{2} + \frac{y}{2(x+z)}. \tag{10}$$

We distinguish two cases

A. $r \leq \frac{n+3}{2}$. Assume that n is even. Then $q_{\frac{n}{2}} = 1$, $q_t = 0, t \neq \frac{n}{2}$ is feasible to P. Therefore if $y = 0$, then $EV = 1$. If $y > 0$, then by Corollary 2 we may set $y = 1$. Define $r = \frac{1}{x+z}$.Thus we have

$$EV \leq \frac{W(n,x,1,z,t^*)}{W(n,x,1,z,\frac{n}{2})} = \frac{((\frac{n}{2})^2 - (\frac{r}{2})^2)(x+z) + (\frac{n}{2} + \frac{r}{2})}{(\frac{n}{2})^2(x+z) + \frac{n}{2}} = \frac{(n+r)^2}{n(n+2r)}.$$

This is an increasing function of r, therefore

$$EV \leq \frac{(n + \frac{n+3}{2})^2}{n(n + 2\frac{n+3}{2})} = \frac{\frac{1}{4}(3n+3)^2}{n(2n+3)}. \tag{11}$$

This ratio is a decreasing function of n. Since $n = 6$ is the smallest even number satisfying $n \geq 5$ we get

$$EV \leq \frac{\frac{1}{4}(3 \cdot 6 + 3)^2}{6(2 \cdot 6 + 3)} = 1,225.$$

If n is odd, then $q_{\frac{n-1}{2}} = \frac{1}{2}$, $q_{\frac{n+1}{2}} = \frac{1}{2}, q_t = 0, t \neq \frac{n-1}{2}, \frac{n+1}{2}$ is feasible to P. Thus

$$EV \leq \frac{W(n, x, 1, z, t^*)}{\frac{1}{2}W(n, x, 1, z, \frac{n-1}{2}) + \frac{1}{2}W(n, x, 1, z, \frac{n+1}{2})} = \frac{(n + r)^2}{n(n + 2r) - 1}.$$

The right-hand side is again an increasing function of r and thus

$$EV \leq \frac{\frac{1}{4}(3n + 3)^2}{n(2n + 3) - 1}.$$

Since this ratio is a decreasing function of n, substituting $n = 5$ we obtain the estimation

$$EV \leq \frac{81}{64} = \left(\frac{9}{8}\right)^2 = 1,265625.$$

B. $r > \frac{n+3}{2}$. Again, if $y = 0$, then $EV = 1$ and we may set $y = 1$. Take the minimum of $C(n, x, 1, z, t)$ with respect to t. The minimumpoint is

$$t' = \frac{(n + 1)x + 2 + (3n - 1)z}{4(x + z)} = \frac{n + 1}{4} + \frac{r}{2} + \frac{(n - 1)zr}{2}. \tag{12}$$

Since $r > \frac{n+3}{2}$ and $zr > 0$, therefore

$$t' - \frac{n}{2} = \frac{n + 1}{4} + \frac{r}{2} + \frac{(n - 1)zr}{2} - \frac{n}{2} > 1. \tag{13}$$

$C(n, x, 1, z, t)$ is a convex quadratic function of t which is symmetric to its minimumpont t'. We know that $C(n, x, 1, z, \frac{n}{2}) < 0$, therefore by (13) we have $C(n, x, 1, z, [t']) < 0$ and $C(n, x, 1, z, [t'] + 1) < 0$, where $[a]$ denotes the integer part of the real number a. Consider first the simple case when $[t'] \leq t^* \leq [t'] + 1$. Then $[t'] = [t^*]$ and since $\max_{t=0,1,\ldots,n} W(n, x, y, z, t) = \max\{W(n, x, 1, z, [t^*]), W(n, x, 1, z, [t^*] + 1\}$, the two solutions $q_{[t^*]} = 1$, $q_t = 0, t \neq [t^*]$ and $q_{[t^*]+1} = 1$, $q_t = 0, t \neq [t^*] + 1$ are both feasible to P implying $EV = 1$.

Thus we may suppose that $t^* \notin [[t'], [t'] + 1]$ and meaning that it is sufficient to prove that

$$EV \leq \sup_{n,x,z} \frac{W(n, x, 1, z, t^*)}{W(n, x, 1, z, t')} \leq \frac{9}{8}. \tag{14}$$

Showing that

$$8W(n,x,z,t^*) - 9W(n,x,z,t') < 0 \tag{15}$$

holds for all possible values of the parameters implies the validity of (14). Using the notation $r = \frac{1}{x+z}$ we can bring (15) to the following form

$$8((\frac{n}{2}+\frac{r}{2})(\frac{n}{2}-\frac{r}{2})\frac{1}{r}+\frac{n}{2}+\frac{r}{2})-$$

$$9[(\frac{n+1}{4}+\frac{r}{2}+\frac{n-1}{2}rz)(\frac{3n-1}{4}-\frac{r}{2}-\frac{n-1}{2}rz)\frac{1}{r}+\frac{n+1}{4}+\frac{r}{2}+\frac{n-1}{2}rz < 0.$$

Multiply both siges by r

$$8(\frac{n^2}{4}-\frac{r^2}{4}+\frac{n}{2}r+\frac{r^2}{2})-$$

$$9[(\frac{n+1}{4}+\frac{r}{2}+\frac{n-1}{2}rz)(\frac{3n-1}{4}-\frac{r}{2}-\frac{n-1}{2}rz)+\frac{n+1}{4}r+\frac{r^2}{2}+\frac{n-1}{2}r^2z] < 0.$$

Multiplying out after simplification and rearrangement we get

$$\frac{5}{16}n^2-\frac{9}{8}n+\frac{9}{16}+r(-\frac{1}{2}n)+r^2(-\frac{1}{4})+rz(-\frac{9}{4}(n-1)^2)+(rz)^2\frac{9}{4}(n-1)^2 < 0. \tag{16}$$

The sum of the last two terms is a convex quadratic function of rz. Since $0 \leq rz < 1$, this cannot be more than 0. The coefficients of r and r^2 are negative. Taking in account that $r > \frac{n+3}{2}$, if (16) holds by substituting $r = \frac{n+3}{2}$and omitting the last two terms, then it holds for all possible values of the parameters. Then (16) reduces to $-\frac{9}{4}n < 0$ which obviously holds for any $n \geq 1$.

