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Axiomatizations of the Shapley Value for Upstream Responsibility Games

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Axiomatizations of the Shapley Value for Upstream Responsibility Games[*]

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Abstract

In this paper the problem of sharing the cost of emission in supply chains is considered. We focus on allocation problems that can be described by rooted trees, called cost-tree problems, and on the induced transferable utility cooperative games, called upstream responsibility games (Gopalakrishnan et al., 2017).

The Shapley value (Shapley, 1953) is probably the most popular value for transferable utility cooperative games. In Radványi (2018) we showed that Shapley (1953)'s and Young (1985)'s axiomatizations of the Shapley value are valid on the class of upstream responsibility games.

According to Gopalakrishnan et al. (2017) we introduce some pollution related properties and we relate them to the axioms from the TU games literature.

Keywords: Upstream responsibility games; Cost sharing; Emission; Supply chain; Shapley value; Rooted tree; Axiomatization of the Shapley value.

JEL Classification: C71.

1 Introduction

In this paper we consider cost sharing problems given by rooted trees, called *cost-tree problems.* We assign transferable utility (henceforth TU) cooperative games (henceforth games) to these cost sharing problems. Particularly,

we consider energy supply chains with a motivated dominant leader, who has the power to assign the suppliers responsibilities for both direct and indirect emissions. The induced games are called *upstream responsibility games* (Gopalakrishnan et al., 2017).

For an example consider a supply chain where we look at the responsibility allocation of greenhouse gas (GHG) emission among the firms in the chain. One of the main questions is how to share the costs related to the emission among the firms. The supply chain and the related firms (or any other actors) can be represented by a rooted tree.

The root of the tree represents the end product produced by the supply chain. The root is connected to only one node which is the leader of the chain. Each further node represents one firm, and the edges of the rooted tree represent the producing process among the firms with the related emissions. Our goal is to share the responsibility of the emission while embodying the principle of upstream emission responsibility.

In this paper we rely on the TU game model of Gopalakrishnan et al. (2017), called GHG Responsibility-Emissions and Environment (GREEN) game. Gopalakrishnan et al. use the Shapley value (Shapley, 1953) as an allocation method, consider some pollution related properties that an emission allocation rule should meet, and provide several axiomatizations as well.

In this paper we translate the axioms used in the cost sharing literature related to allocation of greenhouse gas emission in supply chains (Gopalakrishnan et al., 2017) to the language of transferable utility cooperative games. We also consider the characterizations of the Shapley pollution allocation rule provided by Gopalakrishnan et al. (2017) and show that two of them are equivalent or impling Young's axiomatization of the Shapley value.

The setup of this paper is as follows: in the next section we introduce the concept of upstream responsibility games and characterize the class of them. In Section 3 we present Shapley (1953)'s and Young (1985)'s axiomatizations of the Shapley value, which work on the class of upstream responsibility games. In Section 4 we introduce the properties relying on the pollution problem and relate them to the TU games terminologies. We conclude our results by comparing the characterizations of Gopalakrishnan et al. (2017) to Young's axiomatization.

2 Upstream Responsibility Games

2.1 Preliminaries

Notions, notations: $\#N$ is for the cardinality of set N, and 2^N denotes the class of all subsets of N. $A \subset B$ means $A \subseteq B$, but $A \neq B$. $A \uplus B$ stands for the union of disjoint sets A and B.

A *graph* is a pair $G = (V, E)$, where the elements of V are called *vertices* or nodes, and E stands for the ordered pairs of vertices, called *edges* or arcs. A *rooted tree* is a graph in which any two vertices are connected by exactly one path, and one vertex has been designated the root denoted as node root. In the case of a supply chain consisting several entities, which are cooperating in the production of a final product, the manufacturing process can be modeled with a directed rooted tree. The set of nodes V represents the entities, henceforth players, and a directed arc emanating from node i towards root represents the activity by player i contributing to the manufacturing of the final product. We assume that only one node emanates arc enters the root, this emanates from node 1 which node represents the end consumer.

The tree-order is the partial ordering on the vertices of a rooted tree with $i \leq j$, if the unique path from j to the root passes through i. The *chain* is a rooted tree such that any vertices $i, j \in V$, $i \leq j$ or $j \leq i$. That is, a chain is a rooted tree with only one "branch". For any pair $e \in E$, $e = \overline{ij}$ means e is an edge between vertices $i, j \in V$ such that $i \leq j$. For each $i \in V$, let $S_i(G) = \{j \in V : j \geq i\}$, that is, for any $i \in V$, $i \in S_i(G)$. For each $i \in V$ let $P_i(G) = \{j \in V : j \leq i\}$, that is, for any $i \in V$, $i \in P_i(G)$. Moreover, for any $V' \subseteq V$, let $(P_{V'}(G), E_{V'})$ be the sub-rooted-tree of (V, E), where $P_{V'}(G) = \cup_{i \in V'} P_i(G)$ and $E_{V'} = \{\overline{ij} \in E : i, j \in P_{V'}(G)\}$.

