The Filippov-Ważewski relaxation theorem revisited*

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Abstract

The converse statement of the Filippov-Ważewski relaxation theorem is proven, more precisely, two differential inclusions have the same closure of their solution sets if and only if the right-hand sides have the same convex hull. The idea of the proof is examining the contingent derivatives to the attainable sets.

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1 Introduction

The corner stone in the theory of differential inclusions and their applications (particularly in control theory) is the celebrated Filippov-Ważewski relaxation theorem (see Theorem 2.4.2 in [1], or Theorem 10.3 in [3] for a more general formulation). This result basically states that the solution set of a Lipschitzian differential inclusion is dense in the set of relaxed solutions, i.e. in the solution set of the differential inclusion whose right-hand side is the convex hull of the original set valued map. This implies in particular that the attainable sets of the nonconvexified inclusion are dense in the attainable sets of the convexified inclusion. Therefore, the relaxation theorem can be regarded as a far reaching generalization of the bang-bang principle in linear control theory.

In the present paper we choose a different approach to the problem, namely, given a differential inclusion with convex valued right-hand side, we look for a smaller set valued map which essentially yields the same attainable

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sets. By using the contingent derivative we will obtain a necessary condition for this problem. More precisely, we show that such a smaller set valued map necessarily contains all the extremal points of the convex valued map. This means in particular, that, in a certain sense, the converse of the relaxation theorem holds true.

Summing up, if we want to economize a differential inclusion or a control system (shrinking the right-hand side as much as possible while essentially retaining the attainable sets) the ultimate answer would be the set of extremal points. Unfortunately, we cannot guarantee in general that such an iclusion admits solutions. However, there are positive results in this direction, if the set valued map possesses nonempty interior; for a comprehensive survey of this area we refer to Pianigiani [6].

2 Differential inclusions

In this section we introduce some basic concepts and notations concerning differential inclusions. For proofs and more details we refer to [3].

Let X denote a finite dimensional Euclidean space, and Ω a nonempty open subset of X. Consider a set valued map F defined on Ω with nonempty compact values in X and let x be a point given in Ω.

Definition 1 An absolutely continuous function $\varphi : I \to X$ defined on an open interval I is said to be a *solution for F through x*, if

$$\begin{aligned} &\varphi(t) \in \Omega \quad \text{for every } \; t \in I \\ &\varphi'(t) \in F(\varphi(t)) \quad \text{for a. e. } \; t \in I \\ &0 \in I \quad \text{and} \quad \varphi(0) = x \, . \end{aligned} \tag{1}$$

If F is locally bounded at x that is there exist a neighborhood U of x and a number $\gamma > 0$ such that

$$\|v\| \leq \gamma \tag{2}$$

for every $v \in F(U)$, then we can choose a positive α so that the set

$$M_\alpha = \{y \in X : \|y - x\| \leq \gamma \cdot \alpha\}$$

is contained in U. Let I be the interval $(-\alpha, \alpha)$. It is easy to see that every trajectory for F through x, if any exist, is defined on I.

Throughout the rest of the paper we will always assume that our set valued maps are locally bounded. Let us note that upper semicontinuous maps with compact values defined on locally compact spaces are automatically locally bounded.

We denote by $S_F(x)$ the set of all solutions to (1) defined on the interval $I = (-\alpha, \alpha)$, and by $A_F(t,x)$ the attainable set (solution cross section)

$$A_F(t,x) = \{\varphi(t) : \varphi \in S_F(x)\}$$

from x at $t \in I$.

By $\mathrm{cl}\,\mathrm{co}\,F$ we denote the set valued map whose values are the closed convex hulls of the values of F at every point. Let us note that if the mapping F is upper semicontinuous (resp. continuous, locally Lipschitz), then so is $\mathrm{cl}\,\mathrm{co}\,F$ (cf. [3]).

Let us recall the definition of the contingent derivative to a set valued map Φ defined on a Banach space.

Definition 2 The *contingent derivative* to Φ at (x,y) (where $y \in \Phi(x)$) is defined to be the set valued map $D\Phi(x,y)$, whose graph is the Bouligand contingent cone to the graph of Φ at (x,y). That is

$$\mathrm{graph}\, D\Phi(x,y) = T_{\mathrm{graph}\,\Phi}(x,y)\,,$$

where T stands for the Bouligand contingent cone.