Cases A and B cover all possible values of r, and since $\frac{9}{8} \leq \left(\frac{9}{8}\right)^2$, the proof of the theorem is complete.□

Theorem 2 For the class of two-facility non-increasing simple linear congestion games $EV \geq \frac{9}{8} = 1,125$.

Proof We will consider a series of games where $x = \frac{2}{n-3}, y = 1, z = 0$ (or equivalently $r = \frac{n-3}{2}, y = 1, z = 0$). Assume that n is odd and $\sqrt{n+1}$ is integer. The smallest such $n \geq 5$ is 15. In order to determine the numerator in (9) we have to find the integer maximum of $W(n,x,y,z,t)$. The continuous maximum is attained at $t^* = \frac{n}{2} + \frac{r}{2} = \frac{3(n-1)}{4}$. This is either an integer or half way between two integers. In the first case the continuous and integer maxima coincide. In the latter case, since for fixed n,x,y,z the function W is concave and quadratic in t, the integer maximum of W occurs at either integer neighbor of t^*, say at $t^* - \frac{1}{2} = \frac{3n-5}{4}$. Then the integer maximum of W is

$$\frac{9(n-1)^2}{8(n-3)} \text{ if } t^* \text{ is integer,}$$
$$\frac{(3n-5)(3n-1)}{8(n-3)} \text{if } t^* \text{ is not integer.}$$

Turning to the denominator, we first observe that $C(n,x,y,z,t)$ is a convex quadratic function of t for any fixed n,x,y,z. The two roots of the quadratic equation

$$C(n,x,y,z,t) = C(n, \frac{2}{n-3}, 1, 0, t) = 0$$

are

$$\begin{aligned} t_1 &= \frac{n-1-\sqrt{n+1}}{2} \\ t_2 &= \frac{n-1+\sqrt{n+1}}{2}. \end{aligned}$$

By our assumption both t^1 and t^2 are integers. We claim that $q' = (q'_0, q'_1, ..., q'_n)$, $q'_{t_2} = 1, q_t = 0, t \neq t_2$ is an optimal solution of P. Feasibility is obvious. The objective function value of P at this solution is

$$W(t_2) = \frac{1}{2(n-3)}(2n^2 - 5n + (n-1)\sqrt{n+1} + 1).$$

Problem P is an LP and its dual D is

$$\begin{aligned} D &: \quad \min v \\ v &\geq -C(n, \frac{2}{n-3}, 1, 0, t)u + W(n, \frac{2}{n-3}, 1, 0, t), \text{for all } t = 0, 1, ..., n \\ u &\geq 0. \end{aligned}$$

By substitution and some algebra it can be verified that

$$u = \frac{1}{4n}((n-1)\sqrt{n+1} - (n+1))$$

$$v = \frac{1}{2(n-3)}(2n^2 - 5n + (n-1)\sqrt{n+1} + 1)$$

is a feasible solution to D, v also being the objective function value of D. Since $v = W(t_2)$, by the weak duality theorem of linear programming we have

$$W(t_2) = \max_{q \in L_P} \sum_{t=0}^{t=n} W(n,x,y,z,t) q_t.$$

Thus the following inequalities hold

$$EV \geq \frac{\frac{9(n-1)^2}{8(n-3)}}{\frac{1}{2(n-3)}(2n^2-5n+(n-1)\sqrt{n+1}+1)} \text{ if } t^* \text{ is integer,}$$
$$EV \geq \frac{\frac{(3n-5)(3n-1)}{8(n-3)}}{\frac{1}{2(n-3)}(2n^2-5n+(n-1)\sqrt{n+1}+1)} \text{ if } t^* \text{ is not integer.}$$

In both cases the right-hand side of the inequality goes to $\frac{9}{8}$ if $n \to \infty$ thereby establishing the claim of the theorem.□

From Theorem 1 and 2 we get

Corollary 3 For the class of two-facility non-increasing simple linear congestion games $\frac{9}{8} \leq EV \leq \left(\frac{9}{8}\right)^2$.

Notice that estimation (11) gets tighter as n grows and thus asymptotically we get an exact EV

$$EV \leq \lim_{n\to\infty} \frac{\frac{1}{4}(3n+3)^2}{n(2n+3)} = \frac{9}{8} = 1,125.$$

In non-decreasing simple linear congestion games utility grows as congestion increases. The congestion form for these games is given by the following table

$$\begin{array}{cccc} F1 & & & F2 \\ a_1 = & 0 & b_1 = & y \\ a_2 = & x & b_2 = & y+z \\ & ... & & ... \\ a_t = & (t-1)x & b_t = & y+(t-1)z \\ & ... & & .... \\ a_{n-1} = & (n-2)x & b_{n-1} = & y+(n-2)z \\ a_n = & (n-1)x & b_n = & y+(n-1)z. \end{array}$$

These games are models of situations where the utility of a player grows as the number of players using a facility increases. We may think of two political parties, where each voter's utility (hope of his party winning the election) increases as the number of people casting their votes on his party grows. The complexity of these games, however, does not reach that of the other two types of two-facility simple linear congestion games.

Theorem 3 For non-decreasing simple linear congestion games $EV = 1$.

