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ORIGINAL RESEARCH

On the manipulability of a class of social choice functions: plurality *k*th rules

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Abstract
In this paper we introduce the plurality kth social choice function selecting an alternative, which is ranked kth in the social ranking following the number of top positions of alternatives in the individual ranking of voters. As special case the plurality 1st is the same as the well-known plurality rule. Concerning individual manipulability, we show that the larger k the more preference profiles are individually manipulable. We also provide maximal non-manipulable domains for the plurality kth rules. These results imply analogous statements on the single non-transferable vote rule. We propose a decomposition of social choice functions based on plurality kth rules, which we apply for determining non-manipulable subdomains for arbitrary social choice functions. We further show that with the exception of the plurality rule all other plurality kth rules are group manipulable, i.e. coordinated misrepresentation of individual rankings are beneficial for each group member, with an appropriately selected tie-breaking rule on the set of all profiles.

Keywords Voting rules · Dictatorship · Manipulability

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1 Introduction

The plurality rule selecting the alternative receiving the maximum number of votes is the most commonly used social choice function (or voting rule) in real life. The plurality rule is also supported by several axiomatic characterizations starting with Richelson (1978) and Roberts (1991). Sanver (2009) determined the non-manipulable domains for the plurality rule. Furthermore, the plurality rule has been derived as the solution of reasonable optimization problems by Nitzan (1981), Elkind et al. (2015), and Bednay et al. (2017) etc. In particular, Bednay et al. (2017) introduced and derived the reverse-plurality rule, which selects the alternative receiving the fewest number of votes. The plurality rule and the reverse plurality rules are special cases of the plurality kth rules introduced in this paper. The plurality kth rule selects the alternative receiving the k most number of votes. We resolve ties based on an exogenously given tie-breaking rule.

In multi-winner voting, k alternatives/candidates are selected. For instance, situations where first a shortlist of candidates is created so that in a subsequent round judges can select the winning candidate. Another example is the formation of committees. The problem requires that voters express preferences over the possible winning k-sized sets, which can be too demanding for each voter since the number of k-sized sets becomes easily very large to be ranked by individuals or can require communicating too much information (see for instance Kilgour (2018)). Therefore, multi-winner voting rules usually need from the voters to reveal only their preferences over the set of alternatives as in case of single-winner voting rules. The loss of information makes the investigation of the manipulability of multi-winner voting rules more intriguing. Therefore, more information has to be elicited from the revealed preferences (e.g. Lang and Xia (2016)). Many multi-winner voting rules are simply derived from single-winning voting rules by picking the best k alternatives determined by the respective single-winning voting rules. Choosing the appropriate multi-winner voting rule depends on the problem. Elkind et al. (2017) distinguish between excellence-based election, selecting a diverse committee and proportional representation type multi-winner election problems. For our analysis, the first criterion which focuses on selecting individually best alternatives, is relevant.

The simplest multi-winner voting rule is the so-called single non-transferable vote (SNTV) rule, which can be regarded as an extension of the plurality rule. Plurality based multi-winner voting rules are investigated by Faliszewski et al. (2018). The choices of the plurality, plurality 2nd, . . ., plurality kth rules are the same as that of the SNTV rule selecting a k-sized set. The plurality kth rule basically selects the alternative just making it into the k-sized committee, while the plurality $k + 1$th rule determines the alternative just not making it in there. Clearly, investigating the manipulability of these rules are simpler than that of the SNTV rule. We assume that when comparing two k-sized sets each voter only cares about the alternatives appearing in the symmetric difference of these two sets. We will establish a link between the manipulability of the plurality kth rule and the associated multi-winner voting rule selecting the alternatives obtaining the k highest plurality scores.

We order/arrange the plurality kth rules based on the proportion of manipulable profiles, an approach propagated by Kelly (1985) and Aleskerov and Kurbanov (1999). From the Gibbard–Satterthwaite theorem (1973/75) it follows that the plurality kth rules are manipulable. Concerning individual manipulability, the proportion of manipulable profiles increases in k. The importance of the plurality kth rules is stressed by the observation that any voting rule can be decomposed based on plurality kth rules. Furthermore, we can arrive in this way to a new domain which is maximal with respect to non-manipulability.

