On the completeness of the universal knowledge–belief space *

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Abstract

Meier (2008) shows that the universal knowledge-belief space exists. However, besides the universality there is an other important property might be imposed on knowledge-belief spaces, inherited also from type spaces, the completeness. In this paper we introduce the notion of complete knowledge-belief space, and demonstrate that the universal knowledge-belief space is not complete, that is, some subjective beliefs (probability measures) on the universal knowledge-belief space are not knowledge-belief types.

Keywords and phrases: Belief; Knowledge; Uncertainty; Complete type space; Complete knowledge space; Universal type space; Universal knowledge space; Games of incomplete information.

JEL codes: C70; C72; D80; D82; D83

1 Introduction

For modeling incomplete information situations Harsányi (1967-68) suggests the use of types instead of hierarchies of beliefs. Summing up his concept very roughly, it is the goal to substitute the belief hierarchies by types, to collect all types in an object, and consider the probability measures on this object as the players' (subjective) beliefs.

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In the case of beliefs Harsányi's program above is completed, Heifetz and Samet (1998b) demonstrates the existence of the (purely measurable) universal type space (the object into which all types are collected), Meier (2001) shows that the universal type space is complete (every probability measure on that is a type), and last Pintér (2008) proves that the complete universal type space contains all hierarchies of beliefs.

It is possible to extend type spaces in such a way that those represent not only the beliefs, but the knowledges of the players as well. This extension is called knowledge-belief space.

Meier (2008) introduces the notion of knowledge-beliefs space, and he succeeds in avoiding the traps of the earlier negative results (Heifetz and Samet (1998a) and Brandenburger and Keisler (2006) among others), and shows that the universal knowledge-belief space exists and, naturally, is unique. This important result has been reached by a new, strict, but reasonable definition of knowledge operator (for the details consult with Meier (2008)'s paper).

It is a natural step forward examining the completeness of the universal knowledge-belief space. For type spaces Brandenburger (2003) introduces the notion of completeness, a type space is complete, if for any subjective belief (probability measure) expressible in the model, there is a type which "means" the given subjective belief (see e.g. Pintér (2008)). The completeness is important because it ensures that we can take any probability measure (subjective belief), we need not choose of the possible subjective beliefs, which was Harsányi's original goal too.

In this paper we consider Meier (2008)'s universal knowledge-belief space and conclude that is not complete, that is, there are some probability measures (subjective beliefs) on it which cannot be represented by knowledge-belief types.

Our result can be summarized as follows. In the universal knowledge-belief space there is an event (measurable set) which represents $k_j(k_i(\varphi))$, that is, player j knows that player i knows that event φ happens, and there is an other which represents $k_j(k_i(\neg\varphi))$, that is, player j knows that player i knows that event φ does not happen. However, a probability measure that evaluates these two (disjoint) events equally cannot be in the knowledge-belief space, since $b_i^{\frac{1}{2}}(k_j(k_i(\varphi)))$, that is, player i believes with at least probability of a half that player j knows that player i knows that event φ happens, and $b_i^{\frac{1}{2}}(k_j(k_i(\neg\varphi)))$, that is, player i believes with at least probability of a half that player j knows that player i knows that event φ does not happen, are not consistent knowledge-belief expressions, therefore, they cannot be in any state of the world in any knowledge-belief space.

One can see our result in two ways (at least). First, one can interpret our result as the notion of completeness we recommend for the universal knowledge-belief space is futile, not reasonable, since it goes against the very structure of knowledge-belief spaces. However, the notion of completeness we apply to knowledge-belief spaces is not ours, this a simple variant of Brandenburger (2003)'s concept. Therefore, since Brandenburger (2003)'s concept is widely accepted in the literature, perhaps it is not too unreasonable to check whether this property holds for the universal knowledge-belief space.

On the other hand, one can say that Meier (2008)'s model is futile, not reasonable, since it does not meet such a basic property as completeness. However, it is clear that Meier (2008)'s model has some nice properties (e.g. that there is a universal knowledge-belief space, while e.g. there is no universal topological type space (Pintér, 2010)), so perhaps it is not too unreasonable to use Meier (2008)'s model either.

We do not want to take sides on this matter, we just remark the fact that the universal knowledge-belief space is not complete.

In order to save space we do not introduce every notion we use in this paper, so the reader is kindly asked to consult with Meier (2008)'s paper on the notions not defined in this paper. Needless to say, we give exact reference for any of the used but not defined notion.

Practically, the paper consists of one section, in which we introduce the basic notions and notations and give our non-completeness result.

2 The universal knowledge-belief space is not complete

Throughout the paper, if it is not indicated differently, we use Meier (2008)'s terminologies and notations. Therefore we use Convention 1., Remarks 1., 2., 5., Definitions 1., 6., 7., 8., 10. from his paper. Moreover we assume that the parameter space S is not trivial, that is, the σ-field of S, Σ_S, has cardinality more than two, and that there are at least two players.

