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The Shapley Value for Upstream Responsibility Games

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Abstract

In this paper sharing the cost of emission in supply chains are considered. We focus on allocation problems that can be described by rooted trees, called cost-tree problems, and on the induced transferable utility cooperative games, called upstream responsibility games (Gopalakrishnan et al., 2017). The formal notion of upstream responsibility games is introduced, and the characterization of the class of these games is provided.

The Shapley value (Shapley, 1953) is probably the most popular value for transferable utility cooperative games. Dubey (1982) and Moulin and Shenker (1992) show respectively, that Shapley (1953)'s and Young (1985)'s axiomatizations of the Shapley value are valid on the class of airport games.

We extend Dubey's and Moulin and Shenker's results onto the class of upstream responsibility games, that is, we provide two characterizations of the Shapley value for upstream responsibility games.

Keywords: Upstream responsibility games; Cost sharing; Emission; Supply chain; Shapley value; Rooted tree; Axiomatization of the Shapley value.

JEL Classification: C71.

1 Introduction

In this paper we consider cost sharing problems given by rooted trees, called *cost-tree problems*. We assign transferable utility (TU) cooperative games (henceforth games) to these cost sharing problems. Specializing the problem,

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we consider energy supply chains with a motivated dominant leader, who has the power to assign the suppliers responsibilities for both direct and indirect emissions. The induced games are called *upstream responsibility games* (Gopalakrishnan et al., 2017).

For an example consider a supply chain where we look at the responsibility allocation of greenhouse gas (GHG) emission among the firms in the chain. One of the main questions is how to share the costs related to the emission among the firms. The supply chain and the related firms (or any other actors) can be represented by a rooted tree.

The root of the tree represents the end product produced by the supply chain. The root is connected to only one node which is the leader of the chain. Each further node represents one firm, and the edges of the rooted tree represent the producing process among the firms with the related emissions. Our goal is to share the responsibility of the emission while embodying the principle of upstream emission responsibility.

In this paper we rely on the TU game model of Gopalakrishnan et al. (2017), called GHG Responsibility-Emissions and Environment (GREEN) game. Gopalakrishnan et al. use the Shapley value (Shapley, 1953) as an allocation method, consider some pollution related properties that an emission allocation rule should meet, and provide several axiomatizations as well.

Airport problems and the associated airport games (Littlechild and Thompson, 1977) are defined by chains, a special case of rooted trees. Thomson (2007) gives an overview on the results for airport games. The two main axiomatizations of the Shapley value, Shapley (1953)'s and Young (1985)'s axiomatizations, are considered on airport games by Dubey (1982) and Moulin and Shenker (1992) respectively.

An other extension of the airport games is the well-known class of (standard) fixed-tree games, which is an application of irrigation problems, considered by Aadland and Kolpin (1998) and Márkus et al. (2011) among others.

It is well-known that the validity of a solution concept can vary from subclass to subclass, e.g. Shapley (1953)'s axiomatization of the Shapley value is valid on the class of monotone games but not valid on the class of strictly monotone games. Therefore, we must consider each subclass of games one by one.

In this paper, we consider upstream responsibility games and characterize this class of games. We show that the class of upstream responsibility games is a non-convex cone which is a proper subset of the finite convex cone spanned by the duals of the unanimity games, therefore every upstream responsibility game is concave. Furthermore, as a corollary we show that Shapley (1953)'s and Young (1985)'s axiomatizations work on the class of upstream responsibility games. We also notice that the Shapley value is stable for upstream responsibility games, that is, it is always in the core (Shapley, 1955). This result is a simple corollary of the Ichiishi-Shapley theorem (Shapley, 1971; Ichiishi, 1981) and that every upstream responsibility game is concave. Also by our characterization of the class of upstream responsibility games we get that the Shapley value can be computed efficiently on this class of games (this result is analogue to the one by Megiddo (1978) for fixed-tree games).

The setup of this paper is as follows: in the next section we introduce the concept of upstream responsibility games and characterize the class of them. In Section 3 we present our characterization results: we show that Shapley (1953)'s and Young (1985)'s axiomatizations of the Shapley value work on the classes of upstream responsibility games.

2 Airport and Upstream Responsibility Games

2.1 Preliminaries

Notions, notations: $\#N$ is for the cardinality of set N, and 2^N denotes the class of all subsets of N. $A \subset B$ means $A \subseteq B$, but $A \neq B$. $A \uplus B$ stands for the union of disjoint sets A and B.

