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An LP-based inconsistency monitoring of pairwise comparison matrices

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Abstract

A distance-based inconsistency indicator, defined by the third author for the consistency-driven pairwise comparisons method, is extended to the incomplete case. The corresponding optimization problem is transformed into an equivalent linear programming problem. The results can be applied in the process of filling in the matrix as the decision maker gets automatic feedback. As soon as a serious error occurs among the matrix elements, even due to a misprint, a significant increase in the inconsistency index is reported. The high inconsistency may be alarmed not only at the end of the process of filling in the matrix but also during the completion process. Numerical examples are also provided.

Keywords: linear programming, incomplete data, inconsistency analysis, pairwise comparisons.

1 Introduction

The use of pairwise comparisons (PC) is traced by some scholars to Ramon Llull (1232-1315). However, it is generally accepted that the modern use of PC took place in [12]. It is a natural approach for processing subjectivity although objective data can be also processed this way.

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The consistency-driven approach incorporates the reasonable assumption that by finding the most inconsistent assessments, one is able to reconsider his/her own opinions. This approach can be extended to incomplete data.

Mathematically, an $n \times n$ real matrix $\mathbf{A} = [a_{ij}]$ is a pairwise comparison (PC) matrix if $a_{ij} > 0$ and $a_{ij} = 1/a_{ji}$ for all $i, j = 1, \ldots, n$. Elements a_{ij} represent a result of (often subjectively) comparing the ith alternative (or stimuli) with the jth alternative according to a given criterion. A PC matrix $\mathbf{A}$ is consistent if $a_{ij}a_{jk} = a_{ik}$ for all $i, j, k = 1, \ldots, n$. It is easy to see that a PC matrix $\mathbf{A}$ is consistent if and only if there exists a positive n-vector w such that $a_{ij} = w_i/w_j, i, j = 1, \ldots, n$. For a consistent PC matrix $\mathbf{A}$, the values w_i serve as priorities or implicit weights of the importance of alternatives.

First, let us look at a simple example of a 3×3 reciprocal matrix:

$$\begin{pmatrix} 1 & a & b \\ 1/a & 1 & c \\ 1/b & 1/c & 1 \end{pmatrix}. \tag{1}$$

Koczkodaj defined the inconsistency index in [9] for (1) as

$$CM(a,b,c) = \min\left\{\frac{1}{a}\left|a - \frac{b}{c}\right|, \frac{1}{b}\left|b - ac\right|, \frac{1}{c}\left|c - \frac{b}{a}\right|\right\}. \tag{2}$$

Duszak and Koczkodaj [3] extended this definition (2) for a general $n \times n$ reciprocal matrix $\mathbf{A}$ as the maximum of $CM(a,b,c)$ for all triads (a,b,c), i.e., 3×3 submatrices which are themselves PC matrices, in $\mathbf{A}$:

$$CM(\mathbf{A}) = \max\{CM(a_{ij}, a_{ik}, a_{jk}) \mid 1 \le i < j < k \le n\}. \tag{3}$$

The concept of inconsistency index CM is due to the fact that the consistency of a PC matrix is defined for (all) triads. By definition, a PC matrix is inconsistent if and only if it has as least one inconsistent triad.

It is shown in [2] that CM and some other inconsistency indices are directly related to each other but only in case of 3×3 PC matrices. As the size of PC matrix gets larger than 3×3, this function-like relation between different inconsistency indices does not hold.

Incomplete PC matrices were defined by Harker [6, 7]. Harker's main justification of introducing incomplete PC matrices is the possible claim for reducing the large number of comparisons in case of an e.g., 9×9 matrix ([7], p.838.).

In the incomplete PC matrix below, the missing elements are denoted by $*$:

$$\mathbf{A} = \begin{pmatrix} 1 & a_{12} & * & \dots & a_{1n} \\ 1/a_{12} & 1 & a_{23} & \dots & * \\ * & 1/a_{23} & 1 & \dots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1/a_{1n} & * & 1/a_{3n} & \dots & 1 \end{pmatrix}.$$

We assume that all the main diagonal elements are given and equal to 1. Incomplete matrices were analyzed in [1, 4, 10, 11]. Most previous solutions are based on approaches for deriving weights from complete PC matrices. Weighting is not in the focus of the paper, however, incomplete PC matrices are also used in our approach.

