

A Two-stage Bertrand-Edgeworth Game

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Abstract: In our investigation we are expanding a Bertrand-Edgeworth duopoly into a two-stage game in which during the first stage the firms can select their rationing rule. We will show that under certain conditions the efficient rationing rule is an equilibrium action of the first stage.

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1 Introduction

We will investigate a two-stage extension of the capacity constraint Bertrand-Edgeworth duopoly game. In stage one both firms simultaneously announce a rationing rule, according to which they will serve the consumers, if they become the low-price firm. In stage two they are engaged in a modified capacity constrained Bertrand-Edgeworth game. We will refer to this game as the *rationing game*.

Davidson and Deneckere (1986) already formulated a three-stage extension of the Bertrand-Edgeworth game in that each duopolist can select the way it will serve the consumers, if it becomes the low-price firm. In their model the firms compared to our rationing game additionally can select their capacity levels. They established that in a subgame perfect Nash equilibrium the duopolists will serve the consumers according to the random rationing rule. Their result assumes that the duopolists are risk-neutral. On that point our analyzes will differ.

For a full specification of the Bertrand-Edgeworth game we need a so-called rationing rule. The aggregate demand function and the rationing rule together contain enough information on the determination of the duopolists'

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sales. We will introduce the notion of combined rationing, which contains as special cases the two most frequently used rationing rules, the so-called efficient and random rationing rules. For a description of these rationing rules see for example Tirole (1988).

It has been shown for linear demand curves that when capacities are either small or large, then the Bertrand-Edgeworth duopoly with capacity constraints has an equilibrium in pure strategies (see Wolfstetter, 1993). However, for capacities in an intermediate range, the model only has an equilibrium in mixed strategies. The mixed strategy equilibrium was computed in closed form by Beckmann (1965) for random rationing and by Levitan and Shubik (1972) for efficient rationing. Dasgupta and Maskin (1986b) demonstrated the existence of mixed strategy equilibrium in the case of random rationing for demand curves which intersect both axes.

In Section 2 we will introduce the set of rationing rules from which the firms can choose their first stage action. In Section 3 we will determine the set of those capacity levels for which the Bertrand-Edgeworth game has a pure strategy equilibrium. In Section 4 we will establish that if the firms have special preferences above the set of expected profits and uncertainty, then in the first stage of the rationing game the efficient rationing rule is an equilibrium action.

2 Rationing rules

We impose the following assumptions on the demand curve.

Assumption 2.1 *We shall consider demand curves that are strictly decreasing, continuously differentiable, and intersect both axis.*

Assumption 2.2 *The function $G(p) := pD'(p) + D(p)$ is strictly decreasing.*

A monopolist facing a demand curve satisfying Assumptions 2.1 and 2.2 has a unique positive revenue maximizing price. Let us denote the set of demand curves fulfilling Assumptions 2.1 and 2.2 by \mathcal{D} . The demand facing firm $j \in \{1, 2\}$ is given by a rationing rule. In our model we allow the duopolists only to choose from a special class of rationing rules. We call these combined rationing rules.

Definition 2.3 *A function $\Delta : \mathcal{D} \times \mathbf{R}_+^2 \times \mathbf{R}_+^2 \rightarrow \mathbf{R}_+^2$ is called a combined rationing rule with parameter $\lambda \in [0, 1]$, if the demand firm $j \in \{1, 2\}$ faces is given by*

$$\Delta_j(D, p_1, p_2, q_1, q_2) := \begin{cases} D(p_j) & \text{if } p_j < p_i, i \neq j; \\ \frac{q_j}{q_1 + q_2} D(p_j) & \text{if } p_j = p_i, i \neq j; \\ \max(D(p_j) - \alpha(p_i, p_j)q_i, 0) & \text{if } p_j > p_i, i \neq j; \end{cases}$$

where $\alpha(p_i, p_j) = (1 - \lambda) \frac{D(p_j)}{D(p_i)} + \lambda$.

The efficient and the random rationing rules are also combined rationing rules. We can see this by selecting for λ in Definition 2.3 the values 0 and 1 respectively.

