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RESEARCH ARTICLE

Singularity in the Discrete Dynamic Leontief Model

István Ábel[1], Imre Dobos[2*]

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Abstract
A new wave of applications of the dynamic Leontief model brought into the forefront the singularity problem of the capital matrix. In these applications the singularity of the capital matrix is a common occurrence which complicates the solution of the model. In the singular case the model cannot be transformed in a direct forward recursive form. The method presented in this paper determines the length of a backward system (τ). Several applications stop at observing singularity while referring to the theoretical possibility of the solution. In particular, the singularity of the capital matrix played a prominent role in Bródy's extensive contributions to the input-output literature but he never ventured into the details of its various solutions. We demonstrate that a number of papers dealing with the Leontief model with singular capital matrix based their solutions on similar regularity assumptions. Our formulation in this paper offers a brief overview of the approaches that can be followed in a wide range of applications confronting with the singularity problem.

Keywords
Campbell regularity condition, dynamic Leontief model, Drazin inverse, matrix pencils, singularity, Weierstrass canonical form

[1] Institute of Finance and Accounting, Budapest Business School, H-1149 Budapest, Buzogány u. 10-12., Hungary

[2] Institute of Business Economics, Corvinus University Budapest, H-1093 Budapest, Fővám tér 8., Hungary

[*] Corresponding author, e-mail: imre.dobos@uni-corvinus.hu

1 Introduction

The possibility of singularity in the dynamic Leontief model is well known in the input-output literature. András Bródy has mentioned the singularity problem of the dynamic Leontief model in several papers. Bródy (2000) and revisited in Bródy (2007, p. 904) constructed a *wave matrix* to characterize the dynamics of the economic cycles. His wave matrix leads to a matrix characterization (denoted by S in Bródy's papers) which is *always* singular.

The singularity problem of the dynamic Leontief model often occurs in more recent economic investigations. Jódar and Merrelo (2010a) recalled that singularity is especially relevant when one considers sectors that do not produce significant capital goods, such as agriculture. They also mention that the recent financial crises and the strong changes in the technology sector are showing how unrealistic is to assume non-singularity (Jódar and Merrelo 2010a:p.400).

Okuyama et al. (2006) analyze the Chicago regional economy by using the temporal Leontief inverse analysis[1] and pointed out that the capital structure of the underlying model was singular. This singularity makes it difficult to determine the dynamic Leontief inverse and to calculate the path of the economy (Bródy, 2004; Halkos et al., 2016). Arsenos et al. (2011) examine the phenomenon of singularity in a biregional and multisectoral output growth model similar to the one analysed in Campisi et al. (1993). The results of their paper are demonstrated on the data of the Italian economy (the two regions identified as North and South).

Shao et al. (2012) investigate a dynamic input-output model with singular capital matrix and time delays in investment process. They look for matrix conditions analyzing the controllability of this type of the Leontief model. Jódar and Merello (2010a, 2010b) generalize the dynamic input-output model in two ways. Jódar and Merello (2010a) solve the continuous time dynamic Leontief model with time dependent capital matrix. The other

1 The temporal Leontief inverse was introduced by Sonis and Hewings (1998) as a less complex and more tractable tool to analyze structural change over time.

paper, Jódar and Merello (2010b) uses the Drazin inverse concept to construct the positive solution of the Leontief model with singular capital matrix. Wu (2011) analyzes the stability of the dynamic Leontief model with singular capital matrix. He finds that the dynamic input-output model is stable under certain conditions. The stable solution is calculated with Wu's newly developed simulation toolbox in ADA (a computer software language for scientific and technical computing).

The dynamic Leontief model can be represented by a forward and a backward-looking specification. The backward-looking specification has a balanced growth solution (the system is stable) but the forward-looking version of the model often was found to be unstable (having no non-negative solution) in empirical works (Steenge and Thissen, 2005:p.81). In case of singularity the system can be separated into a forward-looking and a backward looking subsystem of the basic dynamic Leontief model where the household consumption is exogenous. The method we present in this paper determines the output levels in a recursive manner based on a set of past (forward looking) and future (backward looking) consumption levels.

