The existence of an inverse limit of inverse system of measure spaces – a purely measurable case*

Miklós Pintér
Corvinus University of Budapest†

July 26, 2009

Abstract

The existence of an inverse limit of an inverse system of (probability) measure spaces has been investigated since the very beginning of the birth of the modern probability theory. Results from Kolmogorov [10], Bochner [2], Choksi [5], Metivier [14], Bourbaki [3] among others have paved the way of the deep understanding of the problem under consideration. All the above results, however, call for some topological concepts, or at least ones which are closely related topological ones. In this paper we investigate purely measurable inverse systems of (probability) measure spaces, and give a sufficient condition for the existence of a unique inverse limit. An example for the considered purely measurable inverse systems of (probability) measure spaces is also given.

1 Introduction

The existence of an inverse limit of an inverse system of (probability) measure spaces is an important question in probability theory (see e.g. Kolmogorov [10]), in the theory of stochastic processes (see e.g. Rao [18]), and in some sense surprisingly in game theory.

*The author thanks the Hungarian Scientific Research Fund (OTKA) and the János Bolyai Research Scholarship of the Hungarian Academy of Sciences for financial support.

†Department of Mathematics, Corvinus University of Budapest, 1093 Hungary, Budapest, Fővám tér 13-15., miklos.pinter@uni-corvinus.hu

On the field of game theory, actually on the subfield of games with incomplete information, the investigation of the so called hierarchies of beliefs – in the language of mathematics: inverse systems of (probability) measure spaces, see Harsányi [7], Mertens and Zamir [12], Brandenburger and Dekel [4], Heifetz [8], Mertens at al. [13], Pintér [17] among others – calls for the applications of certain existence results.

All the previous results – Kolmogorov, Bochner [2], Choksi [5], Metivier [14], Bourbaki [3] among others – on the existence of an inverse limit of an inverse system of measure spaces, however, calls for topological concepts, various compactness concepts, even if it does not seems at first glance (see e.g. Metivier's result).

The case of hierarchies of beliefs is, however, a special one, it is quite natural to consider these inverse systems of measure spaces as purely measurable ones (see e.g. Heifetz and Samet [9]), so in order to examine the existence of an inverse limit of an inverse system of measure spaces in the purely measurable framework we need a purely measurable existence result.

The main result of this paper is Theorem 7. which provides purely measurable sufficient condition for the existence of a unique inverse limit. Actually, in this paper we introduce the concept of ε-completeness (see Definition 6.), and show that it is sufficient but not necessary condition for the existence of a unique inverse limit of an inverse system of (probability) measure spaces. In our opinion, this result is a common generalization of all the previously cited results (Kolmogorov, Bochner, Choksi, Metivier, Bourbaki), and we discuss it in more detail after Theorem 7.

The mathematical core of a game theoretical problem, Proposition 11., as an example for a class of ε-complete inverse systems of (probability) measure spaces, is also given.

The setup is as follows. In the next section we provide some basic concepts of inverse systems and inverse limits. In Section 3. we present our main result, and Section 4. is about the above mentioned example for the application of our main result. In the last section we mention some obvious generalizations, and conclude briefly.

2 Inverse systems, inverse limits

First some notions and notations. In this paper we work with probability measures, hence if we do not indicate differently we mean every measure is a probability measure.

Let A be an arbitrary set, then $\#A$ is for the cardinality of set A. For any $A \subseteq \mathcal{P}(X)$: $\sigma(A)$ is the coarsest σ-field which contains A. Let $(X, \mathcal{M})$

and $(Y, \mathcal{N})$ be arbitrary measurable spaces. Then $(X \times Y, \mathcal{M} \otimes \mathcal{N})$ or briefly $X \otimes Y$ is the measurable space on the set $X \times Y$ equipped by the σ-field $\sigma(\{A \times B \mid A \in \mathcal{M},\ B \in \mathcal{N}\})$.

Let $\mathcal{A} \subseteq \mathcal{P}(\mathcal{X})$ be a set ring, $\mathcal{C} \subseteq \mathcal{A}$, μ additive set function on $\mathcal{A}$; μ is $\mathcal{C}$-regular if for arbitrary $\epsilon > 0$ and for arbitrary $A \in \mathcal{A}$, $\exists C \in \mathcal{C}$ such that $C \subseteq A$ and $\mu(A \setminus C) < \epsilon$.

Set system $\mathcal{A} \subseteq \mathcal{P}(X)$ is a σ-compact set system, if for any $\{A_n\} \subseteq \mathcal{A}$: $\bigcap_n A_n = \emptyset$ implies that $\exists n^*$ s. t. $\bigcup_{n=1}^{n^*} A_n = \emptyset$.

Let $(X, \mathcal{M}, \mu)$ be an arbitrarily fixed (probability) measure space. Then the set function μ^* on $\mathcal{P}(X)$ is defined as follows: $\forall A \in \mathcal{P}(X)$: $\mu^*(A) \mathrel{\hat=} \inf_{A \subseteq B \in \mathcal{M}} \mu(B)$ (therefore μ^* is an outer measure).

For any topological space (X, τ): $B(X, \tau)$ is for the Borel σ-field.