Proof Denote again by t the number of players choosing $F2$. The absolute maximum of SW is achieved at

$$\begin{array}{rcl} t & = & 0 \text{ if } (n-1)x \geq y+(n-1)z \\ t & = & n \text{ if } (n-1)x \leq y+(n-1)z \end{array}$$

and thus the maximum $SW = \max\{n(n-1)x, n(y+(n-1)z)\}$. Substituting the values from the congestion form and $p_0 = 1, p_i = 0, i \neq 1$ if $(n-1)x \geq y+(n-1)z$ or $p_n = 1, p_i = 0, i \neq n$ if $(n-1)x \leq y + (n-1)z$ into (4) and (5) we see that in both cases we have an SCE, respectively, realizing the absolute maximum of SW. Consequently, $EV = 1$ for this class of games.□

It is worth mentioning that correlation is not necessary for achieving the best possible SW since the two SCE's defined in the proof of Theorem 3 are NE's. If, $n = 2$, and $y < x < y + z$, then we have the "stag hunt" game (see e.g. Osborne and Rubinstein (1994)), where the issue is not the maximization of SW but coordination for realizing the better NE. The scenario of the CE will do for this purpose, there is no need for the "stricter" protocol of the SCE.

4 The exact value of EV for two-facility "chicken-like" linear congestion games

The class of games considered in this section consists of mixed two-facility linear congestion games where utility is lowest when all players use the "decreasing" facility $F1$. We will call this class, for good reason, two-facility "chicken-like" linear congestion games, CH-type games for short. Again, we assume that the lowest utility is normalized to 0 and parameters x, y, z are all nonnegative, $y > 0$, and at least one of x and z is positive. In particular, the congestion form is the following

$$\begin{array}{rcrc}
 & F1 & & F2 \\
a_1 = & (n-1)x & b_1 = & y \\
a_2 = & (n-2)x & b_2 = & y+z \\
 & ... & & ... \\
a_t = & (n-t)x & b_t = & y+(t-1)z \\
 & ... & & .... \\
a_{n-1} = & x & b_{n-1} = & y+(n-2)z \\
a_n = & 0 & b_n = & y+(n-1)z.
\end{array}$$

Assuming that t players choose $F2$, thus $n-t$ players choose $F1$ and substituting in the incentive constraint C and the social welfare function W we get from (4) and (5)

$$C(n, x, y, z, t) = -(t(n-2t+1)x + (2t-n)y + t(2t-n-1)z) \leq 0$$

$$W(n, x, y, z, t) = t(n-t)x + t(y + (t-1)z).$$

Theorem 4 For the class of CH-type games $EV \leq 2$.
Proof We distinguish two cases.

a) $x \leq z$. In this case W is a convex (linear if $x = z$) quadratic function of t on the interval $[0, n]$. Therefore, its maximum is taken at one of the endpoints. Since $W(n, x, y, z, 0) = 0$ and $W(n, x, y, z, n) > 0$, the maximumpoint is $t = n$. Since $C(n, x, y, z, n) = -(n(n-1)(z-x) + ny) < 0$, $t = n$ also satisfies the incentive constraint and thus $EV = 1$.

b) $x > z$. Assume that n is even. Then $q_{\frac{n}{2}} = 1, q_i = 0, i \neq \frac{n}{2}$ is an SCE which can easily be checked by substituting into the incentive constraint. Indeed

$$C(n, x, y, z, \frac{n}{2}) = -\frac{n}{2}(x - z) < 0. \tag{13}$$

The absolute unconstrained continuous maximum of W is attained at

$$t^* = \frac{y + nx - z}{2(x - z)}. \tag{14}$$

Clearly $t^* \geq \frac{n}{2}$. If $t^* \geq n$, then W, being a concave quadratic function of t, attains its maximum on $[0, n]$ at $t = n$. Thus we have the simple estimation

$$EV \leq \frac{W(n, x, y, z, n)}{W(n, x, y, z, \frac{n}{2})} = \frac{n(y + (n-1)z)}{\frac{n^2}{4}(x + z) - \frac{n}{2}z + \frac{n}{2}y} \leq 2.$$

Consider the case when $t^* < n$. Then from (14) we obtain

$$y < nx - (2n-1)z. \tag{15}$$

Look first at the case when $z > 0$. By Corollary 2 we can set $z = 1$. Then we get

$$EV \leq \frac{W(n, x, y, 1, t^*)}{W(n, x, y, 1, \frac{n}{2})} = \frac{\frac{(y+nx-1)^2}{4(x-1)}}{\frac{n^2}{4}(x + 1) - \frac{n}{2} + \frac{n}{2}y} = \frac{(y + nx - 1)^2}{n^2(x^2 - 1) + 2n(x - 1)(y - 1)}.$$

By taking the derivative of the right-hand side with respect to y it is easy to see that it is positive for any $n \geq 2$ that is, it is an increasing function of y over the positive reals for any fixed x. From (15) taking $z = 1$ we obtain

$$y < 1 + nx - 2n. \tag{16}$$

Thus

$$EV \leq \frac{W(n, x, 1 + nx - 2n, 1, n)}{W(n, x, 1 + nx - 2n, 1, \frac{n}{2})} = \frac{4}{3} < 2. \tag{17}$$

If $z = 0$, then $y > 0$ and by Corollary 2 we may set $y = 1$. Then (15) becomes

$$1 < nx$$

and

$$EV \leq \frac{W(n,x,1,0,t^*)}{W(n,x,1,0,\frac{n}{2})} = \frac{\frac{(1+nx)^2}{4x}}{\frac{n^2}{4}x + \frac{n}{2}} = \frac{(1+nx)^2}{n^2x^2+2nx} < \frac{4}{3}.$$

The proof for the case when n is odd is the same, the only difference is that in this case we should work with the SCE $q_{\frac{n}{2}-1} = \frac{1}{2}, q_{\frac{n}{2}+1} = \frac{1}{2}, q_i = 0, i \neq \frac{n}{2}-1, \frac{n}{2}+1$. □

Theorem 5 For the class of CH-type games $EV \geq 2$.

