Viable solutions to nonautonomous inclusions without convexity

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Abstract

The existence of viable solutions is proven for nonautonomous upper semicontinuous differential inclusions whose right-hand side is contained in the Clarke subdifferential of a locally Lipschitz continuous function.

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1 Introduction

In [3], Bressan, Cellina and Colombo (see also Ancona and Colombo [1] for perturbed inclusions) proved the existence of solutions to upper semicontinuous differential inclusions

$$x'(t) \in F(x(t)), \quad x(0) = x_0 \tag{1}$$

without convexity assumptions on the right-hand side. They replaced convexity with cyclical monotonocity, i.e. they assumed the existence of a proper convex potential function V with $F(x) \subset \partial V(x)$ at every point. This condition assures the L^2-norm convergence of the derivatives of approximate solutions thus, no convexity is needed to guarantee that the limit is in fact a solution.

Rossi [7] extended this result to problems with phase constraints (viable solutions), and Staicu [9] considered added perturbations on the right-hand side. Ultimately, both papers followed the method of [3].

The convexity assumption on the potential function V was relaxed by Kánnai and Tallos [6], where lower regular functions were examined. That means a locally Lipschitz continuous function whose upper Dini directional derivatives coincide with the Clarke directional derivatives. Convex analysis subdifferentials were replaced by Clarke subdifferentials.

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Viability problems for nonautonomous inclusions without convexity were discussed by Kánnai and Tallos [5] under continuity assumption on the right-hand side.

In the present paper we prove the existence of viable solutions to nonautonomous inclusions in the presence of phase constraint. The right-hand side of the inclusion is assumed to be measurable in t and upper semicontinuous with respect to x with nonconvex values. A counterexample shows that lower regularity of the potential function cannot be omitted.

2 Lower regular functions

Let X be a real Hilbert space and consider a locally Lipschitz continuous real valued function V defined on X. For every direction $v \in X$ the upper Dini derivative of V at $x \in X$ in the direction v is given by

$$D^+V(x;v) = \limsup_{t \to 0+} \frac{V(x+tv) - V(x)}{t} ,$$

and its generalized (Clarke) directional derivative at x in the direction v is defined by

$$V^\circ(x;v) = \limsup_{y \to x,\, t \to 0+} \frac{V(y+tv) - V(y)}{t} .$$

The directional derivative of V at x in the direction v (if it exists) will be denoted by $DV(x;v)$.

Definition 1 The locally Lipschitz continuous function V is said to be *lower regular* at x if for every direction v in X we have $D^+V(x;v) = V^\circ(x;v)$. We say that V is lower regular if it is lower regular at every point.

Example 1 Let us note here that lower regular functions are not necessarily regular in the sense of Clarke [4]. Take for instance the function $f(x) = \log(1+x)$ on the real positive half line. Now think of a piecewise linear function V with alternating slopes $+1$ and -1, whose graph lies between f and $-f$. Whenever V reaches the graph of f or $-f$, it bounces back. Since for every $x > 0$, $|f'(x)| < 1$, it is obvious that V zigzags infinitely many times in every neighborhood of the origin. Finally, eliminate all corners of V lying on the graph of f by making the derivative turn from 1 into -1 smoothly. Keep the corners on the graph of $-f$. Clearly, such a V is Lipschitz continuous and it can easily be seen that $D^+V(0,1) = V^\circ(0,1) = 1$ and hence, V is lower regular at the origin. However, $DV(0,1)$ does not exist and therefore, V cannot be regular.

The intermediate (or adjacent) cone to the closed subset K at $x \in K$ is

$$I_K(x) = \left\{ v \in X : D^+d_K(x;v) = 0 \right\} ,$$

where d_K denotes the distance function, moreover

$$C_K(x) = \{v \in X : d_K^\circ(x; v) = 0\}$$

is the Clarke tangent cone to K at x. The following characterization of lower regular functions can be verified by a straightforward adaptation of the proof of Theorem 2.4.9 in [4].

Theorem 1 *The following two statements are valid for every x in X.*
(a) $I_{\mathrm{epi}\, V}(x, f(x)) = \mathrm{epi}\, D^+V(x; .)$
(b) V is lower regular at x if and only if $I_{\mathrm{epi}\, V}(x, f(x)) = C_{\mathrm{epi}\, V}(x, f(x))$.

The Bouligand tangent cone to K at $x \in K$ is defined by

$$T_K(x) = \{v \in X : \liminf_{t \to 0+} \frac{1}{t} d_K(x + tv) = 0\}\,.$$

Obviously, $C_K(x) \subset I_K(x) \subset T_K(x)$, while equalities hold if K is convex. For further characterizations we refer to Aubin and Frankowska [2], pp. 239.