In practical applications the dynamic Leontief model (Leontief, 1970) is often represented with its discrete difference equation which offers a recursive method to solve it:

$$x(t) = A \cdot x(t) + B \cdot \left[x(t+1) - x(t)\right] + c(t), \qquad (1)$$

where $x(t)$ is the nonnegative n-dimensional vector of gross industrial outputs in year t; $c(t)$ the n-dimensional vector of final consumption demands for commodities in year t; A the $n \times n$ matrix of input coefficients indicating the amount of goods used to produce one unit of output; and B the $n \times n$ matrix of capital coefficients, where the element b_{ij} is the capital stock of good i needed to produce one unit of output in sector j. These coefficients are constant so they also represent the additional input and stock requirements for the increase of one unit of output. Throughout this paper it is assumed that: (*i*) the matrices A, and B are nonnegative, (*ii*) the matrix A is productive and has a nonnegative Leontief inverse $(I_n - A)^{-1}$, where I_n is the $n \times n$ identity matrix, (*iii*) B is singular, and (*iv*) $c(t)$ is a nonnegative vector.

As B matrix of capital coefficients is singular it is not invertible, which complicates the solution of the model.

The paper is organized, as follows. Section 2 gives a brief survey of the literature about the solution of (1) in case of singularity. Section 3 shows that Campbell's regularity condition used in his solution is very similar to a condition used in regular matrix pencils. Section 4 presents a new solution to the problem using the Weierstrass canonical form of regular matrix pencils. Our approach is based on a transformation resulting in a diagonalization of the system, which expresses the recursive properties of the Leontief model. This transformation is equivalent with the Weierstrass canonical form of regular matrix pencils. Here we concentrate on the backward recursive part of the system. In part 5 we show some special cases, including when the consumption level is constant over time. A summary follows in Section 6.

2 Three approaches to solve the dynamic Leontief model with singular capital matrix *B*

There are at least three approaches in the literature to cope with the problem of singularity of the capital matrix B. The first step to circumvent this difficulty is to use a recursive method. Equation (1) can be transformed in a forward recursive form:

$$B \cdot x(t+1) = \left(I_n - A + B\right) \cdot x(t) - c(t) \quad \left(t = 0, 1, \ldots, T-1\right), \qquad (2)$$

we can calculate $B \cdot x(t+1)$ for $t = 1, 2, \ldots$, but the output levels $x(t+1)$ cannot be expressed because of the singularity of matrix B.

To solve Eq. (2) with singular capital matrix B Campisi et al., (1992; 1993); Kendrick, (1972); Livesey, (1972); Luenberger and Arbel, (1977); and Meyer, (1982) use the same assumption, namely they assume that the matrices can be partitioned in the following way:

$$B = \begin{bmatrix} B_{11} & B_{12} \\ 0 & 0 \end{bmatrix}, A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}.$$

They also assume that the following matrix

$$\begin{bmatrix} B_1 \\ \left(I_n - A\right)_2 \end{bmatrix} = \begin{bmatrix} B_{11} & B_{12} \\ -A_{21} & I_{n-r} - A_{22} \end{bmatrix}$$

is nonsingular. Using this form Eq. (2) can be transformed in the following difference equation, which can be solved:

$$\begin{gathered} \begin{bmatrix} B_{11} & B_{12} \\ -A_{21} & I_{n-r} - A_{22} \end{bmatrix} \cdot \begin{bmatrix} x_1(t+1) \\ x_2(t+1) \end{bmatrix} = \\ \begin{bmatrix} I_r - A_{11} + B_{11} & -A_{12} + B_{12} \\ 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} - \begin{bmatrix} c_1(t) \\ c_2(t+1) \end{bmatrix}. \end{gathered} \qquad (3)$$

From (3) the solution of the difference Eq. (2) is based on the following expression:

$$\begin{gathered} \begin{bmatrix} x_1(t+1) \\ x_2(t+1) \end{bmatrix} = \begin{bmatrix} B_{11} & B_{12} \\ -A_{21} & I_{n-r} - A_{22} \end{bmatrix}^{-1} \cdot \begin{bmatrix} I_r - A_{11} + B_{11} & -A_{12} + B_{12} \\ 0 & 0 \end{bmatrix} \\ \cdot \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} - \begin{bmatrix} B_{11} & B_{12} \\ -A_{21} & I_{n-r} - A_{22} \end{bmatrix}^{-1} \cdot \begin{bmatrix} c_1(t) \\ c_2(t+1) \end{bmatrix}. \end{gathered} \qquad (4)$$