First we take Definition 2. from Meier (2008)'s paper.

Definition 1. *A knowledge–belief space (kb-space) on S for player set I is a 5-tuple $\underline{M} := \langle M, \Sigma, (K_i)_{i\in I}, (T_i)_{i\in I}, \theta\rangle$, where*

1. *M is a non-empty set,*
2. *Σ is a σ-field on M,*

3. $K_i : \Sigma \to \Sigma$ *is a knowledge operator*[1] *on* (M, Σ), *for player* $i \in I$,

4. *for each* $i \in I$*:* T_i *is a* Σ*-measurable function from* M *to* $(\Delta(M, \Sigma), \Sigma_\Delta)$, *the space of probability measures on* (M, Σ),

5. *for each* $m \in M$ *and* $E \in \Sigma$*:* $m \in K_i(E)$ *implies* $T_i(m)(E) = 1$, $i \in I$,

6. *for each* $m \in M$ *and* $E \in \Sigma$*:* $[T_i(m)] \subseteq E$ *implies* $m \in K_i(E)$, *where* $[T_i(m)] := \{m' \in M \mid T_i(m') = T_i(m)\}$, $i \in I$,

7. θ *is a* Σ*-measurable function from* M *to* (S, Σ_S), *the parameter space.*

If we put condition "for each $m \in M$ and $E \in \Sigma$: $[T_i(m)] \subseteq E$ implies $T_i(m)(E) = 1$, $i \in I$" in, and drop Points 3., 5. and 6. out of Definition 1. then we get the concept of type space (Heifetz and Samet, 1998b; Pintér, 2008), that is, a model where only the beliefs of the players are represented, their knowledges are not.

Definition 2. *Knowledge-belief morphism (kb-morphism)* $f : M \to M'$ *between the kb-spaces* $\langle M, \Sigma, (K_i)_{i\in I}, (T_i)_{i\in I}, \theta\rangle$ *and* $\langle M', \Sigma', (K'_i)_{i\in I}, (T'_i)_{i\in I}, \theta'\rangle$ *is a mapping such that*

1. f *is* Σ*-measurable,*

2. *Diagram* (1) *is commutative, that is, for all* $m \in M$*:* $\theta' \circ f(m) = \theta(m)$,

$$\begin{tikzcd} M \arrow[d, "f"'] \arrow[dr, "\theta"] & \\ M' \arrow[r, "\theta'"] & S \end{tikzcd} \tag{1}$$

3. *for all* $i \in N$ *Diagram* (2) *is commutative, that is, for all* $A \in \Sigma'$*:* $K_i \circ f^{-1}(A) = f^{-1} \circ K'_i(A)$,

$$\begin{tikzcd} \Sigma' \arrow[r, "K'_i"] \arrow[d, "f^{-1}"'] & \Sigma' \arrow[d, "f^{-1}"'] \\ \Sigma \arrow[r, "K_i"] & \Sigma \end{tikzcd} \tag{2}$$

[1]The properties of the knowledge operator is listed in Definition 1. in Meier (2008).

4. for each $i \in N$ Diagram (3) *is commutative, that is, for all $m \in M$: $T'_i \circ f(m) = \Delta_f \circ T_i(m)$,*

$$\begin{tikzcd} M \arrow[r, "T_i"] \arrow[d, "f"'] & \Delta(M,\Sigma) \arrow[d, "\Delta_f"] \\ M' \arrow[r, "T'_i"] & \Delta(M',\Sigma') \end{tikzcd} \tag{3}$$

where $\Delta_f : \Delta(M,\Sigma) \to \Delta(M',\Sigma')$ is defined as for each $\mu \in \Delta(M,\Sigma)$ and $A \in \Sigma'$: $\Delta_f(\mu)(A) = \mu(f^{-1}(A))$, $i \in N$.

f kb-morphism is a kb-isomorphism, if f is a bijection and f^{-1} is also a kb-morphism.

It is easy to verify that[2] the kb-spaces as objects and the kb-morphisms as morphisms form a category (the parameter space (S, Σ_S) is fixed).

Definition 3. *The kb-space $\underline{\Omega}$ is universal, if for any kb-space $\underline{M}$ there is a unique kb-morphism f from $\underline{M}$ to $\underline{\Omega}$.*

In the language of category theory the universal kb-space is a terminal (final) object in the category of kb-spaces. Since every terminal object is unique up to isomorphism, hence the universal kb-space is also unique up to kb-isomorphism.

Theorem 4. *The universal kb-space exists.*

Proof. See Theorem 1. in Meier (2008). □

In other words, Theorem 4 shows that there is a terminal object in the category of kb-spaces.