Let $c : E \rightarrow \mathbb{R}_+$, c and (G, c) are called *cost function* and *cost-tree* respectively. An interpretation of cost tree (G, c) might be as follows: there is a given product which is produced by a supply chain. $V = N \cup \{\text{root}\}$, N denotes the set of all entities of the supply process. Node 0, the root denotes the end-consumer, and the leaf nodes represents the most upstream members (that is, firms, etc.) of the supply chain. We assume that only one arc emanates form the root and this emanates from the node 1. Each edge $e \in E$ is associated with a process in the supply chain emitting a pollution c(e). Let e_i denote the unique edge in the tree T emanating from node i (in the direction of the root). In this case $c(e_i)$ represents the direct pollution associated with e_i, the directly created pollution by node (firm) i. Moreover, i also can be responsible for the pollution of other processes in the chain. For each node i, $\mathcal{E}_i$ denotes the set of edges whose associated pollution is the direct or indirect responsibility of node i.

Let $N \neq \emptyset$, $\#N < \infty$, and $v : 2^N \to \mathbb{R}$ be a function such that $v(\emptyset) = 0$. Then N, v are called set of players, and *transferable utility cooperative game* (henceforth game) respectively. The class of games with player set N is denoted by $\mathcal{G}^N$.

A game $v \in \mathcal{G}^N$ is *convex*, if for all $S, T \subseteq N$, $v(S) + v(T) \leq v(S \cup T) + v(S \cap T)$, moreover, it is *concave*, if for all $S, T \subseteq N$, $v(S) + v(T) \geq v(S \cup T) + v(S \cap T)$.

The *dual* of game $v \in \mathcal{G}^N$ is game $\bar{v} \in \mathcal{G}^N$ such that for all $S \subseteq N$, $\bar{v}(S) = v(N) - v(N \setminus S)$. It is well known that the dual of a convex game is a concave game and vice versa.

Let $v \in \mathcal{G}^N$ and $i \in N$, and $v_i'(S) = v(S \cup \{i\}) - v(S)$, where $S \subseteq N$. The vector v_i' is called player i's *marginal contribution function* in game v. Alternatively, $v_i'(S)$ is player i's marginal contribution to coalition S in game v.

Let $v \in \mathcal{G}^N$, the players $i, j \in N$ are *equivalent* in game v, $i \sim^v j$, if for all $S \subseteq N$ such that $i, j \notin S$, $v_i'(S) = v_j'(S)$.

Let N and $T \in 2^N \setminus \emptyset$, and for all $S \subseteq N$, let

$$u_T(S) = \begin{cases} 1, & \text{if } T \subseteq S, \\ 0 & \text{otherwise.} \end{cases}$$

The game u_T is called *unanimity game* on coalition T.

In this paper we use the duals of the unanimity games. For any $T \in 2^N \setminus \emptyset$ and for all $S \subseteq N$,

$$\bar{u}_T(S) = \begin{cases} 1, & \text{if } T \cap S \neq \emptyset, \\ 0 & \text{otherwise.} \end{cases}$$

It is clear that every unanimity game is convex, and the duals of the unanimity games are concave.

Henceforth, we assume that in the considered cost-tree problems there are at least two players, that is, $\#V \geq 3$ and $\#N \geq 2$.

Next we introduce the notion of *upstream responsibility game* (URG) (Gopalakrishnan et al., 2017). Let (G, c) be a cost-tree, representing a supply chain, and N be the set of the members of the chain, henceforth players (the vertices but the root). We denote by a_j the pollution associated with arc j. Let $c(\{i\})$ denote the total pollution emission that player i is directly or indirectly responsible for. $\mathcal{E}_i$ represents the set of edges whose associated pollution is the direct or indirect responsibility of player i, $c(\{i\}) = a(\mathcal{E}_i) \equiv \sum_{j \in \mathcal{E}_i} a_j$. For every $S \subseteq N$ let $\mathcal{E}_S$ denote the collection of edges whose associated pollution is the direct or indirect responsibility of the players in

S, thus $\mathcal{E}_S = \cup_{i \in S} \mathcal{E}_i$ and the pollution which S is directly or indirectly responsible for is $c(S) = a(\mathcal{E}_S) \equiv \sum_{j \in \mathcal{E}_S} a_j$.

The class of upstream responsibility games with player set N is denoted by $\mathcal{G}_{UR}^N$. Let $\mathcal{G}_G$ denote the subclass of upstream responsibility games induced by cost-tree problems on the (specific) rooted tree G.

Definition 1 (Upstream Responsibility Game). *For any cost-tree (G, c), let $N = V \setminus \{root\}$ be the player set, and for any coalition S (the empty sum is 0) let the upstream responsibility game be defined as follows*

$$v_{(G,c)}(S) = \sum_{j \in \mathcal{E}_S} a_j \, .$$

The next example is an illustration of the above definition.

Example 2. Consider the cost-tree in Figure 1, where the rooted tree $G = (V, E)$ is as follows, $V = \{root, 1, 2, 3\}$, $E = \{\overline{root1}, \overline{12}, \overline{13}\}$, and $\mathcal{E}_1 = \{e_1, e_2, e_3\}$, $\mathcal{E}_2 = \{e_2\}$, $\mathcal{E}_3 = \{e_3\}$. Then the cost (pollution) function c is defined as $c(\overline{root1}) = a_1 = 2$, $c(\overline{12}) = a_2 = 4$ and $c(\overline{13}) = a_3 = 1$.
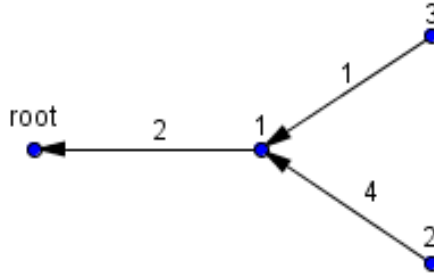

Figure 1: The cost-tree (G, c)

The upstream responsibility game is $v_{(G,c)} = (0, 7, 4, 1, 7, 7, 5, 7)$, that is, $v_{(G,c)}(\emptyset) = 0$, $v_{(G,c)}(\{1\}) = 7$, $v_{(G,c)}(\{2\}) = 4$, $v_{(G,c)}(\{3\}) = 1$, $v_{(G,c)}(\{1, 2\}) = 7$, $v_{(G,c)}(\{1, 3\}) = 7$, $v_{(G,c)}(\{2, 3\}) = 5$ and $v_{(G,c)}(N) = 7$.

