Viability Theory and Economic Modeling

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Abstract

A brief introduction into the theory of differential inclusions, viability theory and selections of set valued mappings is presented. As an application the implicit scheme of the Leontief dynamic input-output model is considered.

1. Introduction

Several problems in economics or biology lead to the investigations of uncertain dynamical systems in which the instantaneous change of the state variable is not necessarily uniquely determined by the current state of the system. Such problems arise for example in the study of large scale dynamical systems in economics or certain type of evolutional systems in biology.

We refer to Aubin [1] (economics) and Hofbauer, Sigmund [5] (biology).

Another area of interest is the use of optimal control theory to model the evolution of large scale dynamical systems in economics. Such an approach assumes the existence of a decision maker who has perfect knowledge about the current state of the system. In addition, this sort of decision maker is supposed to possess complete information about future conditions of the environment. This ideal decision maker is then capable of finding the optimal control that regulates the evolution of the system over a time period.

However, these assumptions are rarely fulfilled in real life systems. Large scale systems seem to have no decision maker, nor do they follow optimal trajectories. But they share one fundamental property: the struggle for staying alive. In many cases we only know that the paths of such systems are subject to certain criteria and they have one thing in common: the search for the path that keeps the system alive.

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Differential inclusions provide a common adequate mathematical tool for both types of problems. Unlike differential equations, the present state of the system does not determine the rate of change uniquely, but instead, a set of possible directions is given at every moment. Trajectories that obey the constraints are called viable evolutions of the system.

Control systems can also be regarded as differential inclusions. In contrast to optimal control problems, in controllability or viability problems the controls are not necessarily known explicitly (in other words, no decision maker is assumed).

2. Differential inclusions

Let X be an Euclidean space, the state space of the system. Let $x_0 \in X$ be given, the initial state of the system at time $t = 0$. Consider a set valued map F defined on $\mathbb{R} \times X$ with nonempty closed images in X. That means at every moment t and every state x the possible directions of the evolution are given by the set $F(t, x) \subset X$. The dynamics of the system is defined by the differential inclusion problem:

$$\begin{aligned} x'(t) &\in F(t, x(t)) \qquad (1) \\ x(0) &= x_0 \end{aligned}$$

An absolutely continuous function the satisfies the above relations on an interval is said to be a solution, i.e. a trajectory of the system.

Some examples will follow below.

Let Y be another Euclidean space and consider the function $g : \mathbb{R} \times X \times X \to Y$. The equation

$$g(t, x(t), x'(t)) = 0 \ , \qquad x(0) = x_0 \qquad (2)$$

is called an implicit differential equation. This is particularly interesting if the derivative $x'(t)$ cannot be expressed explicitly. Such problems are provided for instance by Leontief's dynamic input-output models that we discuss later.

Introduce the set valued map on $\mathbb{R} \times X$ defined by

$$F(t, x) = \{v \in X : g(t, x, v) = 0\}$$

then it is easy to verify that (2) and (1) are equivalent, i.e. they have the same solutions. Continuity assumption on g with respect to v implies the closedness of the images of F. If in addition g is affine with respect to x, the values are convex as well.

Consider now a function $f : [0, T] \times X \times Y \to X$ and $x_0 \in X$. Here X is interpreted as the state space, x_0 stands for the initial condition, while Y is the control space, $[0, T]$ is the time interval. Suppose that a set valued map U is given, defined on $[0, T]$ and with nonempty closed images in Y. This map defines the range tube of controls. The set of admissible controls is given by

$$\hat{U} = \{u : [0, T] \to Y, u \text{ is measurable, } u(t) \in U(t) \text{ a.e. } \}$$

The dynamics of the system is governed by the differential equation:

$$x'(t) = f(t, x(t), u(t)), \qquad x(0) = x_0 \tag{3}$$

where the controls satisfy the constraint $u \in \hat{U}$.

The system is regulated by selecting controls that uniquely determine the corersponding trajectories.

Assume that f is continuous with respect to u and introduce the set valued map

$$F(t, x) = \{f(t, x, u) \in X : u \in U(t)\} \tag{4}$$

with nonempty closed values in X.

Clearly, every solution to the control system (3) also satisfies the differential inclusion (1) with F given above. The opposite direction however, is far from being trivial.

3. Measurable selections

Consider a set valued map G defined on $[0, T]$ with nonempty closed images in an Euclidean space Z.

1 Definition. The map G is said to be *measurable*, if the inverse image

$$G^{-1}(M) = \{t \in [0, T] : G(t) \cap M \neq \emptyset\}$$

of every closed set $M \subset Z$ is Lebesgue-measurable.