For details about contingent derivatives to set valued maps we refer to the comprehensive monograph by Aubin and Frankowska [2].

3 Contingent derivatives to attainable sets

According to the definiton, the contingent derivative to the set valued map $t \to A_F(t,x)$ at the point (t,y) (where $y \in A_F(t,x)$), is the set valued map $DA_F(.,x)(t,y)$, whose graph is the Bouligand contingent cone to the graph of $A_F(.,x)$ at the point (t,y). Namely,

$$\mathrm{graph}\, DA_F(.,x)(t,y) = T_{\mathrm{graph}\,A_F(.,x)}(t,y)\,.$$

Since $A(.,x)$ is locally Lipschitz, the next statement is a special case of Proposition 5.1.4 in [2].

Lemma 1 *The following characterization holds true:* $v \in DA_F(.,x)(t,y)(s)$ *if and only if*

$$\liminf_{h\to 0+} d\left(v, \frac{A_F(t+hs,x)-y}{h}\right) = 0$$

is valid (d denotes the distance function).

If the contingent the derivative is taken at $t = 0$ we will use the simplified notation $DA_F(x)$ instead of $DA_F(.,x)(0,x)(1)$. According to Lemma 1, $v \in DA_F(x)$ if and only if there exist a sequence $t_n \to 0+$ in I and a function $r : I \to X$ such that for every n

$$x + t_n v + r(t_n) \in A_F(t_n, x)\,,$$

where

$$\lim_{n\to+\infty} \frac{1}{t_n}\|r(t_n)\| = 0\,.$$

Since the contingent derivative is the Kuratowski upper limit of the difference quotients, we obtain the following statement.

Lemma 2 *If* F *is locally bounded at* $x \in \Omega$ *and* $S_F(x)$ *is nonempty, then* $DA_F(x)$ *is a nonempty compact subset of* X*.*

Proof. If $S_F(x)$ is nonempty, then so is $A_F(t,x)$ for each $t \in I$. By Lemma 1 we have

$$DA_F(x) = \limsup_{h\to 0+} \frac{A_F(h,x)-x}{h}$$

in the Kuratowski sense. On the other hand, (2) implies that $\|v\| \le \gamma$ for every $v \in DA_F(x)$. Therefore, making use of Theorem 1.1.4 in [2], we get the desired property. □

Lemma 3 *If* F *is upper semicontinuous on* Ω*, then*

$$DA_F(x) \subset \operatorname{cl}\operatorname{co} F(x)$$

for every $x \in \Omega$*.*

Proof. Choose x in Ω and suppose that $DA_F(x)$ is nonempty. Let $v \in DA_F(x)$ and $\varepsilon > 0$ be given. By the upper semicontinuity of $\operatorname{cl}\operatorname{co} F$ there exists a $\delta > 0$ such that $x + \delta B \subset \Omega$ and

$$\operatorname{cl}\operatorname{co} F(y) \subset \operatorname{cl}\operatorname{co} F(x) + \frac{\varepsilon}{2}B$$

for every $y \in x + \delta B$, where B denotes the closed unit ball in X. On the other hand, F is locally bounded, for there exists a positive γ such that $F(y) \subset \gamma B$, if $y \in x + \delta B$. Hence, $|t| < \delta/\gamma$ implies $\|\varphi'(t)\| \leq \gamma$, and

$$\varphi(t) \in x + \delta B$$

for each $\varphi \in S_F(x)$. Consequently,

$$\varphi'(t) \in F(\varphi(t)) \subset \operatorname{cl} \operatorname{co} F(\varphi(t)) \subset \operatorname{cl} \operatorname{co} F(x) + \frac{\varepsilon}{2} B\,.$$

