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Predictor-corrector interior-point algorithm for sufficient linear complementarity problems based on a new type of algebraic equivalent transformation technique

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Abstract

We propose a new predictor-corrector (PC) interior-point algorithm (IPA) for solving *linear complementarity problem* (LCP) with $P_*(\kappa)$-matrices. The introduced IPA uses a new type of *algebraic equivalent transformation* (AET) on the centering equations of the system defining the central path. The new technique was introduced by Darvay et al. [21] for linear optimization. The search direction discussed in this paper can be derived from positive-asymptotic kernel function using the function $\varphi(t) = t^2$ in the new type of AET. We prove that the IPA has $O\left((1+4\kappa)\sqrt{n}\log\frac{3n\mu^0}{4\epsilon}\right)$ iteration complexity, where κ is an upper bound of the handicap of the input matrix. To the best of our knowledge, this is the first PC IPA for $P_*(\kappa)$-LCPs which is based on this search direction.

Keywords: Predictor-corrector interior-point algorithm, $P_*(\kappa)$-linear complementarity problem, new search direction, polynomial iteration complexity
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1. Introduction

The linear complementarity problem (LCP) is a well-known problem which includes linear programming (LP) and linearly constrained (convex) quadratic programming problem (QP), as special cases. Many classical applications of LCPs can be found in different fields, such as optimization theory, game theory, economics, engineering, etc. [25, 8]. For example, bimatrix games can be transformed into LCPs under specific assumptions [40]. Kojima and Saigal [38] used the degree theory in order to study LCPs. Furthermore, the Arrow-Debreu competitive market equilibrium problem with linear and Leontief utility functions can be also given as LCP [68].

More recent work of Brás et al. [6] connected the copositivity testing of matrices and solvability of special LCPs. Darvay et al. [18] published a PC IPA for sufficient LCPs using the function $\bar{\varphi}(t) = t - \sqrt{t}$ for AET, but tested numerically their algorithm beyond the class of sufficient matrices, too. Numerical results produced by the developed PC IPA for testing copositivity of matrices using LCPs were very promising.

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Sloan and Sloan [56] showed that solvability of LCPs related to quitting games ensures the existence of different ε-equilibrium solutions. There is no reported computational study on this type of application of LCPs, yet.

In the LCP we want to find vectors $\mathbf{x}, \mathbf{s} \in \mathbb{R}^n$, that satisfy the constraints

$$-M\mathbf{x} + \mathbf{s} = \mathbf{q},\ \mathbf{xs} = \mathbf{0},\ \mathbf{x}, \mathbf{s} \geq \mathbf{0}, \tag{LCP}$$

where $M \in \mathbb{R}^{n\times n}$ and $\mathbf{q} \in \mathbb{R}^n$. The following notations are used to denote the feasible region, the interior and the solutions set of LCP:

$$\begin{aligned} \mathcal{F} &:= \{(\mathbf{x}, \mathbf{s}) \in \mathbb{R}^n_\oplus \times \mathbb{R}^n_\oplus : -M\mathbf{x} + \mathbf{s} = \mathbf{q}\}, \\ \mathcal{F}^+ &:= \{(\mathbf{x}, \mathbf{s}) \in \mathbb{R}^n_+ \times \mathbb{R}^n_+ : -M\mathbf{x} + \mathbf{s} = \mathbf{q}\}, \text{ and } \mathcal{F}^* := \{(\mathbf{x}, \mathbf{s}) \in \mathcal{F} : \mathbf{xs} = \mathbf{0}\}. \end{aligned}$$

We denoted by $\mathbb{R}^n_\oplus$ the n-dimentional nonnegative orthant and by $\mathbb{R}^n_+$ the positive orthant, respectively. The most important basic results related to LCPs are summarized in the books of Cottle et al. [8] and Kojima et al. [37].

There are several methods for solving LCPs with different matrices, such as simplex [10, 64, 65, 67], criss-cross [11, 12, 26, 27, 24] or other pivot [39, 63] algorithms. However, the IPAs for solving LCPs received more attention in last decades [37]. It should be mentioned that LCPs belong to the class of NP-complete problems [7]. In spite of this fact, due to the results of Kojima et al. [37], if we suppose that the problem's matrix has $P_*(\kappa)$-property, the IPAs solving these kind of LCPs usually have polynomial complexity in the handicap of the problem's matrix, the size of the problem and the bitsize of the data. Beside this, Väliaho [61] showed that the class of P_*-matrices is equivalent to the class of sufficient matrices given by Cottle et al. [9]. Potra and Liu [50] proposed an IPA for sufficient LCPs which uses a wide neighbourhood of the central path and the algorithm does not depend on the handicap of the problem. There are several known IPAs not depending on the handicap of the sufficient matrix, such as the IPAs given by Potra and Sheng [52], Potra and Liu [50], Illés and Nagy [31], Liu and Potra [42] and Lešaja and Potra [55]. The IPAs for solving sufficient LCPs have been also extended to general LCPs [33, 34]. Illés et al. [33, 32] generalized large-update, affine scaling and PC IPAs for solving LCPs with general matrices.

The PC IPAs perform a predictor and one or more corrector steps in a main iteration. The aim of the predictor step is to reach optimality, hence after an affine-scaling step a certain amount of deviation from the central path is allowed. The goal of the corrector step is to return in the neighbourhood of the central path. The PC IPAs turned out to be efficient in practice. The first PC IPA for LO was given by Mehrotra [44] and Sonnevend et al. [57]. Potra and Sheng [51, 52] defined PC IPAs for sufficient LCPs. Mizuno, Todd and Ye [46] gave the first PC IPA for LO which uses only one corrector step in a main iteration and these IPAs were named Mizuno-Todd-Ye (MTY) type PC IPAs. Miao [45] extended the MTY IPA given by [46] to $P_*(\kappa)$-LCPs. Following this result, several MTY type PC IPAs have been proposed among others by Illés and M. Nagy [31], Kheirfam [35] and Darvay et al. [18]. In [18] the authors gave a unified framework to determine the Newton systems and scaled systems in case of PC IPAs using the AET technique.

Barrier functions are oftenly used for the determination of the search directions in case of IPAs. By considering self-regular kernel functions, Peng, Roos and Terlaky [48] reduced the theoretical complexity of large-update IPAs. Later on, Lešaja and Roos [41] provided a unified analysis of IPAs for $P_*(\kappa)$-LCPs that are based on eligible kernel functions. In 2005, Darvay proposed the AET technique for defining search directions in case of IPAs for LO [13, 14]. He applied a continuously differentiable, invertible, monotone increasing function $\bar{\varphi} : (\xi^2, \infty) \to \mathbb{R}$, where $0 \leq \xi < 1$, on the nonlinear equation of the central path problem. The majority of the published IPAs for sufficient LCPs does not use any transformation of the central path, which means that these IPAs use the identity map in the AET technique in order to define the search directions. Darvay [13, 14, 15] used the square root function in the AET technique for LO. In 2016, Darvay et al. [19] introduced an IPA for LO based on the direction using the function $\bar{\varphi}(t) = t - \sqrt{t}$. In her PhD thesis, Rigó [54] presented several IPAs that use the function $\bar{\varphi}(t) = t - \sqrt{t}$ in the AET technique. Recently, Kheirfam and Haghighi [36] have proposed an IPA for $P_*(\kappa)$-LCPs which uses the function $\bar{\varphi}(t) = \frac{\sqrt{t}}{2(1+\sqrt{t})}$ in the AET technique. Haddou et al. [29] have recently introduced a family of smooth concave functions which leads to IPAs with the best known iteration bound. The AET technique has been also extended to LCPs [1, 3, 4, 35, 43].

Zhang and Xu [69] used the equivalent form $\mathbf{v}^2 = \mathbf{v}$ of the centering equation, where $\mathbf{v} = \sqrt{\frac{\mathbf{xs}}{\mu}}$, $\mu > 0$. They considered the $\mathbf{xs} = \mu\mathbf{v}$ transformation. Darvay and Takács [21] introduced a new method for determining class of search directions using a new type of AET of the centering equations. They modified the nonlinear equation $\mathbf{v}^2 = \mathbf{v}$ by applying componentwisely a continuously differentiable function $\varphi : (\xi^2, \infty) \to \mathbb{R}$, $0 \leq \xi < 1$ to the both sides of this equation. The properties of this function φ will be presented in Subsection 2.3. The relationship between the functions φ and $\bar{\varphi}$ will be discussed later as a novelty of this paper. In [21] the authors considered the function $\varphi(t) = t^2$ in order to determine the new search directions. Zhang et al. [70] extended the feasible IPA given in [21] to $P_*(\kappa)$-LCPs. Furthermore, Takács and Darvay [58] generalized the approach for determining search directions proposed in [21] to SO and they showed that the corresponding kernel function is a positive-asymptotic kernel function. The positive-asymptotic kernel function was introduced by Darvay and Takács [20] and differs from the class of kernel functions introduced by Bai et al. [5].

In this paper we introduce a new PC IPA for solving $P_*(\kappa)$-LCPs which uses the new type of AET given in [21] for LO. The proposed IPA applies the function $\varphi(t) = t^2$ on the modified nonlinear equation $\mathbf{v}^2 = \mathbf{v}$ in order to obtain the search directions. In this sense, the corresponding kernel function is a positive-asymptotic kernel function. Similar to [18] we present the method for determining the Newton systems and scaled systems in case of PC IPAs using this new type of AET. We also present the complexity analysis of the proposed PC IPA. Due to the used search direction we have to ensure during the whole process of the IPA that the components of the vector $\mathbf{v}$ are greater than $\frac{\sqrt{2}}{2}$, which makes the analysis more difficult. In spite of this fact, we show that the introduced IPA has $O\left((1+4\kappa)\sqrt{n}\log\frac{3n\mu^0}{4\epsilon}\right)$ iteration complexity, where κ is the upper bound on the handicap of matrix M, μ^0 is the starting, average complementarity gap and ε is the final displacement from the complementarity gap, respectively. This is the first PC IPA for solving $P_*(\kappa)$-LCPs which uses the function $\varphi(t) = t^2$ in the new type of AET.

The paper is organized as follows. In Section 2 we give some basic concepts and useful results about the $P_*(\kappa)$-LCPs and $P_*(\kappa)$-matrices. Furthermore, in Subsection 2.3, depending on the representation of the nonlinear equation of the central path, a new way of applying the AET is discussed and compared to the earlier used AET technique. The usual, but important, scaling technique is discussed together with the unique solvability of the Newton-system, as well. In Subsection 2.5 different types of kernel functions are presented and compared. From our discussion, it is clear that to the function $\varphi(t) = t^2$ used in the new type of AET corresponds a kernel function which is positive-asymptotic kernel function. In Section 3, the new PC IPA is presented. While, Section 4 contains the complexity analysis of the introduced PC IPA with the new search directions. In Section 5 numerical computations are presented and compared to the computational performance of an earlier introduced PC IPA [18] that used different function φ in the AET. In Section 6 some concluding remarks are enumerated.

