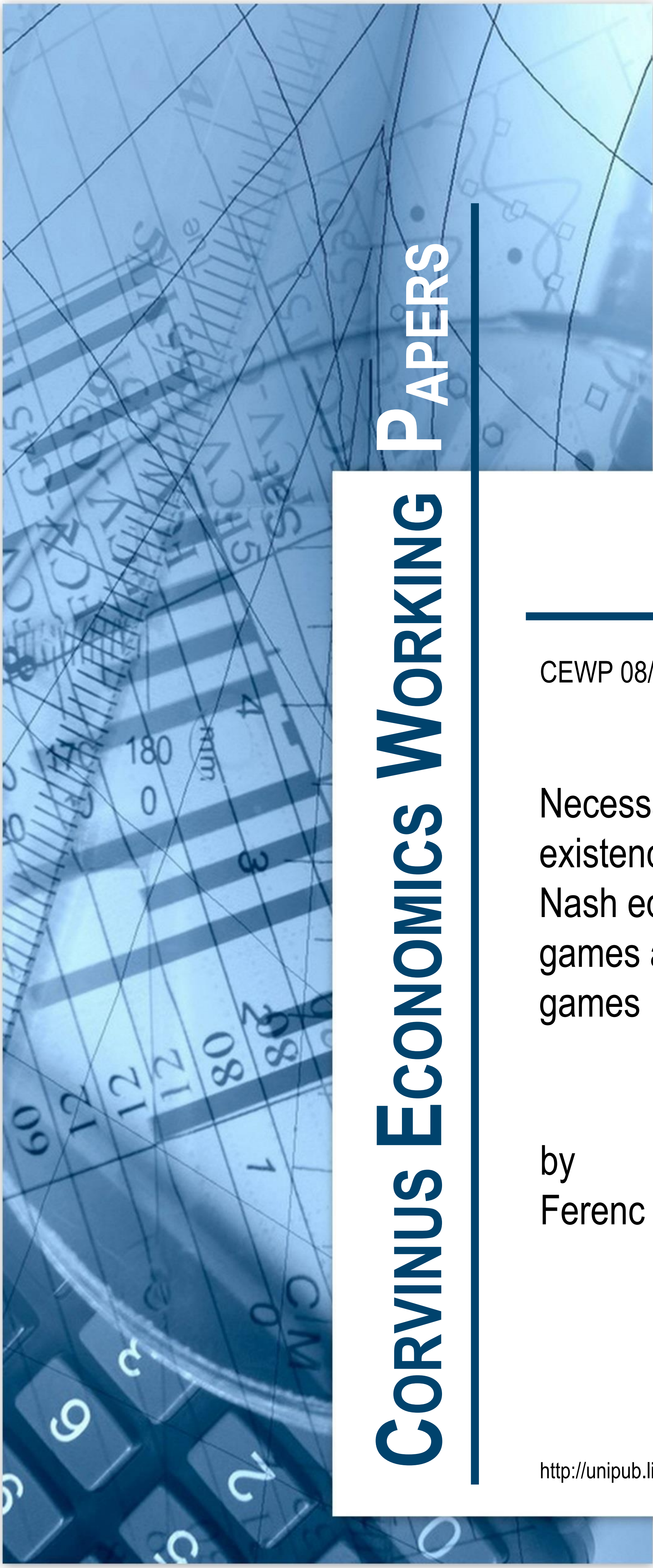

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Necessary conditions on the existence of pure Nash equilibrium in concave games and Cournot oligopoly games

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Necessary conditions on the existence of pure Nash equilibrium in concave games and Cournot oligopoly games

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Abstract

Necessary conditions for the existence of pure Nash equilibria introduced by Joó (A note on minimax theorems, Annales Univ. Sci. Budapest, **39** (1996) 175-179) for concave non-cooperative games are generalized and then applied to Cournot oligopoly games. If for a specified class of games there always exists a pure Nash equilibrium, then cost functions of the firms must be convex. Analogously, if for another specified class of games there always exists a pure Nash equilibrium, then revenue functions of the firms must be concave in their respective variables.

Keywords Nash equilibrium, Cournot oligopoly
JEL-code: L13

1 Introduction

Oligopoly is a market structure where a few competing firms are present and their individual decisions about production and/or selling price influence not only their own profit but everybody else's as well. Thus oligopoly lends itself to being modelled as a non-cooperative game where the players are the firms and payoffs are determined by profit functions usually defined as revenues less costs. Oligopolies have long been in the focus of economic research and practical market design. The ground breaking work of Cournot (1838) had laid down the foundations but intensive research only began when game theory became available to provide the necessary tools for deep analysis. Our focus will be on classical Cournot games where firms make decisions on the production volume of a single homogeneous product subject to capacity constraints.

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Among many other aspects the existence and uniqueness of equilibria of noncooperative games as defined by Nash (1950) has drawn much attention. Beyond direct application of game theoretic existence theorems many papers utilized the special features of an oligopoly game. Excellent reference books on the subject are e.g. Friedman (1977), Okuguchi and Szidarovszky (1990). In game theory much effort has been devoted to weakening conditions imposed on strategy sets/payoff functions to ensure the existence of a (pure) Nash equilibrium point. Staying in finite dimensional spaces, this endeavor is demonstrated by the series of papers marked by the milestone results of von Neumann (1928), Nash (1950), Nikaido and Isoda (1955), Friedman (1977).

These results of course translate to oligopoly games but sufficient conditions directly imposed on the primitives (demand, inverse demand and cost functions) are preferable since their interpretation is more direct and closely related to economic phenomena thus readily embraced by economists. It was realized early that there are limits to generalizations of revenue and/or cost functions if we do not want to lose the desirable property of the existence of a pure Nash equilibrium point. There are examples, a few of them analyzed in Novshek (1985) and quoted in this paper, too, for Cournot games without pure Nash equilibria. These are, however, only examples but not necessary conditions. Necessary conditions in relation to oligopoly games are quite rare.