Therefore, in view of the mean value theorem, we have

$$\int_0^t \varphi'(s)\, ds = \varphi(t) - x \in t \left(\operatorname{cl} \operatorname{co} F(x) + \frac{\varepsilon}{2} B \right) \tag{3}$$

for every $\varphi \in S_F(x)$ and $|t| < \delta/\gamma$. Therefore, if $0 < |t| < \delta/\gamma$, and $y \in A_F(t,x)$ are given, then we can find a trajectory $\varphi \in S_F(x)$ with $\varphi(t) = y$, and by making use of (3) we get

$$\frac{1}{t}(y - x) \in \operatorname{cl} \operatorname{co} F(x) + \frac{\varepsilon}{2} B\,. \tag{4}$$

According to Lemma 1 there exist a sequence $t_n \to 0$ in I with $t_n \neq 0$, and a function $r : I \to X$ such that

$$x + t_n v + r(t_n) \in A_F(t_n, x)$$

for every integer n and

$$\lim_{n \to +\infty} \frac{1}{t_n} \|r(t_n)\| = 0\,.$$

For each n set

$$x_n = x + t_n v + r(t_n)\,,$$

and choose an index n_0 such that for all $n \geq n_0$ we have $|t_n| < \delta/\gamma$ and

$$\left\| \frac{1}{t_n}(x_n - x) - v \right\| = \frac{\|r(t_n)\|}{|t_n|} < \frac{\varepsilon}{2}\,. \tag{5}$$

On the other hand, for $n \geq n_0$, (4) implies that

$$\frac{1}{t_n}(x_n - x) \in \operatorname{cl} \operatorname{co} F(x) + \frac{\varepsilon}{2} B\,. \tag{6}$$

Combining relations (5) and (6), it follows that

$$v \in \operatorname{cl}\operatorname{co} F(x) + \varepsilon B\,.$$

Since ε is arbitrary, this completes the proof. □

It is obvious that the converse inclusion is not true in general even for convex valued mappings. For instance, if $F(x) = \{0\}$ for $x \neq 0$ and $F(0) = [0,1]$, then $DA_F(0) = \{0\}$.

Lemma 4 *If F is lower semicontinuous on Ω, then*

$$F(x) \subset DA_F(x)$$

for every $x \in \Omega$.

Proof. Let $x \in \Omega$, $v \in F(x)$ and $\varepsilon > 0$ be fixed. Since F is lower semicontinuous, there exists a positive δ such that

$$F(y) \cap (v + \varepsilon \operatorname{int} B) \neq \emptyset$$

for every $y \in x + \delta B$. Consider the set valued map $\hat{F}$ on Ω defined by

$$\hat{F}(y) = \operatorname{cl}\,\left(F(y) \cap (v + \varepsilon \operatorname{int} B)\right)\,.$$

Then $\hat{F}$ is also lower semicontinuous (see [1, Proposition 1.1.5]) and therefore, the corresponding solution set $S_{\hat{F}}(x)$ is not empty (cf. [1, Theorem 2.6.1]). Select a solution φ from $S_{\hat{F}}(x)$. Then

$$\int_0^t \varphi'(s)\,ds \in \int_0^t \hat{F}(\varphi(s))\,ds \subset \int_0^t (v + \varepsilon B)\,ds\,,$$

where the last two integrals are taken in the Aumann sense. From these relations we deduce

$$\frac{1}{t}\int_0^t \varphi'(s)\,ds \in v + \varepsilon B$$

if t is sufficiently small. Thus, taking into account that φ is also a solution for F through x, we conclude

$$v \in DA_F(x) + \varepsilon B\,.$$

Since ε is arbitrary, the lemma ensues. □

Again, the straightforward example of $F(x) = \{0,1\}$, if $x \neq 0$ and $F(0) = \{0\}$ shows that the opposite inclusion is generally not valid, since $DA_F(0) = [0,1]$.

The theorem below is the consequence of the preceding lemmas.

Theorem 1 *If F is continuous on Ω, then*

$$F(x) \subset DA_F(x) \subset \operatorname{cl}\operatorname{co} F(x)$$

for every $x \in \Omega$.

4 The relaxation theorem

Consider now two set valued maps F and G defined on Ω with nonempty compact values in X. The following proposition is an easy consequence of the definition of the contingent derivative.

Lemma 5 *Suppose that $A_G(t,x)$ is dense in $A_F(t,x)$ for every $t \in I$ and $x \in \Omega$. Then*

$$DA_G(x) = DA_F(x)$$

for each x in Ω.