A *graph* is a pair $G = (V, E)$, where the elements of V are called *vertices* or nodes, and E stands for the ordered pairs of vertices, called *edges* or arcs. A *rooted tree* is a graph in which any two vertices are connected by exactly one simple path, and one vertex has been designated the root denoted as node root. In the case of a supply chain consisting several entities, which are cooperating in the production of a final product, the manufacturing process can be modeled with a directed rooted tree. The set of nodes V represents the entities, henceforth players, and a directed arc emanating from node i towards root represents the activity by player i contributing to the manufacturing of the final product. We assume that only one node emanates arc enters the root, this emanates from node 1 which node represents the end consumer.

The tree-order is the partial ordering on the vertices of a rooted tree with $i \leq j$, if the unique path from j to the root passes through i. The *chain* is a rooted tree such that any vertices $i, j \in V$, $i \leq j$ or $j \leq i$. That is, a chain is a rooted tree with only one "branch". For any pair $e \in E$, $e = \overline{ij}$ means e is an edge between vertices $i, j \in V$ such that $i \leq j$. For each $i \in V$, let $S_i(G) = \{j \in V : j \geq i\}$, that is, for any $i \in V$, $i \in S_i(G)$. For each $i \in V$ let $P_i(G) = \{j \in V : j \leq i\}$, that is, for any $i \in V$, $i \in P_i(G)$. Moreover, for any $V' \subseteq V$, let $(P_{V'}(G), E_{V'})$ be the sub-rooted-tree of (V, E), where $P_{V'}(G) = \cup_{i \in V'} P_i(G)$ and $E_{V'} = \{\overline{ij} \in E : i, j \in P_{V'}(G)\}$.

Let $c : E \to \mathbb{R}_+$, c and (G, c) are called *cost function* and *cost-tree* respectively. An interpretation of cost tree (G, c) might be as follows: there is a given product which is produced by a supply chain. $V = N \cup \{\text{root}\}$, N denotes the set of all entities of the supply process. Node 1 denotes the end consumer, and the leaf nodes represents the most upstream members (that is, firms, etc.) of the supply chain. Each edge $e \in E$ is associated with a process in the supply chain emitting a pollution c(e). Let e_i denote the unique edge in the tree T emanating from node i (in the direction of the root). In this case $c(e_i)$ represents the direct pollution associated with e_i, the directly created pollution by node (firm) i. Besides i also can be responsible for the pollution of other processes in the chain. For each node i, $\mathcal{E}_i$ denotes the set of edges whose associated pollution is the direct or indirect responsibility of node i.

Let $N \neq \emptyset$, $\#N < \infty$, and $v : 2^N \to \mathbb{R}$ be a function such that $v(\emptyset) = 0$. Then N, v are called set of players, and *transferable utility cooperative game* (henceforth game) respectively. The class of games with player set N is denoted by $\mathcal{G}^N$.

A game $v \in \mathcal{G}^N$ is *convex*, if for all $S, T \subseteq N$, $v(S) + v(T) \leq v(S \cup T) + v(S \cap T)$, moreover, it is *concave*, if for all $S, T \subseteq N$, $v(S) + v(T) \geq v(S \cup T) + v(S \cap T)$.

The *dual* of game $v \in \mathcal{G}^N$ is game $\bar{v} \in \mathcal{G}^N$ such that for all $S \subseteq N$, $\bar{v}(S) = v(N) - v(N \setminus S)$. It is well known that the dual of a convex game is a concave game and vice versa.

Let $v \in \mathcal{G}^N$ and $i \in N$, and $v'_i(S) = v(S \cup \{i\}) - v(S)$, where $S \subseteq N$. The vector v'_i is called player i's *marginal contribution function* in game v. Alternatively, $v'_i(S)$ is player i's marginal contribution to coalition S in game v.

Let $v \in \mathcal{G}^N$, the players $i, j \in N$ are *equivalent*, $i \sim^v j$, if for all $S \subseteq N$ such that $i, j \notin S$, $v'_i(S) = v'_j(S)$.

Let N and $T \in 2^N \setminus \emptyset$, and for all $S \subseteq N$, let

$$u_T(S) = \begin{cases} 1, & \text{if } T \subseteq S, \\ 0 & \text{otherwise.} \end{cases}$$

The game u_T is called *unanimity game* on coalition T.

In this paper we use the duals of the unanimity games. For any $T \in 2^N \setminus \emptyset$ and for all $S \subseteq N$,

$$\bar{u}_T(S) = \begin{cases} 1, & \text{if } T \cap S \neq \emptyset, \\ 0 & \text{otherwise.} \end{cases}$$

It is clear that every unanimity game is convex, and the duals of the unanimity games are concave.

Henceforth, we assume that in the considered cost-tree problems there are at least two players, that is, $\#V \geq 3$ and $\#N \geq 2$.