From now on we assume that for any rooted tree G the associated pollutions sets are as follows $\mathcal{E}_i = P_i(G)$, $i \in V \setminus \{root\}$.

In the papers Márkus et al. (2011) and Radványi (2018) we characterized the classes of airport games and upstream responsibility games. The class of airport games is denoted by $\mathcal{G}_A^N$ and the class of upstream responsibility games by $\mathcal{G}_{UR}^N$. In the following we list such results from the above mentioned two papers that we will use in this paper.

First, we take an obvious observation, for any rooted tree G, $\mathcal{G}_G$ is a cone, that is for any $\alpha \geq 0$, $\alpha\mathcal{G}_G \subseteq \mathcal{G}_G$. Since union of cones is also a cone, both $\mathcal{G}_A^N$ and $\mathcal{G}_{UR}^N$ are cones.

Lemma 3. *For any rooted tree G, $\mathcal{G}_G$ is a cone, therefore, $\mathcal{G}_A^N$ and $\mathcal{G}_{UR}^N$ are cones.*

Lemma 4. *For any chain G, and player $i \in N$, $\bar{u}_{P_i(G)} \in \mathcal{G}_G$. Therefore, $\{\bar{u}_T\}_{T \in 2^N \setminus \{\emptyset\}} \subset \mathcal{G}_A^N \subset \mathcal{G}_{UR}^N$.*

Lemma 5. *For any rooted tree G, $\mathcal{G}_G \subset \text{cone } \{\bar{u}_{P_i(G)}\}_{i \in N}$. Therefore, $\mathcal{G}_A \subset \mathcal{G}_{UR}^N \subset \text{cone } \{\bar{u}_T\}_{T \in 2^N \setminus \{\emptyset\}}$.*

The following example is an illustration of the above result.

Example 6. Consider the upstream responsibility game of Example 2. Then $P_1(G) = \{1\}$, $P_2(G) = \{1,2\}$ and $P_3(G) = \{1,3\}$. Furthermore, $\alpha_{P_1(G)} = 2$, $\alpha_{P_2(G)} = 4$ and $\alpha_{P_3(G)} = 1$. Finally, $v_{(G,c)} = 2\bar{u}_{\{1\}} + 4\bar{u}_{\{1,2\}} + \bar{u}_{\{1,3\}} = \sum_{i \in N} \alpha_{P_i(G)} \bar{u}_{P_i(G)}$.

Further corollaries of Lemmata 4 and 5 are the followings.

Lemma 7. *$\mathcal{G}_A^N$ is not a convex set, moreover $\mathcal{G}_{UR}^N$ is not convex either.*

The following corollary has a key role in Young (1985)'s axiomatization of the Shapley value on the classes of airport and upstream responsibility games. It is well-known that the duals of the unanimity games are linearly independent vectors in $\mathbb{R}^{2^N-1}$. From Lemma 5, for any rooted tree G and $v \in \mathcal{G}_G$, $v = \sum_{i \in N} \alpha_{P_i(G)} \bar{u}_{P_i(G)}$, where weights $\alpha_{P_i(G)}$ are well-defined, that is, those are uniquely determined. The following lemma says that for any game $v \in \mathcal{G}_G$, if we erase the weight of any basis vector (the duals of the unanimity games), then we get a game belonging to $\mathcal{G}_G$.

Proposition 8. *For any rooted tree G and $v = \sum_{i \in N} \alpha_{P_i(G)} \bar{u}_{P_i(G)} \in \mathcal{G}_G$, for each $i^* \in N$, $\sum_{i \in N \setminus \{i^*\}} \alpha_{P_i(G)} \bar{u}_{P_i(G)} \in \mathcal{G}_G$. Therefore, for any airport game $v = \sum_{T \in 2^N \setminus \{\emptyset\}} \alpha_T \bar{u}_T$ and $T^* \in 2^N \setminus \{\emptyset\}$, it holds that $\sum_{T \in 2^N \setminus \{\emptyset, T^*\}} \alpha_T \bar{u}_T \in \mathcal{G}_A^N$, and for any upstream responsibility game $v = \sum_{T \in 2^N \setminus \{\emptyset\}} \alpha_T \bar{u}_T$ and $T^* \in 2^N \setminus \{\emptyset\}$, it holds that $\sum_{T \in 2^N \setminus \{\emptyset, T^*\}} \alpha_T \bar{u}_T \in \mathcal{G}_{UR}^N$.*

The following example is an illustration of the above result.

Example 9. Consider the upstream responsibility game of Example 2, and take player 2. Then

$$c'(e) = \begin{cases} 2, & \text{if } e = \overline{01}, \\ 0, & \text{if } e = \overline{12}, \\ 1, & \text{if } e = \overline{13}. \end{cases}$$

Moreover, $\sum_{i \in N \setminus \{i^*\}} \alpha_{P_i(G)} \bar{u}_{P_i(G)} = 2\bar{u}_{\{1\}} + \bar{u}_{\{13\}} = v_{(G,c')}$ is an upstream responsibility game.