In this paper we will study and apply to the Cournot game a special class of necessary conditions first formulated and proved by Joó (1986, 1996) for general concave games. The main message of our analysis, in loose terms, is that if for a special class of revenue functions there always exists a pure NEP, then the cost functions need to be convex in their respective variables. This can also be reversed: if for a special class of cost functions there always exists a pure Nash equilibrium point, then the revenue functions must be concave in their respective variables.

The paper is organized as follows. In section 2 we set forth a special class of necessary conditions applicable in mathematical programming and game theory. In section 3 we study and generalize necessary conditions for concave games due to Joó (1986, 1996). In section 4 sufficient conditions for the existence of pure Nash equilibria for the Cournot game with general cost functions are discussed. In section 5 the necessary conditions established for concave games are applied to generalized Cournot games. Section 5 concludes.

2 A special class of necessary conditions in mathematical programming and game theory

In mathematical programming the seminal papers of Karush (1939) and Kuhn and Tucker (1951) paved the way for the success of efficient solution methods of convex programming i.e. where the minimum of a convex function is sought over a convex set defined by convex constraints. Pretty soon the natural question was raised: How far can the convexity of the objective and constraint functions be relaxed while preserving most of the nice properties enabling us to apply the efficient methods of convex programming? This research question has led e.g. to replacing concave (convex) objective functions with quasi-concave (quasi-convex) functions (see e.g. Diewert et al (1981)). The conceptual appeal of these notions has made it indispensable in economic analysis and a standard subject of textbooks. On the algorithmic side, mild adjustments of solution methods originally designed for convex programming resulted in efficient methods of quasi-convex programming.

One might ask: Can quasi-convexity further be generalized while maintaining the advantages the availability of local search or other efficient methods provide? Where are the meaningful bounds for these generalizations? While sufficient conditions giving way to generalizations abound in the literature, necessary conditions are much harder to find. This is more so in game theory. Beginning with the ground breaking work of Nash (1950) sufficient conditions to ensure the existence of equilibrium have become less and less restrictive broadening the scope of application of the theory. Necessary conditions, however, are even harder to come by than in mathematical programming. A rare exception is the work of Kolstad and Mathiesen (1987) addressing the uniqueness of the Nash equilibrium point.

The general framework set forth in this paper for necessary conditions is inspired by Martos (1975) in mathematical programming and Joó (1986), (1996) in game theory. While Martos' results are well known those of Joó's have remained practically unnoticed. This is mainly due to the titles of the papers not giving any orientation about what they are really all about. Especially the title of Joó (1996) is misleading claiming the subject of the paper "minimax theorems" when in fact none of the theorems were about minimax.

Define a general mathematical programming problem $P(f, L)$ as

$$\max f(x)$$

subject to

$$x \in L \ ,$$

where $L \subset \mathbb{R}^n$ is the feasible set, $f : \mathbb{R}^n \to \mathbb{R}$ is the objective function.

Let T be a particular property of $P(f, L)$. Let furthermore $\mathcal{L}$ be a family of feasible sets and $\mathcal{F}$ a family of objective functions. The following two statements are said to be Martos-type necessary conditions:

(i) If property T holds for any $P(f, L), L \in \mathcal{L}$, then $f \in \mathcal{F}$.

(ii) If property T holds for any $P(f, L), f \in \mathcal{F}$, then $L \in \mathcal{L}$.

An example of a Martos-type (i) condition is the following.

Theorem 1 (Martos (1975)). *Let L' be a compact, convex subset of $\mathbb{R}^n$. If for any compact, convex set $L \subset L'$ problem $P(f, L)$ has the property that every local maximum point is also a global maximum point, then f is quasi-concave on L'.*

Here $\mathcal{L}$ is the family of all compact, convex subsets of L', $\mathcal{F}$ is the family of quasi-concave functions defined on L', and property T is all local maximum points being also global on a compact, convex set.

We will consider games in normal (strategic) form: $G = \{S_1, ..., S_n; f_1, ..., f_n\}$ or briefly $G = \{S; f\}$ where $S = S_1 \times ..., \times S_n$ is the set of strategy profiles and $f : S \to \mathbb{R}^n$ is the profile of payoff functions. Let T be a property of $G = \{S; f\}$. Let Σ be a family of strategy profiles and F a family of payoff profiles. The following two statements are said to be Joó-type necessary conditions:

(i) If property T holds for any $G = \{S; f\}, S \in \Sigma$, then $f \in F$.

(ii) If property T holds for any $G = \{S; f\}, f \in F$, then $S \in \Sigma$.

An example of Joó-type (i) necessary condition is due to Forgó and Joó (1997). The basic idea is to characterize the functions for which maxmin=minmax holds.

We need a few definitions in order to state the theorems.

Definition 1. $\varphi : \mathbb{R}^2 \to \mathbb{R}$ *is called a submaximum function if for any* $u, v \in \mathbb{R}$, $\varphi(u, v) \leq \max\{u, v\}$.

Let X and Y be compact, convex sets in $\mathbb{R}^n$ and $f : X \times Y \to \mathbb{R}$ a continuous function.