Consider a lower regular function V and let x be a point in X. Suppose $\lambda > 0$ is a Lipschitz constant for V in a neighborhood of x. Let B stand for the closed unit ball in X. By $\partial V(x)$ we denote the Clarke subdifferential of V at x.

Lemma 1 *For every $0 \le \varepsilon \le \lambda$ and $v \in \partial V(x) + \varepsilon B$ the inequality*

$$\|v\|^2 \le D^+V(x; v) + 2\varepsilon\lambda$$

holds true.

Proof. Take $u \in \partial V(x)$ with $\|u - v\| \le \varepsilon$. Since for each $w \in X$ we have $\langle u, w\rangle \le D^+V(x; w)$, by setting $w = v$ it follows

$$\begin{aligned} D^+V(x; v) &\ge \langle u, v\rangle \ge \|v\|^2 + \langle u - v, v\rangle \\ &\ge \|v\|^2 - \varepsilon\|v\| \ge \|v\|^2 - \varepsilon(\varepsilon + \lambda) \ge \|v\|^2 - 2\varepsilon\lambda \end{aligned}$$

that is the desired inequality. □

Lemma 2 *Suppose the function $f(t) = V(x + tv)$ is differentiable at $t = 0$ for some $x \in X$ and $v \in \partial V(x)$. Then $f'(0) = \|v\|^2$.*

Proof. Lower regularity of V at x implies that

$$\langle v, u\rangle \le D^+V(x; u)$$

for each u in X. Applying this inequality with $u = v$ and $u = -v$ the lemma ensues. □

Lemma 3 *If $x : [0,T] \to X$ is absolutely continuous on the interval $[0,T]$ with $x'(t) \in \partial V(x(t))$ a.e., then*

$$(V \circ x)'(t) = \|x'(t)\|^2$$

for a.e. $t \in [0,T]$.

Proof. Let S be a set of measure zero such that both x and $V \circ x$ are differentiable on $[0,T]\setminus S$ moreover $x'(t) \in \partial V(x(t))$ at every $t \in [0,T]\setminus S$. Thus, if $t \in [0,T] \setminus S$ is given, there is a $\delta > 0$ such that $x(t+h) - x(t) - hx'(t) = r(h)$ for every $|h| < \delta$, where $\lim_{h\to 0} \|r(h)\|/h = 0$. Since a locally Lipschitz function on a compact set is globally Lipschitz continuous, we can assume that

$$|V(x(t+h)) - V(x(t) + hx'(t))| \leq \lambda \|r(h)\|$$

whenever $|h| < \delta$. Consequently, the function $h \to V(x(t) + hx'(t))$ is differentiable at $h = 0$, and its derivative is the same as the derivative of $h \to V(x(t+h))$ at $h = 0$. Making use of Lemma 2, we obtain

$$(V \circ x)'(t) = \lim_{h\to 0} \frac{V(x(t) + hx'(t)) - V(x(t))}{h} = \|x'(t)\|^2$$

at each point $t \in [0,T] \setminus S$. $\square$

3 The main result

Let K be a convex and locally compact subset of X and consider a set valued map F defined on $[0,T] \times K$ that is measurable in t and upper semicontinuous with respect to x, with nonempty closed images in X. Let us suppose that there exists a lower regular potential function V on X such that the tangential condition

$$T_K(x) \cap F(t,x) \cap \partial V(x) \neq \emptyset \tag{2}$$

holds true for every $x \in K$, and a.e. $t \in [0,T]$, where $T_K(x)$ denotes the tangent cone to K at x.

Let the point x_0 be given in K and consider the Cauchy problem

$$\begin{aligned} x'(t) &\in F(t,x(t)) \quad \text{a. e.} \\ x(0) &= x_0 \end{aligned} \tag{3}$$

with the phase constraint

$$x(t) \in K, \quad t \geq 0\,. \tag{4}$$

Theorem 2 *Assume that the tangential condition (2) is valid. Then under the above conditions there exists a $T > 0$ such that the problem (3), (4) admits a solution on $[0,T]$.*

Choose $\varrho > 0$ such that $K_0 = K \cap (x_0 + 2\varrho B)$ is compact and V is Lipschitz continuous on $x_0 + 2\varrho B$ with Lipschitz constant $\lambda > 0$. Then $\partial V(x) \subset \lambda B$ for every $x \in K_0$. Set $T = \varrho/\lambda$ and $K_1 = K \cap (x_0 + \varrho B)$. Then no solution x starting from x_0 with

$$x'(t) \in F(t, x(t)) \cap \partial V(x(t)) \quad \text{a. e.} \tag{5}$$

can leave the compact set K_1 on the interval $[0, T]$. Therefore, without loss of generality, we may assume that K is compact. Below we construct a solution to the problem (3), (4) on $[0, T]$ that also solves (5).