Finally, an obvious observation:

Lemma 10. *Every upstream responsibility game is concave.*

Proof. The duals of the unanimity games are concave games, hence by Lemma 5 every upstream responsibility game is concave. □

To sum up the above results we conclude as follows:

Corollary 11. *The class of airport games is a union of finitely many convex cones, but it is not convex, and it is a proper subset of the class of upstream responsibility games. The class of upstream responsibility games is also a union of finitely many convex cones, but is not convex either, and it is a proper subset of the finite convex cone spanned by the duals of the unanimity games, therefore every upstream responsibility game is concave, so every airport game is concave too.*

3 Solutions for upstream responsibility games

In this section we propose solutions for upstream responsibility games.

A *solution* on set $A \subseteq \mathcal{G}^N$ ψ is a set-valued mapping $\psi : A \twoheadrightarrow \mathbb{R}^N$, that is, a solution assign a set of allocations to each game. In the following, we define two solutions.

Let $v \in \mathcal{G}^N$ and

$$p_{Sh}^i(S) = \begin{cases} \dfrac{\#S!(\#(N \setminus S) - 1)!}{\#N!}, & \text{if } i \notin S, \\ 0 & \text{otherwise.} \end{cases}$$

Then $\phi_i(v)$ the Shapley value (Shapley, 1953) of player i in game v is the p_{Sh}^i expected value of v_i'. In other words

$$\phi_i(v) = \sum_{S \subseteq N} v_i'(S) \, p_{Sh}^i(S). \tag{1}$$

Furthermore, let ϕ denote the Shapley value.

It is obvious from its definition that the Shapley solution is a single valued solution, a single valued solution is called value.

Next, we introduce an other solution, the core (Shapley, 1955; Gillies, 1959). Let $v \in \mathcal{G}_{UR}^N$ be an upstream responsibility game. Then the core of upstream responsibility game v is defined as follows

$$\text{core } (v) = \left\{ x \in \mathbb{R}^N : \sum_{i \in N} x_i = v(N), \text{ and for any } S \subseteq N, \sum_{i \in S} x_i \leq v(S) \right\}.$$

The core consists of the stable allocations of the value of the grand coalition, that is, any allocation of the core is such that the allocated cost is the total cost ($\sum_{i \in N} x_i = v(N)$) and no coalition has incentive to deviate from the allocation scheme.

In the following definition we list the axioms we use to characterize the Shapley value.

Definition 12. *A value ψ on $A \subseteq \mathcal{G}^N$ is / satisfies*

- *Pareto optimal (PO), if for all $v \in A$, $\sum_{i \in N} \psi_i(v) = v(N)$,*

- *null-player property (NP), if for all $v \in A$, $i \in N$, $v_i' = 0$ implies $\psi_i(v) = 0$,*

- *equal treatment property (ETP), if for all $v \in A$, $i, j \in N$, $i \sim^v j$ implies $\psi_i(v) = \psi_j(v)$,*

- *additive (ADD), if for all $v, w \in A$ such that $v + w \in A$, $\psi(v + w) = \psi(v) + \psi(w)$,*

- *marginal (M), if for all $v, w \in A$, $i \in N$, $v_i' = w_i'$ implies $\psi_i(v) = \psi_i(w)$.*

Brief interpretations of the above introduced axioms are as follows: Another, commonly used name of axiom *PO* is *Efficiency*. This axiom requires that the total cost must be shared among the players. Axiom *NP* is about that, if a player's marginal contribution is zero, that is, she has no influence, effect on the given situation, then her share (her value) must be zero.

The axiom *ETP* puts the requirement that, if two players have the same impact on the given situation, then their evaluations must be equal. Going back to our example, if two firms are equivalent in regard to their emission costs, then their cost shares must be equal.

A value meets axiom *ADD*, if for any two games, adding up the games first then evaluate the players, or evaluate the players first then adding up their evaluations does not matter. Axiom *M* requires that, if a player in two games produces the same marginal contributions, then her value must be the same in the two games.

First we take an observation:

Proposition 13. *Let $A, B \subseteq \mathcal{G}^N$. If a set of axioms characterizes a solution on both classes of games A and B, and the solution meets the set of axioms on $A \cup B$, then set of axioms characterizes the solution on class $A \cup B$.*

In this section we consider two characterizations of the Shapley value on the classes of upstream responsibility games. The first one is Shapley's original axiomatization (Shapley, 1953).

Theorem 14. *For any rooted tree G, a value ψ on $\mathcal{G}_G$ is PO, NP, ETP and ADD if and only if $\psi = \phi$, that is, if and only if it is the Shapley value. Therefore, a value ψ on the class of upstream responsibility games is PO, NP, ETP and ADD if and only if $\psi = \phi$.*

Proof. For the proof see e.g. Radványi (2018). □

Next we consider Young's axiomatization of the Shapley value (Young, 1985). This was the first axiomatization of the Shapley value not involving the axiom ADD.