Proof Consider an n-player CH-type game with parameters $x = 1 + \frac{2}{n}, y = 0, z = 1, n \geq 4$ and even. First we will determine the exact value of the denominator in (9). We claim that the SCE $q_{\frac{n}{2}} = \frac{n+2}{2n+2}, q_{\frac{n}{2}+1} = \frac{n}{2n+2}, q_i = 0, i \neq \frac{n}{2}, \frac{n}{2}+1$ is an optimal solution of

$$P: \ \max \sum_{t=0}^{t=n} W(n, 1+\frac{2}{n}, 0, 1, t)q_t$$

$$\sum_{t=0}^{t=n} C(n, 1+\frac{2}{n}, 0, 1, t)q_t \leq 0 \tag{18}$$

$$\sum_{t=0}^{t=n} q_t = 1, \quad q_t \geq 0, \ t = 0, 1, ..., n.$$

By substitution, it can be verified that it is feasible and its objective function value is $\frac{n(n+1)}{2} - 1$. The dual of P is

$$\begin{aligned} D \quad &: \quad \min v \\ v \quad &\geq \quad -C(n, 1+\frac{2}{n}, 0, 1, t)u + W(n, 1+\frac{2}{n}, 0, 1, t), \text{for all } t = 0, 1, ..., n \\ u \quad &\geq \quad 0. \end{aligned} \tag{19}$$

We claim that $u = \frac{n}{2} - 1, v = \frac{n(n+1)}{2} - 1$ is a feasible solution of (19). By simple algebra we can determine that the continuous maximum of the concave quadratic function

$$Q(t) = -C(n, 1+\frac{2}{n}, 0, 1, t)u + W(n, 1+\frac{2}{n}, 0, 1, t)$$

is at $t = \frac{n+1}{2}$ which is not an integer but it is half way between the integers $\frac{n}{2}, \frac{n}{2}+1$. By the symmetry of the quadratic function the maximum is attained at both of these integers. The objective function value at both of them is $\frac{n(n+1)}{2} - 1$ which is equal to the objective function value of P at the SCE

$q_{\frac{n}{2}} = \frac{n+2}{2n+2}, q_{\frac{n}{2}+1} = \frac{n}{2n+2}, q_i = 0, i \neq \frac{n}{2}, \frac{n}{2}+1$. Thus by the weak duality theorem of linear programming, this solution is optimal to P. We have just shown that

$$\max_{q \in L_P} \sum_{t=0}^{t=n} W(n, 1+\frac{2}{n}, 0, 1, t) q_t = \frac{n(n+1)}{2} - 1.$$

The absolute maximum of $W(n, 1+\frac{2}{n}, 0, 1, t)$ over $t \in [0, \infty)$ is at

$$t^* = \frac{n(1+\frac{2}{n}) - 1}{\frac{4}{n}},$$

which cannot be less than n if $n \geq 4$ as it is assumed. Thus the maximum of $W(n, 1+\frac{2}{n}, 0, 1, t)$ over $t \in [0, n]$ is $W(n, 1+\frac{2}{n}, 0, 1, n) = n(n-1)$. Therefore we have the following inequality for the EV

$$EV \geq \frac{n(n-1)}{\frac{n(n+1)}{2} - 1} = \frac{2n(n-1)}{n(n+1) - 2}. \tag{20}$$

The right-hand side of the above inequality is an increasing function of n and its limit is 2 as $n \to \infty$ through even n's. □

Consider now the special case when $n = 2$ and

$$y + z < x < 2y + 2z. \tag{21}$$

This is the well known chicken game with the payoffs in bimatrix form

	L	H
L	$y+z, y+z$	y, x
H	x, y	$0, 0$

The first strategy of both players is a low-risk (L) and the second strategy is a high-risk action (H). There are two NE's in pure strategies (L, H) and (H, L), the maximum SW $2(y+z)$ occurs at (L, L) and the minimum at (H, H). This means that if a player chooses H alone, then she gets the highest payoff, whereas both players' choosing H is disastrous, giving the lowest possible SW. For a CH-type game to represent an n-player chicken game (CH-game for short) we need to preserve these properties of the two-player chicken game:

(i) taking H alone gives the highest individual utility,

(ii) highest SW is at the collective choice of L.

Based on the congestion-form model, facility $F1$ plays the role of H, while $F2$ does so for L. Thus, in order to render a CH-type game a CH-game we need to assume that

$$(i) \ (n-1)x > y + (n-1)z$$

$$(ii)\ W(n) > W(t), \quad \text{for all } t = 0, 1, ..., n-1.$$

For (i) to hold it is necessary that $x > z$. For (ii) to hold we need to have

$$t^* = \frac{y + nx - z}{2(x - z)} \geq n$$

or equivalently

$$y \geq nx - (2n-1)z. \qquad (22)$$

The question emerges whether we have a better EV if we constrain ourselves to the class of CH-games? The answer is given in the following theorem.

Theorem 6 For the class of CH-games $EV = 2$.

Proof Since chicken games are CH-type games, by Theorem 4 we have $EV \leq 2$. In order to be able to apply Theorem 5 to prove $EV \geq 2$, it is enough to show that the n-player CH-game with parameters $x = 1 + \frac{2}{n}, y = 0, z = 1$ ($n \geq 4$ and even) is a CH-game. Substituting into (i) and (21) we get

$$(n-1)(1 + \frac{2}{n}) > (n-1),$$

$$0 \geq n(1 + \frac{2}{n}) - 2n + 1$$

which hold if $n \geq 3$.□

For small n's we have a better EV than 2.

Theorem 7 For two-player CH-games $EV = \frac{3}{2}$.