Our latter result can be compared with (Maus et al. 2007a, b). They determine under certain combinations of basic properties the voting rules with the smallest number of manipulable profiles. In particular, for tops-only, anonymous, and surjective social choice functions Maus et al. (2007a) establish that the unanimity rules with status quo have the minimal number of manipulable profiles. Maus et al. (2007b) derive the lower bound on the number of manipulable profiles by imposing assumptions on the number of voters and alternatives (they require that the number of voters is at least as large as the number of alternatives). They also find that social choice functions exhibit a trade off between minimizing manipulability and treating alternatives neutrally.

The paper is organized as follows. Section 2 introduces the framework. Section 3 contains results on individual manipulability. Section 4 describes the maximal non-manipulable subdomains. Section 5 presents the decomposition and Sect. 6 concludes. Finally, the "Appendix" contains further mathematical findings concerning the group manipulability of the plurality kth rules.

2 The framework

The set of alternatives is $A = \{1, \ldots, m\}$ with $|A| = m \geq 2$. The set of voters is $N = \{1, \ldots, n\}$. The set of all strict preference relations (or linear orderings[1]) on A is denoted by $\mathcal{P}$. The set of all preference profiles is denoted by $\mathcal{P}^n$. If $\succ \in \mathcal{P}^n$ and $i \in N$, $\succ_i$ denotes the preference ordering of voter i over A.

Definition 1 A social choice function (SCF) or a voting rule is a mapping $f : \mathcal{P}^n \to A$ that selects a winning alternative for each preference profile.

Note that our definition of a SCF does not allow for possible ties. In case of ties, a fixed tie-breaking rule will be used. A tie-breaking rule $\tau : \mathcal{P}^n \to \mathcal{P}$ maps preference profiles to linear orderings on A. The tie-breaking rule is used only when a unique winner is not determined by the formula. If there are more alternatives chosen by a formula "almost" specifying a SCF, then the highest ranked alternative is selected, based on the given tie-breaking rule among tied alternatives. A tie-breaking rule is called *anonymous* if it is invariant to the ordering of voters' preferences or to the labeling of voters. For instance, the simplest anonymous tie-breaking rule assigns to each profile the same exogenously given ordering of alternatives. It is worth emphasizing that we

[1] A linear ordering is an irreflexive, transitive and total binary relation.

only consider deterministic tie-breaking rules. For some results, we need a special class of tie-breaking rules which we refer to as *tie-order preserving* ones meaning that if at a profile $\succ \in \mathcal{P}^n$ alternatives $T \subseteq A$ are tied at consecutive places and by changing voter i's preference to $\succ_i' \in \mathcal{P}$ at profile $(\succ_i', \succ_{-i})$ alternatives $T' \subseteq A$ are tied at consecutive places, then τ orders the alternatives $T \cap T'$ in the same way at both profiles. For instance, any constant tie-breaking rule (i.e. ties are determined based on an exogenously given linear ordering of alternatives) is tie-order preserving.

The *reverse-plurality rule* f_τ selects the alternative which is the fewest number of times on the top and uses the tie-breaking rule τ to resolve ties when required. The following class of rules range from the reverse-plurality rule to the plurality rule.

Definition 2 Rank the alternatives based on their plurality scores and employ a tie-breaking rule τ in case of ties, to obtain a linear ordering of the alternatives. The *plurality kth rule* $f_\tau^{(k)}$ selects the kth placed alternative in the obtained linear ordering.

Clearly the plurality rule is the plurality 1st rule and the reverse-plurality rule is the plurality mth rule.

We now define multi-winner voting rules, which in our setting choose k alternatives out of the m alternatives. Let $\mathcal{C}^{(k)}$ be set of all k-sized subsets of A.

Definition 3 A mapping $g : \mathcal{P}^n \to \mathcal{C}^{(k)}$ that selects the winning k-sized subset of alternatives is called a *multi-winner voting rule* (MWVR).

We will consider the single non-transferable vote rule (referred to as the SNTV rule), which is a straightforward extension of the plurality rule to a MWVR.

Definition 4 For a given $k = 1, \ldots, m$, the *single non-transferable vote rule* $g_\tau^{(k)} : \mathcal{P}^n \to \mathcal{C}^{(k)}$ selects the first k placed alternatives obtained by ranking the alternatives based on their plurality scores and employing the tie-breaking rule τ to resolve possible ties.