We say the measurable spaces $(X, \mathcal{M})$ and $(Y, \mathcal{N})$ are measurable isomorphic if there is such a bijection f between them that both f and f^{-1} are measurable.

Next, we discuss the basic concepts of inverse systems and inverse limits.

Definition 1. *Let $(I, \leq)$ be a preordered set, $(X_i)_{i \in I}$ be a family of nonvoid sets, and $\forall i, j \in i$ s.t. $i \leq j$: $f_{ij} : X_j \to X_i$. The system $(X_i, (I, \leq), f_{ij})$ is an inverse system if it meets the following points, $\forall i, j, k \in I$ s.t. $i \leq j$ and $j \leq k$:*

1. *$f_{ii} = id_{X_i}$,*
2. *$f_{ik} = f_{ij} \circ f_{jk}$.*

The inverse system (projective system) is a system of sets connected in a certain way.

Definition 2. *Let $((X_i, \mathcal{A}_i, \mu_i), (I, \leq), f_{ij})$ be such an inverse system that $\forall i \in I$: $(X_i, \mathcal{A}_i, \mu_i)$ is a measure space. The inverse system $((X_i, \mathcal{A}_i, \mu_i), (I, \leq), f_{ij})$ is an inverse system of measure spaces if it meets the following points, $\forall i, j \in I$ s.t. $i \leq j$:*

1. *f_{ij} is a $\mathcal{A}_j$-measurable function,*
2. *$\mu_i = \mu_j \circ f_{ij}^{-1}$.*

Definition 3. *Let* $(X_i, (I, \leq), f_{ij})$ *be an inverse system. Let* $X \triangleq \prod_{i \in I} X_i$, *and* $P \triangleq \{x \in X \mid pr_i(x) = f_{ij} \circ pr_j(x),\ \forall (i \leq j)\}$, *where* $\forall i \in I$: pr_i *is the coordinate projection from* X *to* X_i.

Then P *is called the inverse limit of the inverse system* $(X_i, (I, \leq), f_{ij})$, *and it is denoted by* $\varprojlim(X_i, (I, \leq), f_{ij})$.

Moreover, let $p_i \triangleq pr_i|_P$, *so* $\forall i.j \in I$ *s.t.* $i \leq j$: $p_i = f_{ij} \circ p_j$.

The inverse limit is the generalization of the Cartesian product. If $\leq$ in $(I, \leq)$ is the empty relation, then the inverse limit is the Cartesian product.

Definition 4. *Let* $((X_i, \mathcal{A}_i, \mu_i), (I, \leq), f_{ij})$ *be an inverse system of measure spaces, and* $P \triangleq \varprojlim(X_i, (I, \leq), f_{ij})$. *Then the* $(P, \mathcal{A}, \mu)$ *measure space is the inverse limit of the inverse system of measure spaces* $((X_i, \mathcal{A}_i, \mu_i), (I, \leq), f_{ij})$ *and it is denoted by* $(P, \mathcal{A}, \mu) \triangleq \varprojlim((X_i, \mathcal{A}_i, \mu_i), (I, \leq), f_{ij}|_{i \leq j})$. *if it meets the following points,* $\forall i \in I$:

1. $\mathcal{A}$ *is the coarsest* σ*-algebra w.r.t.* p_i *is* $\mathcal{A}$*-measurable,*

2. μ *is such a measure that* $\mu \circ p_i^{-1} = \mu_i$.

Kolmogorov's [10] extension theorem is an inverse limit result. It states the following: let consider a family of finite dimensional distributions on $\mathbb{R}^d$ $(d < \infty)$ s.t. they are consistent, i.e. $\forall n$: μ_n on $(\mathbb{R}^n, B(\mathbb{R}^n)$ is the marginal distribution of μ_{n+1} on $(\mathbb{R}^{n+1}, B(\mathbb{R}^{n+1}))$. Then there exists such a unique distribution μ on $(\mathbb{R}^{\mathbb{N}}, B(\mathbb{R}^{\mathbb{N}}))$ that $\forall n$: μ_n is the marginal distribution of μ. This theorem is called the Kolmogorov Extension Theorem. In this case the inverse system of measure spaces is as follows: $((\mathbb{R}^n, B(\mathbb{R}^n),\ \mu_n), \mathbb{N}, pr_{mn})$, where $pr_{mn} : \mathbb{R}^n \to \mathbb{R}^m$ is a coordinate projection. Therefore, the Kolmogorov Extension Theorem states that $\varprojlim((\mathbb{R}^n, B(\mathbb{R}^n), \mu_n), \mathbb{N}, pr_{mn})$ exists and is unique.

Since if $\varprojlim(X_i, (I, \leq), f_{ij}) \neq \emptyset$ then $\bigcup_i p_i^{-1}(\mathcal{M}_i)$ is an algebra (field), and set function μ defined on $\bigcup_i p_i^{-1}(\mathcal{M}_i)$ as $\forall i \in I$: $\mu \circ p_i^{-1} = \mu_i$ is an additive set function, the main problem of the existence of measure inverse limit is the σ-additivity of μ.

