The Shapley value for shortest path games*

Miklós Pintér[†] and Anna Radványi [‡]
Corvinus University of Budapest

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Abstract

In this paper shortest path games are considered. The transportation of a good in a network has costs and benefit too. The problem is to divide the profit of the transportation among the players. Fragnelli et al (2000) introduce the class of shortest path games, which coincides with the class of monotone games. They also give a characterization of the Shapley value on this class of games.

In this paper we consider further four characterizations of the Shapley value (Shapley (1953)'s, Young (1985)'s, Chun (1989)'s, and van den Brink (2001)'s axiomatizations), and conclude that all the mentioned axiomatizations are valid for shortest path games. Fragnelli et al (2000)'s axioms are based on the graph behind the problem, in this paper we do not consider graph specific axioms, we take TU axioms only, that is, we consider all shortest path problems and we take the view of an abstract decision maker who focuses rather on the abstract problem than on the concrete situations.

Keywords: TU games, Shapley value, Shortest path games, Axiomatizations of the Shapley value

JEL Classification: C71.

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[†]Corresponding author: Department of Mathematics, Corvinus University of Budapest, 1093 Hungary, Budapest, Fővám tér 13-15., miklos.pinter@uni-corvinus.hu.

[‡]Institute of Economics, Hungarian Academy of Sciences, and Department of Mathematics, Corvinus University of Budapest, anna.radvanyi@uni-corvinus.hu

1 Introduction

In this paper we consider the class of shortest path games. There are given some agents, a good, and a network. The agents own the nodes of the network and they want to transport the good from certain nodes of the network to another. The transportation cost depends on the chosen path. The successful transportation of a good means profit. The problem is not only choosing the shortest path (a path with minimum cost, that is, with maximum profit), we also have to divide the profit arising among the players.

Fragnelli et al (2000) introduce the notion of shortest path games and they prove that the class of such games coincides with the well-known class of monotone games. They also give a characterization of the Shapley value (Shapley, 1953) on the class of shortest path games.

In this paper we consider further characterizations of the Shapley value: Shapley (1953)'s, Young (1985)'s, Chun (1989)'s, and van den Brink (2001)'s axiomatizations, and explore whether they are valid on the class of shortest path games. We conclude that all above mentioned characterizations of the Shapley value are valid on the class of shortest path games.

This paper is different from Fragnelli et al (2000) in two points. First, Fragnelli et al (2000) gives a new axiomatization of the Shapley value, but we consider four well-known characterizations. Second, Fragnelli et al (2000)'s axioms are based on the graph behind the problem, in this paper we do not consider graph specific axioms, we take TU axioms only. This means that while Fragnelli et al (2000) consider a fixed graph problem, we consider all shortest path problems, so we take the view of an abstract decision maker (e.g. a minister) who focuses rather on the abstract problem, than on the concrete situations.

The setup of the paper is as follows. In Section 2 we introduce the notions related to transferable utility (TU) games. In Section 3 we discuss the notion of shortest path games and Fragnelli et al (2000)'s result on the coincidence of the classes of shortest path games and monotone games. The last section is about our results.

2 Preliminaries

Notations: $|N|$ is for the cardinality of set N, $\mathcal{P}(N)$ denotes the class of all subsets of N. $\complement A$ is for the complement of set A. $A \subset B$ means $A \subseteq B$ but $A \neq B$. Lin (A) is the smallest linear space which contains A (the linear hull of A). Similarly, cone (A) is the smallest convex cone which contains A.

Let $N \neq \emptyset$, $|N| < \infty$, and $v : \mathcal{P}(N) \to \mathbb{R}$ be a function such that $v(\emptyset) = 0$.

Then N, v are called set of players, and transferable utility cooperative game (henceforth game) respectively. The class of games with players' set N is denoted by $\mathcal{G}^N$.

