CORVINUS ECONOMICS WORKING PAPERS

CORVINUS UNIVERSITY of BUDAPEST

Faculty of Economics

CEWP 03/2019

Bertrand-Edgeworth duopoly with a socially concerned firm

by
Balázs Nagy,
Attila Tasnádi

http://unipub.lib.uni-corvinus.hu/3982

Bertrand-Edgeworth duopoly with a socially concerned firm*

Balázs Nagy† and Attila Tasnádi‡

Department of Mathematics, Corvinus University of Budapest,
H-1093 Budapest, Fővám tér 8.

February 28, 2019

Abstract

The government may regulate a market by obtaining partial ownership in a firm. This type of socially concerned firm behaves as a combined profit and social surplus maximizer. We investigate the presence of a socially concerned firm in the framework of a Bertrand-Edgeworth duopoly with capacity constraints. In particular, we determine the mixed-strategy equilibrium of this game and relate it to both the standard and the mixed versions of the Bertrand-Edgeworth game. In contrast to other results in the literature we find that full privatization is the socially best outcome, that is the optimal level of public ownership is equal to zero.

Keywords: Bertrand-Edgeworth, mixed duopoly, semi-public firm, mixed-strategy equilibrium.

JEL Classification Number: D43, L13.

*This research is granted by the Pallas Athéné Domus Sapientiae Foundation Leading Researcher Program.

†*e-mail:* `balazs.nagy@outlook.com`.

‡*e-mail:* `attila.tasnadi@uni-corvinus.hu`, (`www.uni-corvinus.hu/~tasnadi`).

1 Introduction

In their seminal paper Merrill and Schneider (1966) investigated the welfare effect of a public firm in a quantity-setting oligopoly. The case of a so-called semi-public firm or socially concerned firm with an objective function obtained as a weighted sum of firm's profit and social surplus was analyzed by Matsumura (1998) for which he determined the optimal governmental share and found an interior solution, that is the pure public firm case and the standard profit-maximizing case does not emerge in equilibrium. Similar investigations have been carried out for the heterogeneous goods price-setting duopoly game by Barcéna-Ruiz and Sedano (2011) in which again the optimal governmental share was positive.

The current paper investigates the homogeneous good price-setting semi-public duopoly game. The simpler mixed duopoly game with a purely public firm was investigated by Balogh and Tasnái (2012) for which they found that an equilibrium in pure strategies always exists in contrast to the duopoly with a purely private firm, henceforth referred to as the standard case. However, since in the semi-public setting both firms objective functions have a profit component, there is a capacity region for which a pure-strategy equilibrium does not exist. Hence, the analysis of the semi-public case becomes much more difficult. Tasnádi (2013) gave a necessary and sufficient condition for the existence of a pure-strategy equilibrium and established the existence of a mixed-strategy equilibrium.

In this paper we derive certain properties of the mixed-strategy equilibrium and determine the mixed-strategy equilibrium in explicit form for the case of symmetric capacity constraints and linear demand. Based on our results we conclude that the socially optimal governmental share is zero, and thus the standard Bertrand-Edgeworth game results in higher social surplus than the proper semi-public Bertrand-Edgeworth game. In this respect the Bertrand-Edgeworth framework behaves more market like than the Cournot setting or the differentiated version of the Bertrand framework.

2 The framework

Concerning the demand function, we impose the following assumption.

Assumption 1. (i) D intersects the horizontal axis at quantity a and the vertical axis at price b; (ii) D is strictly decreasing, concave and twice-continuously differentiable on $(0, b)$; (iii) D is right-continuous at 0 and left-continuous at b; and (iv) $D(p) = 0$ for all $p \geq b$.

We shall denote by P the inverse demand function, that is $P(q) = D^{-1}(q)$ for $0 < q \leq a$, $P(0) = b$, and $P(q) = 0$ for all $q > a$.

We denote the set of firms by $\{1, 2\}$, where 1 will be the semi-public firm and 2 the private firm.

Assumption 2. The firms face zero unit cost up to their capacity constraints k_1 és k_2.[1] For simplicity we assume that the semi-public firm is not capable of serving the entire demand, i.e. $k_1 < a$.[2]

We shall denote by p^c the market clearing price, by p^M the price set by a monopolist without capacity constraints, and by p_i^M the price set by a monopolist with capacity constraint k_i, where $i \in \{1, 2\}$, i.e. $p^c = P(k_1 + k_2)$, $p^M = \arg\max_{p\in[0,b]} pD(p)$, and $p_i^M = \arg\max_{p\in[0,b]} p \min\{D(p), k_i\}$. In what follows $p_1, p_2 \in [0, b]$ stand for the prices set by the firms.