The following fundamental result is due to Kuratowski and Ryll-Nardzewski (see [4]).

2 Theorem. *If G is a measurable map on $[0, T]$ with nonempty closed values in Z, then there exists a measurable function g defined on $[0, T]$ such that*

$$g(t) \in G(t)$$

almost everywhere.

Such a function is called a *measurable selection* of G.

3 Theorem. (Filippov's implicit function lemma) *Consider the control system (3) and suppose f is measurable with respect to (x, u). Assume in addition that U is a measurable map with nonempty closed values and consider the set valued map defined by (4). Let us given a continuous function $x : [0, T] \to X$. If $z : [0, T] \to X$ is any measurable function with*

$$z(t) \in F(t, x(t))$$

a.e., then there exists a measurable selection $u(t) \in U(t)$ *with*

$$z(t) = f(t, x(t), u(t))$$

a.e. in $[0, T]$.

Outline of proof: Introduce the set valued map:

$$G(t) = U(t) \cap \{u \in Y : z(t) = f(t, x(t), u\}$$

on $[0, T]$. It can be verified that G is measurable and it admits nonempty closed values. Therefore, based on the selection theorem, we can find a measurable selection u of G. This selection readily fulfills the requirements. $\square$

A rigorous treatment of the theory of set valued maps can be found in [6]. Some recent results are developed in [7].

4. Viability

Suppose K is a given nonempty closed subset of X that contains the states in which the system can stay alive. Whenever the system leaves K it collapses. The natural question to raise is what conditions guarantee that the system possesses a path that never leaves K.

In other words, if $x_0 \in K$, does there exist a solution φ to the Cauchy-problem (1) that satisfies

$$\varphi(t) \in K \tag{5}$$

for every $t \geq 0$? Such a trajectory is called *viable*. The set K is called *viability domain*, if for every initial state $x_0 \in K$ there exists a viable trajectory starting from x_0.

It is worth mentioning that viability problems are different from invariance problems. While invariance of a system depends on the behavior of F outside K, viability entirely depends on the properties of F inside K. This feature is illustrated by the following example.

4 Example. Put

$$F(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ 2\sqrt{x} & \text{if } x > 0 \end{cases}$$

on the real line and take $K = [-1, 0]$. The set K is obviously a viability domain, since from any state the constant function stays in K. But from $x_0 = 0$ we have another solution: $\varphi(t) = t^2$ that leaves K. Clearly, the existence of such a solution cannot be eliminated by requiring more "regularity" of F on K.

5. Tangent cones

If φ is a viable solution to (1), then for every $s, t \geq 0$ we have $\varphi(s), \varphi(t+s) \in K$. Therefore,

$$\frac{\varphi(t+s) - \varphi(s)}{t} \in \frac{1}{t}(K - \varphi(s))$$

that suggests that the derivative $\varphi'(s)$ (whenever it exists) should be tangent to K at $\varphi(s)$ (except for a set of measure zero). This leads us the following definition.

5 Definition. The *tangent cone* to K at $x \in K$ is given by

$$T_K(x) = \{v \in X : \liminf_{t \to 0+} \frac{1}{t} d_K(x + tv) = 0\}$$

where d_K is the distance function from K.

The set valued map F is said to be *integrably bounded* if there exists a locally intagrable function λ on the real line with

$$F(t, x) \subset \lambda(t)(1 + \|x\|)B$$

for a.e. t and each x, where B denotes the closed unit ball in X. The map F is called a *Caratheodory map*, if it is integrably bounded, measurable in t and upper semicontinuous in x with nonempty convex, compact images.

6 Theorem. *Assume that F is a Caratheodory map. Then the tangential condition*

$$F(t, x) \cap T_K(x) \neq \emptyset$$

for a.e. t and each $x \in K$ implies that K is a viability domain.

7 Lemma. *Let D be an $n \times n$ matrix and G be a measurable, integrably bounded set valued map with nonempty convex closed values in $\operatorname{im} D$. Then there exists a constant α (that depends entirely on D) such that the map*

$$H(t) = D^{-1}G(t) \cap \alpha\lambda(t)B$$

is measurable and integrably bounded with nonempty convex and closed images.

Sketch of proof: Obviously H admits convex closed values. Let N denote the othogonal complement to $\ker D$ in $\mathbb{R}^n$, then D is an isomorphism on N. Put $\alpha = \|D^{-1}\|$ on N. For every $v \in G(t)$

$$\|D^{-1}v\| \leq \alpha\|v\| \leq \alpha\lambda(t)$$

thus $D^{-1}v \in H(t)$ and hence $H(t)$ is nonempty.