2. Linear complementarity problems with $P_*(\kappa)$-matrices

In this section we summarize important definitions and results related to LCPs with sufficient matrices. Furthermore, we introduce the AET of the central path. Following the steps of Darvay et al. [21], first we derive a known, equivalent description of the central path and then we apply the AET approach, see Subsection 2.3. The first observation is related to the fact that the same search directions can be obtained in different ways. Another interesting fact is the connection between different AET functions, equivalent forms of the central path and some kernel functions.

2.1. Sufficient matrices

Kojima et al. [37] presented the notion of $P_*(\kappa)$-matrices, which is a generalization of positive semidefinite matrices. We call a problem $P_*(\kappa)$-LCP if the problem's matrix of (LCP) is $P_*(\kappa)$-matrix, i.e.

$$(1+4\kappa)\sum_{i\in I_+(\mathbf{x})} x_i(Mx)_i + \sum_{i\in I_-(\mathbf{x})} x_i(Mx)_i \geq 0, \quad \forall \mathbf{x}\in\mathbb{R}^n, \tag{1}$$

where

$$I_+(\mathbf{x}) = \{1 \leq i \leq n : x_i(Mx)_i > 0\} \text{ and } I_-(\mathbf{x}) = \{1 \leq i \leq n : x_i(Mx)_i < 0\}$$

and $\kappa \geq 0$ is a nonnegative real number. We will assume throughout the paper that $\mathcal{F}^+ \neq \emptyset$, there is an initial point $(\mathbf{x}^0, \mathbf{s}^0) \in \mathcal{F}^+$ and M is a $P_*(\kappa)$-matrix.

We use P_* to denote the set of all $P_*(\kappa)$-matrices for all $\kappa \geq 0$. In [9] Cottle et al. gave the definition of column sufficient, row sufficient and sufficient matrices, respectively. In this sense, a matrix is sufficient if it is both column and row sufficient. Kojima et al. [37] showed that a P_*-matrix is column sufficient and Guu and Cottle [28] proved that it is row sufficient, too. This means, that each P_*-matrix is sufficient. Furthermore, Väliaho [61] showed that the class of P_*-matrices is equivalent to the class of sufficient matrices.

The handicap of M [61] is the smallest value of $\hat{\kappa}(M) \geq 0$ such that M is $P_*(\hat{\kappa}(M))$-matrix. Väliaho [61] also proved that a matrix M is P_* if and only if the handicap $\hat{\kappa}(M)$ of M is finite.

Note that the worst-case iteration complexity of the IPAs for LCP depends on the upper bound of the handicap of the matrix M. Väliaho [60] gave an algorithm which decides whether a matrix M is sufficient or not. Furthermore, Väliaho [62] conjectured that the handicap of a matrix M is a continuous function of the elements of M and he proposed an algorithm which gives the handicap of a sufficient matrix. Tseng [59] proved that deciding whether a square matrix with rational entries is a column sufficient matrix leads to a co-NP-complete problem. Hence, given a square matrix M we can not decide in polynomial time whether $M \in P_*(\kappa)$. De Klerk and E.-Nagy [23] showed that the handicap of a $P_*(\kappa)$-matrix may be exponential in its bit size. This means that if the handicap of the matrix is exponentially large in the size and bit size of the problem, then the known complexity bounds of IPAs may not be polynomial in the input size of the LCP.

2.2. Central path of sufficient LCPs

The central path problem for (LCP) is:

$$-M\mathbf{x} + \mathbf{s} = \mathbf{q}, \quad \mathbf{x}, \mathbf{s} > \mathbf{0}, \quad \mathbf{x}\mathbf{s} = \mu\,\mathbf{e}, \tag{2}$$

where $\mathbf{e}$ denotes the n-dimensional vector of ones and $\mu > 0$. Kojima et al. [37] showed that the sequence $\{(\mathbf{x}(\mu), \mathbf{s}(\mu)) \mid \mu > 0\}$ of solutions lying on the central path parameterised by $\mu > 0$ approach a solution $(\mathbf{x}, \mathbf{s})$ of the (LCP).

T. Illés, C. Roos, and T. Terlaky gave an elementary constructive proof for the existence and uniqueness of the central path for sufficient LCPs in an unpublished manuscript in 1997. The constructive proof of Illés et al. appears in Theorem 3.6 in the PhD thesis of M. E.-Nagy [47].

Similarly to [21], we use $\mathbf{x}, \mathbf{s} > \mathbf{0}$ and $\mu > 0$, hence we obtain:

$$\mathbf{x}\,\mathbf{s} = \mu\, e \Leftrightarrow \frac{\mathbf{x}\,\mathbf{s}}{\mu} = \mathbf{e} \Leftrightarrow \sqrt{\frac{\mathbf{x}\,\mathbf{s}}{\mu}} = \mathbf{e} \Leftrightarrow \frac{\mathbf{x}\,\mathbf{s}}{\mu} = \sqrt{\frac{\mathbf{x}\mathbf{s}}{\mu}}.$$

Now the central path problem for (LCP) can be equivalently stated as

$$-M\mathbf{x} + \mathbf{s} = \mathbf{q}, \quad \mathbf{x}, \mathbf{s} > \mathbf{0}, \quad \frac{\mathbf{x}\,\mathbf{s}}{\mu} = \sqrt{\frac{\mathbf{x}\mathbf{s}}{\mu}}. \tag{3}$$

Different forms of the central path problem (2) and (3) will be used later in the AET context.

Kojima et al. proved an important result in Lemma 4.1 of [37], which plays important role in the solvability of the Newton system. An important corollary of Lemma 4.1 presented in [37] is the following.

Corollary 2.1. *Let $M \in \mathbb{R}^{n\times n}$ be a $P_*(\kappa)$-matrix, $\mathbf{x}, \mathbf{s} \in \mathbb{R}^n_+$. Then, for all $\mathbf{a}_\varphi \in \mathbb{R}^n$ the system*

$$\begin{aligned} -M\Delta\mathbf{x} + \Delta\mathbf{s} &= \mathbf{0} \\ S\Delta\mathbf{x} + X\Delta\mathbf{s} &= \mathbf{a}_\varphi \end{aligned} \tag{4}$$

has a unique solution $(\Delta\mathbf{x}, \Delta\mathbf{s})$, where X and S are the diagonal matrices obtained from the vectors $\mathbf{x}$ and $\mathbf{s}$.

2.3. Algebraic equivalent transformation (AET) of the central path

The goal of the AET technique introduced by Darvay [13, 14] is to represent the central path in a different way and to derive Newton-system from these representations depending on the continuously differentiable, invertible, monotone increasing function $\bar\varphi : (\xi^2, \infty) \to \mathbb{R}$, where $0 \le \xi < 1$.

Now, we can apply the AET to the central path problem in the form (2) or (3). In case of applying the AET method to (2), we obtain the following form of the central path

$$-M\mathbf{x} + \mathbf{s} = \mathbf{q}, \quad \mathbf{x}, \mathbf{s} > \mathbf{0}, \quad \bar\varphi\left(\frac{\mathbf{x}\,\mathbf{s}}{\mu}\right) = \bar\varphi(\mathbf{e}). \tag{5}$$

However, if the AET is applied to (3) using the continuously differentiable function $\varphi : (\xi^2, \infty) \to \mathbb{R}$, where $0 \le \xi < 1$, then we get

$$-M\mathbf{x} + \mathbf{s} = \mathbf{q}, \quad \mathbf{x}, \mathbf{s} > \mathbf{0}, \quad \varphi\left(\frac{\mathbf{x}\,\mathbf{s}}{\mu}\right) = \varphi\left(\sqrt{\frac{\mathbf{x}\,\mathbf{s}}{\mu}}\right). \tag{6}$$

The following interesting question arises: if we use different transformed forms of the central path (say (5) or (6)), is it necessary to use some extra criterion on functions φ? An answer regarding this question will be given at the end of this subsection.

An interesting observation is the connection between systems (5) and (6). For this, let $\bar\varphi : (\xi^2, \infty) \to \mathbb{R}$

$$\bar\varphi(t) = \varphi(t) - \varphi(\sqrt{t}). \tag{7}$$

This leads to

$$\bar\varphi\left(\frac{\mathbf{x}\,\mathbf{s}}{\mu}\right) = \varphi\left(\frac{\mathbf{x}\,\mathbf{s}}{\mu}\right) - \varphi\left(\sqrt{\frac{\mathbf{x}\,\mathbf{s}}{\mu}}\right). \tag{8}$$

Hence, we have

$$\bar{\varphi}\left(\frac{\mathbf{x}\,\mathbf{s}}{\mu}\right)=\bar{\varphi}(\mathbf{e})\Leftrightarrow\varphi\left(\frac{\mathbf{x}\,\mathbf{s}}{\mu}\right)-\varphi\left(\sqrt{\frac{\mathbf{x}\,\mathbf{s}}{\mu}}\right)=\varphi(\mathbf{e})-\varphi(\sqrt{\mathbf{e}})\Leftrightarrow\varphi\left(\frac{\mathbf{x}\,\mathbf{s}}{\mu}\right)=\varphi\left(\sqrt{\frac{\mathbf{x}\,\mathbf{s}}{\mu}}\right).$$

Majority of the published IPAs using the AET, derives the Newton-system from (5), while only few, like Darvay and Takács [21], and Zhang et al. [70] applies the AET to (6). We follow the second approach to derive the corresponding Newton-system.

For an $(\mathbf{x},\mathbf{s})\in\mathcal{F}^+$ our aim is to find search directions $\Delta\mathbf{x}$ and $\Delta\mathbf{s}$ such that

$$\begin{aligned}-M(\mathbf{x}+\Delta\mathbf{x})+(\mathbf{s}+\Delta\mathbf{s}) &= \mathbf{q},\\ \varphi\left(\frac{\mathbf{xs}}{\mu}+\frac{\mathbf{x}\Delta\mathbf{s}+\mathbf{s}\Delta\mathbf{x}+\Delta\mathbf{x}\Delta\mathbf{s}}{\mu}\right) &= \varphi\left(\sqrt{\frac{\mathbf{xs}}{\mu}+\frac{\mathbf{x}\Delta\mathbf{s}+\mathbf{s}\Delta\mathbf{x}+\Delta\mathbf{x}\Delta\mathbf{s}}{\mu}}\right),\end{aligned}$$

We neglect the quadratic terms and apply Taylor's theorem to the function $\bar{\varphi}(t)=\varphi(t)-\varphi(\sqrt{t})$. Hence, after some calculations we obtain (4) with

$$\mathbf{a}_\varphi=\mu\frac{-\varphi\left(\frac{\mathbf{xs}}{\mu}\right)+\varphi\left(\sqrt{\frac{\mathbf{xs}}{\mu}}\right)}{\varphi'\left(\frac{\mathbf{xs}}{\mu}\right)-\frac{1}{2\sqrt{\frac{\mathbf{xs}}{\mu}}}\varphi'\left(\sqrt{\frac{\mathbf{xs}}{\mu}}\right)}. \tag{9}$$

Now, from the denominator of the obtained fractional expression, it is clear that we need extra assumption on the function φ, namely

$$2\,t\,\varphi'(t^2)-\varphi'(t)>0, \tag{10}$$

for all $t>\xi$, with $0\le\xi<1$.