For all $i \in \{1, 2\}$ we shall denote by $p_i^m = \arg\max_{p\in[0,b]} pD_i^r(p)$ the unique revenue maximizing price on the firms' residual demand curves $D_i^r(p) = (D(p) - k_j)^+$, where $j \in \{1, 2\}$ and $j \neq i$, if $D_i^r(0) > 0$. Let $p_i^m = 0$ if $D_i^r(0) = 0$. The inverse residual demand curves will be denoted by R_1 and R_2. Clearly, p^c and p_i^m are well defined whenever Assumptions 1 and 2 are satisfied. We have $p_i^M \geq p^M > p_i^m$. Furthermore, $k_1 < a$ implies $p_2^m > 0$. It can be easily verified that from $k_i > k_j$ it follows that $p_i^m > p_j^m$.

Let us denote by p_i^d the smallest price p_i for which $p_i \min\{k_i, D_i(p_i)\} = p_i^m D_i^r(p_i^m)$, whenever this equation has a solution.[3] Provided that the private firm has 'sufficient' capacity (i.e. $p^c < p_2^m$), then if it is a profit-maximizer, it is indifferent to whether serving residual demand at price level p_i^m or selling $\min\{k_i, D_i(p_i^d)\}$ at the lower price level p_i^d. By Deneckere and Kovenock (1992, Lemma 1) we know that $p_i^d > p_j^d$ if $k_i > k_j$.

Since for the interesting price region the low-price firm cannot satisfy the whole demand, its consumers have to be rationed so that the residual demand of the high-price firm is a function of the consumers served by the low-price firms. The most frequently employed rationing rule is the so-called efficient rationing rule, which is reasonable if there is a secondary market for the duopolists' products.

Assumption 3. We assume efficient rationing on the market.

[1] The main assumption here is that firms have identical unit costs, assuming zero unit costs is just a matter of normalization since firms will produce to order.

[2] In case of $k_1 \geq a$ a pure-strategy equilibrium exists, which is not necessarily unique; however, sales happen only at price zero.

[3] The equation defining p_i^d has a solution if, for instance, $p_i^m \geq p^c$, which will be the case in our analysis when we will refer to p_i^d.

Under efficient rationing the demand faced by the firms $i \in \{1,2\}$ equals

$$\Delta_i\left(D,p_1,k_1,p_2,k_2\right)=\begin{cases} D\left(p_i\right) & \text{if} \quad p_i<p_j, \\ \frac{k_i}{k_1+k_2}D\left(p_i\right) & \text{if} \quad p=p_i=p_j, \\ \left(D\left(p_i\right)-k_j\right)^+, & \text{if} \quad p_i>p_j. \end{cases}$$

In case of equal prices we assume for simplicity that firms split demand in proportion to their capacities. However, we could have admitted a large class of tie-breaking rules, the only tie-breaking rules that have to be avoided are those ones giving priority to one of the two firms.

We turn to specifying the firms payoff functions. Let $\alpha \in (0,1)$ be the weight of the surplus-maximizing component in the payoff function of the semi-public firm, which might be a function of the governmental share in the equity of firm 1. We do not need to analyze the extreme cases of $\alpha = 0$ and $\alpha = 1$, which correspond to the already analyzed cases of the standard Bertrand-Edgeworth game and to the mixed version of the Bertrand-Edgeworth game investigated by Balogh and Tasnádi (2012). The payoff function of the semi-public firm is given by

$$\begin{aligned} \pi_1(p_1,p_2) &= (1-\alpha)p_1\min\left\{k_1,\Delta_1\left(D,p_1,k_1,p_2,k_2\right)\right\}+ \\ &\quad \alpha\int_0^{\min\left\{(D(p_j)-k_i)^+,k_j\right\}}R_j(q)dq+\alpha\int_0^{\min\{a,k_i\}}P(q)dq, \end{aligned} \tag{1}$$

where $0 \leq p_i \leq p_j \leq b$. Observe that because of efficient rationing social surplus is only a function of the largest price at which sales are realized.