Definition 2. *Given a submaximum function* φ, f *is said to be* φ*-concave-like if for all* $\lambda > 0$ *and* $x_1, x_2 \in X$, *there exists* $x_3 \in X$ *such that*

$$y \in Y \Longrightarrow f(x_3, y) \geq \varphi(f(x_1, y), f(x_2, y)) - \lambda \ .$$

Theorem 2 (Forgó and Joó,1997)**.** *If*

(i) for any closed set $K \subset Y$,

$$\max_{x \in X} \min_{y \in K} f(x, y) = \min_{y \in K} \max_{x \in X} f(x, y) \ ,$$

(ii) for any closed set $K \subset Y$ *the set of maximizers*

$$\arg\max_{x \in X} \{\min_{y \in K} f(x, y)\}$$

is convex, then f *is* φ*-concave-like.*

Theorem 3 (Forgó and Joó,1997)**.** *If*

(i) $f(x, .)$ *is linear on* Y,

(ii) φ *is linear on* $\mathbb{R}^2$,

(iii) for any closed, convex set $C \subset Y$,

$$\max_{x \in X} \min_{y \in C} f(x, y) = \min_{y \in C} \max_{x \in X} f(x, y) \ ,$$

(iv) for any closed, convex set $C \subset Y$, the set of maximizers

$$\arg\max_{x\in X}\{\min_{y\in C} f(x,y)\}$$

is convex, then f is φ-concave-like.

The following theorems are also necessary conditions of a somewhat different nature. We will call a function f defined on a convex, compact set $C \subset \mathbb{R}^n$ partially concave if it is concave in each of its variables if the rest of the variables are held fixed.

Theorem 4 (Theorem 1 in Joó (1986))**.** *Let $f_k : [0,1]^n \to \mathbb{R}$ $(k = 1, ..., n)$ be continuous functions, and $f = f_1 \times ... \times f_n$. Let T be the following property: If $f'_k : [0,1]^n \to \mathbb{R}$ $(k = 1, ..., n)$ is continuous and partially concave in the k-th variable, then the game $G = \{[0,1]^n, f + f'\}$ has at least one Nash equilibrium point, where $f' = f'_1 \times ... \times f'_n$. If property T holds, then each function f_k $(k = 1, ..., n)$ is partially concave in its k-th variable.*

Theorem 4 was extended to games with convex, compact strategy sets.

Theorem 5 (Theorem 2 in Joó (1986))**.** *Let $K_1, ..., K_n$ be convex, compact subsets of finite dimensional euclidean spaces, $f_k : K_1 \times ... \times K_n \to \mathbb{R}$ $(k = 1, ..., n)$ be continuous functions and $f = f_1 \times ... \times f_n$. Let T be the following property: If $f'_k : K_1 \times ... \times K_n \to \mathbb{R}$ $(k = 1, ..., n)$ is continuous and partially concave in the k-th variable, then the game $G = \{K_1, ..., K_n; f + f'\}$ has at least one Nash equilibrium point, where $f' = f'_1 \times ... \times f'_n$. If property T holds, then each function f_k $(k = 1, ..., n)$ is partially concave in its k-th variable.*

3 Necessary conditions for concave games

One of the standard existence theorems in noncooperative game theory is due to Nikaido and Isoda (1955):

Theorem 6. *Let $G = \{S_1, ..., S_n; f_1, ..., f_n\}$ be a game in normal form. If*

(i) the strategy sets $S_1, ..., S_n$ are nonempty, compact, convex sets of finite dimensional euclidean spaces,

(ii) the payoff functions $f_k : \times_{j=1}^n S_j \to \mathbb{R}$ $(k = 1, ..., n)$ are continuous and partially concave in the k-th variable,

then G has at least one Nash equilibrium point.

Theorem 4 of Joó (1996) gives a necessary condition for the concavity of the payoff functions when the payoff function is subjected to concave perturbations. We give a generalization of Theorem 4 where the continuity of the payoff functions is relaxed to (partial) upper semicontinuity. Key to the generalization is a characterization of concave functions which we will give in the form of a lemma. We need two propositions to prove the lemma.

Proposition 1. *If the function $f : [a, b] \to \mathbb{R}$ is bounded from above, then the function*

$$\begin{aligned} \Phi &: \quad \mathbb{R} \to \mathbb{R} \\ \Phi(c) &: \quad = \sup_{t \in [a,b]} \Big(f(t) + c \cdot t \Big) \end{aligned}$$

is Lipschitz continuous.

Proof. For any $c, d \in \mathbb{R}$ and $x \in [a, b]$ we have

$$f(x) + d \cdot x \; leq f(x) + c \cdot x + (|a| + |b|) \cdot |c - d| \leq \sup_{t \in [a,b]} \Big(f(t) + c \cdot t \Big) + (|a| + |b|) \cdot |c - d| \; ,$$

or equivalently

$$\sup_{x \in [a,b]} \Big(f(x) + d \cdot x \Big) \leq \sup_{t \in [a,b]} \Big(f(t) + c \cdot t \Big) + (|a| + |b|) \cdot |c - d\| \; .$$

Using the definition of Φ and rearranging we obtain

$$\Phi(d) - \Phi(c) \leq \Big(|a| + |b| \Big) \cdot |d - c| \; .$$

Changing the role of c and d we get

$$|\Phi(d) - \Phi(c)| \leq (|a| + |b|) \cdot |d - c|$$

which was to be proved. □

Proposition 2. *Let $f : [a, b] \to \mathbb{R}$ be a function bounded from above and $a < x < b$. Then there exist $c, d \in \mathbb{R}$ such that*

$$\begin{aligned} \sup_{t \in [a,x]} \Big(f(t) + c \cdot t \Big) &\leq \sup_{t \in [x,b]} \Big(f(t) + c \cdot t \Big) \\ \sup_{t \in [a,x]} \Big(f(t) + d \cdot t \Big) &\geq \sup_{t \in [x,b]} \Big(f(t) + d \cdot t \Big) \; . \end{aligned}$$

Proof. Define

$$c := \max \left\{ 0, \; \frac{\sup\limits_{[a,x]} f \; - \; f(b)}{b \; - \; x} \right\} .$$