Theorem 15. *For any rooted tree G, a single valued solution ψ on $\mathcal{G}_G$ is PO, ETP and M if and only if $\psi = \phi$, that is, if and only if it is the Shapley value. Therefore, a value ψ on the class of upstream responsibility games is PO, ETP and M if and only if $\psi = \phi$.*

Proof. For the proof see e.g. Radványi (2018). □

Furthermore, Lemma 10 and the well-known results of Shapley (1971) implies the following corollary:

Corollary 16. *For any upstream responsibility game v, $\phi(v) \in \text{Core}(v)$, that is, the Shapley value is in the core. Moreover, since every airport game is an upstream responsibility game, for any airport game v, $\phi(v) \in \text{Core}(v)$.*

The above corollary shows that on the two considered classes of games the Shapley value is stable, that is, it can be considered as a core selection.

Finally, as a direct corollary of our characterization of the class of upstream responsibility games, Lemma 5, we have the following result:

Corollary 17. *The Shapley value on the class of the upstream responsibility games can be calculated in polynomial time.*

Proof. Take an arbitrary upstream responsibility game $v \in \mathcal{G}_{UR}^N$. Then

$$v = \sum_{i \in N} \alpha_{P_i(G)} \bar{u}_{P_i(G)} \, .$$

Moreover,

$$\phi_j(\bar{u}_{P_i(G)}) = \begin{cases} \frac{1}{\#P_i(G)}, & \text{if } j \in P_i(G), \\ 0 & \text{otherwise.} \end{cases}$$

Therefore, by $\alpha_{P_i(G)} = c_{\overline{i_{-i}}}$ we have that for all $j \in N$:

$$\phi_j(v) = \sum_{j \in P_i(G)} \frac{c_{\overline{i_{-i}}}}{\#P_i(G)} \, .$$

$\square$

4 Axiomatic basis for the Shapley value relied on pollution related properties

In this section we consider pollution related properties in the case of upstream responsibility games. We rely on the axiomatization of Gopalakrishnan et al. (2017), who introduce six properties an allocation rule should satisfy in the context of allocating the responsibility of pollution. They also introduce three axiomatizations of the Shapley value using these properties. Our goal for this section is to compare those pollution related axioms with axioms arising from the cooperative game theory, which axioms we have introduced in the previous section.

First of all some notions and notations, some of them as a reminder: we consider a supply chain represented with graph $G = (V, E)$, with n firms denoted by $N = \{1, 2, \ldots, n\}$. $M = \{1, 2, \ldots, m\}$ denotes the set of all processes in the chain, where each process j corresponds to an activity by some player of the supply chain. The process j has a related pollution of f_j and a set of firms N_j which bear responsibility for it. For each process j, f_j represents the pollution for arc (j_1, j_2), so $f_j = a_j \geq 0$. The Shapley value was shown to divide the pollution f_j equally among the firms who are responsible for it.

Let $\mathcal{P}_i$ the set of processes that firm i is responsible for, and $\widetilde{\mathcal{P}}_i$ the set of processes with non-zero pollution. A responsibility matrix B shows if firm i is responsible for the pollution of process j, if and only if $b_{i,j} = 1$, and 0 otherwise. A firm i is responsible for the total pollution of $\sum_{j=1}^m f_j b_{i,j}$. For the set of firms S, $b_{S,j} = 1$ if at least one firm in S is responsible for the pollution of process j, and 0 otherwise. S is responsible for the total pollution of $f(\mathcal{P}_S) = \sum_{j=1}^m f_j b_{S,j}$. Let $f = (f_1, f_2, \ldots, f_m)$ be the total footprint set consisting of the pollution of all processes in the supply chain.

We introduce the definition of a pollution allocation rule, according to Gopalakrishnan et al. (2017):

Definition 18. *A pollution allocation rule φ is defined on a supply chain with n firms, set of processes M and a responsibility matrix B as a mapping $\varphi : \mathbb{R}_+^m \to \mathbb{R}_+^n$, which allocates to each firm its responsibility towards the total pollution such that $\sum_{i=1}^n \varphi_i(f) = \sum_{i=1}^m f_j$.*

A pollution allocation rule allocates the total pollution responsibility among the firms (without double counting). It is worth to notice, that a pollution allocation rule is Pareto-optimal by definition.

In the following we introduce some properties that any rule should possess in the context of pollution allocation according to Gopalakrishnan et al. (2017).

Definition 19. *A pollution allocation rule φ on a supply chain with n firms, set of processes M and a responsibility matrix B satisfies the property of*

- *equal sharing of extra pollution (ESEP), if $f' \geq f$ such that $f'_j = f_j$ for all processes $j \in M \setminus \{k\}$ and $f'_k \geq f_k$, then for $p, q \in N$ such that $b_{p,k} = b_{q,k} = 1$ implies $\varphi_p(f') - \varphi_p(f) = \varphi_q(f') - \varphi_q(f)$,*

- *no free riding (NFR), if for any firm i and footprints $f' \geq f$, such that $f'_j = f_j$ for all processess j for which $b_{i,j} = 1$, then $\varphi_i(f') = \varphi_i(f)$,*

- *firm equivalence (FE), if for two firms i and j, $(\widetilde{P_i}) = (\widetilde{P_j})$, then $\varphi_i(f) = \varphi_j(f)$,*

- *firm nullity (FN), if for a firm i, $(\widetilde{P_i}) = \emptyset$, then $\varphi_i(f) = 0$,*

- *process history independence (PHI), if for $f' \geq f$ and $\widetilde{f'} \geq \widetilde{f}$ footprints, such that $f'_j = f_j$ and $\widetilde{f'_j} = \widetilde{f_j}$ for all processes $j \in M \setminus \{k\}$, $f'_k = \widetilde{f'_k}$ and $f_k = \widetilde{f_k}$, then for any firm i, $\varphi_i(f') - \varphi_i(f) = \varphi_i(\widetilde{f'}) - \varphi_i(\widetilde{f})$,*