Next we introduce the notion of *upstream responsibility games* (URG) (Gopalakrishnan et al., 2017). Let (G, c) be a cost-tree, representing a supply chain, and N be the set of the members of the chain, henceforth players (the vertices but the root). We denote by a_j the pollution associated with arc j. Let $c(\{i\})$ denote the total pollution emission that player i is directly or indirectly responsible for. $\mathcal{E}_i$ represents the set of edges whose associated pollution is the direct or indirect responsibility of player i, $c(\{i\}) = a(\mathcal{E}_i) \equiv \sum_{j \in \mathcal{E}_i} a_j$. For every $S \subseteq N$ let $\mathcal{E}_S$ denote the collection of edges whose associated pollution is the direct or indirect responsibility of the players in S, thus $\mathcal{E}_S = \cup_{i \in S} \mathcal{E}_i$ and the pollution which S is directly or indirectly responsible for is $c(S) = a(\mathcal{E}_S) \equiv \sum_{j \in \mathcal{E}_S} a_j$.

The class of upstream responsibility games with player set N is denoted by $\mathcal{G}_{UR}^N$. Let $\mathcal{G}_G$ denote the subclass of upstream responsibility games induced by cost-tree problems on a (specific) rooted tree G.

Definition 1 (Upstream Responsibility Game)**.** *For any cost-tree* (G, c)*, let* $N = V \setminus \{root\}$ *be the player set, and for any coalition* S *(the empty sum is* 0*) let the upstream responsibility game be defined as follows*

$$v_{(G,c)}(S) = \sum_{j \in \mathcal{E}_S} a_j \,.$$

The next example is an illustration of the above definition.

Example 2. Consider the cost-tree in Figure 1, where the rooted tree $G = (V, E)$ is as follows, $V = \{\text{root}, 1, 2, 3\}$, $A = \{\overline{\text{root}1}, \overline{12}, \overline{13}\}$, and the cost (pollution) function c is defined as $c(\overline{\text{root}1}) = 2$, $c(\overline{12}) = 4$ and $c(\overline{13}) = 1$.

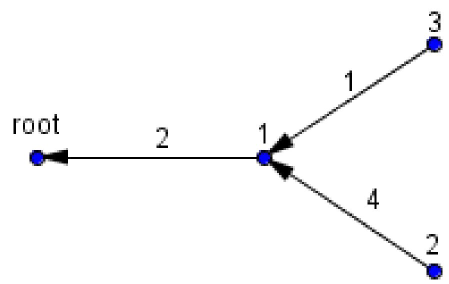

Figure 1: The cost-tree (G, c)

The upstream responsibility game $v_{(G,c)} = (0,7,4,1,7,7,5,7)$, that is, $v_{(G,c)}(\emptyset) = 0$, $v_{(G,c)}(\{1\}) = 7$, $v_{(G,c)}(\{2\}) = 4$, $v_{(G,c)}(\{3\}) = 1$, $v_{(G,c)}(\{1,2\}) = 7$, $v_{(G,c)}(\{1,3\}) = 7$, $v_{(G,c)}(\{2,3\}) = 5$ and $v_{(G,c)}(N) = 7$.

The notion of *airport games* is introduced by Littlechild and Thompson (1977). An airport problem can be illustrated by the following example. There is an airport with one runway, and there are k types of planes. Each type of planes i determines a cost c_i for maintaining the runway. E.g. if i stands for the small planes, then the maintenance cost of a runway for small planes is c_i. If j is the category of big planes, then $c_i < c_j$, since the big planes need longer runway. That is, the player set N is given by a partition: $N = N_1 \uplus \cdots \uplus N_k$, where N_i stands for the planes of category i, and each category i determines a maintenance cost c_i, such that $c_1 < \ldots < c_k$. When we consider a coalition of players (planes) S, then the maintenance cost of coalition S is the maximum maintenance cost of the members' maintenance costs of S. That is, the cost of coalition S is the maintenance cost of the biggest plane of coalition S.

We provide two equivalent definitions of airport games; the first one is as follows:

Definition 3 (Definition of Airport Games I)**.** *Let $N = N_1 \uplus \cdots \uplus N_k$ be the player set, and $c \in \mathbb{R}^k_+$, such that $c_1 < \ldots < c_k \in \mathbb{R}_+$ be an airport problem. Then airport game $v_{(N,c)} \in \mathcal{G}^N$ is defined as follows, $v_{(N,c)}(\emptyset) = 0$, and for each non-empty coalition $S \subseteq N$*

$$v_{(N,c)}(S) = \max_{i:N_i \cap S \neq \emptyset} c_i \,.$$

An alternative definition of airport games is as follows:

Definition 4 (Definition of Airport Games II)**.** *Let $N = N_1 \uplus \cdots \uplus N_k$ be the player set, and $c = c_1 < \ldots < c_k \in \mathbb{R}_+$ be an airport problem. Let $G = (V,E)$ be a chain such that $V = N \cup \{root\}$ and $E = \{\overline{root1}, \overline{12}, \ldots, \overline{\#N-1\#N}\}$, where $N_1 = \{1,\ldots,\#N_1\}, \ldots, N_k = \{\#N - \#N_k + 1, \ldots, \#N\}$. Moreover, for each $\overline{ij} \in E$, let $c(\overline{ij}) = c_{N(j)} - c_{N(i)}$, where $N(i) = \{N^* \in \{N_1, \cdots, N_k\} : i \in N^*\}$.*

For cost-tree (G,c) airport game $v_{(N,c)} \in \mathcal{G}^N$ is defined as follows: let $N = V \setminus \{root\}$ be the player set, and for any coalition S (the empty sum is 0)

$$v_{(N,c)}(S) = \sum_{e \in E_S} c(e) \,.$$

It is obvious that both definitions above give the same games, and let the class of airport games with player set N be denoted by $\mathcal{G}_A^N$. Furthermore, let $\mathcal{G}_G$ denote the subclass of airport games induced by airport problems on chain G. Notice that, the notation $\mathcal{G}_G$ is consistent with the notation introduced in Definition 1, because if G is a chain, then $\mathcal{G}_G \subseteq \mathcal{G}_A$, in other cases, when G is not a chain, $\mathcal{G}_G \setminus \mathcal{G}_A \neq \emptyset$. Since not every rooted tree is a chain, $\mathcal{G}_A^N \subset \mathcal{G}_{UR}^N$.

Example 5. Consider the airport problem (N, c'), where $N = \{\{1\} \uplus \{2,3\}\}$, and $c'_{N(1)} = 5$ and $c'_{N(2)} = c'_{N(3)} = 8$ $(N(2) = N(3))$. Then consider the cost-tree in Figure 2, where the rooted tree $G = (V, A)$ is as follows, $V = \{\text{root}, 1, 2, 3\}$, $A = \{\overline{\text{root}1}, \overline{12}, \overline{23}\}$, and the cost function c is defined as $c(\overline{\text{root}1}) = 5$, $c(\overline{12}) = 3$ and $c(\overline{23}) = 0$.

Figure 2: The cost-tree (G, c)

Then the induced airport game is as follows: $v_{(G,c)} = (0, 5, 8, 8, 8, 8, 8, 8)$, that is $v_{(G,c)}(\emptyset) = 0$, $v_{(G,c)}(\{1\}) = 5$, $v_{(G,c)}(\{2\}) = v_{(G,c)}(\{3\}) = v_{(G,c)}(\{1,2\})$ $= v_{(G,c)}(\{1,3\}) = v_{(G,c)}(\{2,3\}) = v_{(G,c)}(N) = 8$.

Next we characterize the classes of airport games and upstream responsibility games. First, we take an obvious observation, for any rooted tree G, $\mathcal{G}_G$ is a cone, that is for any $\alpha \geq 0$, $\alpha \mathcal{G}_G \subseteq \mathcal{G}_G$. Since union of cones is also a cone, both $\mathcal{G}_A^N$ and $\mathcal{G}_{UR}^N$ are cones.

Lemma 6. *For any rooted tree G, $\mathcal{G}_G$ is a cone, therefore, $\mathcal{G}_A^N$ and $\mathcal{G}_{UR}^N$ are cones.*

In the following lemma we show that the dual of any unanimity game is an airport game.

Lemma 7. *For any chain G, $T \subseteq N$ such that $T = P_i(G)$, $i \in N$, $\bar{u}_T \in \mathcal{G}_G$. Therefore, $\{\bar{u}_T\}_{T \in 2^N \setminus \{\emptyset\}} \subset \mathcal{G}_A^N \subset \mathcal{G}_{UR}^N$.*

Proof. For any $i \in N$, $N = (N \setminus P_i(G)) \uplus P_i(G)$, and let $c_1 = 0$ and $c_2 = 1$, that is, the cost of the members of coalition $N \setminus P_i(G)$ is 0, and the cost of the members of coalition $P_i(G)$ is 1 (see Definition 3). Then the generated airport game $v_{(G,c)} = \bar{u}_{P_i(G)}$.

On the other hand, it is clear that there is an airport game which is not the dual of any unanimity game. □

It is important to see how the classes of airport games and upstream responsibility games are related to the convex cone spanned by the duals of the unanimity games.