Lemma 2.2. *Let $\bar{\varphi}:(\xi^2,\infty)\to\mathbb{R}$ as given in (7). Then, $\bar{\varphi}:(\xi^2,\infty)\to\mathbb{R}$ is monotone increasing if and only if condition (10) is satisfied for the function φ.*

Proof. Using (7) we have $\bar{\varphi}'(t)=\varphi'(t)-\frac{1}{2\sqrt{t}}\varphi'(\sqrt{t})$. Hence,

$$\bar{\varphi}'(t)>0,\ \forall t>\xi^2 \qquad \text{if and only if} \qquad \varphi'(t)-\frac{1}{2\sqrt{t}}\varphi'(\sqrt{t})>0,\ \forall t>\xi^2. \tag{11}$$

Considering change of variable $u:=\sqrt{t}$ in the second part of (11) we obtain condition (10).

Depending on the used functions φ we can have different vectors $\mathbf{a}_\varphi$. In [18] and [54] the authors presented the functions $\bar{\varphi}$ already used in the literature in case of IPAs in order to derive complexity results for different class of optimization problems, including LO and sufficient LCPs, as well.

Now, if a function φ satisfying condition (10) is applied to (6), then using (7) and Lemma 2.2 we immediately obtain an IPA with $\bar{\varphi}$ applied to (5). However, if a function $\bar{\varphi}$ satifying $\bar{\varphi}'(t)>0$ is applied to (5) and we derive an IPA, we do not have guarantee that a correponding function φ exists, due to the fact that the connection between $\bar{\varphi}$ and φ is given as a functional equation (7). Thus, we do not have in this case immediately another description of the IPA. In other words, we should consider the following question: can we find a corresponding function $\varphi:(\xi^2,\infty)\to\mathbb{R}$ for a given $\bar{\varphi}:(\xi^2,\infty)\to\mathbb{R}$, $0\le\xi<1$? To answer this, we give counterexamples. Using the definition of the function $\bar{\varphi}$ given in (7), we have $\lim_{t\to0}\bar{\varphi}(t)=\bar{\varphi}(1)=0$. However, the functions $\bar{\varphi}$ are monotone increasing. Hence, all the functions $\bar{\varphi}$ that are defined on the whole interval $(0,\infty)$, i.e. $\xi=0$, are counterexamples. However, it would be interesting to define a class of monotone increasing functions $\bar{\varphi}$ for which we can assign corresponding functions φ. For this, we should solve the functional equation $\varphi(t)-\varphi(\sqrt{t})=\bar{\varphi}(t)$ for a given function $\bar{\varphi}:(\xi^2,\infty)\to\mathbb{R}$. This leads to further research topics.

2.4. Scaling

Let us consider

$$\mathbf{v} = \sqrt{\frac{\mathbf{x}\,\mathbf{s}}{\mu}}, \quad \mathbf{d} = \sqrt{\frac{\mathbf{x}}{\mathbf{s}}}, \quad \mathbf{d}_x = \frac{\mathbf{d}^{-1}\,\Delta\mathbf{x}}{\sqrt{\mu}} = \frac{\mathbf{v}\,\Delta\mathbf{x}}{\mathbf{x}}, \quad \mathbf{d}_s = \frac{\mathbf{d}\,\Delta\mathbf{s}}{\sqrt{\mu}} = \frac{\mathbf{v}\,\Delta\mathbf{s}}{\mathbf{s}}. \tag{12}$$

From (12) we obtain

$$\Delta\mathbf{x} = \frac{\mathbf{x}\,\mathbf{d}_x}{\mathbf{v}} \quad \text{and} \quad \Delta\mathbf{s} = \frac{\mathbf{s}\,\mathbf{d}_s}{\mathbf{v}}.$$

Hence, if we substitute these in the second equation of system (4) we get

$$\frac{\mathbf{x}\,\mathbf{s}\,\mathbf{d}_x}{\mathbf{v}} + \frac{\mathbf{x}\,\mathbf{s}\,\mathbf{d}_s}{\mathbf{v}} = \mu \frac{2\mathbf{v}\left(\varphi(\mathbf{v}) - \varphi(\mathbf{v}^2)\right)}{2\mathbf{v}\varphi'(\mathbf{v}^2) - \varphi'(\mathbf{v})}. \tag{13}$$

The transformed Newton system (4) with $\mathbf{a}_\varphi$, see (9), obtained from (6) by applying the AET and then scaling it, leads to the following form of the scaled Newton-system:

$$\begin{aligned} -\bar{M}\mathbf{d}_x + \mathbf{d}_s &= \mathbf{0}, \\ \mathbf{d}_x + \mathbf{d}_s &= \mathbf{p}_\varphi, \end{aligned} \tag{14}$$

where $\bar{M} = DMD$, $D = diag(\mathbf{d})$ and

$$\mathbf{p}_\varphi = \frac{2\left(\varphi(\mathbf{v}) - \varphi(\mathbf{v}^2)\right)}{2\,\mathbf{v}\,\varphi'(\mathbf{v}^2) - \varphi'(\mathbf{v})}. \tag{15}$$

From Theorem 3.5 proposed in [37] and Corollary 2.1 it can be proved that system (14) has unique solution.

It should be mentioned that if we use the function $\varphi : \left(\frac{1}{2}, \infty\right) \to \mathbb{R}$, $\varphi(t) = t$, which satisfies condition (10), then we have

$$\mathbf{p}_\varphi = \frac{2\mathbf{v} - 2\mathbf{v}^2}{2\mathbf{v} - \mathbf{e}}. \tag{16}$$

Interestingly enough that exactly the same $\mathbf{p}_\varphi$ vector can be derived if the AET is applied to (5) with function $\bar{\varphi}(t) = t - \sqrt{t}$. For details see papers [19, 16] for LO and [17, 18] for sufficient LCPs. This can be proved by using (7), because in this case we have $\bar{\varphi}(t) = \varphi(t) - \varphi(\sqrt{t}) = t - \sqrt{t}$. Furthermore, if we apply the AET to system (6) using the function $\varphi(t) = t^2$, then we obtain the same system as if we apply $\bar{\varphi}(t) = \varphi(t) - \varphi(\sqrt{t}) = t^2 - t$ to system (5). It should be mentioned, that this function has not been used in the literature in the AET technique. Hence, the function $\varphi(t) = t^2$ used in the AET approach and applied to (6) leads to novel search directions discussed in this paper.

2.5. Search directions based on kernel functions

Peng et al. [48] introduced the class of self-regular barrier functions and defined large-update IPAs for solving LO problems. In [54] the author presented different types of kernel functions that are used in the determination of the search directions.

Definition 2.1. (Bai et al. [5]) A function $\psi : \mathbb{R}_+ \to \mathbb{R}_\oplus$ is called kernel function if it is twice continuously differentiable and if the following conditions hold:

i. $\psi(1) = \psi'(1) = 0$;

ii. $\psi''(t) > 0$, for all $t > 0$;

iii. $\lim_{t \downarrow 0} \psi(t) = \lim_{t \to \infty} \psi(t) = \infty$.

Note that if condition iii. of Definition 2.1 is satisfied, then some authors call the function ψ coercive kernel funcion [66]. Moreover, some authors also consider other conditions for defining a class of kernel functions [41]. A barrier function $\Psi : \mathbb{R}^n_+ \to \mathbb{R}$ can be constructed as $\Psi(\mathbf{v}) = \sum_{i=1}^n \psi(v_i)$, where $\mathbf{v} \in \mathbb{R}^n_+$. Peng et al. [48] considered a modification of the third equation of system (14):

$$\mathbf{d}_x + \mathbf{d}_s = -\nabla\Psi(\mathbf{v}).$$

The search directions for sufficient LCPs using self-regular IPAs are determined as the unique solutions of the system

$$\begin{aligned} -\bar{M}\mathbf{d}_x + \mathbf{d}_s &= \mathbf{0}, \\ \mathbf{d}_x + \mathbf{d}_s &= -\nabla\Psi(\mathbf{v}). \end{aligned} \tag{17}$$

In [20, 53] the authors introduced the notion of the *positive-asymptotic kernel function* and its associated barrier for SO problems. It is clear that the concept of positive-asymptotic kernel function can be used for LCPs, as well, see [54].

Definition 2.2. (Darvay and Takács [20]) Let $0 \leq \xi < 1$ and $D = (\xi, +\infty)$ be an open interval. A function $\psi : D \to [0, +\infty)$ is called *ξ-asymptotic kernel function* if it is twice continuously differentiable and if the following conditions hold:

i. $\psi(1) = \psi'(1) = 0$;

ii. $\psi''(t) > 0$, for all $t > \xi$;

iii. $\lim_{t \downarrow \xi} \psi(t) = \lim_{t \to \infty} \psi(t) = \infty$.

Note that if $\xi = 0$ then the notion of ξ-asymptotic kernel function coincides with the concept of kernel function.

Definition 2.3. (Darvay and Takács [20]) A function is a *positive-asymptotic kernel function* iff it is ξ-asymptotic and $0 < \xi < 1$.

From the second equations of systems (14) and (17), we have

$$\mathbf{d}_x + \mathbf{d}_s = -\nabla\Psi(\mathbf{v}) = \mathbf{p}_\varphi = \frac{2\left(\varphi(\mathbf{v}) - \varphi(\mathbf{v}^2)\right)}{2\,\mathbf{v}\,\varphi'(\mathbf{v}^2) - \varphi'(\mathbf{v})}, \tag{18}$$

Using (18) and $\Psi(\mathbf{v}) = \sum_{i=1}^n \psi(v_i)$ we can associate a corresponding kernel function, see [21], to several functions φ appeared in the new type of AET specified in (6) as:

$$\psi(t) = \int_1^t \frac{2\varphi(\eta^2) - 2\varphi(\eta)}{2\eta\varphi'(\eta^2) - \varphi'(\eta)} d\eta.$$

If we use $\varphi(t) = t^2$ we get the following kernel function:

$$\psi : \left(\frac{\sqrt{2}}{2}, \infty\right) \to \mathbb{R}_\oplus, \quad \psi(t) = \frac{t^2 - 1}{4} - \frac{\log(2t^2 - 1)}{8}.$$

Note that this function is positive-asymptotic kernel function, with $\xi = \frac{\sqrt{2}}{2}$. It should be mentioned that in [54] a more detailed description is given related to the relationship between the approach based on kernel functions and on the classical AET characterized by system (5).

In the following subsection we give a general method of determining the scaled predictor and scaled corrector systems in case of PC IPAs using this new type of AET.