Proof We only have to consider case b) and subcase $t^* = \frac{y+2x-z}{2(x-z)} > 2$ in the proof of Theorem 4 since for $t^* \leq 2$ we know from (17) that $EV \leq \frac{4}{3} < \frac{3}{2}$. The maximum SW is $2(y+z)$ by (21). The maximum SW of SCE's is obtained as the optimal objective function value of the following LP

$$\max (x+y)p_1 + 2(y+z)p_2$$

$$2yp_0 + (z-x)p_1 + 2(x-y-z)p_2 \leq 0$$

$$p_0 + p_1 + p_2 = 1, \ p_0, p_1, p_2 \geq 0. \qquad (23)$$

$p_0 = 0, p_1 = \frac{2}{3}, p_2 = \frac{1}{3}$ is easily seen to be feasible to (23) and thus we have the estimation

$$\frac{2(y+z)}{\frac{2}{3}(x+y) + \frac{2}{3}(y+z)} \leq \frac{3}{2}$$

which must hold since $x > y + z$ by (21).

Take the chicken game with parameters $x = 1 + \varepsilon, y = 0, z = 1$ which obviously satisfy (21). It can be verified that $p_0 = 0, p_1 = \frac{2}{3}, p_2 = \frac{1}{3}$ is an optimal solution of (23) with objective function value $\frac{2}{3}(1+\varepsilon)+\frac{2}{3}$. The absolute maximum of the SW is 2. Thus we have

$$\lim_{\varepsilon \to 0} \frac{2}{\frac{2}{3}(1+\varepsilon)+\frac{2}{3}} = \frac{3}{2}$$

completing the proof.□

It is interesting that the EV does not get any worse if the number of players increases by 1.

Theorem 8 For the class of three-player CH-games $EV = \frac{3}{2}$.

Proof The maximum SW of SCE's is obtained as the optimal value of the following LP

$$\max(2x+y)p_1+2(x+y+z)p_2+(3y+6z)p_3$$

$$3yp_0+(-2x+y+2z)p_1-yp_2+(6x-3y-6z)p_3 \leq 0$$

$$p_0+p_1+p_2+p_3 = 1, \; p_0, p_1, p_2, p_3 \geq 0. \tag{24}$$

The absolute maximum of SW is either $2(x+y+z)$ or $3y+6z$. In the first case $p_2 = 1, p_i = 0, i \neq 2$ is an SCE and $EV = 1$. From $2(x+y+z) < 3y+6z$ we get $x > z + \frac{1}{2}$. Since $p_2 = 1, p_i = 0, i \neq 2$ is feasible, we have the estimation

$$EV \leq \frac{3y+6z}{2x+2y+2z} < \frac{3y+6z}{2(z+\frac{1}{2})+2y+2z} = \frac{3y+6z}{2y+4z+1} < \frac{3}{2}. \tag{25}$$

Consider the CH-game with parameters $x = 1+\varepsilon, y = 0, z = 1$. It is easy to show that the SCE $p_2 = 1, p_i = 0, i \neq 2$ is an optimal solution of (24) with objective value $4+2\varepsilon$. The absolute maximum of the SW is 6. Thus we have

$$\lim_{\varepsilon \to 0} \frac{6}{4+2\varepsilon} = \frac{3}{2}. \tag{26}$$

Combining (25) and (26) we get $EV = \frac{3}{2}$.□

From (20) for $n = 4$ we get the lower bound $\frac{28}{17} = 1,647..$ and by Theorem 5 this grows (through even n's) monotonically towards 2.

An example

Assume that there are four firms which can decide on mitigating the pollution of a common resource (e.g. a lake) that they use for their business activities or do nothing and dump waste in the lake uncontrolled. They have bilateral contracts with each other in which they pledge to control the pollution. Not controlling the pollution means a violation of this contract and is penalized by having to pay a fine. The fine is proportional with the amount of the pollutants damaging the lake which depends on the number of firms violating the contract. So each firm faces the decision problem of choosing between mitigation (M) and uncontrolled pollution (P) knowing the consequences of both. Utilities (based on profits achievable) are such that every firm plays a chicken game with every other. Assume that the utilities are given by the following matrix (Firm i is the row player, firm j is the column player)

$$\begin{array}{ccc} & M & P \\ M & (6,6) & (2,7) \\ P & (7,2) & (0,0) \end{array}.$$

This gives rise to the following congestion game with congestion form

No. of firms	P	M
1	21	6
2	14	10
3	7	14
4	0	18

.

This is a CH-type game, in particular a four-person CH-game. To determine an SW maximizing SCE we solve the following LP:

$$\max 27p_1 + 48p_2 + 63p_3 + 72p_4$$

$$24p_0 + 3p_1 - 6p_2 - 3p_3 + 12p_4 \leq 0$$

$$p_0 + p_1 + p_2 + p_3 + p_4 = 1$$

$$p_0, p_1, p_2, p_3, p_4 \geq 0.$$

The optimal solution is $p_0 = 0, p_1 = 0, p_2 = 0, p_3 = 0,8, p_4 = 0,2$ with $SW = 64,8$. For the best pure NE's $SW = 63$. Thus the best SCE gives a $2,86\%$ improvement relative to the best pure NEP's.

To implement the SCE, the firms may establish a club every player may or may not join. Members of the club commit themselves to follow the instruction of the club official. The official does a lottery and with probability $0,2$ forces every firm to mitigate and with probability $0,8$ forces three of them to mitigate and let the remaining one pollute. Which one to choose to allow to pollute can be chosen arbitrarily. In the spirit of the inherent symmetry of the firms the most acceptable way is a uniform random selection. This policy is stable in the sense that if everybody joins the club, there is no incentive for any player to leave it.