Definition 5. *The kb-space $\underline{M}$ is complete, if for each $i \in I$, $\mu \in \Delta(M, \Sigma_{-i})$ there exists $m \in M$ such that $\mu = T_i(m)|_{(M,\Sigma_{-i})}$, where $\Sigma_{-i} := \sigma(\theta^{-1}(\Sigma_S) \cup \bigcup\limits_{j \in I \setminus \{i\}} K_j(\Sigma) \cup \bigcup\limits_{j \in I \setminus \{i\}} T_j^{-1}(\Sigma_\Delta))$[3].*

The above notion of completeness is the natural variant of that Brandenburger (2003) applies to type spaces.

[2] The reason for that we use commutative diagrams in Definition 2. is to suggest that category theory is the right framework for analyzing kb-spaces.

[3] $\sigma(\cdot)$ is for the smallest σ-field that contains the given set system.

Intuitively, a kb-space is complete, if for any player any probability measure on the descriptions of the other players and the nature represents a kb-type of the given player. Therefore, if a kb-space is complete, then it is enough to take the probability measures on it, they give correct and full descriptions of the beliefs in the kb-types in the model.

As we have already mentioned in the introduction, the (purely measurable) universal type space is complete (Meier, 2001), hence it is very desirable that the universal kb-space be also complete. Unfortunately, it does not happen.

Theorem 6. *The universal kb-space $\langle \Omega, \Sigma_\Omega, (K_i^*)_{i\in I}, (T_i^*)_{i\in I}, \theta^* \rangle$ is not complete.*

Proof. Let $E \in \Sigma_S$ be such that $E, \complement E \neq \emptyset$, and $\varphi \in \Phi$ be the kb-expression that $[\varphi] = {\theta^*}^{-1}(E)$[4], moreover consider players $i, j \in I$, $i \neq j$.

Take the following kb-expressions: $k_j(k_i(\varphi))$ and $k_j(k_i(\neg\varphi))$. Then

$$k_j(k_i(\varphi)) \Rightarrow k_i(\varphi) \Rightarrow \varphi$$

and

$$k_j(k_i(\neg\varphi)) \Rightarrow k_i(\neg\varphi) \Rightarrow \neg\varphi \ ,$$

where $\Rightarrow$ is for the implication. Therefore, $[k_j(k_i(\varphi))], [k_j(k_i(\neg\varphi))] \in K_j(\Sigma_\Omega) \subseteq \Sigma_{-i}$, and $[k_j(k_i(\varphi))] \cap [k_j(k_i(\neg\varphi))] = \emptyset$.

Furthermore, $\underline{\Omega}$ is universal kb-space, therefore $k_i(\varphi) \Rightarrow k_i(k_i(\varphi))$ (positive introspection) and that $k_i(\varphi) \Rightarrow \neg k_j(k_i(\neg\varphi))$ imply that for all $\omega \in \Omega$:

$$\begin{aligned} k_i(\varphi) \in \omega \Rightarrow k_i(\neg k_j(k_i(\neg\varphi))) \in \omega \Rightarrow b_i^1(\neg k_j(k_i(\neg\varphi))) \in \omega \\ \Rightarrow \forall p \in (0,1] : \ \neg b_i^p(k_j(k_i(\neg\varphi))) \in \omega \ . \end{aligned}$$

Analogously, $\neg k_i(\varphi) \Rightarrow k_i(\neg k_i(\varphi))$ (negative introspection) and that $\neg k_i(\varphi) \Rightarrow \neg k_j(k_i(\varphi))$ imply that for all $\omega \in \Omega$:

$$\begin{aligned} \neg k_i(\varphi) \in \omega \Rightarrow k_i(\neg k_j(k_i(\varphi))) \in \omega \Rightarrow b_i^1(\neg k_j(k_i(\varphi))) \in \omega \\ \Rightarrow \forall p \in (0,1] : \ \neg b_i^p(k_j(k_i(\varphi))) \in \omega \ . \end{aligned}$$

Then we get at that for all $\omega \in \Omega$ ($[k_i(\varphi)] \cup [\neg k_i(\varphi)] = \Omega$), $p \in (0,1)$:

$$b_i^p(k_j(k_i(\varphi))) \wedge b_i^{1-p}(k_j(k_i(\neg\varphi))) \notin \omega \ .$$

[4]See Definition 10. in Meier (2008).

Therefore, if $\mu \in \Delta(\Omega, \Sigma_{-i})$ such that $(p = \frac{1}{2})$

$$\mu([k_j(k_i(\varphi))]) = \mu([k_j(k_i(\neg\varphi)))]) = \frac{1}{2} \ ,$$

then $\nexists \omega \in \Omega$: $\mu = T_i(\omega)|_{(M,\Sigma_{-i})}$, that is, the universal kb-space is not complete. $\square$

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