In our model, the use of incomplete PC matrix is rather means than object. The aim of the paper is to provide an LP based monitoring system which is able to compute CM-inconsistency in each step of the filling in process as well as to inform the decision maker if s/he exceeds a given inconsistency threshold. The algorithm is constructive in the sense that it localizes the main root of CM-inconsistency. It is of fundamental importance to assume that the decision maker should be guided or supported but not led by making the comparisons for him/her in a mechanical way by proposing a reduction algorithm.

Decision support tools for controlling or predicting inconsistency during the filling in process have been provided by Wedley [14], Ishizaka and Lusti [8] and Temesi [13].

As the decision maker fills in a PC matrix, an incomplete PC matrix is resulted in by adding each element (except for the $n(n-1)/2$-th one, then the PC matrix becomes complete), whose CM-inconsistency, to be defined, plays an important role in our inconsistency monitoring system.

Let us assume that the number of the missing elements above the main diagonal in $\mathbf{A}$ is d, hence the total number of missing elements in $\mathbf{A}$ is $2d$. Let $(i_l, j_l), l = 1, \dots, d$, where $i_l < j_l$, denote the positions of the missing elements above the diagonal in $\mathbf{A}$.

Let us substitute variables $x_{i_l j_l}, l = 1, \dots, d$, for the missing values above the main diagonal in $\mathbf{A}$. Similarly, missing reciprocal values below the main diagonal are replaced by $1/x_{i_l j_l}, l = 1, \dots, d$. We denote the new matrix by $\mathbf{A}(x_{i_1 j_1}, \dots, x_{i_d j_d})$ to attack the following optimization problem:

$$\begin{array}{ll} \min & CM(\mathbf{A}(x_{i_1 j_1}, \dots, x_{i_d j_d})) \\ \text{s.t.} & x_{i_l j_l} > 0, \ l = 1, \dots, d, \end{array} \tag{4}$$

for replacing the missing values by positives values and their reciprocals so that $CM(\mathbf{A}(x_{i_1j_1},\ldots,x_{i_dj_d}))$ is minimal. It is easy to see the practical ramifications of this approach. There is no need to continue the filling-in process of the missing elements if the optimal value of (4) exceeds a pre-determined inconsistency threshold. Instead, one should concentrate on finding the sources of inconsistency among the already given elements in $\mathbf{A}$.

2 The LP form of the optimization problem

The nonlinear structure of PC matrices is due to the reciprocal property ($a_{ij} = 1/a_{ji}$). However, when the element-wise logarithm of a PC matrix is taken, some properties, e.g. the transitivity rule ($a_{ij}a_{jk} = a_{ik}$) becomes linear. Linearized forms also occur in weighting methods [5].

Let $L = \{(i,j) \mid 1 \leq i < j \leq n, a_{ij} \text{ is given}\}$ and $\bar{L} = \{(i_l, j_l) \mid l = 1,\ldots,d\}$ denote the index sets of the given and missing elements, respectively, above the main diagonal in $\mathbf{A}$. Then, for the sake of a unified notation, introducing variables x_{ij} for all $(i,j) \in L$ as well, and an auxiliary variable u, problem (4) can be written into the following equivalent form:

$$\begin{array}{ll} \min & u \\ \text{s.t.} & CM(x_{ij}, x_{ik}, x_{jk}) \leq u,\ 1 \leq i < j < k \leq n, \\ & x_{ij} = a_{ij},\ (i,j) \in L, \\ & x_{ij} > 0,\ (i,j) \in \bar{L}. \end{array} \tag{5}$$

It is clear that (5) has a feasible solution, variable u is nonnegative for any feasible solution, the optimal values of (4) and (5) coincide, furthermore, $x_{ij} > 0, (i,j) \in \bar{L}$, is an optimal solution of (4) if and only if it is a part of an optimal solution of (5).