Numerous results discuss this problem, the most important ones are as follows: Kolmogorov, Bochner [2], Choksi's [5], Millington and Sion [15], Metivier's [14], Mallory and Sion's [11], Bourbaki's [3]. The next theorem is a variant of Metivier's result, which can be considered as the most general one, at least in this framework, among the above result

Theorem 5 (Metivier). *Let $((X_n, \mathcal{M}_n, \mu_n), \mathbb{N}, f_{mn})$ be an inverse system of (probability) measure spaces*[1]. *If $\forall m, n \in \mathbb{N}$ s.t. $m \leq n$:*

1. *$\mathcal{C}_n \subseteq \mathcal{M}_n$ is σ-compact set system,*
2. *f_{mn} is a surjective (onto) function,*
3. *$f_{mn}(\mathcal{C}_n) \subseteq \mathcal{C}_m$,*
4. *$\forall x_m \in X_m$: $f_{mn}^{-1}(\{x_m\}) \cap \mathcal{C}_n$ is σ-compact set system,*
5. *$\forall C_k \in \mathcal{C}_n$, $k \in \mathbb{N}$: $\bigcap_k C_k \in \mathcal{C}_n$,*
6. *μ_n is $\mathcal{C}_n$-regular,*

then

$$(X, \mathcal{M}, \mu) \circeq \varprojlim((X_n, \mathcal{M}_n, \mu_n), \mathbb{N}, f_{mn}|_{m \leq n})$$

exists and is unique, moreover μ is $\mathcal{C}$-regular, where $\mathcal{C}$ consists of such sets that $\forall C \in \mathcal{C}$: $\exists C_n \in \mathcal{C}_n$ s.t. $\forall n$: $f_{nn+1}^{-1}(C_n) \subseteq C_{n+1}$ and $C = \varprojlim(C_n, \mathbb{N}, f_{mn})$, and $\mathcal{C}$ is σ-compact set system.

Proof. See Metivier 3.2. Theoreme pp. 269–270. Q.E.D.

Notice that in the "classical" Kolmogorov Extension Theorem the inverses system of measure spaces $((\mathbb{R}^n, B(\mathbb{R}^n), \mu_n), \mathbb{N}, pr_{mn})$ meets the condition of the above theorem.

3 The general result

The key notion of the paper:

Definition 6. *The inverse system of measure spaces $((X_n, \mathcal{M}_n, \mu_n), \mathbb{N}, f_{mn})$ is ε - complete, if $\forall \varepsilon \in [0, 1]$, $\forall (m \leq n)$, and $\forall A \subseteq X_m$:*

$$(\mu_n^*(f_{mn}^{-1}(A)) = \varepsilon) \Rightarrow (\mu_m^*(A) = \varepsilon) \ .$$

[1] Metivier's assumption is weaker, he requires that the measures should be non-negative σ-finite measures, and the index set is s.t. that has countable cofinal subset.

It is a slight calculation to verify that Halmos' [6] example (exercise (3) pp. 214–215) is not an ε-complete inverse system of measure spaces. Furthermore, the reader may wonder whether it is enough to weaken ε-completeness for the case of $\varepsilon = 0$. The answer is "not," if we manipulate Halmos' above mentioned example in such a way that the diagonal has $1-\varepsilon$ measure, and the set of the off diagonal elements has ε (we distort Halmos' example by ε), then we get at an 0-complete inverse system of measure spaces having no inverse limit.

A further feature, the ε-completeness is not necessary condition for the existence of inverse limit. For this fact see the following inverse system of measure spaces:

$$(([0,\frac{1}{n+1}),\{\emptyset,[0,\frac{1}{n+1})\},\delta_0),\mathbb{N},id_{0,\frac{1}{n+1})}) \ , \tag{1}$$

where δ_0 is for the Dirac measure concentrated at point 0. It is easy to verify that (1) is an inverse system of measure spaces and its inverse limit is $(\{0\},\{\emptyset,\{0\}\},\delta_0)$. However, (1) is not ε-complete.

The next theorem is the main result of this paper.

Theorem 7. *Let $((X_n,\mathcal{M}_n,\mu_n),\mathbb{N},f_{mn})$ be an ε-complete inverse system of measure spaces. Then $(X,\mathcal{M},\mu) \doteq \varprojlim((X_n,\mathcal{M}_n,\mu_n),\mathbb{N},f_{mn})$ exists and is unique.*

Proof. (1) $X \doteq \varprojlim(X_n,\mathbb{N},\ f_{mn}) \neq \emptyset$: $\forall n$: let

$$Q_n \doteq \{x \in X_0 \mid f_{0n}^{-1}(\{x\}) = \emptyset\} \ .$$

It is clear that $\forall n$: $Q_n \subseteq Q_{n+1}$.

From ε-completeness $\forall n$:

$$\mu_0^*(Q_n) = 0 \ ,$$

hence

$$\mu_0^*(\bigcup_n Q_n) = 0 \ ,$$

therefore

$$\complement(\bigcup_n Q_n) \neq \emptyset \ .$$

Finally, from the definition of $\{Q_n\}_n$, $\forall x_0 \in \complement(\bigcup_n Q_n)$: $\exists x \in X$ such that $x_0 = p_0(x)$, where $p_0 : X \to X_0$ is the canonical projection.