If M is any open set in $\mathbb{R}^n$, then DM is open in $\operatorname{im} D$ and, therefore

$$\{t : D^{-1}(G(t)) \cap M\} = \{t : G(t) \cap DM \neq \emptyset\}$$

which is measurable and so is $D^{-1} \circ G$. This implies that H is measurable since it appears as the intersection of two measurable maps.

6. Leontief-type systems

Consider a time interval $[0, T]$ and the Leontief-system on $[0, T]$

$$x(t) = A(t)x(t) + B(t)x'(t) + c(t) \qquad x(0) = x_0 \tag{6}$$

Here $A(t)$ is an $n \times n$ matrix that stands for the productivity matrix, $B(t)$ is $n \times n$ and denotes the investment matrix and the n-vector $c(t)$ gives the consumption at instant $t \in [0, T]$.

We are interested in the case when $B(t)$ is singular, i.e. $x'(t)$ cannot be expressed explicitly from equation (6). If $\hat{B}(t)$ is nonsingular and approximates $B(t)$ in the sense that $\|\hat{B}(t) - B(t)\|$ tends to zero (this is possible since nonsingular matrices form a dense subset of all $n \times n$ matrices), then replacing $B(t)$ with $\hat{B}(t)$ in (6) we obtain the explicit differential equation

$$x'(t) = \hat{B}(t)^{-1}(I - A(t))x(t) - \hat{B}(t)^{-1}c(t) \tag{7}$$

7. Regularity conditions

The major trouble with this approach is that while $\|\hat{B}(t) - B(t)\| \to 0$, solutions to (7) do not converge to solutions of (6) therefore this approximation is illegal. The reason for that is $\|\hat{B}(t)^{-1}\|$ becomes unbounded.

An immediate necessary condition for the existence of solutions is

$$\operatorname{im}(I - A(t)) \subset c(t) + \operatorname{im} B(t) \tag{8}$$

which may prove to be too restrictive in applications. Therefore, we are lead to a more general model that we discuss in the context of differential inclusions.

Consider the following implicit control system

$$C(t)x(t) - Dx'(t) \in U(t), \qquad x(0) = x_0 \tag{9}$$

where $C(t)$ and D are $n \times m$ matrices, U is an integrably bounded measurable set valued map with nonempty convex closed values in $\mathbb{R}^m$. Motivated by the dynamic Leontief model, D stands for the investment matrix, $C(t) = I - A(t)$ and $c(t)$ is replaced by the set $U(t)$ that can be regarded as the set of controls (interpreted as a budgetary interference).

Clearly, for $U(t) = \{c(t)\}$ inclusion (9) reduces to the classical model (6), while for $U(t) = c(t) + \mathbb{R}^m_+$ or $U(t) = c(t) - \mathbb{R}^m_+$ the model is presented in the form of inequalities.

Suppose that a nonempty closed set K in $\mathbb{R}^n$ is given that defines the viable states of the system. We look for trajectories of the input-output system (9) that satisfy the viability constraint $\varphi(t) \in K$ for $t \geq 0$.

An immediate necessary condition is: $\operatorname{im} C(t) \subset U(t) + \operatorname{im} D$.

If this condition is combined with the tangential condition we have that for every $x \in K$ there exists a $v \in T_K(x)$ with $\|v\| \leq 2\alpha\lambda(t)(1 + \|x\|)$ such that

$$C(t)x - Dv \in U(t) \tag{10}$$

for a.e. $t \in [0, T]$. This condition turns out to be sufficient for the existence of viable solutions to the control system (9)

8 Theorem. *Under the tangential condition (10) for every x_0 in K there exists a viable solution to the control system (9).*

Sketch of proof. Introduce $G(t, x) = \operatorname{im} D \cap (C(t)x - U(t))$. Making use of Lemma 7 it is easy to verify that

$$F(t, x) = D^{-1}(G(t, x)) \cap 2\alpha\lambda(t)(1 + \|x\|)B$$

is a Caratheodory map. Exploiting (10) we have that the tangential condition

$$F(t, x) \cap T_K(x) \neq \emptyset$$

is also fulfilled. Therefore, by Theorem 6 the differential inclusion problem

$$x'(t) \in F(t, x(t)) \qquad x(0) = x_0$$

possesses a viable solution that is obviously the desired viable solution to (9) as well.

8. References

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