- *disaggregation invariance (DI), in the following case: suppose firm i chooses to disaggregate and represent itself as firms i_1 and i_2, with an arc (i_1, i_2) joining them, which corresponds to a process with zero pollution. Each of i_1 and i_2 is responsible for disjoint sets of processes $\mathcal{P}_{i_1}$ and $\mathcal{P}_{i_2}$ respetively, such that $\mathcal{P}_{i_1} \cup \mathcal{P}_{i2} = \mathcal{P}_i \cup \{(i_1, i_2)\}$. The disaggregated supply chain G' has $n + 1$ firms and process set $M \cup \{(i_1, i_2)\}$. Then φ is said to be disaggregated invariant if for a firm $j \neq i$, $\varphi_j(G) = \varphi_j(G')$ and $\varphi_{i_1}(G') + \varphi_{i_2}(G') = \varphi_i(G)$.*

The property of *equal sharing of extra pollution* means, that if the pollution increases at some processes and remains the same at all the others, then any two firms who are responsible for the extra pollution should allocate the

extra burden equally. *No free riding* requires that if the total amount of the pollution increases, but still remains for some firm, then the pollution allocation must be unchanged for those firms who are not responsible for the increase. *Firm equivalence* implies that if two firms are responsible for the exact same set of polluting processes (it means they are equivalent), then their share of the total pollution must be equal. *Firm nullity* states that if a firm is not responsible for any polluting process, then it's allocation must be zero. The property of *process history independence* requires that the change of responsibilities of the firms due to a change in pollution of any process is independent of the pollution levels of other processes. In the case of *disaggregation invariance* if for example a manufacturer decides to disaggregate and instead of selling directly, sells via a distributor who creates no additional pollution, it should not change the pollution allocation to the firms (so the property might avoids manipulation by de-merging while reporting emission).

The Shapley pollution allocation rule allocates the pollution of each process to all the firms held responsible for equally. The following theorem provides three axiomatizations for the Shapley pollution allocation rule, which according to Corollary 17 coincides with the Shapley value.

Theorem 20 (Gopalakrishnan et al., 2017). *The Shapley pollution allocation rule is uniquely characterized by each of the following sets of independent properties:*

1. *Equal sharing of extra pollution and no free riding.*

2. *Firm equivalence, firm nullity and process history independence.*

3. *Firm equivalence, no free riding and disaggregation invariance.*

The above defined properties and the axiomatization results are in some way similar to the cooperative game theoretic axioms and characterizations we introduced in the previous section. The similarity is formalized in the following proposition. But first we define one more axiom used in cooperative game theory, the fairness property (following the definition by van den Brink, 2001).

Definition 21. *A value ϕ on $A \subseteq \mathcal{G}^N$ satisfies the* fairness property *(FP), if for $v, w \in A$ and $i, j \in N$ such that $v + w \in A$ and $i \sim^w j$ it holds that $\phi_i(v + w) - \phi_i(v) = \phi_j(v + w) - \phi_j(v)$.*

Proposition 22. *Let (G, c) be a cost-tree, representing a supply chain, a single valued pollution allocation rule φ defined on the supply chain with n firms, a set of processes M and a responsibility matrix B, a value ψ be defined on $\mathcal{G}_{UR}$ as $\psi(v_{(G,c)}) = \varphi(f)$. Then φ satisfies the property of*

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- *firm nullity, if and only if ψ is NP,*

- *firm equivalence, if and only if ψ is ETP.*

Proof.
- Firm nullity $\Leftrightarrow NP$:

  $\Rightarrow$: We assume that for $i \in N$ $c'_i = 0$, that is for every $S \in N$ $c(S \cup \{i\}) = c(S)$. Therefore $\widetilde{\mathcal{P}}_i = \emptyset$. Firm nullity implies that $\varphi_i(f) = 0$, so $\psi_i(G, c) = 0$, therefore ψ is NP.

  $\Leftarrow$: We assume that ψ is NP, lets take an $i \in N$ with $\widetilde{\mathcal{P}}_i = \emptyset$, which means that $\sum_{j \in \mathcal{E}_i} f_j = 0$. This implies that $\sum_{j \in \mathcal{E}_{S \cup \{i\}}} f_j = \sum_{j \in \mathcal{E}_S} f_j$. Therefore $c(S \cup \{i\}) = c(S) \Leftrightarrow c'_i = 0$. Since ψ is NP, $\psi_i(G, c) = 0$, so $\varphi_i(f) = 0$, therefore φ satisfies the property of firm nullity.

- Firm equivalence $\Leftrightarrow ETP$:

  $\Rightarrow$: We assume that $i \sim^c j$, that is $c'_i(S) = c'_j(S)$, which means for all $S \setminus \{i, j\}$, $c(S \cup \{i\}) - c(S) = c(S \cup \{j\}) - c(S)$, that is, $c(S \cup \{i\}) = c(S \cup \{j\})$. Therefore $\sum_{k \in \mathcal{E}_{S \cup \{i\}}} f_k = \sum_{k \in \mathcal{E}_{S \cup \{j\}}} f_k \Rightarrow \widetilde{\mathcal{P}}_i = \widetilde{\mathcal{P}}_j$. So firm equivalence implies that $\varphi_i = \varphi_j$ which means $\psi_i(G, c) = \psi_j(G, c)$.