(2) $\bigcup_n p_n^{-1}(\mathcal{M}_n)$ is a field, and μ is an additive set function on it: From point (1) $\bigcup_n p_n^{-1}(\mathcal{M}_n)$ is a field.

$\forall A \in \bigcup_n p_n^{-1}(\mathcal{M}_n)$: let

$$\mu(A) \mathrel{\hat{=}} \mu_n(\hat{A}) ,$$

where n and $\hat{A} \in \mathcal{M}_n$ are such that $A = p_n^{-1}(\hat{A})$.

Let $\hat{A} \in \mathcal{M}_m$ and $\hat{B} \in \mathcal{M}_{m'}$ be such that $p_m^{-1}(\hat{A}) = p_{m'}^{-1}(\hat{B})$. Then $\exists n^*$: $n^* \geq m$ and $n^* \geq m'$, and $\forall n \geq n^*$: let

$$Q'_n \mathrel{\hat{=}} \{x \in X_{n^*} \mid f_{n^*n}^{-1}(\{x\}) = \emptyset\} .$$

It is clear that $\forall n \geq n^*$: $Q'_n \subseteq Q'_{n+1}$.

From ε-completeness $\forall n \geq n^*$:

$$\mu^*_{n^*}(Q'_n) = 0 ,$$

therefore

$$\mu^*_{n^*}(\bigcup_{n \geq n^*} Q'_n) = 0 .$$

$p_m^{-1}(\hat{A}) = p_{m'}^{-1}(\hat{B})$ implies that

$$f_{mn^*}^{-1}(\hat{A}) \bigtriangleup f_{m'n^*}^{-1}(\hat{B}) \subseteq \bigcup_{n \geq n^*} Q'_n ,$$

where $\bigtriangleup$ is for the symmetric difference. Put it differently $f_{nn^*}^{-1}(\hat{A}) \bigtriangleup f_{m'n^*}^{-1}(\hat{B}) \in \mathcal{M}_{n^*}$ and $\mu_{n^*}(f_{mn^*}^{-1}(\hat{A}) \bigtriangleup f_{m'n^*}^{-1}(\hat{B})) = 0$. Then

$$\mu_m(\hat{A}) = \mu_{m'}(\hat{B}) ,$$

i.e. μ is well-defined (unique).

By repeating the above reasoning we get at that μ is additive.

(3) μ is σ-additive: Let $\{A_n\}_n$, $\forall n$: $A_n \in \mathcal{M}_n$ be such sets that $f_{nn+1}^{-1}(A_n) \supseteq A_{n+1}$, and $\bigcap_n p_n^{-1}(A_n) = \emptyset$ (i.e. we show that μ is upper σ-continuous at $\emptyset$).

$\forall n$: let

$$L_n \mathrel{\hat{=}} \{x \in A_0 \mid f_{0n}^{-1}(\{x\}) \cap A_n = \emptyset\} .$$

It is clear that $\forall n$: $L_n \subseteq L_{n+1}$.

Since $\forall n$: $f_{nn+1}^{-1}(A_n) \supseteq A_{n+1}$, therefore $\forall n$:

$$f_{0n}^{-1}(L_n) \subseteq f_{0n}^{-1}(A_0) \setminus A_n \ ,$$

hence

$$\mu_n^*(f_{0n}^{-1}(L_n)) \leq \mu_0(A_0) - \mu_n(A_n) \ .$$

ε-completeness implies that $\forall n$:

$$\mu_0^*(L_n) \leq \mu_0(A_0) - \mu_n(A_n) \ .$$

$\{\mu_n(A_n)\}_n$ is a bounded monotonic decreasing sequence of real numbers, hence it is convergent. μ_0 is a probability measure, hence

$$\mu_0^*(\bigcup_n L_n) = \lim_{n\to\infty} \mu_0^*(L_n) \leq \mu_0(A_0) - \lim_{n\to\infty} \mu_n(A_n) \ .$$

From the definition of $\{A_n\}_n$, $\bigcap_n p_n^{-1}(A_n) = \emptyset$, hence $\lim_{n\to\infty} L_n = A_0$: $\mu_0^*(\bigcup_n L_n) = \mu_0(A_0)$, moreover, from point (2): $\forall n$: $\mu(p_n^{-1}(A_n)) \doteq \mu_n(A_n)$, hence

$$\mu(p_n^{-1}(A_n)) \to 0 \ .$$

(4) The extension of μ: From point (3) μ is a probability measure on the field $\bigcup_n p_n^{-1}(\mathcal{M}_n)$, hence it can be uniquely extended onto $\mathcal{M} \doteq \sigma(\bigcup_n p_n^{-1}(\mathcal{M}_n))$.

Q.E.D.

Remark 8. It is worth noticing that the conditions of Metivier's theorem (Theorem 5.) imply that the considered inverse system of measure spaces is ε-complete.

Let $n \in \mathbb{N}$ and $A \subseteq X_n$ be arbitrarily fixed, and $\varepsilon \doteq \mu_{n+1}^*(f_{nn+1}^{-1}(A))$. Since $\mathcal{M}_{n+1}$ is a σ-field, $\exists B \in \mathcal{M}$ s.t. $B \subseteq \complement f_{nn+1}^{-1}(A)$ and $\mu_{n+1}(B) = 1 - \varepsilon$.