The problem (5) is not easy to solve directly because of the non-convexity of $CM(\cdot)$ of (2) in the arguments. However, a useful property of different inconsistency indicators for 3×3 PC matrices was published in [2]. For the 3×3 PC matrix of (1), let us denote

$$T(a,b,c) = \max\left\{\frac{ac}{b}, \frac{b}{ac}\right\}. \tag{6}$$

As shown in [2],

$$CM(a,b,c) = 1 - \frac{1}{T(a,b,c)}, \quad T(a,b,c) = \frac{1}{1 - CM(a,b,c)}. \tag{7}$$

Since the univariate function $f(u) = 1/(1-u)$ is strictly increasing on $(-\infty, 1)$ and $T(a,b,c) = f(CM(a,b,c))$, problem (5) can be transcribed into the equivalent form

$$\begin{array}{ll} \min & f(u) \\ \text{s.t.} & T(x_{ij}, x_{ik}, x_{jk}) \le f(u),\ 1 \le i < j < k \le n, \\ & x_{ij} = a_{ij},\ (i,j) \in L, \\ & x_{ij} > 0,\ (i,j) \in \bar{L}. \end{array} \tag{8}$$

By substitution $t = f(u)$ and applying the definition (6), problem (8) can be written into the next equivalent form:

$$\begin{array}{ll} \min & t \\ \text{s.t.} & x_{ij}x_{jk}/x_{ik} \le t,\ 1 \le i < j < k \le n, \\ & x_{ik}/(x_{ij}x_{jk}) \le t,\ 1 \le i < j < k \le n, \\ & x_{ij} = a_{ij},\ (i,j) \in L, \\ & x_{ij} > 0,\ (i,j) \in \bar{L}. \end{array} \tag{9}$$

Since $t \ge 1$ for any feasible solution of (9), the variables x_{ij} are positive, and the function $\log t$ is strictly increasing over $t > 0$, we can use the old trick of the logarithmic mapping:

$$\begin{array}{l} z = \log t, \\ y_{ij} = \log x_{ij},\ 1 \le i < j \le n, \\ b_{ij} = \log a_{ij},\ (i,j) \in L. \end{array} \tag{10}$$

Then (9) can be written into the following equivalent form:

$$\begin{array}{ll} \min & z \\ \text{s.t.} & y_{ij} + y_{jk} - y_{ik} \le z,\ 1 \le i < j < k \le n, \\ & -y_{ij} - y_{jk} + y_{ik} \le z,\ 1 \le i < j < k \le n, \\ & y_{ij} = b_{ij},\ (i,j) \in L. \end{array} \tag{11}$$

The optimization problem (11) is a linear programming problem. It has a feasible solution with the non-negative objective function over the feasible region. Consequently, (11) has an optimal solution. Furthermore, problem (4) has also an optimal solution because of the chain of equivalent problems established up to this point. The following statements summarize the essence of the equivalent transcriptions applied above.

Proposition 1. Problems (4) and (11) have optimal solutions.

If $\bar{x}_{ij} > 0, (i,j) \in \bar{L}$, is an optimal solution and $\bar{u}$ is the optimal value of (4), then

$$\begin{array}{l} \bar{z} = \log \frac{1}{1-\bar{u}} = -\log(1-\bar{u}), \\ \bar{y}_{ij} = \log \bar{x}_{ij},\ \ (i,j) \in \bar{L}, \\ \bar{y}_{ij} = \log a_{ij},\ (i,j) \in L, \end{array} \tag{12}$$

is an optimal solution of (11).

Conversely, if $\bar{z}$, $\bar{y}_{ij}, 1 \leq i < j \leq n$, is an optimal solution of (11), then

$$\bar{x}_{ij} = e^{\bar{y}_{ij}}, \ (i,j) \in \bar{L}, \tag{13}$$

is an optimal solution and $\bar{u} = 1 - e^{-\bar{z}}$ is the optimal value of (4). □

The most inconsistent triad, which is not necessarily unique, is identified from the active constraints in (11). If $\bar{u}$ is greater than a pre-defined threshold of CM inconsistency, then the decision maker is recommended to reconsider the triad(s) associated with the active constraints. This case is involved in the third numerical example in Section 3.