We denote by S_T the solution set to the problem (3), (4) on the interval $[0, T]$. S_T will be regarded as a subset of the Banach space $W^{1,2}(0, T, X)$ of absolutely continuous functions equipped with the norm

$$\|x\| = \max_{t \in [0,T]} \|x(t)\| + \left(\int_0^T \|x'(t)\|^2 \, dt \right)^{\frac{1}{2}} .$$

Theorem 3 *Under the additional assumption*

$$F(t, x) \subset \partial V(x) \quad \textit{for a.e. } t \textit{ and each } x \in K$$

there exists a $T > 0$ such that S_T is a nonempty compact subset in $W^{1,2}(0, T, X)$.

4 Regularizing the set valued vector field

Consider the viability problem given in (2), (3) and (4). By regularizing the set valued map F on the right-hand side of the Cauchy problem (3) we will reduce the nonautonomous problem to the autonomous case.

Let $\varepsilon > 0$ be given. Then we can find a countable collection of disjoint open subintervals $(a_j, b_j) \subset [0, T]$, $j = 1, 2, \ldots$ such that their total length is less then ε and a set valued map F_ε defined on

$$D = \left([0, T] \setminus \bigcup_{j=1}^{\infty} (a_j, b_j) \right) \times K$$

that is jointly upper semicontinuous and $F_\varepsilon(t, x) \subset F(t, x)$ for each $(t, x) \in D$. Moreover, if u and v are measurable functions on $[0, T]$ such that

$$u(t) \in F(t, v(t)) \quad \text{a.e. on } [0, T]$$

then for a.e. $t \in \left([0, T] \setminus \bigcup_{j=1}^{\infty} (a_j, b_j) \right)$ we have

$$u(t) \in F_\varepsilon(t, v(t))$$

(we refer to Rzeżuchowski [6] for this Scorza-Dragoni type theorem). It is obvious that all trajectories to F are also trajectories to F_ε.

Now we extend F_ε to the whole $[0,T]\times K$ with retaining upper semicontinuity and the tangential condition (2). Let us define

$$\tilde{F}_\varepsilon(t,x) = \begin{cases} F_\varepsilon(t,x) & \text{if } t\in[0,T]\setminus\cup_{j=1}^{\infty}(a_j,b_j) \\ F_\varepsilon(a_j,x) & \text{if } a_j<t<(a_j+b_j)/2 \\ F_\varepsilon(b_j,x) & \text{if } (a_j+b_j)/2<t<b_j \\ F_\varepsilon(a_j,x)\cup F_\varepsilon(b_j,x) & \text{if } t=(a_j+b_j)/2 \end{cases}$$

It is easy to see that $\tilde{F}_\varepsilon$ still fulfills the tangential condition (2).

Lemma 4 *$\tilde{F}_\varepsilon$ is upper semicontinuous on $[0,T]\times K$.*

Proof. Routine calculations show that the graph of $\tilde{F}_\varepsilon$ is closed. On the other hand, the images of $\tilde{F}_\varepsilon$ are contained in a neighborhood of the upper semicontinuous map F. □

5 Proof of the theorems

Proof of Theorem 2. By extending the state space from X to $\mathbb{R}\times X$ we can reduce our problem to the autonomous case.

For every $(t,x)\in[0,T]\times K$ introduce

$$\tilde{V}(t,x) = t + V(x)\,.$$

It can easily be checked that

$$D^+\tilde{V}((t,x),(s,v)) = s + D^+V(x,u) = s + V^\circ(x,u) = \tilde{V}^\circ(x,u)\,.$$

Therefore, $\tilde{V}$ is lower regular and obviously

$$(1,v)\in\partial\tilde{V}(t,x) \quad \text{if and only if} \quad v\in\partial V(x) \tag{6}$$

for all (t,x) in $[0,T]\times K$.