It is easy to verify that $\mathcal{G}^N$ is isomorphic with $\mathbb{R}^{2^{|N|}-1}$. Henceforth, we assume that there is a fixed isomorphism[1] between the two spaces, and regard $\mathcal{G}^N$ and $\mathbb{R}^{2^{|N|}-1}$ as identical.

Let $v \in \mathcal{G}^N$ and $i \in N$, and for each $S \subseteq N$: let $v'_i(S) = v(S \cup \{i\}) - v(S)$. v'_i is called player i's marginal contribution function in game v. Put it differently, $v'_i(S)$ is player i's marginal contribution to coalition S in game v. Furthermore, players $i, j \in N$ are equivalent in game v, $i \sim^v j$, if for each $S \subseteq N \setminus \{i, j\}$: $v'_i(S) = v'_j(S)$.

For a set of players N and coalition $T \subseteq N$, $T \neq \emptyset$, and for each $S \subseteq N$ let:

$$u_T(S) = \begin{cases} 1, & \text{if } T \subseteq S \\ 0 & \text{otherwise} \end{cases} .$$

Then game u_T is called unanimity game on coalition T.

The function ψ is a solution on set $A \subseteq \Gamma^N = \bigcup\limits_{T \subseteq N,\ T \neq \emptyset} \mathcal{G}^T$, if $\forall T \subseteq N$, $T \neq \emptyset$: $\psi|_{\mathcal{G}^T \cap A} : \mathcal{G}^T \cap A \to \mathbb{R}^T$. Therefore in this paper we assume that a solution is single valued (more precisely: the range of a solution consists of singleton sets).

Let $v \in \mathcal{G}^N$, and

$$p^i_{Sh}(S) = \begin{cases} \dfrac{|S|!(|N \setminus S| - 1)!}{|N|!}, & \text{if } i \notin S \\ 0 & \text{otherwise} \end{cases} .$$

Mapping $\phi_i(v)$, the Shapley value (Shapley, 1953) of player i in game v, is the p^i_{Sh} expected value of v'_i. In other words

$$\phi_i(v) = \sum_{S \subseteq N} v'_i(S)\ p^i_{Sh}(S) \ . \tag{1}$$

Furthermore let ϕ denote the Shapley solution.

In the next definition we list the axioms we use to characterize a solution.

Definition 1. *The solution ψ on $A \subseteq \mathcal{G}^N$ is / satisfies*

- *Pareto optimal (PO), if for each game $v \in A$: $\sum\limits_{i \in N} \psi_i(v) = v(N)$,*

[1]The fixed isomorphism is the following: we take an arbitrary complete ordering on N, therefore $N = \{1, \ldots, |N|\}$, and $\forall v \in \mathcal{G}^N$: let $v = (v(\{1\}), \ldots, v(\{|N|\}), v(\{1, 2\}), \ldots, v(\{|N| - 1, |N|\}), \ldots, v(N)) \in \mathbb{R}^{2^{|N|}-1}$.

- *null-player property (NP), if for each game* $v \in A$*, player* $i \in N$*:* $v'_i = 0$ *implies* $\psi_i(v) = 0$*,*

- *equal treatment property (ETP), if for each game* $v \in A$*, players* $i, j \in N$*:* $i \sim^v j$ *implies* $\psi_i(v) = \psi_j(v)$*,*

- *additive (ADD), if for each pair of games* $v, w \in A$ *such that* $v+w \in A$*:* $\psi(v + w) = \psi(v) + \psi(w)$*,*

- *fairness property (FP), if for each games* $v, w \in A$*, players* $i, j \in N$ *such that* $v + w \in A$ *and* $i \sim^w j$*:* $\psi_i(v+w) - \psi_i(v) = \psi_j(v+w) - \psi_j(v)$*,*

- *marginality (M), if for each games* $v, w \in A$*, player* $i \in N$*:* $v'_i = w'_i$ *implies* $\psi_i(v) = \psi_i(w)$*,*

- *coalitional strategic equivalence (CSE), if for each game* $v \in A$*, player* $i \in N$*, coalition* $T \subseteq N$*,* $\alpha > 0$*:* $i \notin T$ *and* $v + \alpha u_T \in A$ *imply* $\psi_i(v) = \psi_i(v + \alpha u_T)$*.*

A brief interpretation of the above axioms is the following.