From point 6. of Theorem 5. $\exists \{C_m\} \subseteq \mathcal{C}_{n+1}$ s.t. $\forall m$: $C_m \subseteq B$ and $\lim_{m\to\infty} \mu_{n+1}(C_m) = 1 - \varepsilon$.

From points 2. and 3. of Theorem 5. $\forall m$: $f_{nn+1}(C_m) \in \mathcal{M}_n$ and $\mu_n(f_{nn+1}(C_m)) \geq \mu_{n+1}(C_m)$.

Since $f_{nn+1}^{-1}(\complement A) = \complement f_{nn+1}^{-1}(A)$, $\forall m$: $f_{nn+1}(C_m) \subseteq \complement A$, and $\lim_{m\to\infty} \mu_{n+1}(C_m) = 1 - \varepsilon$ implies that $\mu^*(A) = \varepsilon$.

At first glance it seems that Theorem 7. requires much less than Theorem 5., but it provides the same result. From the viewpoint of the existence of a unique inverse limit of an inverse system of measure spaces it is true, however, genererally we can say only that while all the previous results require more

(Theorem 5. is included) they also provide more, they state not only that the inverse limit exists and is unique, but they characterize it. On the other hand, the characterization is used in their proofs.

4 An example

First we introduce the measurable structure used in this section.

Definition 9. *Let $(X, \mathcal{M})$ be arbitrarily fixed measurable space, and denote $\Delta(X, \mathcal{M})$ the set of the probability measures on it. Then the σ-field $\mathcal{A}^*$ on $\Delta(X, \mathcal{M})$ is defined as follows:*

$$\mathcal{A}^* \circeq \sigma(\{\{\mu \in \Delta(X, \mathcal{M}) \mid \mu(A) \geq p\},\ A \in \mathcal{M},\ p \in [0,1]\}) \ .$$

In other words

$$\mathcal{A}^* \circeq \sigma(\bigcup_{A \in \mathcal{M}} pr_A^{-1}(B([0,1]))) \ ,$$

where $pr_A : [0,1]^{\mathcal{M}}|_{\Delta(X,\mathcal{M})} \to [0,1]_A$, $[0,1]_A$ is the A copy of $[0,1]^{\mathcal{M}}$, and $B([0,1])$ is for the Borel σ-field on $[0,1]$.

The $\mathcal{A}^*$ σ-field is specially interesting in interactive epistemology. Games with incomplete information are such games where the players are uncertain about some parameters of the game (the description of the game). In such games it is a cardinal problem that what the players believe about the game, and what the players believe about the other players' beliefs about the game an so on. Interactive epistemology is on this problem, it deals with the players' beliefs and knowledge. It is usual in the literature that the players' beliefs are modeled by probability measures.

In order to talk about the players' beliefs we need sentences like player i believe with at least probability α that event A happens (see e.g. Aumann [1], Heifetz and Samet [9]). This requirement is formalized in Definition 9. More precisely, the σ-field in Definition 9. is the weakest σ-field among those meet the above requirement, where the events (of the base space) are of $\mathcal{M}$.

Notice that $\mathcal{A}^*$ is not a fixed σ-field, it depends on the measurable space on which the probability measures are defined. Therefore $\mathcal{A}^*$ is similar to the *weak** topology which depends on the topology of the base (primal) space. Henceforth the non-written σ-fields are the $\mathcal{A}^*$ σ-fields.

The following diagram introduces the mathematical problem considered in this section. Altough Mertens et al.'s [13] consider topological spaces and Borel σ-fields, and compact regular measures, and our model is purely measurable the following formalization of the problem conceptually comes from Mertens et al.'s paper.

Definition 10. *In diagram* (2)

$$
\begin{array}{ccc}
\Theta^i & & \Delta(S \otimes \Theta^{N\setminus\{i\}}) \\
\downarrow p^i_{n+1} & & \downarrow id_S \quad \downarrow p^{N\setminus\{i\}}_n \\
\Theta^i_{n+1} & \doteq & \Delta(S \otimes \Theta^{N\setminus\{i\}}_n) \\
\downarrow q^i_{nn+1} & & \downarrow id_S \quad \downarrow q^{N\setminus\{i\}}_{n-1n} \\
\Theta^i_n & \doteq & \Delta(S \otimes \Theta^{N\setminus\{i\}}_{n-1})
\end{array}
\tag{2}
$$

- *N is an arbitrary set,*
- *$i \in N$ is an arbitrarily element of N,*
- *$n \in \mathbb{N}$,*
- *$(S, \mathcal{A})$ is an arbitrary measurable space,*

  moreover $\forall j \in N$:
- $\#\Theta^j_{-1} = 1$,
- $q^j_{-10} : \Theta^j_0 \to \Theta^j_{-1}$,
- $\forall n \in \mathbb{N} \cup \{-1\}$*:* $\Theta^{N\setminus\{j\}}_n \doteq \bigotimes_{k \in N\setminus\{j\}} \Theta^k_n$,
- $\forall m, n \in \mathbb{N}$, $m \leq n$, $\forall \mu \in \Theta^j_n$*:*

  $$q^j_{mn}(\mu) \doteq \mu|_{S \otimes \Theta^{N\setminus\{j\}}_{m-1}} \ ,$$

  therefore q^j_{mn} is a measurable mapping.
- $\Theta^j \doteq \varprojlim(\Theta^j_n, \mathbb{N} \cup \{-1\}, q^j_{mn})$,
- $\forall n \in \mathbb{N} \cup \{-1\}$*:* $p^j_n : \Theta^j \to \Theta^j_n$ *is canonical projection,*