Equation (4) shows that based on the assumed regularity of A system (2) can be solved in a forward recursive manner. Note that this solution also assumes that we know the consumption level in the next period of the planning horizon. Although this solution is based on a rather trivial regularity (productivity) assumption for the input coefficient matrix A, which is easily satisfied in input-output economics, but there is little reason to assume that we know the future path of the consumption levels.[2]

Other authors, including Kreijger and Neudecker (1976) and Schinnar (1978), suggest another method for solution based on the Moore-Penrose generalized inverse. In this case a unique Moore-Penrose generalized inverse B^P is defined and system (2) can be solved following a recursive *forward looking* approach:

$$x(t+1) = B^P \cdot \left(I_n - A + B\right) \cdot x(t) - B^P \cdot c(t) + \left(I_n - B^P \cdot B\right) \cdot p$$
$$\left(t = 0,1,\ldots,T-1\right), \tag{5}$$

where B^P is the Moore-Penrose inverse of B and vector p is an arbitrary vector.

This method seems straightforward but there is a problem, namely that vector p is independent of the consumption levels, so the solution is not closed for all the parameters of the model. We are not aware of any proposal for a reasonable construction of vector p in the vast input-output literature. This imperfection calls for other approaches, and there is a proposal in Campbell (1979) leading us to the third approach.

A third approach offered in Campbell (1979), and also revisited by Jódar and Merrelo (2010 a,b) applies the concept of the Drazin inverse[3] and uses another regularity assumption. The assumption is that the matrix $\lambda \cdot B + (I_n - A + B)$ is invertible for a scalar l. Campbell had not shown that this regularity condition is easily fulfilled in a productive economy, but it is certainly true for $\lambda = -1$ if the matrix A is productive. For a productive A there exists a nonnegative Leontief inverse, which means that $\lambda \cdot B + (I_n - A + B)$ is invertible for $\lambda = -1$.

Campbell, 1979 solved the singular dynamic Leontief system by using the concept of Drazin inverse by introducing the following formulations:

$$\hat{B} = \left[\lambda \cdot B + \left(I_n - A + B\right)\right]^{-1} \cdot B = R \cdot \begin{bmatrix} C & 0 \\ 0 & N \end{bmatrix} \cdot R,$$

$$\hat{B}^D = R \cdot \begin{bmatrix} C & 0 \\ 0 & 0 \end{bmatrix} \cdot R, \quad \text{and}$$

$$\hat{c}(t) = \left[\lambda \cdot B + \left(I_n - A + B\right)\right]^{-1} \cdot c(t).$$

Where matrix $\hat{B}^D$ is the Drazin inverse of $\hat{B}$, matrix R is an invertible matrix such that matrix C is invertible, and matrix N is nilpotent with $m = index\ N$. Using these notations system (2) can be divided into a forward and a backward recursive system:

$$\hat{B}^D \cdot \hat{B} \cdot x(t+1) = \hat{B}^D \cdot \left(I_n - \lambda \cdot \hat{B}\right) \cdot x(t) - \hat{B}^D \cdot \hat{c}(t),$$

and

$$\left(I_n - \hat{B}^D \cdot \hat{B}\right) \cdot x(t+1)$$
$$= \left(I_n - \lambda \cdot \hat{B}\right)^D \cdot \left(I_n - \hat{B}^D \cdot \hat{B}\right) \cdot \left[\hat{B} \cdot x(t+2) - \hat{c}(t+1)\right]$$

If the economy is assumed to run forever, we get

$$\left(I_n - \hat{B}^D \cdot \hat{B}\right) \cdot x(t+1)$$
$$= -\sum_{i=0}^{m-1} \left\{ \left[\left(I_n - \lambda \cdot \hat{B}\right)^D\right]^{i+1} \cdot \hat{B}^i \cdot \left(I_n - \hat{B}^D \cdot \hat{B}\right) \cdot \hat{c}(t+i+1) \right\}$$

where m = index $\hat{B}$. Campbell proved that the difference equation system (2) is regular if there exists a λ such that $\lambda \cdot B + (I_n - A + B)$ is invertible, and index $\hat{B} \leq 1$. In this paper we offer a fourth solution to Eq. (2).

3 Campbell's regularity condition and the matrix pencils

The regularity condition used by Campbell is reminiscent to the property that matrices B and build regular matrix pencils denoted by $(B, (I_n - A + B))$. This means that the matrix $\lambda \cdot B + (I_n - A + B)$ is nonsingular for the value $\lambda = 1$. For $\lambda = 1$ the expression $\lambda \cdot B + (I_n - A + B)$ is simply $-(I_n - A)$. With the assumption that A is productive, i.e. it has a Leontief inverse, this form is invertible.