Let us consider a network of towns and a set of companies. Let each town host the site of only one company, in this case we say that the company *owns* the city. There is given a good (e.g. a raw material or a finished product) that some of the towns are producing (called *sources*) and some other towns are consuming (called *sinks*). Hereafter we refer to a series of towns as *path*, and we say a path is *owned* by a group of companies if and only if all towns of the path are owned by one of these companies. A group of companies is able to transport the good from a source to a sink if there exists a path connecting the source to the sink which is owned by the same group of companies. The delivery of the good from source to sink results in a fixed value benefit, and a cost depending on the chosen transportation path. The goal is the transportation of the good through a path with minimal cost to achieve a maximal profit.

With the interpretation above let us consider the axioms introduced earlier. The axiom *PO* (commonly referred to as *efficiency*) requires that the total value of the grand coalition must be distributed among the players. In our example *PO* states that the whole profit from the transportation must be shared among the companies.

Axiom *NP* states that if a player's marginal contribution is zero (i.e. she has no influence, effect on the given situation) then her share (her value) must be zero. In the context of our example this means that if a company does not have an effect on the transportation profit then the company's share in the profit must be zero.

On the class of transferable utility games the axiom ETP is equivalent with another well-known axiom, *symmetry*. In our case these axioms require that if two players have the same effect in the given situation then their evaluations must be equal. Going back to our example, if two companies are equivalent with respect to the transportation profit of the good then their shares from the profit must be equal.

A solution meets axiom ADD if for any two games the result is equal if we add up the games first and evaluate the players later, or if we evaluate the players first and add up their evaluations later. Let us modify our example so that we consider the same network of towns (the same structure of companies) in two consecutive years. In this case ADD requires that if we want to evaluate the profit of a company for these two years (that is we sum the shares of a company up to the two years), then the share must be equal to the sum of the shares of the company in the two years separately.

FP puts that if we add up two games such that in one of them two players are equivalent, then the evaluations of the given two players must change equally from the values they get in the game where they are not necessarily equivalent to the values they get in the game we get by adding up the two original games. In our example it means that if the town-network "absorbs" a new company and we consider the network in two consecutive years where the "absorbed" company has the same profit in the years, then the shares of the "new" company must change the total profit of the enlarged networks in the two years (according to the original network) equally. It is worth noting that the origin of this axiom goes back to Myerson (1977).

Axiom M requires that if a given player in two games produces the same marginal contributions then that player must be evaluated equally in those games. Therefore, in our example if we consider profits for two consecutive years and there is given a company providing the same effect on the profit of transportation (e.g. it raises the profit with the same amount) in the two years separately, then the shares in the profit of the company must be equal in the two years.

CSE can be interpreted as follows: let us assume that some companies together (coalition T) are responsible for the change (raise) in the profit of the transportation. Then a CSE solution evaluates the companies in such a way that the shares of the companies which are not responsible for the raise in the profit of the transportation ($\complement T$), from the profit of the transportation do not change.

It is worth noticing that Chun (1989)'s original definition of CSE is different from ours. He defined CSE as "ψ is coalitional strategic equivalence (CSE), if for each $v \in A$, $i \in N$, $T \subseteq N$, $\alpha \in \mathbb{R}$: $i \notin T$ and $v + \alpha u_T \in A$ imply $\psi_i(v) = \psi_i(v + \alpha u_T)$." However if for some $\alpha < 0$: $v + \alpha u_T \in A$ then by

$w = v + \alpha u_T$ we get "$i \notin T$ and $w + \beta u_T \in A$ imply $\psi_i(w) = \psi_i(w + \beta u_T)$", where $\beta = -\alpha > 0$. Therefore the two CSE definitions – Chun (1989)'s and ours – are equivalent.