Then for every $t \in [a, x]$ we have

$$c \cdot (b - t) \geq c \cdot (b - x) \geq \sup_{[a,x]} f - f\,(b) \geq f\,(t) - f\,(b)\,,$$

implying

$$f\,(t) + c \cdot t \leq f\,(b) + c \cdot b,$$

or equivalently

$$\sup_{t\in[a,x]}\Big(f\left(t\right)+c\cdot t\Big)\leq f\left(b\right)+c\cdot b\ ,$$

from which we get the first assertion of the proposition. Define

$$d:=\min\left\{0,\frac{f\left(a\right)\ -\ \sup\limits_{[x,b]} f}{x\ -\ a}\right\}\ .$$

By similar reasoning as before we will arrive at

$$\sup_{t\in[x,b]}\Big(f\left(t\right)+d\cdot t\Big)\leq f\left(a\right)+d\cdot a\ ,$$

leading to the second assertion of the proposition. □

Lemma 1. *Let $f:[a,b]\to\mathbb{R}$ be an upper semicontinuous function. If for any $c\in\mathbb{R}$ the set*

$$\left\{x\in[a,b]:f(x)+c\cdot x=\max_{t\in[a,x]}\left(f(t)+c\cdot t\right)\right\}$$

is a closed interval, then f is concave.

Proof. We will show that at any point $a<x_0<b$ there is a line supporting f from above. Consider the function

$$\begin{aligned}\Psi\ &:\ \mathbb{R}\to\mathbb{R},\\ \Psi\left(c\right)\ &:=\ \max_{t\in[a,x_0]}\Big(f\left(t\right)+c\cdot t\Big)-\max_{t\in[x_0,b]}\Big(f\left(t\right)+c\cdot t\Big)\ .\end{aligned}$$

Ψ is continuous by Proposition 1, and by Proposition 2 there are numbers $c,d\in\mathbb{R}$ such that $\Psi(c)\leq 0\leq\Psi(d)$. Thus by Bolzano's theorem there is a number c_* for which $\Psi(c_*)=0$, i.e.

$$\max_{t\in[a,x_0]}\Big(f\left(t\right)+c_*\cdot t\Big)=\max_{t\in[x_0,b]}(f\left(t\right)+c_*\cdot t).$$

This common maximum is also the maximum of the function $t\to f\left(t\right)+c_*\cdot t$ on the interval $[a,b]$. Therefore, there are numbers $a\leq x_1\leq x_0\leq x_2\leq b$ such that

$$f\left(x_1\right)+c_*\cdot x_1=\max_{t\in[a,b]}\Big(f\left(t\right)+c_*\cdot t\Big)=f\left(x_2\right)+c_*\cdot x_2. \tag{1}$$

By the assumption, the level set H belonging to the maximum of the function $t\to f\left(t\right)+c_*\cdot t$ is a closed interval and by (1) $x_1,x_2\in H$. Thus by $x_1\leq x_0\leq x_2$ we have $x_0\in H$. Therefore for any $t\in[a,b]$,

$$f\left(x_0\right)+c_*\cdot x_0\geq f\left(t\right)+c_*\cdot t$$

holds. After rearrangement we get

$$f(t) \le f(x_0) - c_* \cdot (t - x_0) \ . \tag{2}$$

The expression on the right-hand side of (2) is a straight line which supports f from above at x_0. □

Similar characterization of quasi-convex (and thereby quasi-concave) functions based on the nature of the set of maximum points was given by Forgó (1996).

Theorem 7. *Let $X \subset \mathbb{R}^n$ be a non-empty convex set and $f : \mathbb{R}^n \to \mathbb{R}$ a continuous function. Then f is quasi-convex on X if and only if for any closed interval $I \subset X$ the set of minimum points of f over I is a closed interval.*

Proof. Assume that f is not quasi-convex. Then there are $x^1 \neq x^2 \in X$ and $x^0 \in [\ x^1, x^2]$ such that

$$f(x^0) > \max\{f(x^1), f(x^2)\} \ . \tag{3}$$

We may suppose that $f(x^1) \ \le f(x^2)$. Let

$$H^i = \{\lambda : 0 \le \lambda \le 1, f(\lambda x^0 + (1-\lambda)x^i) \le f(x^2)\}, i = 1, 2$$

and

$$\lambda^i = \max_{\lambda \in H^i} \lambda, i = 1, 2.$$

The sets H^i are non-empty, closed and bounded by (3) and the continuity of f, therefore λ^1, λ^2 are well defined and both are less than 1. Also, by the continuity of f we have

$$f(\lambda x^0 + (1-\lambda)x^i) = f(x^2), i = 1, 2.$$

Let

$$\begin{aligned} y^1 &= \lambda^1 x^0 + (1-\lambda^1)x^1 \ . \\ y^2 &= \lambda^1 x^0 + (1-\lambda^2)x^2 \ . \end{aligned}$$

By the definition of λ^1, λ^2 and since $y^1 \neq y^2$, the problem

$$\begin{aligned} &\min f(x) \\ x \ &\in \ [y^1, y^2] \end{aligned}$$

has exactly two optimal solutions y^1 and y^2, a contradiction. □

Interestingly, the continuity assumption cannot be relaxed to lower semicontinuity as the following example shows.

Example 1. *Let* $X = [-1, 1]$ *and* $f : \mathbb{R} \to \mathbb{R}$.

$$\begin{array}{rcll} f(x) & = & x+1, & \text{if} \quad -1 \le x < 0 \ , \\ f(x) & = & -1, & \text{if} \quad x = 0 \ , \\ f(x) & = & -x+1, & \text{if} \quad 0 < x \le 1. \end{array}$$

f is lower semicontinuous. To see this consider all possible lower level-sets $L(\beta) = \{x \in X : f(x) \le \beta\}$ for different values of β:

$$\begin{array}{cc} (i) & 1 \le \beta < 0 : L(\beta) = \{0\}, \\ (ii) & \beta = 0 : L(\beta) = \{-1, 0, 1\}, \\ (iii) & 0 < \beta < 1 : L(\beta) = [-1, -1+\beta] \cup \{0\} \cup [\beta, 1-\beta], \\ (iv) & \beta \ge 1 : L(\beta) = [-1, 1]. \end{array}$$

In all cases $L(\beta)$ is closed i.e. f is lower semicontinuous.