Now we can show that if a continuous convex valued map is given, then any smaller upper semicontinuous map that essentially retains the same attainable sets, necessarily contains all extremal points of the convex valued map.

Theorem 2 *Assume that F is continuous on Ω with convex compact values and consider an upper semicontinuous compact valued map G such that $G(x) \subset F(x)$ for every $x \in \Omega$. Suppose that $A_G(t,x)$ is dense in $A_F(t,x)$ for all $t \in I$ and $x \in \Omega$. Then*

$$\operatorname{cl}\operatorname{co} G(x) = F(x)$$

for each x in Ω.

Proof. By applying Lemma 5, Lemma 3 and Theorem 1 we get for an arbitrary x in Ω

$$F(x) = DA_F(x) = DA_G(x) \subset \operatorname{cl}\operatorname{co} G(x)\,,$$

and this proves the desired equality. □

As a consequence, we can reformulate the relaxation theorem in the following way. Recall that a set valued map is said to be locally Lipschitz on Ω, if for every $x \in \Omega$ there exists a positive λ such that

$$F(y) \subset F(x) + \lambda \|x - y\| \cdot B$$

for every y in a neighborhood of x, where B denotes the closed unit ball in X. Density of the solution sets will be understood with respect to the C-norm in the Banach space $C(I)$ of continuous functions.

Theorem 3 *Assume that F is locally Lipschitz on Ω with convex compact values, and consider a compact valued locally Lipschitz map G with the same Lipschitz constants such that $G(x) \subset F(x)$ for every $x \in \Omega$. Then $S_G(x)$ is dense in $S_F(x)$ for all $x \in \Omega$ if and only if*

$$\mathrm{cl}\,\mathrm{co}\,G(x) = F(x)$$

for each x in Ω.

Proof. The suffiency is the classical relaxation theorem. The necessity follows from the fact that if $S_G(x)$ and $S_F(x)$ have the same closure with respect to the C-norm, then so do $A_G(t, x)$ and $A_F(t, x)$ in the norm of X for every $t \in I$, hence Theorem 2 can be applied. □

As is well known, Lipschitz-continuity is essential above, see [1] for a counter-example for continuous mappings. Although some approximation results for relaxed solutions can be obtained even for lower semicontinuous set valued maps, see [5] for the precise statements.

It is worth noting here that the relaxation theorem is no longer valid if the Banach space $C(I)$ is replaced with the Sobolev space $W^{1,1}$ equipped with the norm

$$\|x\|_{W^{1,1}} = \|x(0)\| + \int_0^\alpha \|x'(t)\|\,dt\,.$$

In fact, solution sets in $W^{1,1}$ are closed (resp. compact) for closed (resp. compact) valued Lipschitzian maps, while in $C(I)$ the convexity of the values is essential for proving the closedness of the solution sets (see [7] and also [4] with infinite time horizon, for more details).

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References

[1] J.-P. Aubin and A. Cellina, Differential Inclusions; Set Valued Maps and Viability Theory, Springer-Verlag, Berlin, Heidelberg, 1984.

[2] J.-P. Aubin and H. Frankowska, Set-Valued Analysis, Birkhäuser, Basel, Boston, 1990.

[3] K. Deimling, Multivalued Differential Inclusions, Walter de Gruyter, Berlin, New York, 1992.

[4] Z. Kánnai and P. Tallos, Stability of solution sets of differential inclusions, *Acta Sci. Math. (Szeged)*, 61 (1995), 197–207.

[5] A. Ornelas, Approximation of relaxed solutions for lower semicontinuous differential inclusions, *Annales Polon. Math.*, 57 (1991) 117–125.

[6] G. Pianigiani, Differential inclusions; The Baire category method, in *Methods of Nonconvex Analysis*, held at Varenna, Italy, *Lecture Notes in Math.*, vol. 1446, pp. 104–136, Springer-Verlag, Berlin, Heidelberg, New York, 1989.

[7] Qi Ji Zhu, On the solution set of differential inclusions in Banach space, *Journal of Differential Equations*, 93 (1991), 213–237.