The following lemma is on some obvious and well-known relations among the above listed axioms.

Lemma 2. *See the following points:*

1. *If solution ψ is ETP and ADD then it is FP.*
2. *If solution ψ is M then it is CSE.*

Proof. It is left for the reader (for point 1. one can see van den Brink's van den Brink (2001) Proposition 2.3. point (i) p. 311.). □

Finally a well known result, we use later intensively.

Proposition 3. *The Shapley solution is PO, NP, ETP, ADD, FP, M, and CSE.*

3 Shortest path games

In this section we introduce the class of shortest path games. Recently, economists pay more attention to network optimization problems, where the nodes of the network are owned by the agents. The goal is to find a distribution of the costs or of the profits. So in the case of shortest path games we have to allocate the profits generated by a coalition of agents who own the nodes of the network, and who want to transport a good from sources to sinks in the network at a minimum cost. By defining the class of shortest path games we rely on Fragnelli et al (2000).

Definition 4. *A shortest path problem Σ is a tuple (X, A, L, S, T), where*

- *(X, A) is a directed graph without loops, that is, X is a finite set, A is a subset of $X \times X$ such that every $a = (x_1, x_2) \in A$ satisfies that $x_1 \neq x_2$. The elements of X and A are called nodes and arcs, respectively. For each $a = (x_1, x_2) \in A$ we say that x_1 and x_2 are the ends of a.*
- *L is a map assigning to each arc $a \in A$ a non-negative real number $L(a)$. $L(a)$ can be interpreted as the length of a.*
- *S and T are non-empty and disjoint subsets of X. The elements of S and T are called sources and sinks, respectively.*

A path P in Σ connecting two nodes x_0 and x_p is a collection of nodes $\{x_0, \ldots, x_p\}$ with $(x_{i-1}, x_i) \in A$, $i = 1, \ldots, p$. $L(P)$, the length of the path P is the sum $\sum_{i=1}^{p} L(x_{i-1}, x_i)$. We remark that if we write *path* we mean *path connecting a source and a sink*. A path P is *shortest path* if there exists no other path P' with $L(P') < L(P)$. In a shortest path problem we look for such shortest paths.

Now we introduce the relating TU games. There is given a shortest path problem Σ whose nodes are owned by a finite set of players N according to a map $o : X \to N$, such that $o(x) = i$ means that player i is the owner of node x. For each path P, $o(P)$ denotes the set of players who own the nodes of P. We assume that the transportation of a good from a source to a sink produces an income g, and the cost of the transportation is given by the length of the used path. A path P is owned by a coalition $S \subseteq N$, if $o(P) = S$, and we assuma that a coalition S can only transport a good through own paths.

Definition 5. *A* shortest path cooperative situation σ *is a tuple* (Σ, N, o, g). *We can associate with σ the TU game v_σ given by, for each $S \subseteq N$:*

$$v_\sigma(S) = \begin{cases} g - L_S, & \text{if } S \text{ owns a path in } \Sigma \text{ and } L_S < g \\ 0 & \text{otherwise} \end{cases},$$

where L_S is the length of the shortest path owned by S.

Definition 6. *A* shortest path game v_σ *is a game associated with a shortest path cooperative situation σ. Let SPG denote the class of shortest path games.*

See the following example:

Example 7. Let $N = \{1, 2\}$ be the set of players, the graph in Figure 1 represents the shortest path cooperative situation, s_1, s_2 are the sources, t_1, t_2 are the sink nodes. The numbers on the arcs identify their costs or lengths, and $g = 7$. Player 1 owns the nodes s_1, x_1, and t_1, Player 2 owns nodes s_2, x_2, and t_2, and Table 1 gives the induced shortest path game.