  $\Leftarrow$: Assume that ψ is ETP and there exists i and $j \in N$: $\widetilde{\mathcal{P}}_i = \widetilde{\mathcal{P}}_j$. This means that $\sum_{k \in \mathcal{E}_i} f_k = \sum_{k \in \mathcal{E}_j} f_k \Rightarrow \sum_{k \in \mathcal{E}_{S \cup \{i\}}} f_k = \sum_{k \in \mathcal{E}_{S \cup \{j\}}} f_k$ for any $S \setminus \{i, j\}$. Therefore $c(S \cup \{i\}) = c(S \cup \{j\})$, that is $i \sim^c j$. Since ψ is ETP, $\psi_i = \psi_j$, therefore $\varphi_i(f) = \varphi_j(f)$, that is φ satisfies the property of firm equivalence.

  $\square$

Remark 23. From Lemma 5 we know that for any upstream responsibility game $v \in \mathcal{G}^N_{UR}$ it holds that

$$v = \sum_{i \in N} f_i \bar{u}_{P_i(G)},$$

where f_i is the weight of the (only) edge from node i towards to the root, that is, f_i is player i's footprint, $P_i(G) = \{j \in V : j \leq i\}$, $V = N \cup \{root\}$.

Definition 24. *A value ψ meets axiom weak fairness with dual (WFD) on the class of game $A \subseteq \mathcal{G}^N$, if for all $v \in A$, $\alpha > 0$, $T \in \mathcal{P}(N) \setminus \{\emptyset\}$ such that $v + \alpha \bar{u}_T \in A$ it holds that for all $i, j \in T$*

$$\psi_i(v + \alpha \bar{u}_T) - \psi_i(v) = \psi_j(v + \alpha \bar{u}_T) - \psi_j(v).$$

Definition 25. *A value ψ meets axiom weak fairness with dual II (WFD2) on the class of game $A \subseteq \mathcal{G}^N$, if for all $v \in A$, $\alpha > 0$, $T \in \mathcal{P}(N) \setminus \{\emptyset\}$ such that $v + \alpha \bar{u}_T \in A$ it holds that for all $i, j \in T$ or $i, j \notin T$, that is, for all $i, j \in N$ such that $i \sim^{\bar{u}_T} j$*

$$\psi_i(v + \alpha \bar{u}_T) - \psi_i(v) = \psi_j(v + \alpha \bar{u}_T) - \psi_j(v).$$

Corollary 26. *WFD2 implies WFD.*

Theorem 27. *Let $A = \text{ConvexCone } \{\bar{u}_T\}_{T \in C \subseteq \mathcal{P}(N) \setminus \{\emptyset\}}$, then a value ψ meets FP if and only if it meets WFD2.*

Proof. It is clear that FP implies WFD2.

Let $v, w \in \mathcal{G}^N$, $i, j \in N$ be such that $v, v + w \in A$ and $i \sim^w j$. Then

$$w = \sum_{T \in C} \alpha_T \bar{u}_T$$

for some α_Ts.

Let $C_+ = \{T \in C \colon \alpha_T \geq 0\}$, moreover, let

$$w_+ = \sum_{T \in C_+} \alpha_T \bar{u}_T,$$

and

$$w_- = \sum_{T \in C \setminus C_+} \alpha_T \bar{u}_T.$$

Furthermore $i \sim^w j$ if and only if for each $T \in C$ either $i, j \in T$ or $i, j \notin T$ or $\alpha_T = 0$. Then it is clear that $i \sim^w j$ if and only if $i \sim^{w_+} j$ and $i \sim^{w_-} j$.

Then $v + w_+ \in A$, and by applying WFD2 for v and w_+ $|C_+|$ times we get that for all i and j such that $i \sim^{w_+} j$ it holds that

$$\psi_i(v + w_+) - \psi_i(v) = \psi_j(v + w_+) - \psi_j(v). \tag{2}$$

Then by applying WFD2 for $v + w$ and $-w_-$ $|C \setminus C_+|$ times we get that for all i and j such that $i \sim^{w_-} j$ it holds that

$$\psi_i(v + w_+) - \psi_i(v + w) = \psi_j(v + w_+) - \psi_j(v + w),$$

that is,

$$\psi_i(v + w) - \psi_i(v + w_+) = \psi_j(v + w) - \psi_j(v + w_+). \tag{3}$$

Adding up equations (2) and (3), we get that by applying WFD2 for v and $v + w$ $|C|$ times we get that for all i and j such that $i \sim^w j$ it holds that

$$\psi_i(v + w) - \psi_i(v) = \psi_j(v + w) - \psi_j(z).$$

$\square$

Corollary 28. *On the class of $\mathcal{G}^N_{(G,V)}$ FP is equivalent with WFD2, and FP implies WFD.*

Proposition 29. *A rule $\bar{\psi}$ on the class of GREEN problems with a fixed tree meets ESEP if and only if the corresponding value ψ on the class of upstream responsibility game meets WFD.*

Proof. For a weight system f let v_f be the corresponding TU game. Take f and f' as in the definition of ESEP, then

$$v_{f'} - v_f = v_{f'-f} = (f'_k - f_k)\bar{u}_{P_k}.$$

Moreover, $p, q \in P_k$ if and only if $b_{pk} = b_{qk} = 1$, that is, $p \sim^{\bar{u}_{P_k}} q$ if and only if $b_{pk} = b_{qk} = 1$.