Theorem 8. *For any rooted tree G, $\mathcal{G}_G \subset \text{cone }\{\bar{u}_{P_i(G)}\}_{i\in N}$. Therefore, $\mathcal{G}_A \subset \mathcal{G}^N_{UR} \subset \text{cone }\{\bar{u}_T\}_{T\in 2^N\setminus\{\emptyset\}}$.*

Proof. First we show that $\mathcal{G}_G \subset \text{cone }\{\bar{u}_{P_i(G)}\}_{i\in N}$.

Let $v \in \mathcal{G}_G$ be an upstream responsibility game. Since $G = (V, E)$ is a rooted tree, for each $i \in N$, $\#\{j \in V : \overline{ij} \in E\} = 1$, so we can name the node before player i, let $i_- = \{j \in V : \overline{ij} \in E\}$. Then for any $i \in N$, let $\alpha_{P_i(G)} = c_{\overline{i_- i}}$.

Finally, it is easy to see that $v = \sum_{i\in N} \alpha_{P_i(G)} \bar{u}_{P_i(G)}$.

Second we show that $\text{cone }\{\bar{u}_{P_i(G)}\}_{i\in N} \setminus \mathcal{G}_G \neq \emptyset$. Let $N = \{1, 2\}$, then $\sum_{T\in 2^N\setminus\{\emptyset\}} \bar{u}_T \notin \mathcal{G}_G$, that is game $(1, 1, 3)$ is not an upstream responsibility game. □

The following example is an illustration of the above result.

Example 9. Consider the upstream responsibility game of Example 2. Then $P_1(G) = \{1\}$, $P_2(G) = \{1, 2\}$ and $P_3(G) = \{1, 3\}$. Furthermore, $\alpha_{P_1(G)} = 2$, $\alpha_{P_2(G)} = 4$ and $\alpha_{P_3(G)} = 1$. Finally, $v_{(G,c)} = 2\bar{u}_{\{1\}} + 4\bar{u}_{\{1,2\}} + \bar{u}_{\{1,3\}} = \sum_{i\in N} \alpha_{P_i(G)} \bar{u}_{P_i(G)}$.

Next we discuss further corollaries of Lemmata 7 and 8. First we show that even if for any rooted tree G, $\mathcal{G}_G$ is a convex set, the classes of airport games and upstream responsibility games are not convex.

Lemma 10. *$\mathcal{G}^N_A$ is not a convex set, moreover $\mathcal{G}^N_{UR}$ is not convex either.*

Proof. Let $N = \{1, 2\}$. From Lemma 7 $\{\bar{u}_T\}_{T\in 2^N\setminus\{\emptyset\}} \subseteq \mathcal{G}^N_A$, however, $\sum_{T\in 2^N\setminus\{\emptyset\}} \frac{1}{3}\bar{u}_T \notin \mathcal{G}^N_{UR}$, that is, game $(1/3, 1/3, 1)$ is not an upstream responsibility game. □

The following corollary has a key role in Young (1985)'s axiomatization of the Shapley value on the classes of airport and upstream responsibility games. It is well-known that the duals of the unanimity games are linearly independent vectors. From Lemma 8, for any rooted tree G and $v \in \mathcal{G}_G$, $v = \sum_{i\in N} \alpha_{P_i(G)} \bar{u}_{P_i(G)}$, where weights $\alpha_{P_i(G)}$ are well-defined, that is, those are uniquely determined. The following lemma says that for any game $v \in \mathcal{G}_G$, if we erase the weight of any basis vector (the duals of the unanimity games), then we get a game belonging to $\mathcal{G}_G$.

Proposition 11. *For any rooted tree G and $v = \sum_{i \in N} \alpha_{P_i(G)} \bar{u}_{P_i(G)} \in \mathcal{G}_G$, for each $i^* \in N$, $\sum_{i \in N \setminus \{i^*\}} \alpha_{P_i(G)} \bar{u}_{P_i(G)} \in \mathcal{G}_G$. Therefore, for any airport game $v = \sum_{T \in 2^N \setminus \{\emptyset\}} \alpha_T \bar{u}_T$ and $T^* \in 2^N \setminus \{\emptyset\}$, $\sum_{T \in 2^N \setminus \{\emptyset, T^*\}} \alpha_T \bar{u}_T \in \mathcal{G}_A^N$, and for any upstream responsibility game $v = \sum_{T \in 2^N \setminus \{\emptyset\}} \alpha_T \bar{u}_T$ and $T^* \in 2^N \setminus \{\emptyset\}$, $\sum_{T \in 2^N \setminus \{\emptyset, T^*\}} \alpha_T \bar{u}_T \in \mathcal{G}_{UR}^N$.*

Proof. Let $v = \sum_{i \in N} \alpha_{P_i(G)} \bar{u}_{P_i(G)}$ and $i^* \in N$. Then let the cost function c' be defined as follows, for any $e \in E$, (see the proof of Lemma 8)

$$c'_e = \begin{cases} 0, & \text{if } e = \overline{i^*_- i^*}, \\ c_e & \text{otherwise.} \end{cases}$$

Then game $\sum_{i \in N \setminus \{i^*\}} \alpha_{P_i(G)} \bar{u}_{P_i(G)} = v_{(G,c')}$, that is, $\sum_{i \in N \setminus \{i^*\}} \alpha_{P_i(G)} \bar{u}_{P_i(G)} \in \mathcal{G}_G$. □

The following example is an illustration of the above result.