Finally, we present Fragnelli et al (2000)'s result on the relation of the classes of shortest path games and monotone games.

Definition 8. *A $v \in \mathcal{G}^N$ is a monotone game if $\forall S, T \in N$, $S \subseteq T$ implies $v(S) \leq v(T)$.*

Theorem 9. *$SPG = MO$, where MO is for the class of monotone games.*

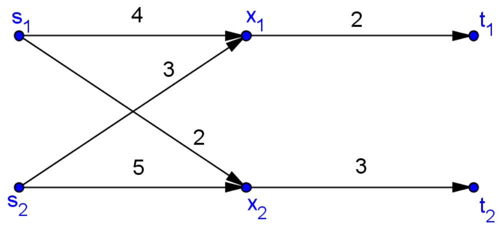

Figure 1: The graph of the shortest path cooperative situation of Example 7

S	Shortest path owned by S	$L(S)$	$v(S)$
$\{1\}$	$\{s_1, x_1, t_1\}$	6	1
$\{2\}$	$\{s_2, x_2, t_2\}$	8	0
$\{1, 2\}$	$\{s_1, x_2, t_2\} \sim \{s_2, x_1, t_1\}$	5	2

Table 1: The induced shortest path game of Example 7

4 Results

In this section we organize our results into thematic subsections.

4.1 The potential

In this subsection we turn our attention to the potential characterization (Hart and Mas-Colell, 1989) of the Shapley value on the class of monotone games.

Definition 10. *Let $v \in \mathcal{G}^N$ and $T \subseteq N$, $T \neq \emptyset$. Then the subgame of v on coalition T, $v^T \in \mathcal{G}^T$, is defined as follows, for each $S \subseteq T$:*

$$v^T(S) = v(S) \ .$$

It is clear that v^T must be defined only on the subsets of T.

Definition 11. *Let $A \subseteq \Gamma^N$, $P : A \to \mathbb{R}$ be a function, and for each game $v \in \mathcal{G}^T \cap A$ and player $i \in T$: $|T| = 1$ or $v^{T \setminus \{i\}} \in A$:*

$$P'_i(v) = \begin{cases} P(v), & \text{if } |T| = 1 \\ P(v) - P(v^{T \setminus \{i\}}) & \text{otherwise} \end{cases} . \tag{2}$$

Furthermore, if for each game $v \in \mathcal{G}^T \cap A$ such that either $|T| = 1$ or for each player $i \in T$: $v^{T \setminus \{i\}} \in A$:

$$\sum_{i\in T} P_i'(v) = v(T) \ ,$$

then P *is called potential on set* A.

Definition 12. *Set* $A \subseteq \Gamma^N$ *is subgame closed, if for each coalition* $T \subseteq N$ *such that* $|T| > 1$*, game* $v \in \mathcal{G}^T \cap A$*, and player* $i \in T$*:* $v^{T\setminus\{i\}} \in A$.

The concept of subgame is meaningful only if the original game has at least two players. Therefore in the above definition we require that for each player i: $v^{T\setminus\{i\}}$ be in the set under consideration only if there are at least two players in T.

Theorem 13. *Let* $A \subseteq \Gamma^N$ *be a subgame closed set of games. Then function* P *on* A *is a potential, if and only if for each game* $v \in \mathcal{G}^T \cap A$ *and player* $i \in T$*:* $P_i'(v) = \phi_i(v)$.

Proof. See e.g. Peleg and Sudhölter (2003) Theorem 8.4.4. on pp. 216-217. □

Next we focus on the class of monotone games.

Corollary 14. *A function* P *on the class of monotone games is a potential, if and only if for each monotone game* $v \in \mathcal{G}^T$ *and player* $i \in T$*:* $P_i'(v) = \phi_i(v)$*, that is, if and only if* P_i' *is the Shapley value,* $i \in N$.