The private firm maximzes its profits:

$$\pi_2(p_1,p_2)=p_2\min\left\{k_2,\Delta_2\left(D,p_1,k_1,p_2,k_2\right)\right\}. \tag{2}$$

In an analogous way to price p_2^m, which maximizes the private firms profits when serving residual demand, we define the payoff maximizing price p_1^s for the semi-public firm when it faces residual demand:

$$p_1^s=\arg\max_{p_1\in[0,b]}\left\{(1-\alpha)p_1D_1^r\left(p_1\right)+\alpha\int_0^{D(p_1)}P(q)dq\right\}.$$

It can be checked that p_1^s is determined uniquely and that $p_1^s < p_1^m$ under Asummptions 1-3.

Concerning the pure-strategy equilibrium of the capacity constrained Bertrand-Edgeworth game with a socially concerned firm, henceforth called the semi-public Bertrand-Edgeworth game, the following statement has been established by Tasnádi (2013).

Established fact 1. *Under Assumption 1-3, the semi-public Bertrand-Edgeworth game has a pure-strategy equilibrium if and only if* $\max\{p_1^s, p_2^m\} \leq p^c$. *If a pure-strategy equilibrium exists, then it is given by*

$$p_1^* = p_2^* = p^c = P(k_1 + k_2). \tag{3}$$

The existence of a mixed-strategy equilibrium can be established by employing a recent existence theorem demonstrated by Prokopovych and Yannelis (2014, Theorem 3).

If a pure-strategy equilibrium exists, the standard, the mixed and the semi-public Bertrand-Edgeworth games all result in the same outcome in which the firms produce at their capacity constraints and the equilibrium price is the market clearing price. Therefore, in what follows we focus on the case in which a pure-strategy equilibrium does not exist.

3 Some properties of the mixed-strategy equilibrium

In this section we assume that a pure-strategy equilibrium does not exist, i.e. $\max\{p_1^s, p_2^m\} > p^c$. We shall denote by (φ_1, φ_2) be an arbitrary mixed-strategy equilibrium. Let $\overline{p}_i = \sup \operatorname{supp}(\varphi_i)$ and $\underline{p}_i = \inf \operatorname{supp}(\varphi_i)$, where $i \in \{1, 2\}$.

Observe that $p_2^m > p^c$ implies $\underline{p}_2 \geq p_2^d > p^c$ because the private firms profits at price p_2^m are at least as large as at price p_2^d. Hence, $\underline{p}_1 \geq p_2^d$. Furthermore, if $p_1^s > p^c \geq p_2^m$, then $\underline{p}_1 > p^c$ and $\underline{p}_2 > p^c$.

We present some lemmas concerning the mixed-strategy equilibrium.

Lemma 1. *Under Assumptions 1, 2, 3, and* $\max\{p_1^s, p_2^m\} > p^c$, *we obtain that* φ_1 *and* φ_2 *cannot have both an atom at the same price.*

Proof. Suppose that there exists a price $p \in [0, b]$ for which $\varphi_1(p) > 0$ and $\varphi_2(p) > 0$. However, this would imply because of $\underline{p}_1 > p^c$ and $\underline{p}_2 > p^c$ that both firms $i \in \{1, 2\}$ would be better off by unilaterally shifting probability mass from price p to $p - \varepsilon$; a contradiction. □

Lemma 2. *Under Assumptions 1, 2, 3, and* $\max\{p_1^s, p_2^m\} > p^c$, *for any mixed-strategy equilibrium* (φ_1, φ_2) *we have* $\overline{p}_1 = p_1^s > \overline{p}_2$ *or* $\overline{p}_1 < \overline{p}_2 = p_2^m$ *or* $\min\{p_1^s, p_2^m\} \leq \overline{p}_1 = \overline{p}_2$.

Proof. Let $\overline{p}_1 > \overline{p}_2$. If $\overline{p}_1 > p_1^s$, then the semi-public firm could benefit from setting a price below $\overline{p}_1$ because of the strict concavity of its residual payoff

function. If $\overline{p}_1 < p_1^s$, then the semi-public firm would make more profits by setting price p_1^s than setting any other in $(\overline{p}_2, p_1^s)$; a contradiction. Hence, in case of $\overline{p}_1 > \overline{p}_2$ we must have $\overline{p}_1 = p_1^s$.

In an analogous way it can be shown that if $\overline{p}_1 < \overline{p}_2$, we must have $\overline{p}_2 = p_2^m$.