The set of optimum points of

$$\begin{array}{rcl} & & \min f(x) \\ x & \in & [y^1, y^2] \end{array}$$

is closed and convex for any intervals $[y^1, y^2] \subset X$. Indeed the list of all possible intervals and optimum sets L are the following

$$\begin{array}{cccl} (i) & y^1 & < & y^2 < 0 : L = \{y^1\}, \\ (ii) & y^1 & < & y^2 = 0 : L = \{0\}, \\ (iii) & 0 & = & y^1 < y^2 : L = \{0\}, \\ (iv) & 0 & < & y^1 < y^2 : L = \{y^2\}, \\ (v) & y^1 & < & 0 < y^2 : L = \{0\}. \end{array}$$

However, f is not quasi-convex since if $x^1 = -1, x^2 = 1, x^0 = \frac{1}{2}$, then

$$f\left(\frac{1}{2}\right) > \max\{f(-1), f(1)\}.$$

Now we turn to the main result of this section: the generalization of Theorem 4.

We will call a function f defined on a convex, compact set $C \subset \mathbb{R}^n$ partially upper semicontinuous if it is upper semicontinuous in each of its variables if the rest of the variables are held fixed and continuous in the rest of the variables.

Theorem 8. *Let $f_k : [0, 1]^n \to \mathbb{R}$ $(k = 1, ..., n)$ be partially upper semicontinuous functions, and $f = f_1, ..., f_n$. Let T be the following property: If $f'_k : [0, 1]^n \to \mathbb{R}$ $(k = 1, ..., n)$ is continuous and partially concave in the k-th variable, then the game $G = \{[0, 1]^n, f + f'\}$ has at least one Nash equilibrium point, where $f' = f'_1 \times ... \times f'_n$. If property T holds, then each function f_k $(k = 1, ..., n)$ is partially concave in its k-th variable.*

Proof. The proof goes along the lines of the proof of Theorem 1 in Joó (1996). We use proof by contradiction. Assume that the theorem is not true and there exists at least one k such that f_k is not partially concave, i.e. there is a $y_0 = (y_{0,1}, ..., y_{0,k-1}, y_{0,k+1}, ..., y_{0,n})$ such that $f_k(., y_0)$ is not concave in its kth variable over $[0,1]$. Without loss of generality take $k = 1$. Taking into account that $f_1(y_{0,1}, .) : [0,1]^{n-1} \to \mathbb{R}$ is continuous by assumption and being not concave is invariant under small perturbations, we may assume that

$$0 < y_{0,k} < 1, \qquad k = 2, ..., n.$$

Adding a linear term to $f_1(., y_0)$ does not alter concavity (or non-concavity for that matter) and so by Lemma 1 there exist real numbers a, b to satisfy

$$\begin{aligned} 0 &\leq a < b \leq 1, \\ f_1(a, y_0) &= f_1(b, y_0) = m, \\ f_1(u, y_0) &< m \text{ for } a < u < b, \end{aligned}$$

where $m = \max_{u \in [0,1]} f_1(u, y_0)$. The maximum exists by the partial semicontinuity of f_1.

Define the function $f_1' : [0,1]^n \to \mathbb{R}$

$$\begin{aligned} f_1'(x_1, y) &= -\alpha \|x_1 - a\| \|y_2 - y_{0,2}\|, \text{ if } y_2 \geq y_{0,2} \\ f_1'(x_1, y) &= -\alpha \|x_1 - b\| \|y_2 - y_{0,2}\|, \text{ if } y_2 \leq y_{0,2}. \end{aligned}$$

Let $g_1 = f_1 + f_1'$. It is easy to see that f_1' is continuous and partially concave in all of its variables.

Define the sets

$$M_1 = \{(x_1^*, y) : y \in [0,1]^{n-1}, g_1(x_1^*, y) = \max_{x_1 \in [0,1]} g_1(x_1, y)\},$$

$$L = \left[(0, ..., 0), (\frac{a+b}{2}, y_0)\right] \cup \left[(\frac{a+b}{2}, y_0), (1, ..., 1)\right].$$

In M_1 we have collected the maximum points of $g_1$with respect to the variable x_1 and L is composed of two straight lines through the points $(0, ..., 0), (\frac{a+b}{2}, y_0)$ and $(\frac{a+b}{2}, y_0), (1, ..., 1)$, respectively.

We insert here a lemma that is crucial in the proof of the theorem.

Lemma 2. *If α is a large enough positive number, then $M_1 \cap L = \emptyset$.*

Proof. Observe that M_1 is closed since $g_1$is partially upper semicontinuous, in particular it is upper semicontinuous in the first variable and continuous in the rest of the variables. By the definition of a and b the point $(\frac{a+b}{2}, y_0)$ does not belong to M_1.

We claim that there exists $\delta > 0$ such $(x_1, y) \in L$ and $\|y_2 - y_{0,2}\| \leq \delta$ implies $(x_1, y) \notin M_1$. Assume that this implication does not hold. Then for any $\delta > 0$ there exists $(x_1, y)(\delta) \in L$ and $\|y_2(\delta) - y_{0,2}\| \leq \delta$ such that $(x_1, y)(\delta) \in M_1$.