Proof. It is easy to verify that the class of monotone games is a subgame closed set of games. Therefore we can apply Theorem 13. □

4.2 Shapley's characterization

In this subsection we look in Shapley (1953)'s classical characterization. The next theorem fits into the sequence of more and more enhanced results of Shapley (1953), Dubey (1982), Peleg and Sudhölter (2003).

Theorem 15. *Let* $A \subseteq \mathcal{G}^N$ *be such that* $\mathrm{cone}\ (\{u_T\}_{T\subseteq N,\ T\neq\emptyset}) \subseteq A$*. Then a solution* ψ *on* A *is PO, NP, ETP and ADD if and only if* $\psi = \phi$.

Proof. if: See Proposition 3.

only if: Let $v \in A$ be a game and ψ a solution on A be PO, NP, ETP and ADD. If $v = 0$ then NP implies that $\psi(v) = \phi(v)$, therefore w.l.o.g. we can assume that $v \neq 0$.

We know that there exist weights $\{\alpha_T\}_{T\subseteq N,\ T\neq\emptyset} \subseteq \mathbb{R}$ such that

$$v = \sum_{T \subseteq N,\ T \neq \emptyset} \alpha_T u_T \ .$$

Let $Neg = \{T : \alpha_T < 0\}$. Then

$$\left(- \sum_{T \in Neg} \alpha_T u_T \right) \in A \ ,$$

and

$$\left(\sum_{T \in 2^N \setminus (Neg \cup \{\emptyset\})} \alpha_T u_T \right) \in A \ .$$

Furthermore

$$v + \left(- \sum_{T \in Neg} \alpha_T u_T \right) = \sum_{T \in 2^N \setminus (Neg \cup \{\emptyset\})} \alpha_T u_T \ .$$

Since for each unanimity game u_T and $\alpha \geq 0$ Axioms PO, NP and ETP imply $\psi(\alpha u_T) = \phi(\alpha u_T)$, and since Axiom ADD:

$$\psi \left(- \sum_{T \in Neg} \alpha_T u_T \right) = \phi \left(- \sum_{T \in Neg} \alpha_T v_T \right)$$

and

$$\psi \left(\sum_{T \in 2^N \setminus (Neg \cup \{\emptyset\})} \alpha_T u_T \right) = \phi \left(\sum_{T \in 2^N \setminus (Neg \cup \{\emptyset\})} \alpha_T u_T \right) \ .$$

Then Proposition 3. and Axiom ADD imply

$$\psi(v) = \phi(v) \ .$$

Therefore the proof is complete. $\square$

By Theorem 15 we can conclude on the class of monotone games.

Corollary 16. *A solution ψ on the class of monotone games is PO, NP, ETP and ADD if and only if $\psi = \phi$, that is, if and only if it is the Shapley solution.*

Proof. The class of monotone games contains the convex cone spanned by the unanimity games $\{u_T\}_{T \subseteq N,\ T \neq \emptyset}$, hence we can apply Theorem 15. $\square$

4.3 van den Brink's axiomatization

In this subsection we discuss van den Brink (2001)'s characterization of the Shapley value on the class of monotone games.

The next lemma is a slight generalization of van den Brink (2001)'s Proposition 2.4. (point (ii) p. 311).

Lemma 17. *Let $A \subseteq \mathcal{G}^N$ be such that $0 \in A$, and solution ψ on A be NP and FP. Then ψ is ETP.*

Proof. Let $v \in A$ be such that $i \sim^v j$, and $w = 0$, then NP implies $\psi(0) = 0$. From that ψ is FP

$$\psi_i(v + w) - \psi_i(w) = \psi_j(v + w) - \psi_j(w) \ ,$$

hence $\psi_i(v + w) = \psi_j(v + w)$. From FP again

$$\psi_i(v + w) - \psi_i(v) = \psi_j(v + w) - \psi_j(v) \ .$$

Then $\psi_i(v + w) = \psi_j(v + w)$ implies that

$$\psi_i(v) = \psi_j(v) \ .$$

□

The next proposition is the key result of this subsection.