2.6. Search directions in case of PC IPAs

Darvay et al. [18] gave a general framework to determine the scaled systems in case of PC IPAs for sufficient LCPs. Following the steps of their method, we give firstly the scaled corrector system, which coincides with system (14). This system has the unique solution: $\mathbf{d}_x^c = (I+\bar{M})^{-1}\mathbf{p}_\varphi$, $\mathbf{d}_s^c = \bar{M}(I+\bar{M})^{-1}\mathbf{p}_\varphi$. From $\Delta^c\mathbf{x} = \frac{\mathbf{x}\,\mathbf{d}_x^c}{\mathbf{v}}$ and $\Delta^c\mathbf{s} = \frac{\mathbf{s}\,\mathbf{d}_s^c}{\mathbf{v}}$ we can calculate search directions $\Delta^c\mathbf{x}$ and $\Delta^c\mathbf{s}$. The difference between this method and the one presented in [18] is that we have different value of the vector $\mathbf{p}_\varphi$ due to the used function $\varphi(t) = t^2$ in the AET technique. In the transformed Newton system (4) we decompose $\mathbf{a}_\varphi$ given in (9) in the following way [18]:

$$\mathbf{a}_\varphi = f(\mathbf{x},\mathbf{s},\mu) + g(\mathbf{x},\mathbf{s}), \tag{19}$$

where $f : \mathbb{R}^n_+ \times \mathbb{R}^n_+ \times \mathbb{R}_\oplus \to \mathbb{R}^n$ with $f(\mathbf{x},\mathbf{s},0) = \mathbf{0}$ and $g : \mathbb{R}^n_+ \times \mathbb{R}^n_+ \to \mathbb{R}^n$. We set $\mu = 0$ in (19), because we would like to make as greedy predictor step as possible. From [18] we obtain

$$\begin{aligned} -\bar{M}\mathbf{d}_x + \mathbf{d}_s &= \mathbf{0}, \\ \mathbf{d}_x + \mathbf{d}_s &= \frac{\mathbf{v}g(\mathbf{x},\mathbf{s})}{\mathbf{x}\mathbf{s}}, \end{aligned} \tag{20}$$

where $\bar{M} = DMD$. The unique solution of system (20) is $\mathbf{d}_x^p = (I+\bar{M})^{-1}\frac{\mathbf{v}g(\mathbf{x},\mathbf{s})}{\mathbf{x}\mathbf{s}}$ and $\mathbf{d}_s^p = \bar{M}(I+\bar{M})^{-1}\frac{\mathbf{v}g(\mathbf{x},\mathbf{s})}{\mathbf{x}\mathbf{s}}$. The difference between this approach and the one given in [18] lies in the different value of the vector $\mathbf{a}_\varphi$ and of $g(\mathbf{x},\mathbf{s})$. From $\Delta^p\mathbf{x} = \frac{\mathbf{x}\,\mathbf{d}_x^p}{\mathbf{v}}$ and $\Delta^p\mathbf{s} = \frac{\mathbf{s}\,\mathbf{d}_s^p}{\mathbf{v}}$ the $\Delta^p\mathbf{x}$ and $\Delta^p\mathbf{s}$ search directions can be easily calculated. It should be mentioned that the decomposition (19) is not trivial and we have no guarantee that such decomposition exists for all functions φ suitable for AET.

3. New PC IPA for $P_*(\kappa)$-LCPs based on a new search direction

In this section we introduce a PC IPA using the AET technique presented in Subsection 2.3. We deal with the function $\varphi : \left(\frac{1}{2}, \infty\right) \to \mathbb{R}$, $\varphi(t) = t^2$, so we obtain

$$\mathbf{p}_\varphi = \frac{\mathbf{v} - \mathbf{v}^3}{2\mathbf{v}^2 - \mathbf{e}}. \tag{21}$$

It should be mentioned that the condition $2t\varphi'(t^2) - \varphi'(t) > 0, \forall t > \xi$ is satisfied in this case, where $\xi = \frac{\sqrt{2}}{2}$. Let us define the centrality measure $\delta : \mathbb{R}^n_+ \times \mathbb{R}^n_+ \times \mathbb{R}_+ \to \mathbb{R} \cup \{\infty\}$ as

$$\delta(\mathbf{x},\mathbf{s},\mu) := \delta(\mathbf{v}) := \frac{\|\mathbf{p}_\varphi\|}{2} = \frac{1}{2}\left\|\frac{\mathbf{v} - \mathbf{v}^3}{2\mathbf{v}^2 - \mathbf{e}}\right\|. \tag{22}$$

Beside this, we give the τ-neighbourhood of a fixed point of the central path as

$$\mathcal{N}_2(\tau,\mu) := \{(\mathbf{x},\mathbf{s}) \in \mathcal{F}^+ : \delta(\mathbf{x},\mathbf{s},\mu) \le \tau\} = \left\{(\mathbf{x},\mathbf{s}) \in \mathcal{F}^+ : \frac{1}{2}\left\|\frac{\mathbf{v} - \mathbf{v}^3}{2\mathbf{v}^2 - \mathbf{e}}\right\| \le \tau\right\}, \tag{23}$$

where τ is a threshold parameter and $\mu > 0$ is fixed.

First, we need to find the decomposition of $\mathbf{a}_\varphi$ as it is given in (19):

$$\mathbf{a}_\varphi = \frac{\mu\,\mathbf{x}\,\mathbf{s}}{2\,(2\,\mathbf{x}\,\mathbf{s}-\mu\,\mathbf{e})} - \frac{\mathbf{x}\,\mathbf{s}}{2},$$

hence $f(\mathbf{x},\mathbf{s},\mu) = \frac{\mu\,\mathbf{x}\,\mathbf{s}}{2\,(2\,\mathbf{x}\,\mathbf{s}-\mu\,\mathbf{e})}$, which satisfies the condition $f(\mathbf{x},\mathbf{s},0)=\mathbf{0}$ and $g(\mathbf{x},\mathbf{s}) = -\frac{\mathbf{x}\,\mathbf{s}}{2}$. In this case, the transformed Newton system (4) with (9) is the following:

$$\begin{aligned} -M\Delta\mathbf{x} + \quad \Delta\mathbf{s} &= \mathbf{0}, \\ S\Delta\mathbf{x} + X\Delta\mathbf{s} &= \frac{\mu\,\mathbf{x}\,\mathbf{s}}{2\,(2\,\mathbf{x}\,\mathbf{s}-\mu\,\mathbf{e})} - \frac{\mathbf{x}\,\mathbf{s}}{2}. \end{aligned} \tag{24}$$

Note that some IPAs use firstly corrector steps and after that predictor step, see Potra [49]. Our algorithm also performs firstly a corrector step if the initial interior point is not well centered and after that a predictor one. The PC IPA starts with $(\mathbf{x},\mathbf{s}) \in \mathcal{N}_2(\tau,\mu)$, which holds in case of $(\mathbf{x}^0,\mathbf{s}^0)$, because $\delta(\mathbf{x}^0,\mathbf{s}^0,\mu) \le \tau$. In a corrector step we obtain $\mathbf{d}_x^c$ and $\mathbf{d}_s^c$ by solving

$$\begin{aligned} -\bar{M}\mathbf{d}_x^c + \mathbf{d}_s^c &= \mathbf{0}, \\ \mathbf{d}_x^c + \mathbf{d}_s^c &= \frac{\mathbf{v}-\mathbf{v}^3}{2\mathbf{v}^2-\mathbf{e}}, \end{aligned} \tag{25}$$

where we used the scaling notations considered in Section 2.4, $\bar{M} = DMD$ and $D = diag(\mathbf{d})$. From Theorem 3.5 given in [37] and Corollary 2.1 it can be proved that system (25) has unique solution:

$$\mathbf{d}_x^c = (I+\bar{M})^{-1}\frac{\mathbf{v}-\mathbf{v}^3}{2\mathbf{v}^2-\mathbf{e}}, \quad \mathbf{d}_s^c = \bar{M}(I+\bar{M})^{-1}\frac{\mathbf{v}-\mathbf{v}^3}{2\mathbf{v}^2-\mathbf{e}}.$$

Algorithm 3.1 : **PC IPA for sufficient LCPs based on a new type of AET**

Let $\epsilon > 0$ be the accuracy parameter, $0 < \theta < 1$ the update parameter and τ the proximity parameter. Furthermore, a known upper bound κ of the handicap $\hat{\kappa}(M)$ is given. Assume that for $(\mathbf{x}^0,\mathbf{s}^0)$ the $\left(\mathbf{x}^0\right)^T\mathbf{s}^0 = n\mu^0$, $\mu^0 > 0$ holds such that $\delta(\mathbf{x}^0,\mathbf{s}^0,\mu^0) \le \tau$ and $\frac{\mathbf{x}^0\mathbf{s}^0}{\mu^0} > \frac{1}{2}\mathbf{e}$.

begin

$k := 0$;

while $\left(\mathbf{x}^k\right)^T\mathbf{s}^k > \epsilon$ **do begin**

(corrector step)
compute $(\Delta^c x^k, \Delta^c s^k)$ from system (25) using (26);
let $(\mathbf{x}^c)^k := \mathbf{x}^k + \Delta^c x^k$ and $(\mathbf{s}^c)^k := \mathbf{s}^k + \Delta^c s^k$;

(predictor step)
compute $(\Delta^p x^k, \Delta^p s^k)$ from system (27) using (29);
let $(\mathbf{x}^p)^k := (\mathbf{x}^c)^k + \theta\Delta^p\mathbf{x}^k$ and $(\mathbf{s}^p)^k := (\mathbf{s}^c)^k + \theta\Delta^p\mathbf{s}^k$;

(update of the parameters and the iterates)
$(\mu^p)^k = \left(1-\frac{\theta}{2}\right)\mu^k$;
$\mathbf{x}^{k+1} := (\mathbf{x}^p)^k, \quad \mathbf{s}^{k+1} := (\mathbf{s}^p)^k, \quad \mu^{k+1} := (\mu^p)^k$;
k:=k+1;

end

end.

From

$$\Delta^c \mathbf{x} = \frac{\mathbf{x}\,\mathbf{d}_x^c}{\mathbf{v}} \quad \text{and} \quad \Delta^c \mathbf{s} = \frac{\mathbf{s}\,\mathbf{d}_s^c}{\mathbf{v}} \tag{26}$$

the $\Delta^c \mathbf{x}$ and $\Delta^c \mathbf{s}$ search directions can be easily obtained. Let

$$\mathbf{x}^c = \mathbf{x} + \Delta^c \mathbf{x}, \quad \mathbf{s}^c = \mathbf{s} + \Delta^c \mathbf{s}.$$

Consider the following notations:

$$\mathbf{v}^c = \sqrt{\frac{\mathbf{x}^c \mathbf{s}^c}{\mu}}, \quad \mathbf{d}^c = \sqrt{\frac{\mathbf{x}^c}{\mathbf{s}^c}}, \quad D^+ = diag(\mathbf{d}^c), \quad \bar{M}^+ = D^+ M D^+.$$

Then, the scaled predictor system is

$$\begin{aligned} -\bar{M}^+ \mathbf{d}_x^p + \mathbf{d}_s^p &= \mathbf{0}, \\ \mathbf{d}_x^p + \mathbf{d}_s^p &= -\frac{\mathbf{v}^c}{2}, \end{aligned} \tag{27}$$

which has the solution

$$\mathbf{d}_x^p = -(I + \bar{M}^+)^{-1} \frac{\mathbf{v}^c}{2}, \quad \mathbf{d}_s^p = -\bar{M}^+ (I + \bar{M}^+)^{-1} \frac{\mathbf{v}^c}{2}. \tag{28}$$

Then, using

$$\Delta^p \mathbf{x} = \frac{\mathbf{x}^c}{\mathbf{v}^c} \mathbf{d}_x^p \quad \text{and} \quad \Delta^p \mathbf{s} = \frac{\mathbf{s}^c}{\mathbf{v}^c} \mathbf{d}_s^p, \tag{29}$$

the search directions $\Delta^p \mathbf{x}$ and $\Delta^p \mathbf{s}$ can be easily calculated. The point after a predictor step is

$$\mathbf{x}^p = \mathbf{x}^c + \theta \Delta^p \mathbf{x}, \quad \mathbf{s}^p = \mathbf{s}^c + \theta \Delta^p \mathbf{s}, \quad \mu^p = \left(1 - \frac{\theta}{2}\right) \mu,$$

where $\theta \in (0,1)$ is the update parameter.