We describe two different markets in which a combined rationing rule can be implemented. First, suppose that there are n consumers with identical individual demand functions $d(\cdot)$, who are served by the low-price firm in order of their arrival. Let n be sufficiently large, so that the amount purchased by the marginal consumer, who still obtains a positive level of the product, can be neglected. Let $p_1 < p_2$ and $q_1 \leq D(p_1) = nd(p_1)$. The low-price firm can serve $m := \lfloor q_1/d(p_1) \rfloor$ consumers totally. Fix an arbitrary value $0 \leq \lambda \leq 1$. Assume that firm 1 serves $m_1 := \lfloor (1 - \lambda)q_1/d(p_1) \rfloor$ consumers completely. Each remaining consumer obtains $\frac{q_1 - m_1 d(p_1)}{n - m_1}$ amount of the product. In the described case the residual demand is

$$D^r(p_2) \approx D(p_2) - (1 - \lambda)q_1 \frac{D(p_2)}{D(p_1)} - \lambda q_1,$$

if n is sufficiently large. The way how the low-price firm serves the consumers combines the two different methods, how on the market with identical consumers the efficient and the random rationing rule can be achieved (see Davidson and Deneckere, 1986). However, on the same market a combined rationing rule can also be implemented, if each consumer can purchase $\frac{q_1}{n}(\lambda + (1 - \lambda)d(p_2)/d(p_1))$ amount of the product.

Second, we assume that $D(p)$ is the summation of inelastic demands of heterogenous consumers, all of whom want to purchase one unit of the good, provided the price is below their reservation price. Suppose the low-price firm begins with selling $(1 - \lambda)q_1$ output on a first-come-first-served basis. The consumers served in that way are a random sample of the consumer population. Hence, the demand of the so far unsatisfied consumers at price p_2 is $D(p_2) - (1 - \lambda)q_1 D(p_2)/D(p_1)$. Thereafter, it sells the remaining λq_1 output to the consumers with the highest reservation values first. This leads to a combined rationing rule with parameter λ .

It is worthwhile to mention that if the demand side of the market can be described by a representative consumer having a Cobb-Douglas utility function $u(x, m) = Ax^{(1-\lambda)}m^\lambda$ where x is the amount purchased from the duopolists' product and m is the consumption from a composite commodity, then we obtain a combined rationing rule with parameter λ on the market (for details see Tasnádi, 1998).

3 Pure strategy equilibrium

For given λ_1 and λ_2 we determine the set of those capacity levels to which pure strategy equilibrium exists in the capacity constraint Bertrand-Edgeworth game. Let us remark that the existence of mixed strategy equilibrium follows easily from Dasgupta's and Maskin's Theorem 5 (1986a).

We assume without loss of generality that the marginal costs of the firms are zero. We consider the capacity constraints k_1 and k_2 of the two firms as given.

We restrict ourselves to capacities from the set

$$L := \{(k_1, k_2) \in \mathbf{R}_{++}^2 \mid k_1 + k_2 \leq D(0)\}$$

because for capacities not in L the Bertrand-Edgeworth game reduces to the Bertrand duopoly, or it will not have a pure strategy equilibrium for any rationing rules. To any $\lambda_1, \lambda_2 \in [0, 1]$ parameters describing the rationing rules of the firms, we introduce the set $K(\lambda_1, \lambda_2) \subset L$ containing those capacity levels for which the corresponding Bertrand-Edgeworth game possesses Nash equilibrium in pure strategies. Assumption 2.2 assures that $K(\lambda_1, \lambda_2)$ will not be empty.

Proposition 3.1 *The set $K(\lambda_1, \lambda_2)$ increases strictly if $\min\{\lambda_1, \lambda_2\}$ increases so far as $K(\lambda_1, \lambda_2) \neq L$. If $(k_1, k_2) \in K(\lambda_1, \lambda_2)$, then the pure strategy Nash equilibrium is given by*

$$q_i^* = k_i \quad \text{and} \quad p^* = p_1^* = p_2^* = D^{-1}(k_1 + k_2). \quad (1)$$

Proof: First, we show that only (1) can be an equilibrium. No equilibrium can exist with $p_1 < p_2$ because, if $D(p_1) > k_1$, firm 1 will want to increase its price, and if $D(p_1) \leq k_1$, firm 2 will wish to reduce its price below p_2 . Similarly, no equilibrium is possible with $p_2 > p_1$. There cannot be an equilibrium with $p_1 = p_2 > p^*$, since both firms have the incentive to lower their prices slightly. It is obvious that a price below p^* cannot be rational for any firm.