On the other hand, straightforward arguments show that

$$(1,v)\in T_{[0,T]\times K}(t,x) \quad \text{if and only if} \quad v\in T_K(x)\,. \tag{7}$$

Combining (6) and (7), the tangential condition (2) implies that

$$T_{[0,T]\times K}(t,x)\cap\tilde{F}_\varepsilon(t,x)\cap\partial\tilde{V}(t,x)\neq\emptyset \tag{8}$$

at every point in $[0,T]\times K$. By exploiting Theorem 2. in [6], we infer the existence of a solution x_ε to

$$\begin{aligned} x_\varepsilon'(t) &\in \tilde{F}_\varepsilon(t,x_\varepsilon(t)) \quad \text{a. e.} \\ x_\varepsilon(0) &= x_0 \end{aligned} \tag{9}$$

satisfying the phase constraint

$$x_\varepsilon(t) \in K, \quad t \geq 0 \tag{10}$$

on $[0, T]$ for each $\varepsilon > 0$. Assume that λ is a Lipschitz constant for V on the compact set K. Since by the tangential condition (8) for every solution x_ε to (9) we have $x'_\varepsilon(t) \in \partial V(x_\varepsilon(t))$, we deduce

$$\|x'_\varepsilon(t)\| \leq \lambda + 1 \tag{11}$$

almost everywhere on $[0, T]$ for each $\varepsilon > 0$.

Set $\varepsilon = 1/n$ and consider a sequence of solutions x_n. Making use of (10), graph x_n is contained in K and x_n is also a solution to the inclusion (3) except for a set E_n of measure not exceeding $1/n$ for each n. Therefore, in view of (11), we can select a subsequence, again denoted by x_n, which uniformly converges to an absolutely continuous function x on $[0, T]$, moreover $x'_n \to x'$ weakly in $L^2(0, T, X)$.

By passing to the limit, standard arguments show that $x'(t) \in \partial V(x(t))$ a. e. Thus, taking advantage of Lemma 3, we obtain

$$\int_0^T \|x'_n(t)\|^2 \, dt = \int_0^T (V \circ x_n)'(t) \, dt = V(x_n(T)) - V(x_0) \, .$$

Hence, by the continuity of V, we get

$$\lim_{n \to \infty} \int_0^T \|x'_n(t)\|^2 \, dt = V(x(T)) - V(x_0) = \int_0^T \|x'(t)\|^2 \, dt \, ,$$

or in other words

$$\lim_{n \to \infty} \|x'_n\|_{L^2} = \|x'\|_{L^2} \, .$$

This latter relation combined with the weak convergence implies the L^2-norm convergence of the derivative sequence. Consequently, we aasume that x'_n converges to x' almost everywhere. This tells us that x is a solution to the problem (2), (3), (4) on $[0, T]$. $\square$

Proof of Theorem 3. Consider a sequence x_n in S_T. Since the derivatives are uniformly bounded, without loss of generality we may assume that $x'_n \to x'$ weakly in $L^2(0, T, X)$ and $x_n \to x$ uniformly on $[0, T]$. By Lemma 3 we have

$$\int_0^T \|x(t)\|^2 \, dt = V(x_n(T)) - V(x_0) \, .$$

Since the right hand side of the above equality converges to $V(x(T)) - V(x_0)$, and by standard arguments $x'(t) \in \partial V(x(t))$, a repeated application of Lemma 3 gives us

$$\lim_{n \to \infty} \int_0^T \|x'_n(t)\|^2 \, dt = \int_0^T \|x'(t)\|^2 \, dt \, ,$$

and hence, $x_n' \to x'$ with respect to the $L^2(0,T,X)$-norm. From this point we can follow the patterns of the proof to Theorem 2 to get that x lies in S_T. This proves that S_T is a compact subset of $W^{1,2}(0,T,X)$. $\Box$

Example 2 It is worth mentioning here that our Theorem 2 generalizes the result of [3]. Indeed, take the lower regular function V on the real line described in Example 1. Consider the differential inclusion problem

$$x'(t) \in F(x(t)), \quad x(0) = 0\,, \tag{12}$$

where the set valued map F is given by

$$F(x) = \begin{cases} \{V'(x)\}, & \text{if the derivative exists} \\ [-1,1], & \text{if } x = 0 \\ \{-1,1\} & \text{otherwise.} \end{cases}$$

It is easy to verify that F is upper semicontinuous, admits nonconvex values in every neighborhood of the origin and $F(x) \subset \partial V(x)$ at every point. However, it is obvious that there is no proper convex continuous function W with $F(x) \subset \partial W(x)$ since F is not monotone.

Finally, let us note that the lower regularity of the potential function V cannot be omitted. Consider for instance the Cauchy problem (12) with

$$F(x) = \begin{cases} \{1\}, & \text{if } x < 0 \\ \{-1,1\}, & \text{if } x = 0 \\ \{-1\}, & \text{if } x > 0 \end{cases}$$

that is the common example of an upper semicontinuous map with no solutions. Although we have $F(x) \subset \partial V(x)$ at every point for $V(x) = -|x|$, the potential function V is clearly not lower regular at the origin.

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