Suppose that $\min\{p_1^s, p_2^m\} > \overline{p}_1 = \overline{p}_2$. Then since in equilibrium at least one of the mixed strategies cannot have an atom at $\overline{p}_1 = \overline{p}_2$, say $\varphi_i(\{\overline{p}_i\}) = 0$, firm $j \neq i$ could increase its payoff by setting either price p_1^s or p_2^m; a contradiction.

□

Lemma 3. *Let Assumptions 1, 2, and 3 be satisfied and let* (φ_1, φ_2) *be a mixed-strategy equilibrium. If* $\max\{p_1^s, p_2^m\} > p^c$*, then* $\underline{p}_1 = \underline{p}_2$ *and* $\varphi_1(\underline{p}_1) = \varphi_2(\underline{p}_2) = 0$.

Proof. First, we establish that $\overline{p}_i \leq p_i^M$. Clearly, the semi-public firm's prices above p_1^M would be strictly dominated by price p_1^M (i.e. $\pi_1(p_1^M, \varphi_2) > \pi_1(p_1, \varphi_2)$ for any $p_1 > p_1^M$ and any mixed strategy φ_2 played by the private firm). The case of $\overline{p}_2 \leq p_2^M$ is even more obvious. Hence, the firms' do not set 'extremely' high prices.

Second, we demonstrate that $\underline{p}_1 \leq \underline{p}_2$. Suppose to the contrary that $\underline{p}_1 > \underline{p}_2$. Then by $\underline{p}_2 < \overline{p}_2 \leq p_2^M$ the private firm would benefit from switching from φ_2 to any price $p_2 \in (\underline{p}_2, \underline{p}_1)$; a contradiction.

Third, we demonstrate that $\underline{p}_1 \geq \underline{p}_2$. Suppose to the contrary that $\underline{p}_1 < \underline{p}_2$. Then by $\underline{p}_1 < \overline{p}_1 \leq p_1^M$ the public firm would benefit from switching from φ_1 to any price $p_1 \in (\underline{p}_1, \underline{p}_2)$ since the profit component of its payoff function would increase and the social surplus component of its payoff function would not change; a contradiction.

Forth, suppose that $\varphi_1(\underline{p}_1) > 0$. Then for a sufficiently small $\varepsilon > 0$ price $\underline{p}_1 - \varepsilon$ would strictly dominate price $\underline{p}_1 + \varepsilon$ for the private firm; a contradiction.

Finally, suppose that $\varphi_2(\underline{p}_2) > 0$. Then for a sufficiently small $\varepsilon > 0$ price $\underline{p}_2 - \varepsilon$ would strictly dominate price $\underline{p}_2 + \varepsilon$ for the semi-public firm since its profit component would be radically larger at the former price than at the latter one by its discontinuity at $\underline{p}_2$, while the social surplus component would be just slightly lower by the continuity of the social surplus component; a contradiction. □

4 Determining the mixed-strategy equilibrium in case of linear demand

Determining the mixed-strategy equilibrium of the standard Bertrand-Edgeworth duopoly under fairly general conditions is a difficult task. The semi-public version of this game appears to be even more difficult. Therefore, in this section we focus on the case of linear demand and symmetric capacities. Let $D(p) = (1-p)^+$, $P(q) = (1-q)^+$ and $k = k_1 = k_2$. Then we have a non-existence of an equilibrium in pure strategies if and only if $k \in (1/3, 1)$. For the latter capacity region we get:

$$p_1^m = p_2^m = \frac{1-k}{2}, \quad p_1^d = p_2^d = \frac{(1-k)^2}{4k}, \quad \text{and} \quad p_1^s = \frac{1-\alpha}{2-\alpha}(1-k).$$

Social surplus equals:

$$SW(p_1, p_2) = \begin{cases} \frac{1}{2}(1+p_1)(1-p_1) = \frac{1}{2}(1-p_1^2) & \text{if} \quad p_1 \geq p_2; \\ \frac{1}{2}(1+p_2)(1-p_2) = \frac{1}{2}(1-p_2^2) & \text{if} \quad p_1 < p_2. \end{cases}$$

Assume that there exists a mixed-strategy equilibrium such that $\varphi_1(\overline{p}_2) = 0$, which in turn implies $\overline{p}_2 = p_2^m$ and $\underline{p}_2 = p_2^d$. Then we find the mixed-strategy equilibrium in a similar way as in the standard case of the game. In equilibrium the private firm's equilibrium profit equals $\overline{\pi}_2 = p_2^d k = p_2^m(1 - p_2^m - k)$, which must be the case if the second case, i.e. $\overline{p}_2 = p_2^m$ in the statement of Lemma 2 holds true. We shall denote the cumulative distribution functions of the semi-public and the private firms by F and G, respectively.