4. Analysis of the PC IPA

In the first part of the analysis we deal with the corrector step. Consider the following wide neighbourhood

$$\mathcal{D}(\beta, \mu) = \{(\mathbf{x}, \mathbf{s}) \in \mathcal{F}^+ : \mathbf{x}\mathbf{s} \geq \beta \mu \mathbf{e}\},$$

where $0 < \beta < 1$ and $\mu > 0$. In this way, Algorithm 3.1 works in a neighbourhood which is obtained by the intersection of $\mathcal{N}_2(\tau, \mu)$ given in (23) and $\mathcal{D}\left(\frac{1}{2}, \mu\right)$.

4.1. The corrector step

The corrector part of the proposed PC IPA is similar to the classical small-update IPAs. Therefore, the results of M. Zhang et. al [70] can be used to analyse the corrector steps of the proposed PC IPA. In the next theorem the strict feasibility of the full-Newton IPA is proved, where $\mathbf{v}^c = \sqrt{\frac{\mathbf{x}^c \mathbf{s}^c}{\mu}}$.

Theorem 4.1. *(Theorem 1 in [21], and Lemma 3 in [70]) Let* $\delta := \delta(\mathbf{x}, \mathbf{s}, \mu) < \frac{1}{\sqrt{1+4\kappa}}$ *and* $\mathbf{v} > \frac{\sqrt{2}}{2}\mathbf{e}$*. Then, we have* $(\mathbf{x}^c, \mathbf{s}^c) \in \mathcal{F}^+$ *and* $\mathbf{v}^c \geq \sqrt{1 - (1+4\kappa)\delta^2}\ \mathbf{e}$*. Moreover, if we choose* $\delta := \delta(\mathbf{x}, \mathbf{s}, \mu) < \frac{1}{\sqrt{2(1+4\kappa)}}$*, then we have* $\mathbf{v}^c > \frac{\sqrt{2}}{2}\mathbf{e}$.

Theorem 4.1 shows that after the corrector step $(\mathbf{x}^c, \mathbf{s}^c) \in \mathcal{D}\left(\frac{1}{2}, \mu\right)$ holds. The next lemma shows the quadratic convergence of the corrector step.

Lemma 4.2. *(Theorem 2 in [70]) Let* $\delta := \delta(\mathbf{x}, \mathbf{s}, \mu) < \frac{1}{\sqrt{2(1+4\kappa)}}$ *and* $\mathbf{v} > \frac{\sqrt{2}}{2}\mathbf{e}$. *Then,*

$$\delta(\mathbf{x}^c, \mathbf{s}^c, \mu) \leq \frac{5(1+4\kappa)\delta^2}{1-2(1+4\kappa)\delta^2}\sqrt{1-(1+4\kappa)\delta^2}.$$

Corollary 4.3. *Let* $\delta := \delta(\mathbf{x}, \mathbf{s}, \mu) \leq \frac{1}{2\sqrt{1+4\kappa}}$ *and* $\mathbf{v} > \frac{\sqrt{2}}{2}\mathbf{e}$. *Then,* $\delta^c \leq 10(1+4\kappa)\delta^2$.

Proof. *From* $\delta(\mathbf{x}, \mathbf{s}, \mu) < \frac{1}{2\sqrt{1+4\kappa}}$ *we have*

$$1 - 2(1+4\kappa)\delta^2 \geq \frac{1}{2}.$$

Using this, Lemma 4.2 and $\sqrt{1-(1+4\kappa)\delta^2} \leq 1$ *we obtain*

$$\delta(\mathbf{x}^c, \mathbf{s}^c, \mu) \leq \frac{5(1+4\kappa)\delta^2}{1-2(1+4\kappa)\delta^2} \leq 10(1+4\kappa)\delta^2,$$

which yields the result.

Next lemma provides an upper bound for the duality gap after a full-Newton step.

Lemma 4.4. *(Lemma 4 in [70]) Let* $\delta := \delta(\mathbf{x}, \mathbf{s}, \mu)$ *given as in (22). Then,*

$$(\mathbf{x}^c)^T \mathbf{s}^c < \mu(n + 9\delta^2).$$

4.2. Technical lemmas

In this subsection we present important results that will be used in the next part of the analysis. We assume that M is a $P_*(\kappa)$-matrix for a given $\kappa \geq \hat{\kappa}(M) \geq 0$. From $-M\Delta^p\mathbf{x} + \Delta^p\mathbf{s} = \mathbf{0}$, we have

$$(1+4\kappa)\sum_{i\in I_+} \Delta^p x_i \Delta^p s_i + \sum_{i\in I_-} \Delta^p x_i \Delta^p s_i \quad \geq \quad 0, \tag{30}$$

where $I_+ = \{i : \Delta^p x_i \Delta^p s_i > 0\}$ and $I_- = \{i : \Delta^p x_i \Delta^p s_i < 0\}$. Using (12) we obtain $\mathbf{d}_x^p \mathbf{d}_s^p = \frac{\Delta^p\mathbf{x}\Delta^p\mathbf{s}}{\mu}$. Hence, (30) can be written as

$$(1+4\kappa)\sum_{i\in I_+} d_{x_i}^p d_{s_i}^p + \sum_{i\in I_-} d_{x_i}^p d_{s_i}^p \quad \geq \quad 0. \tag{31}$$

The following lemma is similar to that of Lemma 1 in the paper of Kheirfam [35] and Lemma 5.3 in [18]. However, we use another type of AET tranformation and different function φ.

Lemma 4.5. *Let* $\delta^c = \delta(\mathbf{x}^c, \mathbf{s}^c, \mu) = \frac{1}{2}\left\|\frac{\mathbf{v}^c - (\mathbf{v}^c)^3}{2(\mathbf{v}^c)^2 - \mathbf{e}}\right\|$. *Then, the following inequality holds*

$$\|\mathbf{d}_x^p \mathbf{d}_s^p\| < \frac{n(2+\kappa)(1+4\delta^c)^2}{4}$$

PROOF. Using the second equation of the scaled predictor system (27) we obtain

$$\sum_{i\in I_+} d^p_{x_i} d^p_{s_i} \le \frac{1}{4}\|\mathbf{d}^p_x + \mathbf{d}^p_s\|^2 = \frac{\|\mathbf{v}^c\|^2}{16}.$$

Using the proof of Lemma 5.3 given in [18] and from the relation (31) we have

$$\frac{\|\mathbf{v}^c\|^2}{4} \ge \|\mathbf{d}^p_x\|^2 + \|\mathbf{d}^p_s\|^2 - 8\kappa \sum_{i\in I_+} d^p_{x_i} d^p_{s_i} \ge \|\mathbf{d}^p_x\|^2 + \|\mathbf{d}^p_s\|^2 - \frac{1}{2}\kappa\|\mathbf{v}^c\|^2. \tag{32}$$

Hence, $\|\mathbf{d}^p_x\|^2 + \|\mathbf{d}^p_s\|^2 \le \left(\frac{1}{4} + \frac{1}{2}\kappa\right)\|\mathbf{v}^c\|^2 < \left(1 + \frac{1}{2}\kappa\right)\|\mathbf{v}^c\|^2$. Similar to the proof of Lemma 5.3 of [18], we give an upper bound for $\|\mathbf{v}^c\|$. Consider the notation $\sigma^c = \|\mathbf{e} - \mathbf{v}^c\|$, which is the centrality measure used in [14, 35]. Using the relation (5.6) given in [18] we have

$$\|\mathbf{v}^c\| \le \sqrt{n}(\sigma^c + 1). \tag{33}$$

Moreover,

$$\delta^c = \frac{1}{2}\left\|\frac{\mathbf{v}^c - (\mathbf{v}^c)^3}{2(\mathbf{v}^c)^2 - \mathbf{e}}\right\| = \frac{1}{2}\left\|\frac{\mathbf{v}^c(\mathbf{e} + \mathbf{v}^c)}{2(\mathbf{v}^c)^2 - \mathbf{e}}(\mathbf{e} - \mathbf{v}^c)\right\| > \frac{1}{4}\|\mathbf{e} - \mathbf{v}^c\| = \frac{\sigma^c}{4}, \tag{34}$$

where we used that the function $\bar{h}(t) = \frac{t^2+t}{2t^2-1} > \frac{1}{2}$, for $t > \frac{\sqrt{2}}{2}$. Hence, we have $\sigma^c < 4\delta^c$. Using (33) and (34) we get

$$\|\mathbf{v}^c\| < \sqrt{n}(1 + 4\delta^c). \tag{35}$$

Thus,

$$\|\mathbf{d}^p_x \mathbf{d}^p_s\| \le \|\mathbf{d}^p_x\|\|\mathbf{d}^p_s\| \le \frac{1}{2}\left(\|\mathbf{d}^p_x\|^2 + \|\mathbf{d}^p_s\|^2\right) \le \frac{1}{2}\left(1 + \frac{1}{2}\kappa\right)\|\mathbf{v}^c\|^2 < \frac{n(2+\kappa)(1+4\delta^c)^2}{4},$$

which proves the lemma.

Consider

$$\mathbf{q}_v = \mathbf{d}^c_x - \mathbf{d}^c_s. \tag{36}$$

Then, we have

$$\mathbf{d}^c_x = \frac{\mathbf{p}_\varphi + \mathbf{q}_v}{2}, \quad \mathbf{d}^c_s = \frac{\mathbf{p}_\varphi - \mathbf{q}_v}{2} \quad \text{and} \quad \mathbf{d}^c_x \mathbf{d}^c_s = \frac{\mathbf{p}_\varphi^2 - \mathbf{q}_v^2}{4}. \tag{37}$$

We give an upper bound for the norm of $\mathbf{q}_v$ depending on the centrality measure. The proof technique is similar to the one given in [2] for $P_*(\kappa)$-LCPs over Cartesian product of symmetric cones.

Lemma 4.6. *(c.f. Lemma 5.4 in [18] and Lemma 5.1 in [2]) The following inequality holds:*

$$\|\mathbf{q}_v\| \le 2\sqrt{1+4\kappa}\,\delta^2,$$

where $\delta = \delta(\mathbf{x}, \mathbf{s}, \mu)$ is the proximity measure given in (22).

The next subsection contains the analysis of the predictor step.

4.3. The predictor step

Lemma 4.7 gives a sufficient condition for the strict feasibility of the predictor step.