Since both L and M_1 are closed, there is a sequence $\{\delta_k\}$, $k = 1, 2, ..., \lim \delta_k = 0$, such that $\{(x_1, y)(\delta)\}$ converges to a point $(x_1^*, y^*) \in M_1 \cap L$. By the definition of L we have $y^* = y_0$. and thus $(x_1^*, y_0) \in L$ can only hold if $x_1^* = \frac{a+b}{2}$ which is impossible because $(\frac{a+b}{2}, y_0) \notin M_1$.

Take now a $\delta > 0$ such that there is no point $(x_1, y) \in L$ and $\|y_2 - y_{0,2}\| \leq \delta$ which belongs to M_1. Notice that δ does not depend on α. If $y_2 > y_{0,2} + \delta$, then by the definition of g_1 and f_1' we have $g_1(x_1, y) = f_1(x_1, y) - \alpha\|x_1 - a\|\|y_2 - y_{0,2}\|$ and for the points $(x_1, y) \in L$ we have $x_1 > \frac{a+b}{2}$ since along L all coordinates are monotone increasing. It is easy to see that for $x_1 > \frac{a+b}{2}$ we get $g_1(x_1, y) \leq f_1(x_1, y) - \alpha\frac{b-a}{2}\delta$. The function f_1 is upper semicontinuous in x_1 and is therefore bounded from above. Thus we can choose α so large that $g_1(x_1, y)$ cannot be maximal for any $x_1 > \frac{a+b}{2}$ i.e. for the points in L.

Likewise, if $y_2 < y_{0,2} - \delta$, then by the definition of g_1 and f_1' we have $g_1(x_1, y) = f_1(x_1, y) - \alpha\|x_1 - b\|\|y_2 - y_{0,2}\|$ and for the points $(x_1, y) \in L$ we have $x_1 < \frac{a+b}{2}$ and $g_1(x_1, y) \leq f_1(x_1, y) - \alpha\frac{b-a}{2}\delta$ with the same conclusion as above. □

Proceeding with the proof of the theorem construct the functions f_k' ($k = 2, ..., n$) in the following way. Parametrize L by its first coordinate x_1

$$L = \{(x_1, y_2(x_1), ..., y_2(x_1) : x_1 \in [0, 1]\}.$$

We do not need to know the functional form of the piecewise linear functions $y_k(x_1)$ ($k = 2, ..., n$). Let now $x_1 \in [0, 1], y \in [0, 1]^{n-1}$ and define the concave, continuous functions

$$f_k'(x_1, y) = -\alpha\|y_k - y_k(x_1)\|, \quad k = 2, ..., n.$$

Denote

$$(x_k, y) = (y_1, ..., y_{k-1}, x_k, y_{k+1}, ..., y_n), x_k \in [0, 1], y_j \in [0, 1] (j \neq k).$$

Let now $g_k = f_k + f_k'$ and

$$M_k = \{(x_k^*, y) : y \in [0, 1]^{n-1}, g_1(x_1^*, y) = \max_{x_1 \in [0,1]} g_1(x_1, y)\} .$$

By the construction of g_k we have for every $k = 2, ..., n$

$$\begin{aligned} g_k(y_k(x_1), y) &= f_k(y_k(x_1), y), \text{ if } x_k = y_k(x_1) \\ g_k(x_k, y) &\leq f_k(x_k, y)) - \alpha\delta, \text{ if } \|x_k - y_k(x_1)\| \geq \delta. \end{aligned}$$

Since f_k is bounded, for any $\delta > 0$ there exists a large enough α so that $g_k(x_k, y)$ can only be maximal in x_k if $\|x_k - y_k(x_1)\| < \delta$. This means that the points of $M_2 \cap ... \cap M_n$ are uniformly close to those of L. M_1 and $L = \emptyset$ are compact sets and by Lemma 2 $M_1 \cap L = \emptyset$. Therefore the points of M_1 cannot be arbitrarily close to those of L. Since $\delta > 0$ is arbitrary, we have

$$M_1 \cap M_2 \cap ... \cap M_n = \emptyset$$

which means that the game $G = \{[0,1]^n, g\}$ has no Nash equilibrium, contradicting the assumption that $G = \{[0,1]^n, f + f'\}$ has a Nash equilibrium for any continuous and concave function f'. □

4 Sufficient conditions for Cournot oligopoly games to have a pure Nash equilibrium

In Cournot oligopolies firms make decisions about volume of production of a homogeneous product. Production may have capacity bounds other than the natural lower bound 0. Selling price is determined by the production of the entire industry via an inverse demand function. Cost of production may vary from firm to firm. Gross profit is defined as revenue (volume times selling price) minus cost. This model gives rise to a game, called the Cournot game, defined by strategy sets $S_i = [a_i, b_i]$ for firm $i = 1, ..., n$ ($b_i = \infty$ is allowed for some or all i), payoff (profit) functions $f_i(q) = q_i P(Q) - C_i(q_i)$, where $P : \mathbb{R}_+ \to \mathbb{R}_+$ is the inverse demand function assigning to total industry output the highest price the market clears at, $C_i \to \mathbb{R}$ is the cost function assigning to the production q_i of firm i the total cost incurred at that level of production and $Q = \sum_{i=1}^{i=n} q_i$ the total industry output. So the Cournot game G in normal form is given as $G = \{S_1, ..., S_n; f_1, ..., f_n\}$.

It has long been a major line of research in economics in general and industrial organization in particular, to give ever weaker sufficient conditions imposed on the ingredients of the Cournot game that ensure the existence (uniqueness) of a pure Nash equilibrium.