The objective function of the semi-public firm, supposed that the private firm plays its mixed strategy G, is given by

$$\begin{aligned} \pi_1(p_1, G) \quad = \quad & (1-\alpha)p_1 k(1-G(p_1)) + (1-\alpha)p_1(1-p_1-k)G(p_1) + \\ & \alpha\frac{1}{2}(1-p_1^2)G(p_1) + \alpha\frac{1}{2}\int_{p_1}^{p_2^m}(1-p_2^2)dG(p_2) = \overline{\pi}_1, \end{aligned} \tag{4}$$

where the first line of (4) contains the profit component and the second line of (4) the social surplus component of the semi-public firm' s payoff function.

The private firm's objective function, supposed that the semi-public firm plays its mixed strategy F, is given by

$$\pi_2(F, p_2) = p_2 k(1-F(p_2)) + p_2(1-p_2-k)F(p_2) = \overline{\pi}_2. \tag{5}$$

Rearranging (5) we get

$$F(p_2) = \frac{p_2 k - \overline{\pi}_2}{p_2(2k-1+p_2)}. \tag{6}$$

It can be verified that $F(p_2^d) = 0$, $F(p_2^m) = 1$ and F is strictly increasing on $[p_2^d, p_2^m]$.

The private firm's equilibrium strategy can be obtained by solving $\frac{\partial \pi_1}{\partial p_1}(p_1, G) = 0$ and $G(p_2^d) = 0$. By differentiation we get

$$\begin{aligned}
\frac{\partial \pi_1}{\partial p_1}(p_1, G) &= (1-\alpha)k(1-G(p_1)) - (1-\alpha)p_1 k g(p_1) \\
&\quad +(1-\alpha)\left[(1-p_1-k)G(p_1) - p_1 G(p_1) + p_1(1-p_1-k)g(p_1)\right] \\
&\quad -\alpha p_1 G(p_1) + \frac{1}{2}\alpha(1-p_1^2)g(p_1) \\
&\quad -\frac{1}{2}\alpha(1-p_1^2)g(p_1) \\
&= \left[(1-\alpha)(1-2p_1-2k) - \alpha p_1\right] G(p_1) \\
&\quad +(1-\alpha)p_1(1-p_1-2k)g(p_1) + (1-\alpha)k = 0,
\end{aligned}$$

where g is the derivative of G, and the expression is just defined where G is differentiable. Solving the first-order linear differential equation we get[4]

$$G(p_1) = C\frac{1}{p_1}\left(\frac{1}{2k+p_1-1}\right)^{\frac{1}{1-\alpha}} + \frac{k(1-\alpha)}{p_1} \tag{7}$$

and employing $G(p_2^d) = 0$ we arrive to

$$C = -k(1-\alpha)\left(\frac{3}{2}\sqrt{k} - \frac{1}{2\sqrt{k}}\right)^{\frac{2}{1-\alpha}}.$$

Unfortunately, G does not specify a cumulative probability distribution function because it fails to be increasing on $[p_2^d, p_2^m]$. However, it is at least increasing at p_2^d. We shall denote by $p_0 \in (p_2^d, p_2^m)$ the price for which $g(p_0) = 0$ and p_0 is a local maximum of G.[5]

Proposition 1. *$\overline{F}(p)$ and $\overline{G}(p)$ given by*

$$\overline{F}(p) = \begin{cases} 0 & \text{if } p \in [0, p_2^d], \\ F(p) & \text{if } p \in (p_2^d, p_0], \\ 1 & \text{if } p \in (p_0, b] \end{cases} \quad \text{and} \quad \overline{G}(p) = \begin{cases} 0 & \text{if } p \in [0, p_2^d], \\ G(p) & \text{if } p \in (p_2^d, p_0], \\ G(p_0) & \text{if } p \in (p_0, p_2^m], \\ 1 & \text{if } p \in (p_2^m, b], \end{cases}$$

where F and G stand for the functions determined by (6) and (7), respectively. In particular, $\overline{F}$ has an atom at p_0, while $\overline{G}$ has an atom at p_2^m.

[4]Under our assumptions we have that $2k+p-1 = p-p^c > 0$ if $k < 1/2$ and $2p+k-1 > 0$ if $k \geq 1/2$.