3 Numerical examples

Three numerical examples are provided in this section. Let $\mathbf{A}$ be a 4×4 incomplete PC matrix as follows:

$$\mathbf{A} = \begin{pmatrix} 1 & * & 3.5 & 5 \\ * & 1 & 3 & 2.5 \\ 1/3.5 & 1/3 & 1 & * \\ 1/5 & 1/2.5 & * & 1 \end{pmatrix}$$

There are no complete triads in $\mathbf{A}$. The optimization problem (4) can be written for this example as:

$$\begin{array}{ll} \min & CM(\mathbf{A}(x_{13}, x_{24})) \\ \text{s.t.} & x_{13}, x_{24} > 0. \end{array} \tag{14}$$

Reformulate (14) in the same way as (11) is derived from (4), the LP has a unique solution. Apply (12)-(13), $CM^* = 0.236$ is resulted in as the optimal value of (14) and it is still less than the acceptable inconsistency threshold, assumed by Koczkodaj in [9] as $CM \leq 1/3$. Consequently, acceptable inconsistency can still be reached by a suitable filling-in of the missing elements of $\mathbf{A}$.

Let $\mathbf{B}$ be a 5×5 incomplete PC matrix as follows:

$$\mathbf{B} = \begin{pmatrix} 1 & * & 1.5 & 2 & * \\ * & 1 & 1/2 & * & 4 \\ 1/1.5 & 2 & 1 & * & * \\ 1/2 & * & * & 1 & 1/3 \\ * & 1/4 & * & 3 & 1 \end{pmatrix}$$

Now $CM^* = 0.62$. The above example with incomplete data demonstrates there is no way of completing it with data to bring the inconsistency below 1/3 hence no need to even collect the missing data. This may be a helpful way of eliminating data collection which may be sometimes time consuming hence expensive. It is also noted that matrix $\mathbf{B}$ does not seem to be so bad at first sight. Let C_1, C_2, C_3, C_4, C_5 denote the criteria whose importancess are presented in $\mathbf{B}$, and let $C_i \succ C_j$ denote that 'C_i is more important than C_j'. One can check that the pairwise comparisons of $\mathbf{B}$ reflect ordinally transitive relations: $C_1 \succ C_3 \succ C_2 \succ C_5 \succ C_4$. The roots of inconsistency are of cardinal nature rather than ordinal.

Let $\mathbf{D}$ be a 7×7 incomplete PC matrix filled in in a sequential order $d_{12}, d_{13}, \ldots, d_{23}, d_{24}, \ldots$. Assume that the decision maker intends to write $d_{45} = 4$ but s/he happens to mistype it by $d_{45} = \mathbf{1/4}$. The optimization problem(11) is solved after entering each matrix element and it contains $2 \times \binom{7}{3} = 70$ inequality constraints and 1-16 (= number of known entries in $\mathbf{D}$) equality constraints.

$$\mathbf{D} = \begin{pmatrix} 1 & 3 & 9 & 3/2 & 6 & 5 & 2 \\ 1/3 & 1 & 3 & 1/2 & 2 & 3/2 & 1/2 \\ 1/9 & 1/3 & 1 & \underline{1/6} & \underline{2/3} & 1/2 & 1/5 \\ 2/3 & 2 & \underline{6} & 1 & \underline{\mathbf{1/4}} & * & * \\ 1/6 & 1/2 & \underline{3/2} & \underline{\mathbf{4}} & 1 & * & * \\ 1/5 & 2/3 & 2 & * & * & 1 & * \\ 1/2 & 2 & 5 & * & * & * & 1 \end{pmatrix}$$

	d_{1i} $(i = 2, 3, \ldots, 7)$	d_{23}	d_{24}	d_{25}	d_{26}	d_{27}	d_{34}	d_{35}	d_{36}	d_{37}	d_{45}
CM^*	0	0	0	0	1/10	1/4	1/4	1/4	1/4	1/4	**15/16** (!)