Lemma 4.7. *Let* $(\mathbf{x}^c, \mathbf{s}^c) > \mathbf{0}$, $0 < \theta < 1$ *and* $\mu > 0$ *such that* $\delta^c := \delta(\mathbf{x}^c, \mathbf{s}^c, \mu) < \frac{1}{4}$. *Consider* $\mathbf{x}^p = \mathbf{x}^c + \theta \Delta^p \mathbf{x}$ *and* $\mathbf{s}^p = \mathbf{s}^c + \theta \Delta^p \mathbf{s}$. *Let*

$$z(\delta^c, \theta, n) := (1 - 4\delta^c)^2 - \frac{n(2+\kappa)\theta^2(1+4\delta^c)^2}{2(2-\theta)}.$$

If $z(\delta^c, \theta, n) > 0$, *then* $\mathbf{x}^p > \mathbf{0}$ *and* $\mathbf{s}^p > \mathbf{0}$.

Proof. Let us consider $\mathbf{x}^p(\alpha) = \mathbf{x}^c + \alpha\, \theta\, \Delta^p \mathbf{x}$ and $\mathbf{s}^p(\alpha) = \mathbf{s}^c + \alpha\, \theta\, \Delta^p \mathbf{s}$, for $0 \le \alpha \le 1$. Then, $\mathbf{x}^p(\alpha) = \frac{\mathbf{x}^c}{\mathbf{v}^c}(\mathbf{v}^c + \alpha\, \theta\, \mathbf{d}_x^p)$ and $\mathbf{s}^p(\alpha) = \frac{\mathbf{s}^c}{\mathbf{v}^c}(\mathbf{v}^c + \alpha\, \theta\, \mathbf{d}_s^p)$. Using relation (5.17) given in [18] and from the second equation of system (27) we obtain:

$$\begin{aligned} \mathbf{x}^p(\alpha)\mathbf{s}^p(\alpha) &= \mu \left((\mathbf{v}^c)^2 + \alpha\theta \mathbf{v}^c(\mathbf{d}_x^p + \mathbf{d}_s^p) + \alpha^2\, \theta^2\, \mathbf{d}_x^p \mathbf{d}_s^p \right) \\ &= \mu \left(\left(1 - \frac{1}{2}\alpha\theta\right)(\mathbf{v}^c)^2 + \alpha^2\, \theta^2\, \mathbf{d}_x^p \mathbf{d}_s^p \right). \end{aligned} \tag{38}$$

Hence, we obtain

$$\min \left(\frac{\mathbf{x}^p(\alpha)\mathbf{s}^p(\alpha)}{\mu\left(1 - \frac{\alpha\theta}{2}\right)} \right) = \min \left((\mathbf{v}^c)^2 + \frac{\alpha^2\theta^2}{1 - \frac{\alpha\theta}{2}} \mathbf{d}_x^p \mathbf{d}_s^p \right) \ge \min \left((\mathbf{v}^c)^2 \right) - \frac{2\alpha^2\theta^2}{2 - \alpha\theta} \|\mathbf{d}_x^p \mathbf{d}_s^p\|_\infty.$$

We have $1 - \sigma^c \le v_i^c \le 1 + \sigma^c, \ \ \forall i = 1, \dots, n$. Using these bounds, (34) and $\delta^c < \frac{1}{4}$ we have

$$\min (\mathbf{v}^c)^2 \ge (1 - \sigma^c)^2 \ge (1 - 4\delta^c)^2. \tag{39}$$

We will use that the real valued function $f(\alpha) = \frac{2\alpha^2\theta^2}{2 - \alpha\theta}$ is strictly increasing for $0 \le \alpha \le 1$ and each fixed $0 < \theta < 1$. Moreover, from Lemma 4.5 and (39) we obtain

$$\min \left(\frac{\mathbf{x}^p(\alpha)\mathbf{s}^p(\alpha)}{\mu\left(1 - \frac{\alpha\theta}{2}\right)} \right) \ \ge \ (1 - 4\delta^c)^2 - \frac{2n(2+\kappa)\theta^2(1+4\delta^c)^2}{4(2-\theta)} = z(\delta^c, \theta, n) > 0. \tag{40}$$

Hence, we have $\mathbf{x}^p(\alpha)\mathbf{s}^p(\alpha) > 0$ for $0 \le \alpha \le 1$. Therefore, $\mathbf{x}^p(\alpha)$ and $\mathbf{s}^p(\alpha)$ do not change sign on $0 \le \alpha \le 1$. Using $\mathbf{x}^p(0) = \mathbf{x}^c > \mathbf{0}$ and $\mathbf{s}^p(0) = \mathbf{s}^c > \mathbf{0}$, we obtain $\mathbf{x}^p(1) = \mathbf{x}^p > \mathbf{0}$ and $\mathbf{s}^p(1) = \mathbf{s}^p > \mathbf{0}$, which yields the result.

Let us introduce

$$\mathbf{v}^p = \sqrt{\frac{\mathbf{x}^p \mathbf{s}^p}{\mu^p}},$$

where $\mu^p = \left(1 - \frac{\theta}{2}\right)\mu$. If we substitute $\alpha = 1$ in (38) and (40) we have

$$(\mathbf{v}^p)^2 = (\mathbf{v}^c)^2 + \frac{2\theta^2}{2 - \theta}\mathbf{d}_x^p \mathbf{d}_s^p \quad \text{and} \quad \min (\mathbf{v}^p)^2 \ge z(\delta^c, \theta, n) > 0. \tag{41}$$

The next lemma analyses the effect of a predictor step and the update of μ on the proximity measure.

Lemma 4.8. *Let* $\delta^c := \delta(\mathbf{x}^c, \mathbf{s}^c, \mu) < \frac{1}{4}$, $\mu^p = \left(1 - \frac{\theta}{2}\right)\mu$, *where* $0 < \theta < 1$, $z(\delta^c, \theta, n) > \frac{1}{2}$ *and consider* $\delta := \delta(\mathbf{x}, \mathbf{s}, \mu)$ *given in (22). The iterates after a predictor step are denoted as* $\mathbf{x}^p$ *and* $\mathbf{s}^p$. *Then, we have* $\mathbf{v}^p > \frac{\sqrt{2}}{2}\mathbf{e}$ *and*

$$\delta^p := \delta(\mathbf{x}^p, \mathbf{s}^p, \mu^p) \le \frac{\sqrt{z(\delta^c, \theta, n)}\left(10(1+4\kappa)\delta^2 + (1-4\delta^c)^2 - z(\delta^c, \theta, n)\right)}{4z(\delta^c, \theta, n) - 2}.$$

PROOF. Using $z(\delta^c, \theta, n) > \frac{1}{2} > 0$, from Lemma 4.7 we get $\mathbf{x}^p > \mathbf{0}$ and $\mathbf{s}^p > \mathbf{0}$, thus the predictor step is strictly feasible. From (41) we obtain

$$\min(\mathbf{v}^p) \ge \sqrt{z(\delta^c, \theta, n)} > \frac{\sqrt{2}}{2},$$

which yields the first part of the result. Beside this,

$$\delta^p := \frac{1}{2}\left\|\frac{\mathbf{v}^p - (\mathbf{v}^p)^3}{2(\mathbf{v}^p)^2 - \mathbf{e}}\right\| = \frac{1}{2}\left\|\frac{\mathbf{v}^p\left(\mathbf{e} - (\mathbf{v}^p)^2\right)}{2(\mathbf{v}^p)^2 - \mathbf{e}}\right\|. \tag{42}$$

Consider $h : \left(\frac{\sqrt{2}}{2}, \infty\right) \to \mathbb{R}$, $h(t) = \frac{t}{2t^2-1}$, which is a decreasing function with respect to t. Using this, (41) and (42) we get

$$\begin{aligned}\delta^p &\le \frac{\min(\mathbf{v}^p)}{4\min(\mathbf{v}^p)^2 - 2}\left\|\mathbf{e} - (\mathbf{v}^p)^2\right\| \le \frac{\sqrt{z(\delta^c, \theta, n)}}{4z(\delta^c, \theta, n) - 2}\left\|\mathbf{e} - (\mathbf{v}^c)^2 - \frac{2\theta^2}{2-\theta}\mathbf{d}_x^p\mathbf{d}_s^p\right\| \\ &\le \frac{\sqrt{z(\delta^c, \theta, n)}}{4z(\delta^c, \theta, n) - 2}\left(\left\|\mathbf{e} - (\mathbf{v}^c)^2\right\| + \frac{2\theta^2}{2-\theta}\left\|\mathbf{d}_x^p\mathbf{d}_s^p\right\|\right). \end{aligned} \tag{43}$$

Using the proof of Lemma 2 given in [21] we obtain the following upper bound for $\left\|\mathbf{e} - (\mathbf{v}^c)^2\right\|$:

$$\left\|\mathbf{e} - (\mathbf{v}^c)^2\right\| \le \left\|\frac{\mathbf{q}_v^2}{4}\right\| + \left\|\frac{9\mathbf{v}^2 - 4\mathbf{e}}{\mathbf{v}^2} \cdot \frac{\mathbf{p}_\varphi^2}{4}\right\| \tag{44}$$

Hence, using (44) and Lemma 4.6 we may write

$$\begin{aligned}\left\|\mathbf{e} - (\mathbf{v}^c)^2\right\| &\le \left\|\frac{\mathbf{q}_v^2}{4}\right\| + \left\|\frac{9\mathbf{v}^2 - 4\mathbf{e}}{\mathbf{v}^2} \cdot \frac{\mathbf{p}_\varphi^2}{4}\right\| \\ &< \frac{\|\mathbf{q}_v\|^2}{4} + 9\frac{\|\mathbf{p}_\varphi\|^2}{4} = (1+4\kappa)\delta^4 + 9\delta^2 \le 10(1+4\kappa)\delta^2. \end{aligned} \tag{45}$$

Using (43), (45), Lemma 4.5 and the definition of the function z we get:

$$\begin{aligned}\delta^p &\le \frac{\sqrt{z(\delta^c, \theta, n)}}{4z(\delta^c, \theta, n) - 2}\left(\left\|\mathbf{e} - (\mathbf{v}^c)^2\right\| + \frac{2\theta^2}{2-\theta}\left\|\mathbf{d}_x^p\mathbf{d}_s^p\right\|\right) \\ &\ge \frac{\sqrt{z(\delta^c, \theta, n)}\left(10(1+4\kappa)\delta^2 + (1-4\delta^c)^2 - z(\delta^c, \theta, n)\right)}{4z(\delta^c, \theta, n) - 2}, \end{aligned} \tag{46}$$

which proves the second statement of the lemma.

It should be mentioned that in Lemma 4.8 the condition $z(\delta^c, \theta, n) > \frac{1}{2}$ should hold, because due to the used function $\varphi(t) = t^2$ in the new type of AET technique for the determination of the search directions, we have to ensure that in each iteration of the algorithm, the components of the vector $\mathbf{v}$ are greater than $\frac{\sqrt{2}}{2}$. Moreover, from Lemma 4.8 follows that $(\mathbf{x}^p, \mathbf{s}^p) \in \mathcal{D}\left(\frac{1}{2}, \mu^p\right)$.