[5]If $g(p) = 0$ has multiple solutions within $p_0 \in (p_2^d, p_2^m)$, then pick the smallest one.

Proof. First, we establish that $G(p) \leq 1$ for any $p \in [p_2^d, p_2^m]$ by showing that

$$G(p) \leq F(p) \quad \text{for any } p \in [p_2^d, p_2^m]. \tag{8}$$

Note that (8) holds with equality in case of $p = p_2^d$. (8) is equivalent with

$$C\left(\frac{1}{2k+p-1}\right)^{\frac{1}{1-\alpha}} + k(1-\alpha) \leq \frac{pk - \overline{\pi}_2}{(2k+p-1)}, \tag{9}$$

which we show by verifying that the derivative of the LHS is smaller than that of the RHS for any $p \in [p_2^d, p_2^m]$:

$$\begin{aligned} C\frac{1}{1-\alpha}\left(\frac{1}{2k+p-1}\right)^{\frac{1}{1-\alpha}-1}\frac{-1}{(2k+p-1)^2} &\leq \frac{k(2k+p-1) - \left(pk - \frac{(1-k)^2}{4}\right)}{(2k+p-1)^2} \\ k\left(\frac{3}{2}\sqrt{k} - \frac{1}{2\sqrt{k}}\right)^{\frac{2}{1-\alpha}}\left(\frac{1}{2k+p-1}\right)^{\frac{\alpha}{1-\alpha}} &\leq \left(\frac{3}{2}k - \frac{1}{2}\right)^2, \end{aligned}$$

where the LHS of the last inequality achieves its maximum value on $[p_2^d, p_2^m]$ for a given $k \in [1/3, 1]$ at $p = p_2^d = (1-k)^2/(4k)$, and therefore

$$\begin{aligned} \left(\frac{3}{2}\sqrt{k} - \frac{1}{2\sqrt{k}}\right)^{\frac{2}{1-\alpha}}\left(\frac{1}{2k + \frac{(1-k)^2}{4k} - 1}\right)^{\frac{\alpha}{1-\alpha}} &\leq \left(\frac{3}{2}\sqrt{k} - \frac{1}{2\sqrt{k}}\right)^2 \\ \left(\frac{3}{2}\sqrt{k} - \frac{1}{2\sqrt{k}}\right)^{\frac{2}{1-\alpha}}\left(\frac{1}{\frac{3}{2}\sqrt{k} - \frac{1}{2\sqrt{k}}}\right)^{\frac{2\alpha}{1-\alpha}} &\leq \left(\frac{3}{2}\sqrt{k} - \frac{1}{2\sqrt{k}}\right)^2, \end{aligned}$$

and we can see that the last inequality holds with equality, and (8) follows. Verifying that $0 \leq G(p)$ for any $p \in [p_2^d, p_2^m]$, can be obtained through simple rearrangements and by employing again $p \geq p_2^d = (1-k)^2/(4k)$.

By (4) and (5) firms 1 and 2 achieve $\overline{\pi}_1$ and $\overline{\pi}_2$ payoffs, respectively, when playing any of their pure strategies $p \in [p_2^d, p_0]$ against their opponents' above specified strategies ($\overline{G}$ and $\overline{F}$). Clearly, playing a price below p_2^d results in less payoff than $\overline{\pi}_1$ and $\overline{\pi}_2$, respectively. It is straightforward that the private firm makes less profit by setting a price $p_2 \in (p_0, p_2^m) \cup (p_2^m, b]$ than by setting price p_2^m, when playing against mixed strategy $\overline{F}$, since for prices in $(p_0, b]$ it serves residual demand with probability one.

We check that the socially concerned firm achieves less than $\overline{\pi}_1$ payoff when playing any pure strategy $p_1 \in (p_0, p_2^m]$ against mixed strategy $\overline{G}$. Starting with (4) and as a slight modification of its solution G, we define $\widetilde{G}$ for any pure strategy $p_1 \in [0, b]$ by $\widetilde{G}(p) = 0$ on $[0, p_2^d]$, by $G(p)$ on $(p_2^d, p_1]$, by