Now we demonstrate the analogy between regular matrix pencils and the closed dynamic Leontief model. The closed dynamic Leontief model is often represented with the following eigenvalue-eigenvector problem:

$$\left[\lambda \cdot B - \left(I_n - A + B\right)\right] \cdot x = 0 \tag{1'}$$

This representation brings the matrix pencils in the picture. This is the same as the generalized eigenvalue problem for the closed form continuous dynamic Leontief model:

$$\lambda \cdot B \cdot x = \left(I_n - A + B\right) \cdot x \tag{1''}$$

Now we turn to the solution of (1') and its difference equation form (2) by using the Weierstrass canonical form (Gantmacher, 1959; Mehrmann et al., 2008). In this approach we use the Campbell regularity condition and the solution takes a form similar to the solution in Campbell's paper.

4 A new approach using the Weierstrass canonical form

The regular matrix pencils $(B, (I_n - A + B))$ can be written in the following Weierstrass canonical form

$$\lambda \cdot B - \left(I_n - A + B\right) = P \cdot \left\{ \lambda \cdot \begin{bmatrix} I_{n-p} & 0 \\ 0 & N \end{bmatrix} - \begin{bmatrix} J & 0 \\ 0 & I_p \end{bmatrix} \right\} \cdot Q \tag{6}$$

where

$$B = P \cdot \begin{bmatrix} I_{n-p} & 0 \\ 0 & N \end{bmatrix} \cdot Q, \quad I_n - A + B = P \cdot \begin{bmatrix} J & 0 \\ 0 & I_p \end{bmatrix} \cdot Q$$

2 Zhang, 2001, outlined an iterative approach to the dynamic Leontief model extended it to a CGE model. He is not dealing with the problem of singularity directly but his approach might offer an interesting alternative.

3 A comparison of Moore-Penrose and Drazin inverses is found in Appendix I. of this paper.

and J is a matrix in Jordan canonical form and N is a nilpotent matrix also in Jordan canonical form. The eigenvalues of the problem (1') or (2) are in the diagonal of the Jordan matrix J. Matrices P and Q are the transformation matrices of the Weierstrass canonical form. They include the left- and right-hand side eigenvectors of the matrix pencils $(B, (I_n - A + B))$. (see more details in Appendix II.) Take t for the degree of the nilpotency of matrix N. We will show in Proposition 1 that the degree of nilpotency of N is equal to the degree of the nilpotency of matrix $(I_n - A)^{-1} \cdot B$. Note that this matrix $(I_n - A)^{-1} \cdot B$ features prominently in determining the eigenvalues of the dynamic Leontief model.

The Weierstrass canonical form (6) of the matrix pencil offers an interesting insight into the characteristics of the Leontief model.[4] The following proposition summarizes the results detailed in Appendix II.

Proposition 1. The degree of the nilpotency of the dynamic Leontief model is identical to the degree of the nilpotency of the matrix $(I_n - A)^{-1} \cdot B$, i.e. the algebraic multiplicity of the eigenvalue zero of matrix $(I_n - A)^{-1} \cdot B$.

The balanced growth rate in the Leontief economy is determined by the smallest positive eigenvalue of matrix $(I_n - A)^{-1} \cdot B$ (Bródy, 1970). Proposition 1 states that the number of time intervals of future consumption levels needed to determine the output $x(t+1)$ in the a forward-looking version of the Leontief model is determined by the degree of nilpotency (τ) of $(I_n - A)^{-1} \cdot B$.

Our next step is to solve problem (2) with the help of regular matrix pencils. Substituting the transformed form of expression (1'') in the equation (2) and using (6) we get:

$$P^{-1} \cdot P \cdot B \cdot Q \cdot Q^{-1} \cdot x(t+1) =$$
$$P^{-1} \cdot P \cdot (I_n - A + B) \cdot Q \cdot Q^{-1} \cdot x(t) - c(t).$$

Using this transformation from Eq. (2) we arrive at two separate difference equations, a forward looking and a backward looking system:

$$\begin{bmatrix} I_{n-p} & 0 \\ 0 & N \end{bmatrix} \cdot \begin{bmatrix} z_1(t+1) \\ z_2(t+1) \end{bmatrix} = \begin{bmatrix} J & 0 \\ 0 & I_p \end{bmatrix} \cdot \begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix} - \begin{bmatrix} \tilde{c}_1((t) \\ \tilde{c}_2((t) \end{bmatrix},$$

where

$$z(t) = \begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix} = Q^{-1} \cdot x(t), \quad \begin{bmatrix} \tilde{c}_1(t) \\ \tilde{c}_2(t) \end{bmatrix} = P^{-1} \cdot c(t).$$