The price p^* is the only candidate for a pure strategy equilibrium price. The profit function of firm i for $p^* < p \leq \bar{p}$ is:

$$\pi_i(p) = pD^r(p) = p \left(D(p) - \lambda_j k_j - (1 - \lambda_j) k_j \frac{D(p)}{D(p_j^*)} \right),$$

where $j \neq i$ and $D^r(\bar{p}) = 0$. For prices greater than \bar{p} the residual profit function is zero. Hence, setting prices unilaterally above \bar{p} is not rational, because prices p^* yield positive profits. The profit function is nonincreasing for prices $p^* < p < \bar{p}$ because of Assumption 2.2, if

$$\frac{d\pi_i}{dp}(p^*) = (p^* D'(p^*) + D(p^*)) \left(1 - (1 - \lambda_j) \frac{k_j}{k_i + k_j} \right) - \lambda_j k_j \leq 0 \quad (2)$$

holds. Rearranging (2) we obtain

$$G(D^{-1}(k_i + k_j)) \leq \frac{\lambda_j k_j}{1 - (1 - \lambda_j) \frac{k_j}{k_i + k_j}} = \frac{\lambda_j k_j (k_i + k_j)}{k_i + \lambda_j k_j}. \quad (3)$$

We have that $K(\lambda_1, \lambda_2)$ increases if $\min\{\lambda_1, \lambda_2\}$ increases, because (3) must hold for both firms and $\frac{\lambda_j k_j (k_i + k_j)}{k_i + \lambda_j k_j}$ is strictly increasing in λ_j .

It remains to show that the set $K(\lambda_1, \lambda_2)$ increases strictly. Let us introduce the following notations:

$$\begin{aligned} K^\alpha(\lambda_1, \lambda_2) &:= \{(k, \alpha k) \in L \mid (k, \alpha k) \in K(\lambda_1, \lambda_2)\}, \\ K_*^\alpha(\lambda_1, \lambda_2) &:= \{(k, \alpha k) \in L \mid k \in (0, D(0)/(1 + \alpha))\} \end{aligned}$$

for any $\alpha > 0$. Rearranging (3) and substituting equal capacities we obtain

$$\frac{G(D^{-1}((1 + \alpha)k))}{k} \leq \frac{(1 + \alpha)\lambda_j}{\alpha + \lambda_j}. \quad (4)$$

Obviously, an analogous condition to (4) must hold for firm j . Thus, $(k, \alpha k) \in K^\alpha(\lambda_1, \lambda_2)$ if and only if

$$\frac{G(D^{-1}((1 + \alpha)k))}{k} \leq \min \left\{ \frac{(1 + \alpha)\lambda_1}{\alpha + \lambda_1}, \frac{(1 + \alpha)\lambda_2}{\alpha + \lambda_2} \right\}. \quad (5)$$

Furthermore, the left side of (5) is continuous for all $k \in (0, \frac{D(0)}{1 + \alpha}]$, therefore the set $K^\alpha(\lambda_1, \lambda_2)$ increases strictly if $\min\{\lambda_1, \lambda_2\}$ increases, as long as $K^\alpha(\lambda_1, \lambda_2) \neq K_*^\alpha(\lambda_1, \lambda_2)$, because the function $\frac{(1 + \alpha)\lambda}{\alpha + \lambda}$ is strictly increasing in λ for $\lambda \in [0, 1]$.

If $K(\lambda_1, \lambda_2) \neq L$, then there is an $\alpha > 0$ so that $K^\alpha(\lambda_1, \lambda_2) \neq K_*^\alpha(\lambda_1, \lambda_2)$. Suppose that $\lambda_1 < \lambda'_1 < \lambda_2$, then $K^\alpha(\lambda_1, \lambda_2)$ is a proper subset of $K^\alpha(\lambda'_1, \lambda_2)$. Finally, since $K^\alpha(\lambda_1, \lambda_2) \subset K(\lambda_1, \lambda_2)$ and $K^\alpha(\lambda'_1, \lambda_2) \setminus K^\alpha(\lambda_1, \lambda_2)$ is nonempty and disjoint from $K(\lambda_1, \lambda_2)$, therefore $K(\lambda_1, \lambda_2)$ is a proper subset of $K(\lambda'_1, \lambda_2)$. We can argue similarly in the case of $\lambda_1 > \lambda_2$ and $\lambda_1 = \lambda_2$. \square

If the demand curve is linear and if we restrict ourselves to symmetric capacities, then $K(\lambda_1, \lambda_2)$ has a simple structure, as we will establish in Proposition 3.2. We have to mention that in case of a linear demand curve the price and quantity units can be chosen so that the demand curve has the form $D(p) = 1 - p$. Let $H(\lambda_1, \lambda_2) := \{k \in (0, D(0)/2] \mid (k, k) \in K^1(\lambda_1, \lambda_2)\}$.