$\widetilde{G}(p) = G(p_1)$ on $(p_1, p_2^m]$, and by $\widetilde{G}(p) = 1$ on $(p_2^m, b]$ (which is not necessarily a mixed strategy). Assuming that the mixed strategy of the private firm would be mainly determined by G, but remains constant on $[p_1, p_2^m)$ and jumps up to 1 at p_2^m, which gives us function $\widetilde{G}$, we determine the price level p^* for the socially concerned firm at which its payoff is maximized. However, since G is not necessarily a cumulative density function because it might have decreasing parts before p_1, we just call the payoff function, which we are maximizing based on $\widetilde{G}$, as a 'virtual payoff' function and we will find out that G is indeed increasing on $[p_2^d, p^*]$, and therefore $\widetilde{G}$ specifies a cumulative density function $p_1 = p^*$. The virtual payoff function of the socially concerned firm is given by

$$\begin{aligned} z(p_1) = \pi_1^g(p_1, G(p_1)) &= (1-\alpha)p_1 k(1-G(p_1)) \\ &\quad +(1-\alpha)p_1(1-p_1-k)G(p_1) \\ &\quad +\alpha\frac{1}{2}(1-p_1^2)G(p_1) \\ &\quad +\alpha\frac{1}{2}(1-(p_2^m)^2)(1-G(p_1)), \end{aligned} \tag{10}$$

where $\pi_1^g(p_1, G(p_1))$ equals its real payoff if G is increasing until p_1.

First, we will show that the sign of g equals the sign of $z'(p_1) = d\pi_1^g(p_1, G(p_1))/dp_1$, which then would imply that $p^* = p_0$ and a function of type $\widetilde{G}$ cannot be cumulative density function in case of $p_1 > p_0$. To establish that $p^* = p_0$ we consider the derivative of the difference of (10) and (4)

$$z(p_1) - \pi_1(p_1, G) = \frac{\alpha}{2}(1-G(p_1))\left(1-(p_2^m)^2\right) - \frac{\alpha}{2}\int_{p_1}^{p_2^m}\left(1-p_2^2\right)dG(p_2), \tag{11}$$

which equals

$$\begin{aligned} \frac{d}{dp_1}\left(z(p_1) - \pi_1(p_1, G)\right) &= -\frac{\alpha}{2}g(p_1)\left(1-(p_2^m)^2\right) + \frac{\alpha}{2}\left(1-p_1^2\right)g(p_1) \\ &= -\frac{\alpha}{2}g(p_1)\left[\left(1-(p_2^m)^2\right) - \left(1-p_1^2\right)\right], \end{aligned}$$

and therefore, it follows that the signs of g and z are identical since $\pi_1(p_1, G) = \overline{\pi}_1$ and $[(1-(p_2^m)^2) - (1-p_1^2)] < 0$ in case of $p_1 < p_2^m$.

Finally, we really turn to proving that setting prices above p^* results in less payoff than $\overline{\pi}_1$ for the socially concerned firm. For any $p \in [p^*, p_2^m]$ we have

$$\begin{aligned} \pi_1(p_1, \overline{G}) &= (1-\alpha)p_1\left[k(1-G(p^*)) + (1-p_1-k)G(p^*)\right] \\ &\quad +\frac{\alpha}{2}\left[(1-p_1^2)G(p^*) + (1-(p_2^m)^2)(1-G(p^*))\right] \end{aligned} \tag{12}$$

and

$$\begin{aligned}\frac{d}{dp_1}\pi_1(p_1,\overline{G}) &= (1-\alpha)\left[k(1-G(p^*))+(1-2p_1-k)G(p^*)\right] \\ &\quad -\alpha p_1 G(p^*). \end{aligned} \tag{13}$$

In order to employ the results obtained so far we need the derivative of z:

$$\begin{aligned} z'(p_1) &= (1-\alpha)\left[k(1-G(p_1))+(1-2p_1-k)G(p_1)\right]-\alpha p_1 G(p_1) \\ &\quad +(1-\alpha)\left[-p_1 k g(p_1)+p_1(1-p_1-k)g(p_1)\right] \\ &\quad +\frac{\alpha}{2}\left[(1-p_1^2)g(p_1)-(1-(p_2^m)^2)g(p_1)\right]. \end{aligned} \tag{14}$$

By employing that p^* maximizes z, and therefore $z'(p^*)=0$, and $g(p^*)=0$ we obtain from (14) that

$$z'(p^*)=(1-\alpha)\left[k(1-G(p^*))+(1-2p^*-k)G(p^*)\right]-\alpha p^* G(p^*)=0. \tag{15}$$

Since the function appearing in (12) is strictly concave it has a unique maximum point, which then equals $p^*=p_0$ by taking (13) and (15) into consideration. □

The following two corollaries can be verified based on Proposition 1. The first one states that the standard Bertrand-Edgeworth duopoly can be obtained as a limiting case of semi-public Bertrand-Edgeworth duopolies.