Example 12. Consider the upstream responsibility game of Example 2, and take player 2. Then

$$c'(e) = \begin{cases} 2, & \text{if } e = \overline{01}, \\ 0, & \text{if } e = \overline{12}, \\ 1, & \text{if } e = \overline{13}. \end{cases}$$

Moreover, $\sum_{i \in N \setminus \{i^*\}} \alpha_{P_i(G)} \bar{u}_{P_i(G)} = 2\bar{u}_{\{1\}} + \bar{u}_{\{13\}} = v_{(G,c')}$ is an upstream responsibility game.

Finally, an obvious observation:

Lemma 13. *Every upstream responsibility game is concave.*

Proof. The duals of the unanimity games are concave games, hence Lemma 8 implies the statement. □

To sum up our results we conclude as follows:

Corollary 14. *The class of airport games is a union of finitely many convex cones, but it is not convex, and it is a proper subset of the class of upstream responsibility games. The class of upstream responsibility games is also a union of finitely many convex cones, but is not convex either, and it is a proper subset of the finite convex cone spanned by the duals of the unanimity games, therefore every upstream responsibility game is concave, so every airport game is concave too.*

3 Solutions for upstream responsibility games

In this section we propose solutions for upstream responsibility games.

A *solution* on set $A \subseteq \mathcal{G}^N$ ψ is a set-valued mapping $\psi : A \twoheadrightarrow \mathbb{R}^N$, that is, a solution assign a set of allocations to each game. In the following, we define two solutions.

Let $v \in \mathcal{G}^N$ and

$$p^i_{Sh}(S) = \begin{cases} \dfrac{\#S!(\#(N \setminus S) - 1)!}{\#N!}, & \text{if } i \notin S, \\ 0 & \text{otherwise.} \end{cases}$$

Then $\phi_i(v)$ the Shapley value (Shapley, 1953) of player i in game v is the p^i_{Sh} expected value of v'_i. In other words

$$\phi_i(v) = \sum_{S \subseteq N} v'_i(S)\ p^i_{Sh}(S)\,. \tag{1}$$

Furthermore, let ϕ denote the Shapley value.

It is obvious from its definition that the Shapley solution is a single valued solution, a single valued solution is called value.

Next, we introduce an other solution, the core (Shapley, 1955). Let $v \in \mathcal{G}^N_{UR}$ be an upstream responsibility game. Then the core of upstream responsibility game v is defined as follows

$$\text{core } (v) = \left\{ x \in \mathbb{R}^N : \sum_{i \in N} x_i = v(N), \text{ and for any } S \subseteq N, \sum_{i \in S} x_i \leq v(S) \right\}.$$

The core consists of the stable allocations of the value of the grand coalition, that is, any allocation of the core is such that the allocated cost is the total cost ($\sum_{i \in N} x_i = v(N)$) and no coalition has incentive to deviate from the allocation scheme.

In the following definition we list the axioms we use to characterize the Shapley value.

Definition 15. *A value ψ on $A \subseteq \mathcal{G}^N$ is / satisfies*

- *Pareto optimal (PO), if for all $v \in A$, $\sum\limits_{i \in N} \psi_i(v) = v(N)$,*
- *null-player property (NP), if for all $v \in A$, $i \in N$, $v'_i = 0$ implies $\psi_i(v) = 0$,*

- *equal treatment property (ETP), if for all* $v \in A$, $i, j \in N$, $i \sim^v j$ *implies* $\psi_i(v) = \psi_j(v)$,
- *additive (ADD), if for all* $v, w \in A$ *such that* $v + w \in A$, $\psi(v + w) = \psi(v) + \psi(w)$,
- *marginal (M), if for all* $v, w \in A$, $i \in N$, $v'_i = w'_i$ *implies* $\psi_i(v) = \psi_i(w)$.

Brief interpretations of the above introduced axioms are as follows: Another, commonly used name of axiom PO is *Efficiency*. This axiom requires that the total cost must be shared among the players. Axiom NP is about that, if a player's marginal contribution is zero, that is, she has no influence, effect on the given situation, then her share (her value) must be zero.