In textbooks one usually starts with the linear case, when the inverse demand function and all cost functions are linear. For identical cost functions (the symmetric case) the Nash equilibrium can be computed, and thereby the existence of a pure Nash equilibrium point constructively proved, by elementary methods. This does not mean that the linear case poses no problems if we raise other questions than the computation of a Nash equilibrium point. Sometimes the linearity of the inverse demand and cost functions causes the problem. This is the situation if we want to have correlated equilibria a la Aumann (1974) other than the Nash equilibrium in the linear oligopoly game. Liu (1996) and Yi (1997) proved that the only correlated equilibrium for Cournot games is the unique Nash equilibrium. Ui (2008) extended this result to general concave games. For linear duopolies, even the coarse correlated equilibrium a la Moulin and Vial (1978) cannot give higher social welfare when it is defined as the sum of the payoffs of the players, Ray and Sen Gupta (2013).

Taking the linear case as starting point significant generalization can be achieved if we keep the assumption of linearity for one of the basic ingredients of the Cournot game and allow complete generality for the other ingredient. In particular, we have the following theorem.

Theorem 9. *Consider a symmetric Cournot oligopoly game where the firms choose positive outputs* $q_{1,...}, q_n$ *and the inverse demand function* $P : \mathbb{R}_{++} \to \mathbb{R}$

assigns the price $P(Q) > 0$ to the overall industry output. Every firm has the same linear cost function with marginal cost c. The Cournot game thus defined has at least one pure Nash equilibrium.

Proof. The game is an ordinal potential game. Consider the potential function

$$F : \mathbb{R}_{++} \to \mathbb{R}, F(q_{1,\dots}, q_n) = q_1 \cdots q_n (P(Q) - c).$$

It is easy to see that F is an ordinal potential function belonging to the Cournot game, (see Monderer and Shapley (1996)). Every minimum point of F is a pure Nash equilibrium. □

The condition that the game is symmetric is indispensable as the following example of Novshek (1985) shows.

Example 2. *There are two firms with linear cost functions. Marginal cost for firm 1 is $\frac{881}{800}$ and marginal cost for firm 2 is $\frac{381}{400}$. Inverse demand is*

$$P(Q) = \begin{cases} 2 - Q & Q \in [0, \frac{99}{100}] \\ \frac{8219}{8119} - \frac{19}{8119} Q & Q \in (\frac{99}{100}, \frac{100}{19}] \\ \frac{10019}{19} - 100Q & Q \in (\frac{100}{19}, \frac{1900}{19}] \\ 0 & Q \in (\frac{1900}{19}, \infty) \end{cases}.$$

Determine the two firms' best-reply correspondences B_1, B_2

$$B_1(y) = \begin{cases} \{\frac{719}{1600} - \frac{y}{2}\} & y \in [0, \frac{719}{800}] \\ \{0\} & y \in (\frac{719}{800}, \infty) \end{cases},$$

$$B_2(y) = \begin{cases} \{\frac{419}{600} - \frac{y}{2}\} & y \in [0. \frac{21}{400}) \\ \{\frac{398}{300}, \frac{100}{19} - \frac{21}{400}\} & y = \frac{21}{400} \\ \{\frac{100}{19} - y\} & y \in (\frac{21}{400}, \frac{3999639}{760000}] \\ \{\frac{8000722}{3040000} - \frac{y}{2}\} & y \in (\frac{3999639}{760000}, \frac{8000722}{1520000}] \\ \{0\} & y \in (\frac{8000722}{1520000}, \infty) \end{cases}.$$

Drawing their graphs it can be observed that they have no points in common i.e. there is no Nash equilibrium.

Theorem 10. *Consider a Cournot oligopoly game with a linear inverse demand function $P(Q) = a - bQ, a, b > 0$, and arbitrary cost functions $C_i(q_i)$, $(i = 1, ..., n)$, $q = (q_1, ..., q_n) \in \mathbb{R}^n_+$. The Cournot game thus defined has at least one pure Nash equilibrium.*

Proof. The game can easily be shown to be a potential game with potential function $F : \mathbb{R}^n_+ \to \mathbb{R}$ (see Monderer and Shapley (1996))

$$F(q) = a \sum_{j=1}^{j=n} q_j - b \sum_{j=1}^{j=n} q_j^2 - b \sum_{1 \le k < j \le n} q_k q_j - \sum_{j=1}^{j=n} C_j(q_j).$$

Again, every minimum point of F is a pure Nash equilibrium point. □

A landmark in departing from linearity is the theorem of Szidarovszky and Yakowitz (1977). Concavity/convexity and smoothness of the inverse demand and cost functions makes it possible to relax linearity.

Theorem 11 (Szidarovszky and Yakowitz, 1977)**.** *Given an industry with n firms, an inverse demand function P and cost functions $C_1, ..., C_n$, if*

(i) *$P : \mathbb{R}_+ \to \mathbb{R}_+$ is nonincreasing, twice continuously differentiable and concave where it has positive value, and*

(ii) *for all i $(i = 1, ..., n)$, $C_i : \mathbb{R}_+ \to \mathbb{R}_+$ is nondecreasing, twice continuously differentiable and convex, then the Cournot game has at least one Nash equilibrium.*

The following important result assumes about the cost functions nothing but being nondecreasing, a natural assumption, and lower semicontinuity, allowing for fix set-up costs.

Theorem 12 (Novshek, 1985)**.** *Given an industry with n firms, an inverse demand function P and cost functions $C_1, ..., C_n$, if*

(i) *$P : \mathbb{R}_+ \to \mathbb{R}_+$ is continuous,*

(ii) *there exists $Q' < \infty$ such that $P(Q') = 0$ and P is twice continuously differentiable and strictly decreasing on $[0, Q')$,*

(iii) *for all $Q \in [0, Q'), P'(Q) + QP"(Q) \leq 0$, and*

(iv) *for all $i = 1, ..., n$, $C_i : \mathbb{R}_+ \to \mathbb{R}_+$ is a nondecreasing, lower semicontinuous function,*

then there exists a Nash equilibrium for the Cournot game.