Proposition 3.2 *If the demand curve is $D(p) = 1 - p$, then*

$$H(\lambda_1, \lambda_2) = \left(0, \frac{1}{2} \min \left\{ \frac{1 + \lambda_1}{2 + \lambda_1}, \frac{1 + \lambda_2}{2 + \lambda_2} \right\} \right]. \quad (6)$$

Proof: Regarding the proof of Proposition 3.1 we only have to determine those capacity constraints for which (2) holds for both firms in the case of equal capacities. Therefore, for firm $i \in \{1, 2\}$ the following inequality has to be satisfied.

$$(-p^* - 1 - p^*)(1 - \frac{1}{2}(1 - \lambda_i)) - \lambda_i k = (4k - 1)\frac{1}{2}(1 + \lambda_i) - \lambda_i k \leq 0 \quad (7)$$

Rearranging (7) and regarding that it has to hold for both firms, we obtain (6). \square

If both firms are serving the consumers according to the efficient rationing rule, then by Proposition 3.2 we get $H(1, 1) = (0, 1/4]$. This well-known result can be found for instance in Wolfstetter (1993). Additionally, if both firms select the random rationing rule, then $H(0, 0) = (0, 1/3]$. This result can be found in Tirole (1988) for example.

4 The rationing game

In this section we only want to indicate that in the two-stage game the efficient rationing rule is under certain conditions an equilibrium first-stage action.

The action sets of both firms in stage one is $[0, 1]$ and in stage two it is the set of price distributions with finite variances above the set $[0, \hat{p}]$, where we denote by \hat{p} the smallest price for that $D(\hat{p}) = 0$. A degenerated probability distribution corresponds to a pure strategy in stage two. Now we modify the payoff functions by assuming that the firms have preferences above the space of expected profits and profit variances, which can be determined by the chosen rationing rule and probability distributions.

Davidson and Deneckere (1986) found that random rationing is the equilibrium action of the appropriate stage in the case when both firms preferences depend only on their expected profits. This means that the firms are risk neutral. We investigate another extreme case in that both firms are extremely risk averse. Let the firms have the following lexicographic preferences $\succ \subset \mathbf{R}_+^2$

$$(e, v) \succ (e', v') \Leftrightarrow v < v' \text{ or } (v = v' \text{ and } e > e'),$$

where e, e' denote expected profits and v, v' denote variances.

We introduce the set valued function $\Lambda : L \rightarrow \mathcal{P}([0, 1] \times [0, 1])$ as follows

$$\Lambda(k_1, k_2) := \{(\lambda_1, \lambda_2) \mid \exists(k_1, k_2) \in K(\lambda_1, \lambda_2)\}.$$

Proposition 4.1 *If the two-stage game has a pure strategy subgame perfect Nash equilibrium, then choosing the efficient rationing rule in the first-stage is a subgame perfect Nash equilibrium action for both firms.*

Proof: If the two-stage game has a pure strategy subgame perfect Nash equilibrium, then $\Lambda(k_1, k_2) \neq \emptyset$. After any first stage action $(\lambda_1, \lambda_2) \in \Lambda(k_1, k_2)$ both firms will set their price to $p^* = D(k_1 + k_2)$ because of Proposition 3.1. Firms are indifferent between any rationing rule pair from set $\Lambda(k_1, k_2)$, because in equilibrium they all guarantee the same profits without uncertainty. The efficient rationing rule is always an equilibrium action of stage one because $(k_1, k_2) \in K(\lambda_1, \lambda_2)$ implies that $(k_1, k_2) \in K(1, 1)$ regarding Proposition 3.1. \square

5 Concluding remarks

These results indicate that the equilibrium rationing rule may lie between the efficient and random rationing rule depending on the firms' preferences above expected profits and profit variances. This conjecture deserves further analyzes, although in general the expected values and variances cannot be determined in closed form since in general the mixed strategy equilibrium cannot be expressed in closed form either.

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