- $\forall m, n \in \mathbb{N} \cup \{-1\}$, $m \leq n$: $q_{mn}^{N \setminus \{j\}}$ *is the product of the mappings* q_{mn}^k, $k \in N \setminus \{j\}$, *and so is* $p_n^{N \setminus \{j\}}$ *of* p_n^k, $k \in N \setminus \{j\}$, *therefore both mappings are measurable,*

- $\Theta^{N \setminus \{j\}} \circeq \bigotimes_{k \in N \setminus \{j\}} \Theta^k$.

Let me give some intuitions what the above definition is about. Let i be an arbitrary player and $(S, \mathcal{A})$ be the parameter space. The paramer space can be interpreted as the space that contains every parameter of the game (e.g. payoffs, etc.). Then Θ_1^i is the space of the so called first order beliefs of player i, i.e. this consists of player i's beliefs about the parameters of the game.

We assume that every player knows her own beliefs, so does player i, therefore it is enough to focus on the beliefs of players not i. Then Θ_2^i is the space of the so called second order beliefs of player i, i.e. this consists of player i's beliefs about the other players' beliefs about parameters of the game. In the same way we can interpret Θ_n^i as the space of player i's nth order beliefs. Finally Θ^N consists of every player's every hierarchy of beliefs i.e. of every player's arbitrary high order beliefs.

In this framework the goal is to "construct" an object, so called type space (see Heifetz and Samet), which consists of the parameters and all the hierarchies of all players, and for any player i, and for any probability measure μ on $S \otimes \Theta^{N \setminus \{i\}}$ there exists $\theta^i \in \Theta^i$ s.t.

$$(S \otimes \Theta^{N \setminus \{i\}}, \mu) = ((S \otimes \Theta_n^{N \setminus \{i\}}, p_{n+1}^i(\theta^i)), \mathbb{N} \cup \{-1\}, (id_S, q_{mn}^{N \setminus \{i\}})) \ .$$

An object having the above property is an ideal one for the propurses of game theory.

The following proposition formalizes the above discussed goal. Since every previous result on this problem – Mertens and Zamir [12], Brandenburger and Dekel [4], Heifetz [8], Mertens at al., Pintér [17] among others – applies topological assumptions and ours do not, the next result is new.

Proposition 11. $\forall i \in N$: $\Theta^i = \Delta(S \otimes \Theta^{N \setminus \{i\}})$, *i.e. they are measurable isomorphic.*

It is easy to verify that all we need to prove is that $\forall i \in N$, $\forall \theta^i \in \Theta^i$: the inverse system of measure spaces

$$((S \otimes \Theta_n^{N \setminus \{i\}}, p_{n+1}^i(\theta^i)), \mathbb{N} \cup \{-1\}, (id_S, q_{mn}^{N \setminus \{i\}})) \tag{3}$$

admits a unique inverse limit, where $(id_S, q_{mn}^{N\setminus\{i\}})$ is the product of the given mappings.

In order to apply Theorem 7. we need the following result.

Lemma 12. *Let $(X, \mathcal{M}_1)$ and $(X, \mathcal{M}_2)$ be such measurable spaces that $\mathcal{M}_1 \subseteq \mathcal{M}_2$, and $f : \Delta(X, \mathcal{M}_2) \to \Delta(X, \mathcal{M}_1)$ be defined as $\forall \nu \in \Delta(X, \mathcal{M}_2)$:*

$$f(\nu) \circeq \nu|_{(X,\mathcal{M}_1)} \ .$$

Moreover, let μ_2 be an arbitrary measure on $\Delta(X, \mathcal{M}_2)$ and $\mu_1 \circeq \mu_2|_{\Delta(X,\mathcal{M}_1)}$.

Then $\forall A \subseteq \Delta(X, \mathcal{M}_1)$, $\forall \varepsilon \in [0, 1]$:

$$(\mu_2^*(f^{-1}(A)) = \varepsilon) \Rightarrow (\mu_1^*(A) = \varepsilon) \ .$$