5 The exact value of EV for two-facility "prisoners' dilemma-like" linear congestion games

In this section we deal with the other class of mixed two-facility simple linear congestion games where utility is lowest when all players use the "increasing" $F1$ facility. We will call this class two-facility "prisoners'dilemma-like" linear congestion games, PD-type games for short. As it will turn out both the classical two-person prisoners' dilemma and a generalization due to Hamburger (1973) are special cases of PD-type games. In Forgó (2016) a bound $EV \leq 4$ was established for the class of PD-type games and it was conjectured that this bound can significantly be decreased. Again, we assume that the lowest utility is normalized to 0 and parameters x, y, z are all nonnegative, $y > 0$, and at least one of x and z is positive. The congestion form is the following

$$\begin{array}{rcrc} & F1 & & F2 \\ a_1 = & 0 & b_1 = & y+(n-1)z \\ a_2 = & x & b_2 = & y+(n-2)z \\ & ... & & ... \\ a_t = & (t-1)x & b_t = & y+(n-t)z \\ & ... & & .... \\ a_{n-1} = & (n-2)x & b_{n-1} = & y+z \\ a_n = & (n-1)x & b_n = & y \end{array}.$$

In the language of the prisoners' dilemma $F1$ can be thought of as the "co-operator" facility whereas $F2$ represents the "defector" facility. The two most important properties of a (two-person) PD from which many others can be deduced (Hamburger, 1994) are the following:

$P1$ Each player has a dominant strategy ($F2$),

$P2$ $(F2, F2)$ is the only NE.

There are many ways to generalize the PD to n players, see Carrol (1988). The minimum requirement for the generalization is to preserve $P1$ and $P2$. Following Hamburger (1973) we define the "cooperators' function" $C(k), k = 1, ..., n$ which is interpreted as the payoff to an $F1$-chooser provided there are k of them and the "defectors' function" $D(k), k = 0, 1, ..., n-1$ which gives the payoff an $F2$-chooser gets provided there are k $F1$-choosers. $C(0)$ and $D(n)$ are undefined. We assume that

$(Q1)$ $C(k) < D(k-1), k = 1, ..., n-1$,

$(Q2)$ $C(n) > D(0)$.

Assumptions $Q1, Q2$ are meant to ensure that $P1$ and $P2$ carry over to the n-person case. $Q1$ means that for a single player it is profitable to leave the set of cooperators no matter how many of them there are. $Q2$ makes collective cooperation preferable to collective defection.

Let t denote the number of players playing $F2$, $t = 0, 1, ..., n$. Then from the congestion form and (4),(5) we can construct the following LP whose optimal objective function value gives the maximum SW achievable in an SCE for fixed n, x, y, z

$$P: \ \max \sum_{t=0}^{t=n} W(n,x,y,z,t) q_t$$

$$\sum_{t=0}^{t=n} C(n,x,y,z,t) q_t \leq 0 \tag{27}$$

$$\sum_{t=0}^{t=n} q_t = 1, \ q_t \geq 0, \quad t = 0,1,...,n.$$

where

$$C(n,x,y,z,t) = -((ty - t(n-t)(x-z) + (n-t)((n-t-1)(x-z) - y)) \tag{28}$$

$$W(n,x,y,z,t) = t(y + (n-t)z) + (n-t-1)(n-t)x. \tag{29}$$

In order for this game to represent an n-person PD, assumptions $Q1$ and $Q2$ must be satisfied. If t players play $F2$, then $n-t$ play $F1$. By $Q1$, $a_{t+1} \leq b_{n-t}$ for $t = 1,...,n-1$ and thus the parameters should satisfy

$$y + tz > tx, \ t = 1,...,n.$$

All these inequalities are implied by the single inequality

$$y + (n-1)z > (n-1)x.$$

Taking assumption $Q2$ in account, we get $C(n) = (n-1)x > D(0) = y$. So, for a two-facility simple mixed linear congestion games to represent an n-person PD it is necessary that the parameters x, y, z satisfy

$$0 < \frac{1}{n-1}y < x < \frac{1}{n-1}y + z \ \text{ if } n \geq 2. \tag{30}$$

We will call a PD-type game a PD-game if (30) is satisfied.

Theorem 9 For PD-type games $EV \leq 2$.

Proof. Extend the domain of the quadratic function W to the interval $[0,n]$. The maximum $W^*$of $W(t)$ over $[0,n]$ is an upper bound of the absolute maximum of SW which is attained at some integer point in $[0,n]$. The coefficient of the quadratic term in (29) is $x-z$. Depending on the sign of $x-z$, we distinguish two cases.

a) $x \geq z$. In this case

$$W^* = \max\{W(0), W(n)\} = \begin{array}{ll} ny & \text{if } x \leq \frac{1}{n-1}y \\ n(n-1)x & \text{if } \frac{1}{n-1}y < x \end{array}.$$

The probabilities $q_t = 0, t = 0, 1, ..., n-1, q_n = 1$ constitute an SCE since $y > 0$, therefore we trivially have $EV = 1$ if $W^* = ny$. Consider now the subcase when $W^* = n(n-1)x$. The set of probabilities $q_0 = q_n = \frac{1}{2}, q_t = 0, t \neq 0, n$ is easily seen to be an SCE by substituting into (27) thus getting

$$-(\frac{1}{2}(-y + (n-1)(x-z)) + \frac{1}{2}y) = -(n-1)(x-z) \leq 0.$$

The SW of this SCE is

$$\frac{1}{2}n(n-1)x + \frac{1}{2}ny.$$

Thus we get the estimation

$$EV \leq \frac{n(n-1)x}{\frac{1}{2}n(n-1)x + \frac{1}{2}ny} < 2.$$

b) $x < z$. Assume first that n is even. An SCE can be obtained by setting the probabilities $q_{\frac{n}{2}} = 1, q_i = 0, i \neq \frac{n}{2}$. This satisfies (27) because

$$-(\frac{n}{2}y - \frac{n}{2}(n-\frac{n}{2})(x-z) + (n-\frac{n}{2})((n-\frac{n}{2}-1)(x-z)-y)) = -\frac{n}{2}(z-x) < 0. \quad (31)$$