Therefore, ESEP implies WFD, and WFD implies ESEP. $\square$

Proposition 30. *A rule $\bar{\psi}$ on the class of GREEN problems with a fixed tree meets NFR if and only if the corresponding value ψ on the class of upstream responsibility game meets M.*

Proof. Let $i \in N$, $f' \geq f$ such that $f'_j = f_j$, if $b_{ij} = 1$.

For all $j \in P_i$ it holds that $f'_j = f_j$.

Let $S \subseteq N \setminus \{i\}$, then

$$v_{f'}(S \cup \{i\}) - v_{f'}(S) = \sum_{j \in P_{S \cup \{i\}}} f'_j - \sum_{j \in P_S} f'_j = \sum_{j \in P_i \setminus P_S} f'_j,$$

and

$$v_f(S \cup \{i\}) - v_f(S) = \sum_{j \in P_{S \cup \{i\}}} f_j - \sum_{j \in P_S} f_j = \sum_{j \in P_i \setminus P_S} f_j.$$

Therefore, for all $S \subseteq N \setminus \{i\}$ it holds that

$$v_{f'}(S \cup \{i\}) - v_{f'}(S) = v_f(S \cup \{i\}) - v_f(S).$$

Therefore NFR is equivalent with M. $\square$

Definition 31. *A value ψ meets axiom equal change (EC) on the class of game $A \subseteq \mathcal{G}^N$, if for all $v, w \in A$, $\alpha > 0$, $T \in \mathcal{P}(N) \setminus \{\emptyset\}$ such that $v + \alpha \bar{u}_T, w + \alpha \bar{u}_T \in A$ it holds that for all $i \in N$*

$$\psi_i(v + \alpha \bar{u}_T) - \psi_i(v) = \psi_i(w + \alpha \bar{u}_T) - \psi_i(w).$$

Proposition 32. *A rule $\bar{\psi}$ on the class of GREEN problems with a fixed tree meets PHI if and only if the corresponding value ψ (which meets PO by definition) on the class of upstream responsibility game meets EC.*

Proof. By that for all $j \in N \setminus \{k\}$ it holds that $f'_j = f_j$ and $\tilde{f}'_j = \tilde{f}_j$ we have that for all $j \in N \setminus \{k\}$

$$\tilde{f}'_j - f'_j = \tilde{f}_j - f_j.$$

Moreover, by that $\tilde{f}'_k = f'_k$ and $\tilde{f}_k = f_k$ we have that

$$\tilde{f}'_k - \tilde{f}_k = f'_k - f_k.$$

Then

$$v_{\tilde{f}'} - v_{\tilde{f}} = (\tilde{f}'_k - \tilde{f}_k)\bar{u}_{P_k} = (f'_k - f_k)\bar{u}_{P_k} = v_{f'} - v_f.$$

Therefore PHI is equivalent with EC.

$\square$

Lemma 33. *If a value ψ meets axioms PO, M and WFD on the class of games $A \subseteq \mathcal{G}^N$, then it meets EC on A.*

Proof. By that ψ meets M if $i \notin T$, then

$$\varphi_i(v + \alpha \bar{u}_T) - \varphi_i(v) = 0 = \varphi_i(w + \alpha \bar{u}_T) - \varphi_i(w).$$

Moreover, if $i \in T$, then by that ψ meets PO and WFD it holds that

$$\varphi_i(v + \alpha \bar{u}_T) - \varphi_i(v) = \frac{\alpha}{|T|} = \varphi_i(w + \alpha \bar{u}_T) - \varphi_i(w).$$

$\square$

Lemma 34. *If a value ψ meets axioms NP and EC on the class of games $A = \text{ConvexCone } \{\bar{u}_T\}_{T \in C \subseteq \mathcal{P}(N) \setminus \{\emptyset\}}$, then it meets M on A.*

Proof. Notice that $0, \alpha \bar{u}_T \in A$. Moreover, by EC for all $i \in N$ it holds that

$$\varphi_i(v + \alpha \bar{u}_T) - \varphi_i(v) = \varphi_i(\alpha \bar{u}_T) - \varphi_i(0).$$

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Then by NP $\varphi_i(\alpha \bar{u}_T) = \varphi_i(0) = 0$, $i \notin T$, that is, for all $i \notin T$

$$\varphi_i(v + \alpha \bar{u}_T) = \varphi_i(v).$$

$\square$

Theorem 35. *Based upon the above results and Theorem 20 we get the following characterizations of the Shapley value for the class of upstream responsibility games (note that a pollution allocation rule φ is PO by definition):*

1. *The characterization of the Shapley pollution allocation rule by axioms (FE + FN + PHI) holds for the upstream responsibility problem if and only if the characterization of the Shapley value by axioms (PO + ETP + NP + EC) holds for the class of upstream responsibility games, therefore the characterization of the Shapley value by axioms (PO + ETP + M) holds for the class of upstream responsibility games.*

2. *The characterization of the Shapley pollution allocation rule by axioms (FE + NFR + DI) holds for the upstream responsibility problem if and only if the characterization of the Shapley value by axioms (PO + ETP + M) holds for the class of upstream responsibility games. This also means, that the axiom of DI is not needed, the axioms of FE and NFR together already characterize the Shapley value for this class of games.*

We get to the conclusion, that the 2. characterization of Theorem 20 implies the 3. characterizations, therefore FN + PHI $\Rightarrow$ NFR. The 2. characterization implies Young's axiomatization of the Shapley value, which is equivalent to the 3. characterization.

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