The solution for the forward system:

$$z_1(t+1) = J^{t+1} \cdot z_1(0) - \sum_{i=0}^{t} J^{t-i} \cdot \tilde{c}_1(i),$$

and the solution for the backward system:

$$z_2(t) = \sum_{j=0}^{\tau-1} N^j \cdot \tilde{c}_2(t+j).$$

We note, that the length of the backward system is τ indicating that to be able to solve the problem we need to know the consumption levels in the next τ periods.

5 Special cases of the model

Two special cases of the model are of special interest. *First* we take the model with constant consumption levels, i.e. $c(t) = c_0$.

$$B \cdot x(t+1) = (I_n - A + B) \cdot x(t) - c_0, \quad (t = 0, 1, \ldots, T-1). \tag{2'}$$

Using the diaginalization transformations applied in (6), the solution of this system is given by the following form:

$$x(t+1) = Q \cdot \begin{bmatrix} J^{t+1} & 0 \\ 0 & 0 \end{bmatrix} \cdot Q^{-1} \cdot x(0) + \sum_{i=0}^{\tau-1} Q \cdot \begin{bmatrix} 0 & 0 \\ 0 & N^i \end{bmatrix} \cdot P^{-1} \cdot c_0 -$$
$$\sum_{j=0}^{t} Q^{-1} \cdot \begin{bmatrix} J^{t-j} & 0 \\ 0 & 0 \end{bmatrix} \cdot P^{-1} \cdot c_0$$

Using the known relations $\sum_{i=0}^{\tau-1} N^i = (I_p - N)^{-1}$, and $\sum_{j=0}^{t} J^{t-j} = (I_{n-p} - J)^{-1} - J^{t+1} \cdot (I_{n-p} - J)^{-1}$ in this expression we get the following equation:

$$x(t+1) = Q \cdot \begin{bmatrix} J^{t+1} & 0 \\ 0 & 0 \end{bmatrix} \cdot \left\{ Q^{-1} \cdot x(0) + \begin{bmatrix} (I_{n-p} - J)^{-1} & 0 \\ 0 & I_p \end{bmatrix} \cdot P^{-1} \cdot c_0 \right\}$$
$$+ Q \cdot \begin{bmatrix} -(I_{n-p} - J)^{-1} & 0 \\ 0 & (I_p - N)^{-1} \end{bmatrix} \cdot P^{-1} \cdot c_0.$$

This equation can be simplified as

$$x(t+1) = Q \cdot \begin{bmatrix} J^{t+1} & 0 \\ 0 & 0 \end{bmatrix} \cdot \left\{ Q^{-1} \cdot x(0) + \begin{bmatrix} (I_{n-p} - J)^{-1} & 0 \\ 0 & I_p \end{bmatrix} \cdot P^{-1} \cdot c_0 \right\}$$
$$+ (I_n - A)^{-1} \cdot c_0.$$

The *second special case* concentrates on the degree of nilpotency (τ). The recursive solutions for (2) mentioned in the first part of Section 2 assumed that the matrix $\begin{bmatrix} B_{11} & B_{12} \\ -A_{21} & I_{n-r} - A_{22} \end{bmatrix} = \begin{bmatrix} B_1 \\ (I_n - A)_2 \end{bmatrix}$ was nonsingular and invertible. With this assumption the solution also assumed that we know the consumption levels for period t and $t+1$, i.e. the degree of nilpotency of the system is $\tau = 1$. Here we will show that the value of τ for our case using the Weierstrass canonical form is equal to one ($\tau = 1$), i.e. the consumption level must be known only for one period forward. We prove this property with the diagonalization method detailed above. After some elementary manipulation we get

4 The derivation of the Weierstrass canonical form and the transformation matrices P and Q are found in Appendix II. of this paper.