The economic interpretation of all but assumption (iii) are obvious. Given assumption (ii), assumption (iii) is equivalent to the assumption that for all nonnegative Y and y with $Y + y < Q', P'(Y + y) + yP"(Y + y) \leq 0$, so each firm's marginal revenue is decreasing in the aggregate output of the rest of the firms.

In the long line of contributions towards weakening the conditions under which there is a pure Nash equilibrium, the paper of Ewerhart (2014) stands out as one bringing most of them under the umbrella of the unifyining concept of biconcavity.

Consider a family of monotone transformations given by

$$\begin{aligned} \varphi_\alpha(x) &= \frac{x^\alpha}{\alpha} \text{ if } \alpha \neq 0 \\ \varphi_\alpha(x) &= \ln x \text{ if } \alpha = 0. \end{aligned}$$

An inverse demand function $P = P(Q)$ is called (α, β)-biconcave if P becomes concave (in the interval where inverse demand is positive) after transforming the price scale by φ_α, and simultaneously, the quantity scale by φ_β, where $\alpha, \beta \in \mathbb{R}$.

The following theorem is an existence result for pure Nash equilibria in Cournot games formulated in terms of biconcavity.

Theorem 13 (Ewerhart, 2014)**.** *Assume that the inverse demand function P is continuous, nonincreasing, nonconstant, and $(\alpha, 1-\alpha)$ biconcave for some $\alpha \in [0,1]$. Furthermore, the cost functions c_i are lower semicontinuous, and nondecreasing for $i = 1, ..., n$. Then the associated Cournot-game has at least one pure Nash equilibrium.*

Notice that condition (iii) in Theorem 11 corresponds to $(1,0)$ biconcavity and thus Theorem 13 is a generalization of Theorem 11.

5 Necessary conditions for Cournot oligopoly games to have a pure Nash equilibrium

In the efforts to get ever weaker sufficient conditions for the existence of pure Nash equilibria, after Novshek's and Ewerhart's it has become clear that in the conventional Cournot model there is not much room for generalizations especially as far as the cost function is concerned. In order to more clearly see the limitations of generalizations we will consider more general Cournot games. It turns out that if we allow more general revenue functions not just the conventional "quantity times price" form, then the existence of a pure Nash equilibrium necessitates the convexity of the cost function.

Let us redefine the Cournot oligopoly game $G = \{S_1, ..., S_n; f_1, ..., f_n\}$ where $S_i = [0,1], f_i(x) = R_i(x) - C_i(x), i = 1, ..., n$. Here $R_i, C_i : S = \times_{j=1}^{j=n} S_j \to \mathbb{R}$ are the (generalized) revenue and cost functions. Notice that in this set-up revenues and costs of each firm may depend on the industry's production profile. Revenue in the classical model does depend on the production profile of the industry, specifically on the firm's own level of production and the total industry production. In case of a generalized revenue function this is not necessarily so, other functional dependence of the revenue on the production profile of the industry is allowed. For cost functions, as opposed to the classical form, the cost of each firm may depend not only on its own production volume but on the production profile of the whole industry.

The general revenue function allows for getting different levels of revenue for two production profiles with the same total production. Indeed, an evenly distributed production profile gives less chance for the firm to get extra leverage by utilizing its position marked by a dominant market share. Also, a general revenue function can take into account other market forces than price (discounts, all sorts of promotions, etc.). By not assuming anything a priori about the monotonicity and the shape of the inverse demand function, unusual markets, such as markets of Giffen and Veblen goods (see Varian (1992)) can be studied in the same model.

Costs can also depend on the whole production profile. Overuse of natural resources may incur costs that increase much faster as industry output increases compared to the situation when only an individual firm uses more of the resource. Even monotony can be violated in special cases. In some countries zero-level production in agriculture is rewarded by subsidies which disappear as production moves away from zero. This is also an example of the presence of discontinuities as well.

The following theorem emphasizes the importance of convexity of the cost functions if we want to ensure the existence of a pure Nash equilibrium point.

Theorem 14. *Let all the cost functions C_i of a generalized Cournot game be partially lower semicontinuous. If the generalized Cournot oligopoly game $G = \{S_1, ..., S_n, R_1 - C_1, ..., R_n - C_n\}$ has a pure Nash equilibrium point for any partially concave continuous revenue function R_i $i = 1, ..., n$, then all C_i $i = 1, ..., n$ are partially convex.*

Proof. By Theorem 8 $-C_i$ is partially concave implying that C_i is partially convex for all $i = 1, ..., n$. □

The following question comes naturally to mind: If we only consider Cournot games (not generalized!) which means that we only require the existence of a pure Nash equilibrium point for a special class of revenue functions, what can be said about the cost function? Surely less than convexity. Maybe quasi-convexity?

The role of the revenue and cost functions can be reversed in a natural way. We then obtain the following necessary condition.

Theorem 15. *Let all the revenue functions R_i of a generalized Cournot game be partially upper semicontinuous. If the generalized Cournot oligopoly game $G = \{S_1, ..., S_n, R_1 - C_1, ..., R_n - C_n\}$ has a pure Nash equilibrium for any partially convex continuous cost function C_i $i = 1, ..., n$, then all R_i $i = 1, ..., n$ are partially concave.*

Theorems similar to Theorem 14 and 15 can be stated for multiproduct oligopolies as defined in Forgo et al. (1999) page 67-72. In this case Theorem 5 has to be invoked in order to arrive at the same results.

6 Conclusion

Necessary conditions for the existence of pure Nash equilibria were derived for generalized Cournot oligopoly games. If for all revenue functions there exists at least one pure Nash equilibrium point for a fixed partially lower semicontinuous cost function, then the cost function must be convex. The question of how to characterize cost functions within the framework of the classical Cournot game where revenues are calculated as the product of volume and price determined by the total production of the industry through an appropriately conditioned inverse demand function remains open.

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