The SW belonging to this SCE is

$$W(\frac{n}{2}) = \frac{n}{2}(y + \frac{n}{2}z) + \frac{n}{2}(\frac{n}{2} - 1)x) = \frac{n}{2}(y + \frac{n}{2}z + \frac{n}{2}x - x).$$

In this case the quadratic function $W(t)$ attains its continuous absolute maximum at $r = \frac{y+nz-(2n-1)x}{2(z-x)}$. If $r > n$, then just as in a), we have $EV = 1$. If $r < 0$, then $W(0) > W(t)$ for all $t \in [0, n]$ and we have the estimation

$$EV \leq \frac{W(0)}{W(\frac{n}{2})} = \frac{n(n-1)x}{\frac{n}{2}(y + \frac{n}{2}z + \frac{n}{2}x - x)} = 4\frac{(n-1)x}{2y + nz + (n-2)x} =$$

$$4\frac{(n-1)x}{2y + nz + (n-2)x} < 2.$$

which truly holds since $z > x$.

Consider now $r \in (0, n)$. We claim that the coefficient of every q_t in (27) is negative if $\frac{n}{2} \leq t \leq n$. To see this, observe that if t is considered a continuous variable, then the coefficient of q_t is a convex quadratic function of t since the coefficient $x - z$ of the quadratic term is positive. The coefficient $-y$ of q_n is negative by assumption, so is the coefficient of $q_{\frac{n}{2}}$ by (31). The negativity at the endpoints of an interval implies negativity at all points by convexity thus establishing our claim.

So, if the continuous maximumpoint r of the function W falls in the interval $[\frac{n}{2}, n]$, so does the integer maximumpoint t^* (being one of the neighboring integers of r). The coefficient of q_{t^*} being negative, $q_{t^*} = 1, q_t = 0, t \neq t^*$ is an SCE and $EV = 1$. Then we have to consider only the case when $0 \leq r < \frac{n}{2}$. The continuous maximum can be bounded from above by

$$W^* = W(r) = n(n-1)x + r^2(z-x) < n(n-1)x + \frac{n^2}{4}(z-x)$$

because $r < \frac{n}{2}$ and $z - x > 0$. For the EV we have

$$EV < \frac{W^*}{W(\frac{n}{2})} = \frac{n(n-1)x + \frac{n^2}{4}(z-x)}{\frac{n}{2}(y + \frac{n}{2}z + \frac{n}{2}x - x)} = \frac{4(n-1)x + n(z-x)}{2y + nz + (n-2)x} < 2.$$

Now we turn to the case when n is odd, $n \geq 3$. We have already seen that the coefficients of q_t in inequality (27) are negative for $\frac{n+1}{2} \leq t \leq n$ and if the integer maximumpoint t^* of W falls in this interval, then $EV = 1$. Therefore it is enough to deal with the case when $0 \leq r < \frac{n+1}{2}$. Consider the $SCE = q_{\frac{n-1}{2}} = \frac{1}{2}, q_{\frac{n+1}{2}} = \frac{1}{2}, q_i = 0, i \neq \frac{n-1}{2}, \frac{n+1}{2}$. The SW of this SCE is

$$W = \frac{1}{2}W(\frac{n-1}{2}) + \frac{1}{2}W(\frac{n+1}{2}) =$$

$$\frac{1}{2}(\frac{n-1}{2}(y + \frac{n+1}{2}z) + \frac{n-1}{2}\frac{n+1}{2}x + \frac{n+1}{2}(y + \frac{n-1}{2}z) + \frac{n-1}{2}\frac{n-3}{2}x).$$

Then we have the estimation

$$\begin{aligned} EV \quad \leq \quad & \frac{n(n-1)x + r^2(z-x)}{\frac{1}{2}W(\frac{n-1}{2}) + \frac{1}{2}W(\frac{n+1}{2})} \leq \\ & \frac{n(n-1)x + \frac{(n+1)^2}{4}(z-x)}{\frac{1}{2}(\frac{n-1}{2}(y + \frac{n+1}{2}z) + \frac{n-1}{2}\frac{n+1}{2}x + \frac{n+1}{2}(y + \frac{n-1}{2}z) + \frac{n-1}{2}\frac{n-3}{2}x)}. \end{aligned}$$

After multiplying the numerator and the denominator by 4 and deleting positive terms from the denominator we get

$$EV \leq \frac{4n(n-1)x + (n+1)^2(z-x)}{(n-1)(n+1)z + \frac{(n-1)(n+1)}{2}x + \frac{(n-1)(n-3)}{2}x}.$$

We would like to prove that this ratio is no more than 2. This means that the following inequality must hold

$$4n(n-1)x + (n+1)^2z - (n+1)^2x \leq 2(n^2-1)z + 2(n-1)^2x.$$

By rearranging we get

$$(n-3)(n+1)x \leq (n-3)(n+1)z$$

which obviously holds by the assumption $x < z$. Thus we have that for PD-type games $EV \leq 2$.□

In Forgó (2016) it is shown that for the two-person PD-game $EV = 2$ which together with Theorem 9 imply the following two corollaries.

Corollary 4 For PD-type games $EV = 2$.

Corollary 5 For PD-games $EV = 2$.

6 Conclusion

All four classes of two-facility simple linear congestion games were considered and for the soft correlated equilibrium (SCE) the enforcement value (EV), an indicator how close one can get to the absolute maximum of social welfare (measured as the sum of the utilities of the players) by applying the special protocol of SCE. For non-increasing utilities it was found $1,125 \leq EV \leq 1,265625$, for non-decreasing utilities $EV = 1$ (the best possible) and for both classes of mixed non-increasing/non-decreasing utilities $EV = 2$. Mixed utility cases contain n-player generalizations of two important social dilemmas: chicken and prisoners' dilemma. For both $EV = 2$, though for the two- and three-person chicken games $EV = 1,5$. The technique used for finding these values is parametric linear programming where parameters are in one row of the coefficient matrix and in the objective function. Further research may take various courses. Just to mention a few: changing utilitarian social welfare to egalitarian, replacing the benefit-model with a cost-model, increasing the number of facilities, examining what happens if the worst-case approach is replaced by the average-case approach, abandon the assumption of linearity, etc. Of course, finding the exact EV for the non-increasing case remains a challange. Actual application of the theory for concrete problems in economics, business, sociology and other social sciences would also do good to enhance the relevance of the models studied.