The axiom ETP puts the requirement that, if two players have the same effects in the given situation, then their evaluations must be equal. Going back to our example, if two firms are equivalent in regard to their emission costs, then their cost shares must be equal.

A value meets axiom ADD, if for any two games, adding up the games first then evaluate the players, or evaluate the players first then adding up their evaluations does not matter. Axiom M requires that, if a given player in two games produces the same marginal contributions, then the player's value must be the same in the two games.

First we take an obvious observation:

Proposition 16. *Let* $A, B \subseteq \mathcal{G}^N$. *If a set of axioms characterizes a solution on both classes of games* A *and* B, *and the solution meets the set of axioms on* $A \cup B$, *then set of axioms characterizes the solution on class* $A \cup B$.

In this section we consider two characterizations of the Shapley value on the classes of airport games and upstream responsibility games. The first one is Shapley's original axiomatization (Shapley, 1953).

Theorem 17. *For any rooted tree* G, *a value* ψ *on* $\mathcal{G}_G$ *is PO, NP, ETP and ADD if and only if* $\psi = \phi$, *that is, if and only if it is the Shapley value. Therefore, a value* ψ *on the class of airport games is PO, NP, ETP and ADD if and only if* $\psi = \phi$, *and a value* ψ *on the class of upstream responsibility games is PO, NP, ETP and ADD if and only if* $\psi = \phi$.

Proof. if: It is well known that the Shapley value meets axioms PO, NP, ETP and ADD, see e.g. Peleg and Sudhölter (2003).

only if: From Lemmata 6 and 7 ψ is defined on the cone spanned by $\{\bar{u}_{P_i(G)}\}_{i \in N}$.

Take $i^* \in N$. Then for any $\alpha \geq 0$ and players $i, j \in P_{i^*}(G)$, $i \sim^{\alpha \bar{u}_{P_{i^*}(G)}} j$, and for any player $i \notin P_{i^*}(G)$, $i \in NP(\alpha \bar{u}_{P_{i^*}(G)})$.

Then the axiom NP implies that for any player $i \notin P_{i^*}(G)$, $\psi_i(\alpha \bar{u}_{P_{i^*}(G)}) = 0$. Moreover, from the axiom ETP for any players $i, j \in P_{i^*}(G)$, $\psi_i(\alpha \bar{u}_{P_{i^*}(G)}) = \psi_j(\alpha \bar{u}_{P_{i^*}(G)})$. Finally, the axiom PO implies $\sum_{i \in N} \psi_i(\alpha \bar{u}_{P_{i^*}(G)}) = \alpha$.

Therefore $\psi(\alpha \bar{u}_{P_{i^*}(G)})$ is well-defined (unique), therefore, since the Shapley value meets the axioms PO, NP and ETP, $\psi(\alpha \bar{u}_{P_{i^*}(G)}) = \phi(\alpha \bar{u}_{P_{i^*}(G)})$.

It is also well known that $\{u_T\}_{T \in 2^N \setminus \emptyset}$ is a basis of $\mathcal{G}^N$, and that so is $\{\bar{u}_T\}_{T \in 2^N \setminus \emptyset}$.

Let $v \in \mathcal{G}_G$ be an upstream responsibility game. Then Lemma 8 implies that

$$v = \sum_{i \in N} \alpha_{P_i(G)} \bar{u}_{P_i(G)} \ ,$$

where for any $i \in N$, $\alpha_{P_i(G)} \geq 0$.

From axiom ADD $\psi(v)$ is well-defined (unique), therefore, since the Shapley value meets the axiom ADD and for any $i \in N$, $\alpha_{P_i(G)} \geq 0$, $\psi(\alpha_{P_i(G)} \bar{u}_{P_i(G)}) = \phi(\alpha_{P_i(G)} \bar{u}_{P_i(G)})$, $\psi(v) = \phi(v)$.

Finally, we can apply Proposition 16. □

In the proof of Theorem 17 we have applied a modified version of Shapley's original proof. In his proof Shapley uses the unanimity games as the basis of $\mathcal{G}^N$. In the proof above we consider the duals of the unanimity games as a basis and use Proposition 16 and Lemmata 6, 7, 8. It is worth noticing that (we discuss it in the next section) for the airport games Theorem 17 is also proved by Dubey (1982), so in this sense our result is also an alternative proof for Dubey (1982)'s result.

Next we consider Young's axiomatization of the Shapley value (Young, 1985). This was the first axiomatization of the Shapley value not involving the axiom ADD.