Corollary 1. *If $\alpha\to 0$, then $\overline{G}(p)\to\overline{F}(p)$ for any $p\in[0,b]$.*

Proof. As we could see above, in equilibrium the private firm's profit equals

$$\overline{\pi}_2 = p_2^d k = \frac{(1-k)^2}{4k}k = \frac{(1-k)^2}{4}.$$

From (6) we get

$$F(p)=\frac{pk-\overline{\pi}_2}{p(2k-1+p)}=\frac{pk-\frac{(1-k)^2}{4}}{p(2k-1+p)}=-\frac{1}{4}\frac{k^2-4pk-2k+1}{p(2k-1+p).}$$

If $\alpha\to 0$, then from Proposition 1 and equation (7) we get for all $p\in[p_2^d,p_2^m]$ that

$$\begin{aligned} G(p) &= -k\left(\frac{3}{2}\sqrt{k}-\frac{1}{2\sqrt{k}}\right)^2\frac{1}{p}\left(\frac{1}{2k+p-1}\right)+\frac{k}{p} \\ &= -\frac{1}{4}\left[(3k-1)^2\frac{1}{p}\left(\frac{1}{2k+p-1}\right)-\frac{4k}{p}\right] \\ &= -\frac{1}{4}\frac{(3k-1)^2-4k(2k+p-1)}{p(2k+p-1)} \\ &= -\frac{1}{4}\frac{k^2-4pk-2k+1}{p(2k+p-1)}. \end{aligned}$$

□

The second corollary states that as the other limiting case one obtains the mixed Bertrand-Edgworth duopoly investigated by Balogh and Tasnádi (2012), where from the two or three pure-strategy equilibria appearing there the NE_2-type equilibrium will be approached.

Corollary 2. *If $\alpha \to 1$, then*

(i) $\overline{F}(p) \to 1$ *for any* $p \in (p_2^d, p_2^m]$, *and*

(ii) $\overline{G}(p) \to 0$ *for any* $p \in [p_2^d, p_2^m]$.

Proof. From Proposition 1 and equation (15) it can be seen that $p_0 \to p_2^d$ as $\alpha \to 1$ which immediately implies *(i)* and *(ii)*. □

Now we state our main result on the optimal level of public ownership.

Proposition 2. *The standard Bertrand-Edgeworth game yields the highest social surplus, which would mean that even partial privatization would be harmful in the semi-public framework.*

Proof. First, observe that for the relevant price region $[p_2^d, p_2^m]$ social surplus is determined by the higher price set by the two firms. Furthermore, social surplus is negatively related to the higher price. It can be verified that $\overline{F}(p)\overline{G}(p)$ is decreasing in α, where $\overline{F}(p)$ and $\overline{G}(p)$ are the mixed strategies of the firms given in Proposition 1, from which the statement of the proposition follows. □

References

[1] *Balogh, T. – Tasnádi, A.* (2012): Does timing of decisions in a mixed duopoly matter? *Journal of Economics (Zeitschrift für Nationalökonomie)*, **106**: 233–249.

[2] *Bárcena-Ruiz, J. C. – Sedano, M.* (2011): Endogenous timing in mixed duopolies: Weighted welfare and price competition. *Japanese Economic Review*, **62**: 485–503.

[3] *Deneckere, R. – Kovenock, D.* (1992): Price Leadership. *Review of Economic Studies*, **59**: 143–162.

[4] *Matsumura, T.* (1998): Partial privatization in mixed duopoly. *Journal of Public Economics*, **70**: 473–483.

[5] *Merrill, W. – Schneider, N.* (1966): Government Firms in Oligopoly Industries: A Short-run Analysis. *Quarterly Journal of Economics*, **80**: 400–412.

[6] *Prokopovych, P. – Yannelis, N.C.* (2014): On the existence of mixed strategy Nash equilibria. *Journal of Mathematical Economics*, **52**: 87–97.

[7] *Tasnádi, A* (2013): Duopólium részben állami tulajdonú vállalattal. In: Matematikai közgazdaságtan: elmélet, modellezés, oktatás - Tanulmányok Zalai Ernőnek, szerk. Matematikai Közgazdaságtan és Gazdaságelemzés Tanszékének munkatársai, Műszaki Könvykiadó, Budapest, 177-186 (in Hungarian).