$$\lambda \cdot \begin{bmatrix} B_1 \\ 0 \end{bmatrix} - \begin{bmatrix} (I_n - A)_1 + B_1 \\ (I_n - A)_2 \end{bmatrix} = (\lambda - 1) \cdot \begin{bmatrix} B_1 \\ 0 \end{bmatrix} - \begin{bmatrix} (I_n - A)_1 \\ (I_n - A)_2 \end{bmatrix} =$$

$$\left\{ (\lambda - 1) \cdot \begin{bmatrix} B_1 \\ 0 \end{bmatrix} \cdot [G, F] - I_n \right\} \cdot (I_n - A), \tag{7}$$

where $(I_n - A)^{-1} = [G, F]$. Two expressions are constructed with this equality:

$$\begin{bmatrix} (I_n - A)_1 \\ (I_n - A)_2 \end{bmatrix} \cdot [G, F] =$$

$$\begin{bmatrix} (I_n - A)_1 \cdot G & (I_n - A)_1 \cdot F \\ (I_n - A)_2 \cdot G & (I_n - A)_2 \cdot F \end{bmatrix} = \begin{bmatrix} I_{n-r} & 0 \\ 0 & I_r \end{bmatrix},$$

and using the regularity condition

$$\begin{bmatrix} B_1 \\ (I_n - A)_2 \end{bmatrix} \cdot [G, F] = \begin{bmatrix} B_1 \cdot G & B_1 \cdot F \\ (I_n - A)_2 \cdot G & (I_n - A)_2 \cdot F \end{bmatrix}$$

$$= \begin{bmatrix} B_1 \cdot G & B_1 \cdot F \\ 0 & I_r \end{bmatrix}. \tag{8}$$

It is clear in (8) that matrix $B_1 \cdot G$ is nonsingular, i.e. it has no zero eigenvalue.

Continuing the diagonalization method we get after the transformation of (8)

$$\left\{ (\lambda - 1) \cdot \begin{bmatrix} B_1 \cdot G & B_1 \cdot F \\ 0 & 0 \end{bmatrix} - I_n \right\} \cdot (I_n - A) =$$

$$= \begin{bmatrix} M_1^{-1} & -M_1^{-1} \cdot M_2 \\ 0 & I_{n-r} \end{bmatrix} \cdot$$

$$\left\{ (\lambda - 1) \cdot \begin{bmatrix} M_1 \cdot (B_1 \cdot G) \cdot M_1^{-1} & M_1 \cdot B_1 \cdot F - M_1 \cdot (B_1 \cdot G) \cdot M_1^{-1} \cdot M_2 \\ 0 & 0 \end{bmatrix} - I_n \right\} \cdot$$

$$\begin{bmatrix} M_1 & M_2 \\ 0 & I_{n-r} \end{bmatrix} \cdot (I_n - A). \tag{9}$$

Where the transformation matrix is

$$\begin{bmatrix} M_1 & M_2 \\ 0 & I_{n-r} \end{bmatrix}^{-1} = \begin{bmatrix} M_1^{-1} & -M_1^{-1} \cdot M_2 \\ 0 & I_{n-r} \end{bmatrix}.$$

Here we can choose matrix M_1 so that it diagonalizes the nonsingular matrix $B_1 \cdot G$. Matrix M_2 can be calculated, as follows

$$M_2 = M_1 \cdot (B_1 \cdot G)^{-1} \cdot B_1 \cdot F.$$

Then we can determine the diagonal form of the matrices

$$\begin{bmatrix} M_1^{-1} & -M_1^{-1} \cdot M_2 \\ 0 & I_{n-r} \end{bmatrix} \cdot$$

$$\left\{ (\lambda - 1) \cdot \begin{bmatrix} M_1 \cdot (B_1 \cdot G) \cdot M_1^{-1} & M_1 \cdot B_1 \cdot F - M_1 \cdot (B_1 \cdot G) \cdot M_1^{-1} \cdot M_2 \\ 0 & 0 \end{bmatrix} - I_n \right\} \cdot$$

$$\begin{bmatrix} M_1 & M_2 \\ 0 & I_{n-r} \end{bmatrix} \cdot (I_n - A) =$$

$$= \begin{bmatrix} M_1^{-1} & -M_1^{-1} \cdot M_2 \\ 0 & I_{n-r} \end{bmatrix}$$

$$\cdot \left\{ \lambda \cdot \begin{bmatrix} J_1 & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} J_1 + I_r & 0 \\ 0 & I_{n-r} \end{bmatrix} \right\} \cdot$$