Proof. Let $A \subseteq \Delta(X, \mathcal{M}_1)$ and $\varepsilon \in [0, 1]$ be arbitrarily fixed. Then it is slight calculation to verify that

$$f^{-1}(A) = \Big(\bigcap_{B \in \mathcal{M}_1} pr_B^{-1}(\{\nu(B) \mid \nu \in A\}) \Big) \cap \Big(\bigcap_{B \in \mathcal{M}_2 \setminus \mathcal{M}_1} pr_B^{-1}([(l_B, u_B)]) \Big) \ , \quad (4)$$

where $l_B \circeq \sup\limits_{B \supseteq C \in \mathcal{M}_1} \inf\limits_{\nu \in A} \nu(C)$, $u_B \circeq \inf\limits_{B \subseteq D \in \mathcal{M}_1} \sup\limits_{\nu \in A} \nu(D)$,

$$[(l_B, u_B)] \circeq \begin{cases} (l_B, u_B), & \text{if } \forall \nu \in A,\ \forall C, D \in \mathcal{M}_1,\ C \subseteq B,\ B \subseteq D : \\ & \nu(C) < l_B \text{ and } u_B < \nu(D) \\ [l_B, u_B), & \text{if } \exists \nu \in A,\ \exists C \in \mathcal{M}_1,\ C \subseteq B :\ l_B = \nu(C), \\ & \text{and } \forall \nu \in A,\ \forall D \in \mathcal{M}_1,\ B \subseteq D :\ u_B < \nu(D) \\ (l_B, u_B], & \text{if } \forall \nu \in A,\ \forall C \in \mathcal{M}_1, C \subseteq B :\ \nu(C) < l_B, \\ & \text{and } \exists \nu \in A,\ \exists D \in \mathcal{M}_1,\ B \subseteq D :\ u_B = \nu(D) \\ [l_B, u_B], & \text{if } \exists \nu, \nu' \in A,\ \exists C, D \in \mathcal{M}_1,\ C \subseteq B,\ B \subseteq D : \\ & l_B = \nu(C) \text{ and } u_B = \nu'(D) \end{cases} ,$$

and $pr_B : [0, 1]^{\mathcal{M}_2}|_{\Delta(X,\mathcal{M}_2)} \to [0, 1]_B$.

$\forall B \in \mathcal{M}_2 \setminus \mathcal{M}_1$: let $L_B, U_B \in \mathcal{M}_1$ be such that $L_B \subseteq B$, $B \subseteq U_B$, and $l_B = \inf\limits_{\nu \in A} \nu(L_B)$, $u_B = \sup\limits_{\nu \in A} \nu(U_B)$ ($\mathcal{M}_1$ is a σ-field, so there are such sets).

Then $\forall B \in \mathcal{M}_2 \setminus \mathcal{M}_1$:

$$f^{-1}(A) \subseteq pr_{L_B}^{-1}([(l_B, 1]) \cap pr_{U_B}^{-1}([0, u_B)]) \subseteq pr_B^{-1}([(l_B, u_B)]) \ . \quad (5)$$

Put the above term differently, $pr_{L_B}^{-1}([(l_B, 1]) \cap pr_{U_B}^{-1}([0, u_B)])$ is not worse in approximating $f^{-1}(A)$ via B than $pr_B^{-1}([(l_B, u_B)])$.

Therefore, from Definition 9. and equation (4): if $\mu_2^*(f^{-1}(A)) = \varepsilon$ then $\exists \{C_n\}_n \subseteq \mathcal{M}_1$ and $\exists \{D_m\}_m \subseteq \mathcal{M}_2 \setminus \mathcal{M}_1$:

$$f^{-1}(A) \subseteq (\bigcap_n pr_{C_n}^{-1}(B_{C_n})) \cap (\bigcap_m pr_{D_m}^{-1}([(l_{D_m}, u_{D_m})])) \ ,$$

and

$$\mu_2(\bigcap_n pr_{C_n}^{-1}(B_{C_n})) \cap (\bigcap_m pr_{D_m}^{-1}([(l_{D_m}, u_{D_m})])) = \varepsilon \ ,$$

where $B_{C_n} \in B([0,1])$ (Borel set of $[0,1]$).

(5) implies that $\exists \{K_n\}_n \subseteq \mathcal{M}_1$:

$$f^{-1}(A) \subseteq \bigcap_n pr_{K_n}^{-1}(B_{K_n}) \ ,$$

($B_{K_n} \in B([0,1])$) and

$$\mu_2(\bigcap_n pr_{K_n}^{-1}(B_{K_n})) = \varepsilon \ ,$$

i.e.

$$A \subseteq \bigcap_n prr_{K_n}^{-1}(B_{K_n}) \quad \text{and} \quad \mu_1(\bigcap_n prr_{K_n}^{-1}(B_{K_n})) = \varepsilon \ ,$$

where $prr_{K_n} : [0,1]^{\mathcal{M}_1}|_{\Delta(X,\mathcal{M}_1)} \to [0,1]_{K_n}$. Therefore $\mu_2^*(f^{-1}(A)) = \varepsilon$ implies that

$$\mu_1^*(A) = \varepsilon \ .$$

Q.E.D.

The proof of Proposition 11. Lemma 12., $j \in$ and $n \in \mathbb{N}$ are arbitrarily fixed, $(X, \mathcal{M}_1) \doteq S \otimes \Theta_{n-1}^{N\setminus\{j\}}$, $(X, \mathcal{M}_2) \doteq S \otimes \Theta_n^{N\setminus\{j\}}$, μ_2 is an arbitrary measure of $\Theta_{n+1}^j \doteq \Delta(X, \mathcal{M}_2)$, $f \doteq q_{nn+1}^j$. The case of q_{-10}^j is trivial, finally if $\forall j \in N$: q_{nn+1}^j meets the given property then so does $(id_S, q_{nn+1}^{N\setminus\{i\}})$., implies that the inverse system of measure spaces (3) is ε-complete, so from Theorem 7. it has a unique inverse limit, i.e. $\forall i \in N$:

$$\Theta^i = \Delta(S \otimes \Theta^{N\setminus\{i\}}) \ .$$

Q.E.D.