Proposition 18. *Let ψ, a solution on the convex cone spanned by the unanimity games, that is, on* cone $(\{u_T\}_{T \subseteq N,\ T \neq \emptyset})$*, be PO, NP and FP. Then ψ is ADD.*

Proof. First we show that ψ is uniquely determined on set cone $(\{u_T\}_{T \subseteq N,\ T \neq \emptyset})$.

Let $v \in$ cone $(\{u_T\}_{T \subseteq N,\ T \neq \emptyset})$ be a (monotone) game, in other words, $v = \sum_{T \subseteq N,\ T \neq \emptyset} \alpha_T u_T$, and let $I(v) = \{T : \alpha_T > 0\}$. The proof goes by induction on $|I(v)|$.

$|I(v)| \leq 1$: By NP and Lemma 17 $\psi(v)$ is uniquely determined.

Suppose that for some $1 \leq k < |I(v)|$, for each $A \subseteq I(v)$ such that $|A| \leq k$: $\psi(\sum_{T \in A} \alpha_T u_T)$ is well defined. Let $C \subseteq I(v)$ be such that $|C| = k + 1$, and $z = \sum_{T \in C} \alpha_T u_T$.

Case 1: There exist $u_T, u_S \in C$ such that there exist $i^*, j^* \in N$: $i^* \sim^{u_T} j^*$, but $i^* \nsim^{u_S} j^*$. In this case, Axiom FP and that $z - \alpha_T u_T, z - \alpha_S u_S \in$ cone $(\{u_T\}_{T \subseteq N,\ T \neq \emptyset})$ imply that for each player $i \in N \setminus \{i^*\}$ such that $i \sim^{\alpha_S u_S} i^*$:

$$\psi_{i^*}(z) - \psi_{i^*}(z - \alpha_S u_S) = \psi_i(z) - \psi_i(z - \alpha_S u_S) \ , \tag{3}$$

and for each player $j \in N \setminus \{j^*\}$ such that $j \sim^{\alpha_S u_S} j^*$:

$$\psi_{j^*}(z) - \psi_{j^*}(z - \alpha_S u_S) = \psi_j(z) - \psi_j(z - \alpha_S u_S) \ , \tag{4}$$

and

$$\psi_{i^*}(z) - \psi_{i^*}(z - \alpha_T u_T) = \psi_{j^*}(z) - \psi_{j^*}(z - \alpha_T u_T) \ . \tag{5}$$

Moreover, PO implies that

$$\sum_{i \in N} \psi_i(z) = z(N) \ . \tag{6}$$

From the induction hypothesis the system of linear equations (3), (4), (5), (6) consists of $|N|$ variables ($\psi_i(z)$, $i \in N$), $|N|$ equations, and it has a unique solution. Therefore $\psi(z)$ is well-defined.

Case 2: $z = \alpha_T u_T + \alpha_S u_S$ such that $S = N \setminus T$. Then $z = m(u_T + u_S) + (\alpha_T - m)u_T + (\alpha_S - m)u_T$, where $m = \min\{\alpha_T, \alpha_S\}$. W.l.o.g. we can assume that $m = \alpha_T$. Then by $i \sim^{m(u_T+u_S)} j$, $\psi((\alpha_S - m)u_S)$ is well-defined (induction hypothesis) and Axiom PO: $\psi(z)$ is well-defined.

To sum up, ψ is well-defined on cone $(\{u_T\}_{T \subseteq N,\ T \neq \emptyset})$. Then Proposition 3 implies that ψ is ADD on cone $(\{u_T\}_{T \subseteq N,\ T \neq \emptyset})$. □

The following theorem, which generalizes van den Brink (2001)'s Theorem 2.5. (pp. 311–315.), is the main result of this subsection.