$$\begin{bmatrix} M_1 & M_2 \\ 0 & I_{n-r} \end{bmatrix} \cdot (I_n - A) =$$

$$= \begin{bmatrix} M_1^{-1} & -M_1^{-1} \cdot M_2 \\ 0 & I_{n-r} \end{bmatrix} \cdot \begin{bmatrix} J_1 & 0 \\ 0 & I_{n-r} \end{bmatrix} \cdot$$

$$\left\{ \lambda \cdot \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} J_1^{-1} + I_r & 0 \\ 0 & I_{n-r} \end{bmatrix} \right\} \cdot \begin{bmatrix} M_1 & M_2 \\ 0 & I_{n-r} \end{bmatrix} \cdot (I_n - A) =$$

$$= \begin{bmatrix} M_1^{-1} & -M_1^{-1} \cdot M_2 \\ 0 & I_{n-r} \end{bmatrix} \cdot \begin{bmatrix} J_1 & 0 \\ 0 & I_{n-r} \end{bmatrix} \cdot \begin{bmatrix} V & 0 \\ 0 & I_{n-r} \end{bmatrix} \cdot$$

$$\left\{ \lambda \cdot \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} J & 0 \\ 0 & I_{n-r} \end{bmatrix} \right\} \cdot$$

$$\begin{bmatrix} V^{-1} & 0 \\ 0 & I_{n-r} \end{bmatrix} \cdot \begin{bmatrix} M_1 & M_2 \\ 0 & I_{n-r} \end{bmatrix} \cdot (I_n - A),$$

where matrix V diagonalizes the last matrix $V^{-1} \cdot (J_1^{-1} + I_r) \cdot V = J$. Here we see that the nilpotent block is a zero matrix so the degree of nilpotency is one ($\lambda = 1$), and we have proved the next proposition.

Proposition 2. If the capital matrix has zero row blocks, and matrix $\begin{bmatrix} B_{11} & B_{12} \\ -A_{21} & I_{n-r} - A_{22} \end{bmatrix}$ is non-singular, then the degree of the nilpotency of the dynamic Leontief model is identical to one, i.e. it is enough to know the consumption vector only one period ahead of the planning horizon.

6 Conclusions

In this paper alternative approaches to the solution of the singularity problem of the dynamic Leontief model were discussed. To solve the problem we used the theory of regular matrix pencils. This approach utilizes the fact that the matrices of the dynamic Leontief model build a regular matrix pencil. The regularity of the model follows from the productivity of matrix A. It was shown that the nature of the recursive solution of the dynamic Leontief model depends on the degree of the nilpotent block of the Weierstrass canonical form.

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Appendix I. Penrose inverse and Drazin inverse of a singular quadratic matrix

Matrix A^{P} is a Penrose inverse of matrix A, if the following matrix equalities hold:

(P1) $A A^{\mathrm{P}} A = A$

(P2) $A^{\mathrm{P}} A A^{\mathrm{P}} = A^{\mathrm{P}}$

(P3) $(A A^{\mathrm{P}})^{\mathrm{T}} = A A^{\mathrm{P}}$

(P4) $(A^{\mathrm{P}} A)^{\mathrm{T}} = A^{\mathrm{P}} A$

where index $^{\mathrm{T}}$ denotes the transpose of a matrix. The Penrose inverse is unique.

Drazin inverse of a quadratic matrix A is defined as

(D1) $A^k A^{\mathrm{D}} A = A^k$

(D2) $A^{\mathrm{D}} A A^{\mathrm{D}} = A^{\mathrm{D}}$

(D3) $A A^{\mathrm{D}} = A^{\mathrm{D}} A$

where matrix A^{D} is the Drazin inverse, and number k is the smallest positive integer for which rank(A^k) = rank(A^{k+1}). The Drazin inverse is also unique.

Appendix II. The Weierstrass canonical form of singular dynamic Leontief model

The diagonalization process starts from (1'') with the following transformations:

$$\lambda \cdot B - (I_n - A + B) = (\lambda - 1) \cdot B - (I_n - A) =$$
$$(I_n - A) \cdot \left[(\lambda - 1) \cdot (I_n - A)^{-1} B - I_n \right]$$