Lemma 4.9. *Let $\delta \leq \frac{1}{16(1+4\kappa)}$. Then, we have $\delta^c < \frac{1}{4}$.*

Proof. *Using $\frac{1}{16(1+4\kappa)} \leq \frac{1}{2\sqrt{1+4\kappa}}$, by applying Corollary 4.3 and from $\kappa \geq 0$ we have*

$$\delta^c \leq 10(1+4\kappa)\delta^2 \leq \frac{10}{256(1+4\kappa)} < \frac{1}{4},$$

which proves the lemma.

In the following lemma we give an upper bound for the duality gap after a main iteration.

Lemma 4.10. *Let $0 < \theta < 1$. If $\delta \leq \frac{1}{16(1+4\kappa)}$, $\mathbf{x}^p$ and $\mathbf{s}^p$ are the iterates obtained after the predictor step of the algorithm, then*

$$(\mathbf{x}^p)^T \mathbf{s}^p \leq \left(1 - \frac{\theta}{2} + \frac{\theta^2}{8}\right) (\mathbf{x}^c)^T \mathbf{s}^c < \frac{3n\mu^p}{2(2-\theta)}.$$

Proof. Using (38) with $\alpha = 1$ and the definition of $\mathbf{v}^p$ we have

$$\begin{aligned} (\mathbf{x}^p)^T \mathbf{s}^p &= \mu^p \mathbf{e}^T (\mathbf{v}^p)^2 = \mu \mathbf{e}^T \left(\left(1 - \frac{\theta}{2}\right) (\mathbf{v}^c)^2 + \theta^2 \mathbf{d}_x^p \mathbf{d}_s^p \right) \\ &= \left(1 - \frac{\theta}{2}\right) (\mathbf{x}^c)^T \mathbf{s}^c + \mu\theta^2 (\mathbf{d}_x^p)^T \mathbf{d}_s^p. \end{aligned} \tag{47}$$

We multiply the second equation of (27) by $(\mathbf{d}_x^p)^T$ and by $(\mathbf{d}_s^p)^T$, respectively. After that, we sum the obtained two equations, hence

$$(\mathbf{d}_x^p)^T \mathbf{d}_s^p = \frac{(\mathbf{x}^c)^T \mathbf{s}^c}{8\mu} - \frac{\|\mathbf{d}_x^p\|^2 + \|\mathbf{d}_s^p\|^2}{2} \leq \frac{(\mathbf{x}^c)^T \mathbf{s}^c}{8\mu}. \tag{48}$$

Using (47) and (48) we get

$$(\mathbf{x}^p)^T \mathbf{s}^p \leq \left(1 - \frac{\theta}{2} + \frac{\theta^2}{8}\right) (\mathbf{x}^c)^T \mathbf{s}^c.$$

If $0 < \theta < 1$, then

$$1 - \frac{\theta}{2} + \frac{\theta^2}{8} < 1. \tag{49}$$

Furhermore, if $\delta \leq \frac{1}{16(1+4\kappa)}$ and $n \geq 1$, then

$$\delta^2 \leq \frac{n}{256(1+4\kappa)^2}.$$

Using this, $\mu^p = \left(1 - \frac{\theta}{2}\right)\mu$, (49) and Lemma 4.4 we have

$$\begin{aligned} (\mathbf{x}^p)^T \mathbf{s}^p &\leq \left(1 - \frac{\theta}{2} + \frac{\theta^2}{8}\right) (\mathbf{x}^c)^T \mathbf{s}^c < (\mathbf{x}^c)^T \mathbf{s}^c < \mu(n + 9\delta^2) \\ &< \frac{\mu^p}{1 - \frac{\theta}{2}} \left(n + \frac{9n}{256(1+4\kappa)^2}\right) < \frac{2\mu^p n}{2-\theta} \left(1 + \frac{9}{256}\right) = \frac{265n\mu^p}{256(2-\theta)} < \frac{3n\mu^p}{2(2-\theta)}, \end{aligned}$$

which yields the result.

4.4. Determination of the values of the proximity and update parameters

We choose the values of the parameters τ and θ in such a way that after a corrector and a predictor step, the proximity measure will not exceed the proximity parameter. Let $(\mathbf{x}, \mathbf{s}) \in \mathcal{N}_2(\tau, \mu)$. Using Lemma 4.2, after a corrector step we have

$$\delta^c := \delta(\mathbf{x}^c, \mathbf{s}^c, \mu) \leq \frac{5(1+4\kappa)\delta^2}{1-2(1+4\kappa)\delta^2}\sqrt{1-(1+4\kappa)\delta^2},$$

which is monotonically increasing with respect to δ, where $\delta < \frac{1}{\sqrt{2(1+4\kappa)}}$. In this way,

$$\delta^c \leq \frac{5(1+4\kappa)\tau^2}{1-2(1+4\kappa)\tau^2}\sqrt{1-(1+4\kappa)\tau^2} =: \omega(\tau).$$

From $\delta \leq \frac{1}{16(1+4\kappa)}$ and using Lemma 4.9 we have $\delta^c < \frac{1}{4}$. Using Lemma 4.8, after a predictor step and a μ-update we have

$$\delta^p := \delta(\mathbf{x}^p, \mathbf{s}^p, \mu^p) \leq \frac{\sqrt{z(\delta^c,\theta,n)}\left(10(1+4\kappa)\delta^2 + (1-4\delta^c)^2 - z(\delta^c,\theta,n)\right)}{4z(\delta^c,\theta,n)-2},$$

where $\delta := \delta(\mathbf{x}, \mathbf{s}, \mu)$ is the proximity measure given in (22). The function $z(\delta^c, \theta, n)$ is decreasing with respect to δ^c. Thus, $z(\delta^c, \theta, n) \geq z(\omega(\tau), \theta, n)$. In Lemma 4.8 we have seen that the function $h(t) = \frac{t}{2t^2-1}$, $t > \frac{\sqrt{2}}{2}$ is decreasing with respect to t, hence

$$h(\sqrt{z(\delta^c,\theta,n)}) \leq h(\sqrt{z(\omega(\tau),\theta,n)}).$$

Note that $(1-4\delta^c)^2 - z(\delta^c,\theta,n) = \frac{2n(2+\kappa)\,\theta^2\,(1+4\delta^c)^2}{4(2-\theta)}$ is increasing with respect to δ^c. Using this and $\delta < \tau$, $\delta^c < \omega(\tau)$, we obtain

$$\begin{aligned}
&\frac{\sqrt{z(\delta^c,\theta,n)}\left(10(1+4\kappa)\delta^2 + (1-4\delta^c)^2 - z(\delta^c,\theta,n)\right)}{4z(\delta^c,\theta,n)-2}\\
\leq\ &\frac{\sqrt{z(\omega(\tau)),\theta,n)}\left(10(1+4\kappa)\tau^2 + (1-4\omega(\tau))^2 - z(\omega(\tau),\theta,n)\right)}{4z(\omega(\tau),\theta,n)-2}.
\end{aligned} \tag{50}$$

Our aim is to to keep $\delta^p \leq \tau$. For this, it suffices that

$$\frac{\sqrt{z(\omega(\tau)),\theta,n)}\left(10(1+4\kappa)\tau^2 + (1-4\omega(\tau))^2 - z(\omega(\tau),\theta,n)\right)}{4z(\omega(\tau),\theta,n)-2} \leq \tau.$$

Setting $\tau = \frac{1}{16(1+4\kappa)}$ and $\theta = \frac{1}{4(1+4\kappa)\sqrt{n}}$, the above inequality holds. Thus, $\mathbf{x}, \mathbf{s} > \mathbf{0}$ and $\delta(\mathbf{x}, \mathbf{s}, \mu) \leq \frac{1}{16(1+4\kappa)} < \frac{1}{\sqrt{2(1+4\kappa)}}$ are maintained during the algorithm. This means that the proposed IPA is well-defined. Furthermore, we have

$$\begin{aligned}
z(\delta^c,\theta,n) &= (1-4\delta^c)^2 - \frac{2n(2+\kappa)\theta^2(1+4\delta^c)^2}{4(2-\theta)}\\
&\geq (1-4\omega(\tau))^2 - \frac{2n(2+\kappa)\theta^2(1+4\omega(\tau))^2}{4(2-\theta)} > \frac{1}{2},
\end{aligned}$$

hence the predictor step is strictly feasible. The way we have chosen the neighbourhood parameter shows that $(\mathbf{x}^p, \mathbf{s}^p) \in \mathcal{N}_2(\tau, \mu^p)$. Therefore,

$$(\mathbf{x}^p, \mathbf{s}^p) \in \mathcal{N}_2(\tau, \mu^p) \cap \mathcal{D}\left(\frac{1}{2}, \mu^p\right). \tag{51}$$

This is important, because it shows that the vector obtained after an iteration of the PC IPA given in Algorithm 3.1 remains in the neighbourhood obtained by the intersection of a small and a wide neighbourhood.

4.5. Complexity bound

The next lemma gives an upper bound for the number of iterations produced by the PC IPA.

Lemma 4.11. *Let* $\mathbf{x}^0$ *and* $\mathbf{s}^0$ *be strictly feasible,* $\theta = \frac{1}{4(1+4\kappa)\sqrt{n}}$, $\mu^0 = \frac{\left(\mathbf{x}^0\right)^T \mathbf{s}^0}{n}$ *and* $\delta(\mathbf{x}^0, \mathbf{s}^0, \mu^0) \leq \tau = \frac{1}{16(1+4\kappa)}$. *Moreover, let* $\mathbf{x}^k$ *and* $\mathbf{s}^k$ *be the iterates obtained after* k *iterations. Then,* $\left(\mathbf{x}^k\right)^T \mathbf{s}^k \leq \epsilon$ *for*

$$k \geq 1 + \left\lceil \frac{2}{\theta} \log \frac{3\left(\mathbf{x}^0\right)^T \mathbf{s}^0}{4\epsilon} \right\rceil.$$

Proof. Using Lemma 4.10 we have

$$\left(\mathbf{x}^k\right)^T \mathbf{s}^k < \frac{3n\mu^k}{4\left(1-\frac{\theta}{2}\right)} = \frac{3n\left(1-\frac{\theta}{2}\right)^{k-1}\mu^0}{4} = \frac{3\left(1-\frac{\theta}{2}\right)^{k-1}\left(\mathbf{x}^0\right)^T \mathbf{s}^0}{4}.$$

The inequality $\left(\mathbf{x}^k\right)^T \mathbf{s}^k \leq \epsilon$ holds if $\frac{3\left(1-\frac{\theta}{2}\right)^{k-1}\left(\mathbf{x}^0\right)^T \mathbf{s}^0}{4} \leq \epsilon$. We take logarithms, hence

$$(k-1)\log\left(1-\frac{\theta}{2}\right) + \log\frac{3\left(\mathbf{x}^0\right)^T \mathbf{s}^0}{4} \leq \log\epsilon.$$

From $\log(1+\theta) \leq \theta$, $\theta \geq -1$, it follows that the above inequality holds if

$$-\frac{\theta}{2}(k-1) + \log\frac{3\left(\mathbf{x}^0\right)^T \mathbf{s}^0}{4} \leq \log\epsilon.$$

This yields the desired result.