Although the CM threshold of acceptability for 7×7 matrices has not been defined yet, 1/3 is assumed to be applicable again. As CM^* exceeds 1/3 (significantly) when $d_{45} = \mathbf{1/4}$ is entered, the inconsistency control identifies it as a possible mistype and asks the decision maker for verification. If the decision maker finds $d_{45} = \mathbf{1/4}$ appropriate and the most inconsistent triad in unique, then an error might occur before typing d_{45}. In our example the most inconsistent triad in not unique, there are three of them, one is underlined above. However, all three triads contains d_{45}. This fact suggests that the root of high inconsistency is in $d_{45} = \mathbf{1/4}$.

The LP optimization problem (11) has been implemented in Maple 13 by using command *LPSolve* on a personal computer with 3.4 GHz processor and 2 GB memory. CPU time, measured by command *time()* , remains under 0.05 seconds as the size of matrices varies between 3×3 and 9×9 and the number of missing elements varies between 1 and 28.

4 Conclusions and final remarks

In this study, we have demonstrated that the distance-based inconsistency can be used to handle incomplete PC matrices as a natural extension of the complete case. It is shown that the determination of the minimal inconsistency level of the possible extensions of an incomplete PC matrix is equivalent to a linear programming problem.

In real life, gathering complete data may be difficult or time-consuming. Incomplete PC matrices occur in each step of the filling in process even the PC matrix is completely given in the end. Thus the proposed approach is a useful tool for signalling the impossibility of completion the given incomplete matrix for an assumed inconsistency threshold as well as for improving the level of inconsistency.

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References

[1] Bozóki, S., Fülöp, J., Rónyai, L., On optimal completions of incomplete pairwise comparison matrices, Mathematical and Computer Modelling, 52: 318–333, 2010.

[2] Bozóki, S., Rapcsák, T., On Saaty's and Koczkodaj's inconsistencies of pairwise comparison matrices, Journal of Global Optimization, 42(2): 157–175, 2008.

[3] Duszak, Z., Koczkodaj, W.W., Generalization of a new definition of consistency for pairwise comparisons, Information Processing Letters, 52: 273–276, 1994.

[4] Fedrizzi, M., Giove, M., Incomplete pairwise comparison and consistency optimization, European Journal of Operational Research 183(1): 303–313, 2007.

[5] Golden, B.L., Wang, Q., An alternative measure of consistency, in: Golden, B.L., Wasil, E.A., Hacker, P.T.(eds.) Analytic Hierarchy Process: Applications and Studies, Springer-Verlag, 68-81, 1990.

[6] Harker, P.T., Alternative modes of questioning in the analytic hierarchy process, Mathematical Modelling, 9(3): 353–360, 1987.

[7] Harker, P.T., Incomplete pairwise comparisons in the analytic hierarchy process, Mathematical Modelling, 9(11): 837–848, 1987.

[8] Ishizaka, A., Lusti, M., An expert module to improve the consistency of AHP matrices, International Transactions in Operational Research 11: 97-105, 2004.

[9] Koczkodaj, W.W., A New Definition of Consistency of Pairwise Comparisons, Mathematical and Computer Modelling, 18(7): 79–84, 1993.

[10] Koczkodaj, W.W., Herman, M.W., Orlowski, M., Managing Null Entries in Pairwise Comparisons, Knowledge and Information Systems, (1)1: 119–125, 1999.

[11] Kwiesielewicz, M., van Uden, E., Ranking decision variants by subjective paired comparisons in cases with incomplete data, in Kumar et al. (Eds), Computational Science and Its Applications – ICCSA, Lecture Notes in Computer Science, Springer, 2669: 208–215, 2003.

[12] Saaty, T.L., A scaling method for priorities in hierarchical structures, Journal of Mathematical Psychology, 15: 234–281, 1977.

[13] Temesi, J., Pairwise comparison matrices and the error-free property of the decision maker, Central European Journal of Operations Research, in press, first published online: 16 April 2010, DOI: 10.1007/s10100-010-0145-8.

[14] Wedley, W.C., Consistency prediction for incomplete AHP matrices, Mathematical and Computer Modelling, 17(4-5): 151–161, 1993.