Now apply a diagonalizing transformation to the components of this equation. First using an appropriate matrix U we can write the matrix $(I_n - A)^{-1} \cdot B$ in a Jordan form, and separate the zero and nonzero eigenvalues:

$$(I_n - A)\cdot\left[(\lambda-1)\cdot(I_n - A)^{-1} B - I_n\right] =$$
$$(I_n - A)\cdot U\cdot\begin{bmatrix}(\lambda-1)\cdot J_1 - I_{n-p} & 0 \\ 0 & (\lambda-1)\cdot N_1 - I_p\end{bmatrix}\cdot U^{-1} =$$
$$(I_n - A)\cdot U\cdot\left\{\lambda\cdot\begin{bmatrix}J_1 & 0 \\ 0 & N_1\end{bmatrix} - \begin{bmatrix}J_1 + I_{n-p} & 0 \\ 0 & N_1 + I_p\end{bmatrix}\right\}\cdot U^{-1}.$$

Transformation matrices U and U^{-1} contain the left- and righthand side eigenvectors of matrix $(I_n - A)^{-1} \cdot B$. This matrix is often examined in the input-output literature to determine balanced growth paths of the dynamic Leontief model. Because of the further non-singular transformations followed after, these matrices contain the eigenvectors of the regular matrix pencils of the dynamic Leontief model. The matrices $J_1 + I_{n-p}$ and $N_1 + I_n$ are now nonsingular and so they are invertible:

$$(I_n - A)\cdot U\cdot\left\{\lambda\cdot\begin{bmatrix}J_1 & 0 \\ 0 & N_1\end{bmatrix} - \begin{bmatrix}J_1 + I_{n-p} & 0 \\ 0 & N_1 + I_p\end{bmatrix}\right\}\cdot U^{-1} =$$
$$(I_n - A)\cdot U\cdot\begin{bmatrix}J_1 & 0 \\ 0 & N_1 + I_p\end{bmatrix}\cdot$$
$$\left\{\lambda\cdot\begin{bmatrix}I_{n-p} & 0 \\ 0 & (N_1 + I_p)^{-1}\cdot N_1\end{bmatrix} - \begin{bmatrix}J_1^{-1} + I_{n-p} & 0 \\ 0 & I_p\end{bmatrix}\right\}\cdot U^{-1}.$$

Let us continue with diagonalizing the coefficient matrix of scalar λ:

$$(I_n - A)\cdot U\cdot\begin{bmatrix}J_1 & 0 \\ 0 & N_1 + I_p\end{bmatrix}\cdot$$
$$\left\{\lambda\cdot\begin{bmatrix}I_{n-p} & 0 \\ 0 & (N_1 + I_p)^{-1}\cdot N_1\end{bmatrix} - \begin{bmatrix}J_1^{-1} + I_{n-p} & 0 \\ 0 & I_p\end{bmatrix}\right\}\cdot U^{-1} =$$
$$(I_n - A)\cdot U\cdot\begin{bmatrix}J_1 & 0 \\ 0 & N_1 + I_p\end{bmatrix}\cdot\begin{bmatrix}V_{n-p} & 0 \\ 0 & V_p\end{bmatrix}\cdot$$
$$\left\{\lambda\cdot\begin{bmatrix}I_{n-p} & 0 \\ 0 & N\end{bmatrix} - \begin{bmatrix}J & 0 \\ 0 & I_p\end{bmatrix}\right\}\cdot\begin{bmatrix}V_{n-p}^{-1} & 0 \\ 0 & V_p^{-1}\end{bmatrix}\cdot U^{-1}. \tag{E}$$

In this last step matrix N stands for the following expression:

$$N = V_p\cdot\left[(N_1 + I_p)^{-1}\cdot N_1\right]\cdot V_p^{-1}$$

Matching these forms with the notations of the transformation matrices used in (6) we get:

$$P = (I_n - A)\cdot U\cdot\begin{bmatrix}J_1 & 0 \\ 0 & N_1 + I_p\end{bmatrix}\cdot\begin{bmatrix}V_{n-p} & 0 \\ 0 & V_p\end{bmatrix},$$
$$Q = \begin{bmatrix}V_{n-p}^{-1} & 0 \\ 0 & V_p^{-1}\end{bmatrix}\cdot U^{-1}.$$

For a nilpotent N_1 matrix we can write that $(N_1 + I_p)^{-1}\cdot N_1 = N_1\cdot\sum_{i=0}^{p-1}(-1)^i\cdot N_1^i$. After substitution this expression in (E) we get the result that the transformed matrix will remain nilpotent and can be written as a product comprising of τ components.

$$N^\tau = V_p\cdot\left[(N_1 + I_p)^{-1}\cdot N_1\right]^\tau\cdot V_p^{-1} = 0$$