Theorem 18. *For any rooted tree G, a single valued solution ψ on $\mathcal{G}_G$ is PO, ETP and M if and only if $\psi = \phi$, that is, if and only if it is the Shapley value. Therefore, a value ψ on the class of airport games is PO, ETP and M if and only if $\psi = \phi$, and a value ψ on the class of upstream responsibility games is PO, ETP and M if and only if $\psi = \phi$.*

Proof. if: It is well known that the Shapley value meets the axioms PO, ETP and M, see e.g. Peleg and Sudhölter (2003).

only if: The proof goes, as that Young's proof does, by induction. For any upstream responsibility game $v \in \mathcal{G}_G$, let $\mathcal{B}(v) = \#\{\alpha_{P_i(G)} > 0 : v = \sum_{i \in N} \alpha_{P_i(G)} \bar{u}_{P_i(G)}\}$. It is clear that $\mathcal{B}(\cdot)$ is well-defined.

If $\mathcal{B}(v) = 0$, then the axioms PO and ETP imply that $\psi(v) = \phi(v)$.

Assume that for any game $v \in \mathcal{G}_G$ such that $\mathcal{B}(v) \leq n$, $\psi(v) = \phi(v)$. Furthermore, let $v = \sum_{i \in N} \alpha_{P_i(G)} \bar{u}_{P_i(G)} \in \mathcal{G}_G$ be such that $\mathcal{B}(v) = n + 1$.

Let $i^* \in N$ be a player such that there exists $i \in N$ such that $\alpha_{P_i(G)} \neq 0$ and $i^* \notin P_i(G)$. Then Lemmata 8 and 11 imply that $\sum_{j \in N \setminus \{i\}} \alpha_{P_j(G)} \bar{u}_{P_j(G)} \in \mathcal{G}_G$, and

$$\left(\sum_{j \in N \setminus \{i\}} \alpha_{P_j(G)} \bar{u}_{P_j(G)} \right)'_{i^*} = v'_{i^*} \ ,$$

therefore from the axiom M

$$\psi_{i^*}(v) = \psi_{i^*} \left(\sum_{j \in N \setminus \{i\}} \alpha_{P_j(G)} \bar{u}_{P_j(G)} \right) \ ,$$

that is, $\psi_{i^*}(v)$ is well-defined (uniquely determined).

Assume that $i^*, j^* \in N$ are such that for any $i \in N$ such that $\alpha_{P_i(G)} \neq 0$, $i^*, j^* \in P_i(G)$. Then $i^* \sim^v j^*$, hence the axiom ETP implies that $\psi_{i^*}(v) = \psi_{j^*}(v)$.

By the axiom PO, $\sum_{i \in N} \psi_i(v) = v(N)$. Therefore, $\psi(v)$ is well-defined (uniquely determined), therefore, since the Shapley solution meets the three considered axioms (PO, ETP and M), $\psi(v) = \phi(v)$.

Finally, we can apply Proposition 16. □

In the above proof we applied the idea of Young's proof, so we did not need any of the alternative proofs for Young (1985)'s axiomatization of the Shapley value by Moulin (1988) and Pintér (2015). We could do so because Lemma 11 ensures that when we apply the induction step in the only if branch we do not leave the considered classes of games. It is also worth noticing that for the airport games Theorem 18 is also proved by Moulin and Shenker (1992), so in this sense our result is also an alternative proof for Moulin and Shenker (1992)'s result.

Furthermore, Lemma 13 and the well-known results of Shapley (1971) and Ichiishi (1981) imply the following corollary:

Corollary 19. *For any upstream responsibility game v, $\phi(v) \in$ Core (v), that is, the Shapley value is in the core. Moreover, since every airport game is an upstream responsibility game, for any airport game v, $\phi(v) \in$ Core (v).*

The above corollary shows that on the two considered classes of games the Shapley value is stable, that is, it can be considered as a core selection.

Finally, as a direct corollary of our characterization of the class of upstream responsibility games, Theorem 8, we have the following result:

Corollary 20. *The Shapley value on the class of the upstream responsibility games can be calculated in polynomial time.*

Proof. Take an arbitrary upstream responsibility game $v \in \mathcal{G}^N_{UR}$. Then

$$v = \sum_{i \in N} \alpha_{P_i(G)} \bar{u}_{P_i(G)} \, .$$

Moreover,

$$\phi_j(\bar{u}_{P_i(G)}) = \begin{cases} \frac{1}{\#P_i(G)}, & \text{if } j \in P_i(G), \\ 0 & \text{otherwise.} \end{cases}$$

Therefore, by $\alpha_{P_i(G)} = c_{\overline{i\_i}}$ we have that for all $j \in N$:

$$\phi_j(v) = \sum_{j \in P_i(G)} \frac{c_{\overline{i\_i}}}{\#P_i(G)} \, .$$

□

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