Theorem 4.12. *Let* $\tau = \frac{1}{16(1+4\kappa)}$ *and* $\theta = \frac{1}{4(1+4\kappa)\sqrt{n}}$. *Then, Algorithm 3.1 is well defined and the algorithm requires at most*

$$O\left((1+4\kappa)\sqrt{n}\log\frac{3n\mu^0}{4\epsilon}\right)$$

iterations. The output is a pair $(\mathbf{x}, \mathbf{s})$ *satisfying* $\mathbf{x}^T\mathbf{s} \leq \epsilon$.

5. Numerical results

We implemented a variant of the proposed PC IPA in the C++ programming language using [22]. We did all computations on a desktop computer with Intel quad-core 2.11 GHz processor and 16 GB RAM. It should be mentioned that the value of the parameter κ can be very large, which leads to a very small value of the parameter θ, see Theorem 4.12. This motivated us to make some modifications in the implementation of the proposed PC IPA.

Algorithm 5.1 : PC IPA from the implementation point of view

Let $\epsilon = 10^{-5}$, $\mathbf{x}^0 = \mathbf{s}^0 = \mathbf{e}$, $\mu^0 = 1$, $0 < \rho < 1$ *and* $lb = \frac{1}{2}$.

begin

$k := 0$;

while $\left(\mathbf{x}^k\right)^T \mathbf{s}^k > \epsilon$ **do begin**

predictor step

$\mu_p^k = \rho \frac{\min\{\mathbf{x}_i^k \mathbf{s}_i^k :\, 1 \leq i \leq n\}}{lb}$;

compute $(\Delta^p x^k, \Delta^p s^k)$ *from system (52);*

$\alpha_x^p = \min\left\{-\frac{x_i^k}{\Delta^p x_i^k} | \Delta^p x_i^k < 0, 1 \leq i \leq n\right\}$;

$\alpha_s^p = \min\left\{-\frac{s_i^k}{\Delta^p s_i^k} | \Delta^p s_i^k < 0, 1 \leq i \leq n\right\}$;

$\alpha^p = \min\{\alpha_x^p, \alpha_s^p\}$;

$(\mathbf{x}^p)^k := \mathbf{x}^k + \rho\alpha^p \Delta^p \mathbf{x}^k; \quad (\mathbf{s}^p)^k := \mathbf{s}^k + \rho\alpha^p \Delta^p \mathbf{s}^k$;

corrector step

$\mu_{aff}^k = \rho \frac{\min\{\left(\mathbf{x}_i^p\right)^k \left(\mathbf{s}_i^p\right)^k :\, 1 \leq i \leq n\}}{lb}$;

$\mu_c^k = \frac{(\mu_{aff}^k)^3}{(\mu_p^k)^2}$;

compute $(\Delta^c x^k, \Delta^c s^k)$ *from system (53);*

$\Delta x^k = \Delta^p x^k + \Delta^c x^k;\ \Delta s^k = \Delta^p s^k + \Delta^c s^k$;

$\alpha_x^c = \min\left\{-\frac{(x_i^p)^k}{\Delta x_i^k} | \Delta x_i^k < 0, 1 \leq i \leq n\right\}$;

$\alpha_s^c = \min\left\{-\frac{(s_i^p)^k}{\Delta s_i^k} | \Delta s_i^k < 0, 1 \leq i \leq n\right\}$;

$\alpha^c = \min\{\alpha_x^c, \alpha_s^c\}$;

$(\mathbf{x}^c)^k := (\mathbf{x}^p)^k + \rho\alpha^c \Delta x^k; \quad (\mathbf{s}^c)^k := (\mathbf{s}^p)^k + \rho\alpha^c \Delta s^k$;

$\mathbf{x}^{k+1} := (\mathbf{x}^c)^k, \quad \mathbf{s}^{k+1} := (\mathbf{s}^c)^k; \quad k := k+1$;

end

end.

Algorithm 5.1 illustrates the computational version of the theoretical PC IPA given in Algorithm 3.1. Due to the performed modifications it may happen that the new points generated by the Algorithm 5.1 can leave the feasibility region. That is why we determined the search directions by considering residual vector in first equation of the Newton system. Hence, we calculated the predictor search directions by solving the following system:

$$\begin{aligned} -M\Delta^p \mathbf{x} + \Delta^p \mathbf{s} &= \mathbf{q} + M\mathbf{x} - \mathbf{s}, \\ S\Delta^p \mathbf{x} + X\Delta^p \mathbf{s} &= g(\mathbf{x}, \mathbf{s}), \end{aligned} \tag{52}$$

where $g(\mathbf{x}, \mathbf{s})$ is given in (19) and it can have different values for different functions φ. In our case $g(\mathbf{x}, \mathbf{s}) = -\frac{\mathbf{xs}}{2}$. Similarly, the corrector search directions were obtained by solving

$$\begin{aligned} -M\Delta^c \mathbf{x} + \Delta^c \mathbf{s} &= \mathbf{q} + M\mathbf{x} - \mathbf{s}, \\ S\Delta^c \mathbf{x} + X\Delta^c \mathbf{s} &= \mathbf{a}_\varphi, \end{aligned} \tag{53}$$

where $\mathbf{a}_\varphi$ is given in (9). In our case $\mathbf{a}_\varphi = \frac{\mu\mathbf{xs}}{2(2\mathbf{xs} - \mu\mathbf{e})} - \frac{\mathbf{xs}}{2}$. The value of the parameter μ in the predictor step was calculated as $\mu_p^k = \rho \frac{\min\{\mathbf{x}_i^k \mathbf{s}_i^k :\, 1 \leq i \leq n\}}{lb}$, where $0 < \rho < 1$, lb denotes a given lower bound, which in our case is $\frac{1}{2}$. In our case the value of ρ was 0.95. The way of determining the value of the parameter μ_p^k ensures that the components of the vector $\mathbf{v}$ are greater than a positive constant, which is important in our case due to the used search direction. After that we calculated the maximal step size α_x^p and α_s^p to the boundary of nonnegative orthant by using minimal ratio test. We considered the minimum value of these step sizes and we determined the vectors $\mathbf{x}^p$ and $\mathbf{s}^p$ without modifying the actual points $\mathbf{x}^k$ and $\mathbf{s}^k$. Note that the vectors $\mathbf{x}^p$ and $\mathbf{s}^p$ were used in the computation of the parameter μ_{aff}, which was calculated in a similar way as the value of the parameter μ^p at the beginning of an iteration. In Algorithm 5.1 we can see that in the calculation of the parameter μ before the corrector step we used Mehrotra's heuristics [44]. It should be mentioned that we considered the search directions obtained by the sum of the predictor and the corrector directions. In the determination of the step length in case of the corrector step we used the same strategy as in case of the predictor step.

We tested the PC IPA on LCPs with sufficient matrices having positive κ parameters generated by Illés and Morapitiye [30]. We generated the test problems in the following way: $\mathbf{q} := -M\mathbf{e} + \mathbf{e}$. We considered $\mathbf{x}^0 = \mathbf{e}$ and $\mathbf{s}^0 = \mathbf{e}$ as starting points for our PC IPA.

We have tested the PC IPA for all 61 $P_*(\kappa)$-LCPs from the selection given in [30]. We could easily obtain results for variants of the PC IPA using different functions φ in this new type of AET technique by changing the right hand side of the Newton-system. In our computational study we compared our PC IPA using the function $\varphi(t) = t^2$ in system (6) with the variant of the IPA which uses the $\varphi(t) = t$ in the new type of AET technique characterized by system (6). Note that in the case when $\varphi(t) = t$ is used, then the value of lb is $\frac{1}{4}$, $g(\mathbf{x}, \mathbf{s}) = -\mathbf{xs}$ and $\mathbf{a}_\varphi = \frac{\sqrt{\mu}\mathbf{xs}}{2\sqrt{\mathbf{xs}} - \sqrt{\mu}\mathbf{e}}$. This yields the same direction as the one used in [18], where system (5) was considered with $\bar{\varphi}(t) = t - \sqrt{t}$. Table 2 contains the average of iteration numbers and CPU times (in seconds) for 10 given LCPs for each size n listed in the table. We can observe that the results are similar for both variants of the PC IPA using the different search directions.

n	$\varphi(t) = t^2$		$\varphi(t) = t$	
	Avg. Iter.	CPU	Avg. Iter.	CPU
10	7	0.0125	7	0.0119
20	7.5	0.0434	7.6	0.0398
50	5.6	0.0963	5.2	0.0898
100	6.1	0.4339	5.6	0.3946
200	6.7	2.8455	6.0	2.6184
500	7	39.6898	6.7	38.2645

Table 1: Numerical results for $P_*(\kappa)$-LCPs from [30] having positive handicap.

De Klerk and E.-Nagy [23] proved that the handicap of the matrix can be exponential in the size of the problem. They considered the following matrix which was proposed by Zs. Csizmadia:

$$M = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ -1 & 1 & 0 & \cdots & 0 \\ -1 & -1 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & -1 & \cdots & 1 \end{pmatrix}, \tag{54}$$

and they proved that $\hat{\kappa}(M) \geq 2^{2n-8} - 0.25$. However, in our computational study we obtained promising results for the two variants of PC IPAs. The results are summarized in Table 2.

n	$\varphi(t) = t^2$		$\varphi(t) = t$	
	Nr. of Iter.	CPU (s)	Nr. of Iter.	CPU (s)
10	12	0.022	12	0.024
20	15	0.049	15	0.069
50	25	0.782	25	0.842
100	43	5.485	43	5.407
200	78	29.04	78	29.345
300	113	132.247	113	133.697
400	149	396.283	149	403.419

Table 2: Numerical results for $P_*(\kappa)$-LCPs with matrix given in (54)

The obtained results can be further analysed, because it seems that the practical iteration complexity is significantly better than the theoretical (worst case) guarantee for the special class of LCPs with the lower triangular P-matrix M, introduced by Zs. Csizmadia.

6. Conclusions and further research

In this paper we proposed a new PC IPA for solving $P_*(\kappa)$-LCPs which uses the new type of AET given in [21] for LO. The presented IPA applies the function $\varphi(t) = t^2$ on the nonlinear equation $\mathbf{v}^2 = \mathbf{v}$ in order to determine the new search directions. The corresponding kernel function is a positive-asymptotic kernel function. Furthermore, similar to [18], we presented the method for determining the Newton systems and scaled systems in case of PC IPAs using this new type of AET. Due to the used search direction we had to ensure during the whole process of the IPA that the components of the vector $\mathbf{v}$ were greater than $\frac{\sqrt{2}}{2}$. In spite of this fact, we proved that the PC IPA retains polynomial iteration complexity in the handicap of the problem's matrix, the size of the problem, the bitsize of the data and the deviation from the complementarity gap. This is the first PC IPA for solving $P_*(\kappa)$-LCPs which uses the function $\varphi(t) = t^2$ in the new type of AET. Moreover, we also provided numerical results where we compared our PC IPA to another variant of this algorithm using $\varphi(t) = t$ in the new type of AET technique. As further research, it would be interesting to find a class of monotone increasing functions $\bar{\varphi}$ for which we can assign corresponding functions φ. This would lead to a case where we can establish equivalence between the two approaches of the AET presented in this paper.

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