5 Final remarks

We give two straightforward generalizations of Theorem 7. Henceforth we assume that $(I, \leq)$ is a right directed set.

A notion: Bochner [2] introduced the concept of sequential maximality, later Millington and Sion [15] "weakened" it, and got at the concept of almost sequential maximality.

Definition 13. *The inverse system of measure spaces* $((X_i, \mathcal{M}_i, \mu_i), (I, \leq), f_{ij})$ *is almost sequentially maximal (a.s.m.) , if* $\forall i_1 \leq i_2 \leq \ldots \in I$ *chains* $\exists A_{i_n} \subseteq X_{i_n}$*:*

- $\forall (n \leq m)$*:* $f^{-1}_{i_n i_m}(A_{i_n}) \subseteq A_{i_m}$,
- $\forall n$*:* $\mu^*_{i_n}(A_{i_n}) = 0$,
- $(\forall n : \ x_{i_n} \in (X_{i_n} \setminus A_{i_n})$ *and* $x_{i_n} = f_{i_n i_{n+1}}(x_{i_{n+1}})) \Rightarrow (\exists x \in \varprojlim(X_i, (I, \leq), f_{ij})$ *such that* $\forall n : \ x_{i_n} = p_{i_n}(x))$.

The first generalization:

Theorem 14. *Let* $((X_i, \mathcal{M}_i, \mu_i), (I, \leq), f_{ij})$ *be such an inverse system of measure spaces that is*

1. *almost sequentially maximal,*
2. ε*-complete.*

Then $\varprojlim((X_i, \mathcal{M}_i, \mu_i), (I, \leq), f_{ij})$ *exists and unique.*

An other generalization:

Theorem 15. *Let* $((X_i, \mathcal{M}_i, \mu_i), (I, \leq), f_{ij})$ *be such an inverse system of measure spaces that (is)*

1. $(I, \leq)$ *has countable cofinal subset,*
2. ε*-complete.*

Then $\varprojlim((X_i, \mathcal{M}_i, \mu_i), (I, \leq), f_{ij})$ *exists and unique.*

The two above theorems are direct corollaries of Theorem 7.

References

[1] Aumann, R.J.: "Interactive epistemology II., Probability" *International Journal of Game Theory* **28**, 301–314. (1999)

[2] Bochner S.: *Harmonic Analysis and the Theory of Probability*, University of California Press (1955)

[3] Bourbaki, N.: *Éléments de Mathématque, Intégration, Livre VI, Chapitre IX, Intégration sur Les Espaces Topologiques Séparés*, Hermann (1969)

[4] Brandenburger, A., E. Dekel: "Hierarchies of beliefs and common knowledge" *Journal of Economic Theory* **59**, 189–198. (1993)

[5] Choksi, J. R.: "Inverse limits of measure spaces" *Proc. London Math. Soc.* **8(Ser 3)**, 321–342. (1958)

[6] Halmos, P. R.: *Measure Theory*, Springer-Verlag (1974)

[7] Harsányi, J.: "Games with incomplete information played by bayesian players part I., II., III." *Management Science* **14**, 159–182., 320–334., 486–502. (1967-1968)

[8] Heifetz, A.: "The bayesian formulation of incomplete information - the non-compact case" *International Journal of Game Theory* **21**, 329–338. (1993)

[9] Heifetz, A., D. Samet: "Topology-free typology of beliefs" *Journal of Economic Theory* **82**, 324–341. (1998)

[10] Kolmogorov, A. N.: *Foundations of the Theory of Probability, Second English Edition, translation edited by Nathan Morrison*, Chelsea Publishing Company (1956)

[11] Mallory, D. J., M Sion.: "Limits of inverse systems of measures" *Ann. Inst. Fourier* **21**, 25–57. (1971)

[12] Mertens, J. F., S. Zamir: "Formulations of bayesian analysis for games with incomplete informations" *International Journal of Game Theory* **14**, 1–29. (1985)

[13] Mertens, J. F., S. Sorin, S. Zamir: "Repeated games part A" *CORE Discussion Paper No. 9420* (1994)

[14] Metivier, M.: "Limites projectives de measures, martingales, applications" *Annali di Matematica* **63**, 225–352. (1963)

[15] Millington, H., M. Sion: "Inverse systems of group-valued measures" *Pacific Journal of Mathematics* **44**, 637–650. (1973)

[16] v. Neumann J., Morgenstern O.: *Theory of Games and Economic Behavior*, Princeton University Press, (1953)

[17] Pintér, M.: "Type space on a purely measurable parameter space" *Economic Theory* **26**, 1239–139. (2005)

[18] Rao, M. M.: *Stochastic Processes and Integration*, Sijthoff & Noordhoof (1979)