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Appendix

When determining the EV, the efficient maximization of SW over the set of SCE's plays a crucial role. Due to the special structure of the n-player two-facility simple congestion games (not necessarily linear) this maximization requires the solution of an LP with $n+1$-constraints. An optimal solution of this LP can be obtained by reducing the problem to an LP with only two constraints.

In Forgó (2014) this reduction was not complete and made certain proofs more complicated than necessary. In this paper we used the reduced problem as a complete substitute for the original one. The proof of why this simplification works has been relegated to this appendix. We use the terminology and notation introduced in Section 2 of this paper.

The set of SCE's is defined by incentive constraints. The incentive constraint of player j is of the form

$$\sum_{(i_1,...,i_n)\in S} f_j(i_1,...,i_n)p_{i_1,...,i_n} \geq \sum_{(i_1,...,i_n)\in S} g_j(i_1,...,i_n)p_{i_1,...,i_n}.$$

The expected (utilitarian) social welfare (SW) is

$$\sum_{j=1}^{n} \sum_{(i_1,...,i_n)\in S} f_j(i_1,...,i_n)p_{i_1,...,i_n}.$$

If we maximize SW over the set of SCE's, then we have an LP with $n+1$ constraints. In addition to the n incentive constraints there are non-negativity constraints and the normalizing equality for the probabilities. We call this the full-size LP.

The following problem will be referred to as the small-size LP (see (1), (2), (3))

$$\max SW = \sum_{t=0}^{n} \binom{n}{t} (b_t t + a_{n-t}(n-t))p_t$$

$$(a_n - b_1)p_0 + \sum_{t=1}^{n-1} (\binom{n-1}{t-1}(b_t - a_{n-t+1}) + \binom{n-1}{t}(a_{n-t} - b_{t+1}))p_t + (b_n - a_1)p_n \geq 0$$

$$\sum_{t=0}^{n} \binom{n}{t} p_t = 1 \ , \ \ p_t \geq 0, \quad t = 0, 1, ..., n.$$

For determining the EV we are primarily interested in the objective value of the full-size LP. For the sake of a more simple exposition, in order to establish a relationship between the full-size LP and the small-size LP we put the problem in a more general setting.

Let $\mathbf{a}_{ij}$ be a row vector of size r_j and c_j be scalars ($i = 1, ..., l; j = 1, ..., k$). Assume that $\mathbf{a}_{ij}\mathbf{1}$ is identical for all $i = 1, ..., l$ where $\mathbf{1}$ is a vector of 1's. Denote $b_j = \mathbf{a}_{ij}\mathbf{1}, j = 1, ..., k$.

Consider the LP with r_j-vectors of variables $\mathbf{x}_j, j = 1, ..., k$ and denote it by FS (full size)

$$FS : \max \sum_{j=1}^{k} c_j \mathbf{1}\mathbf{x}_j$$

$$\sum_{j=1}^{k} \mathbf{a}_{ij}\mathbf{x}_j \geq 0 \quad i = 1, ..., l$$

$$\sum_{j=1}^{k} \mathbf{x}_j \mathbf{1} = 1$$

$$\mathbf{x}_j \geq 0, \quad j = 1, ..., k.$$

Define another LP with scalar variables y_j which will be referred to as SS (small-size)

$$SS: \ \max \sum_{j=1}^{k} c_j r_j y_j$$

$$\sum_{j=1}^{k} b_j y_j \geq 0$$

$$\sum_{j=1}^{k} r_j y_j = 1, \qquad y_j \geq 0, \quad j = 1, ..., k.$$

The following two propositions are straightforward and can be proved by simple substitution. In particular, set $k = n+1, x_1 = p_0, \mathbf{x}_j = \{p_{i_1,...,i_n}, (i_1, ..., i_n) \in S_j\}$ i.e. a vector of variables $p_{i_1,...,i_n}$ for which the number of players choosing $F2$ is j,$j = 1, ..., n-1, x_n = p_n,\ a_{i1} = (a_n - b_1), a_{ij} = \binom{n-1}{j-1}(b_j - a_{n-j+1}) + \binom{n-1}{j}(a_{n-j} - b_{j+1}), j = 1, ..., n-1, a_{i(n+1)} = b_n - a_1,\ r_j = \binom{k}{j-1}, j = 1, ..., k.$

Proposition 1 If $\mathbf{x}_1^0, ..., \mathbf{x}_k^0$ is feasible for FS, then $y_j^0 = \mathbf{x}_j^0 \mathbf{1}, j = 1, ..., k$ is feasible for SS and the two solutions have the same objective value.

Proposition 2 If $y_1^0, ..., y_k^0$ is feasible for SS, then $\mathbf{x}_j^0 = \frac{1}{r_j} y_j^0 \mathbf{1}, j = 1, ..., k$ is feasible for FS and the two solutions have the same objective value.

Corollary 5 If $y_1^0, ..., y_k^0$ is optimal to SS, then $\mathbf{x}_j^0 = \frac{1}{r_j} y_j^0 \mathbf{1}, j = 1, ..., k$ is optimal to FS and the two solutions have the same objective value.

Clearly, the full-size and small-size LP's defined for the maximization of the SW over the set of SCE's can be identified as FS and SS, therefore Corollary 5 holds for them. Thus, if we are only interested in the objective value of the full-size problem, then it is enough to solve the much simpler small-size problem.