Theorem 19. *A solution ψ on the class of monotone games is PO, NP and FP if and only if $\psi = \phi$, that is, if and only if it is the Shapley solution.*

Proof. if: See Proposition 3.

only if: From Theorem 15 and Proposition 18 on cone $(\{u_T\}_{T \subseteq N,\ T \neq \emptyset})$ $\psi = \phi$. Let $v = \sum_{T \subseteq N,\ T \neq \emptyset} \alpha_T u_T$ be a monotone game and $w = (\alpha + 1) \sum_{T \subseteq N,\ T \neq \emptyset} u_T$, where $\alpha = \max\{-\min_T \alpha_T, 0\}$.

Then $v + w \in$ cone $(\{u_T\}_{T \subseteq N,\ T \neq \emptyset})$, for each players $i, j \in N$: $i \sim^w j$, so Axioms PO and FP imply that $\psi(v)$ is well-defined. Then we can apply Proposition 3. □

4.4 Chun's and Young's approaches

In this subsection Chun (1989)'s and Young (1985)'s approaches are discussed. In the case of Young (1985)'s axiomatization we only refer to an external result, in the case of Chun (1989)'s we connect it to Young (1985)'s characterization.

The next result is from Pintér (2011).

Proposition 20. *A solution ψ on the class of monotone games is PO, ETP and M if and only if $\psi = \phi$, that is, if and only if it is the Shapley solution.*

In the game theory literature there is some confusion about the relation of Chun (1989)'s and Young (1985)'s characterizations. van den Brink (2007) suggests that CSE is equivalent to M. However, that argument is not true, e.g. on the class of assignment games this does not hold.

Unfortunately, the class of monotone games does not bring to surface the difference between Axioms M and CSE. The next lemma is about this.

Lemma 21. *On the class of monotone games Axioms M and CSE are equivalent.*

Proof. $CSE \Rightarrow M$: Let monotone games v, w and player $i \in N$ be such that $v'_i = w'_i$. It is easy to verify that $(v-w)'_i = 0$, $v - w = \sum_{T \subseteq N,\ T \neq \emptyset} \alpha_T u_T$, and for each $T \subseteq N$, $T \neq \emptyset$: if $i \in T$, then $\alpha_T = 0$. Therefore, $v = w + \sum_{T \subseteq N \setminus \{i\}, T \neq \emptyset} \alpha_T u_T$.

Let $T^+ = \{T \subseteq N \mid \alpha_T > 0\}$. Then from that for each monotone game z, $\alpha > 0$, and unanimity game u_T: $z + \alpha u_T$ is a monotone game, we get $w + \sum_{T \in T^+} \alpha_T u_T$ is a monotone game, and $w'_i = (w + \sum_{T \in T^+} \alpha_T u_T)'_i$. Furthermore, from CSE: $\psi_i(w) = \psi_i(w + \sum_{T \in T^+} \alpha_T u_T)$.

Moreover, from that for each monotone game z, $\alpha > 0$, and unanimity game u_T: $z + \alpha u_T$ is a monotone game, we get $v + \sum_{T \notin T^+} -\alpha_T u_T$ is a monotone game, and $v'_i = (v + \sum_{T \notin T^+} -\alpha_T u_T)'_i$. Furthermore, CSE implies that: $\psi_i(v) = \psi_i(v + \sum_{T \notin T^+} -\alpha_T u_T)$.

Then $w + \sum_{T \in T^+} \alpha_T u_T = v + \sum_{T \notin T^+} -\alpha_T u_T$, therefore

$$\psi_i(w) = \psi_i\left(w + \sum_{T \in T^+} \alpha_T u_T\right) = \psi_i\left(v + \sum_{T \notin T^+} -\alpha_T u_T\right) = \psi_i(v)\ .$$

$M \Rightarrow CSE$: See Lemma 2. $\square$

Therefore:

Corollary 22. *A solution ψ on the class of monotone games is PO, ETP and CSE if and only if $\psi = \phi$, that is, if and only if it is the Shapley solution.*

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