Note that $g_\tau^{(k)} = \{f_\tau^{(1)}, f_\tau^{(2)}, \ldots, f_\tau^{(k)}\}$. We define below the individual manipulability of SCFs. Consider a preference profile $(\succ_1, \ldots, \succ_n) \in \mathcal{P}^n$.

Definition 5 An SCF $f : \mathcal{P}^n \to A$ is *manipulable by voter $i \in N$* at $(\succ_1, \ldots, \succ_n) \in \mathcal{P}^n$ if

$$\exists \succ_i' \in \mathcal{P} \text{ such that } f(\succ_i', \succ_{-i}) \succ_i f(\succ_i, \succ_{-i}).$$

An SCF $f : \mathcal{P}^n \to A$ is *manipulable at* $(\succ_1, \ldots, \succ_n) \in \mathcal{P}^n$ if there exists a voter $i \in N$ who can manipulate f at $(\succ_1, \ldots, \succ_n) \in \mathcal{P}^n$.

We now define the manipulability of a MWVR. This requires a notion about how a voter compares two possible sets of winners of cardinality k. One possibility is that each voter compares two possible sets of winners *lexicographically* with respect to its preference relation. We believe this is a natural assumption, for instance in situations where a voter is represented by a single candidate or when the MWVR determines a preselection process and the single winner is chosen in another independent round.[2]

[2] Allowing the voters to express their preferences on k-sized sets easily results in an intractable problem.

We will denote by $\succ^L$ the lexicographic extension of $\succ$ to the class of k-sized sets of alternatives. In particular, for any $B = \{b_1, \ldots, b_k\} \subseteq A$ and any $C = \{c_1, \ldots, c_k\} \subseteq A$, where we assume without loss of generality that $b_1 \succ b_2 \succ \ldots \succ b_k$ and $c_1 \succ c_2 \succ \ldots \succ c_k$, we have $B \succ^L C$ if and only if there exists an $l \in M$ such that $b_l \succ c_l$ and $b_j = c_j$ for all $j = 1, \ldots, l-1$. Finally, we will carry out the comparison of two k-sized sets B and C by comparing $B \setminus C$ and $C \setminus B$ based on the lexicographic extension of $\succ$. Formally, we write $B \succ^D C$ if $B \setminus C \succ^L C \setminus B$. In our framework the sets $B \setminus C$ and $C \setminus B$ to be compared will be singletons, which correspond to an alternative entering the selected k-sized set and an alternative exiting the k-sized set through manipulations. It may be reasonable in many contexts to assume that the comparison has to be carried out based on the changes in a set. Since then only singletons have to be compared the investigation of the manipulability of the SNTV rule will become quite simple and very similar to the investigation of the manipulability of the plurality kth rules. It may be reasonable in many contexts to assume that the comparison has to be carried out based on the changes in a set and can be considered as a kind of independence or separability property.

For MWVRs we also introduce individual manipulability.

Definition 6 A MWVR $g : \mathcal{P}^n \to \mathcal{C}^{(k)}$ is *manipulable by voter* $i \in N$ *at* $(\succ_1, \ldots, \succ_n) \in \mathcal{P}^n$ if

$$\exists \succ_i' \in \mathcal{P} \text{ such that } g(\succ_i', \succ_{-i}) \succ_i^D g(\succ_i, \succ_{-i}).$$

A MWVR $g : \mathcal{P}^n \to \mathcal{C}^{(k)}$ is *manipulable at* $(\succ_1, \ldots, \succ_n) \in \mathcal{P}^n$ if there exists a voter $i \in N$ who can manipulate g at $(\succ_1, \ldots, \succ_n) \in \mathcal{P}^n$.

Concerning individual manipulability, we will investigate the manipulability of subdomains for which we introduce the following notion.

Definition 7 An SCF $f : \mathcal{P}^n \to A$ or a MWVR $g : \mathcal{P}^n \to \mathcal{C}^{(k)}$ is *non-manipulable on a subdomain* $\mathcal{D} \subseteq \mathcal{P}^n$ if for voters' preference profiles in $\mathcal{D}$ none of them can manipulate by misrepresenting its preference relation without leaving the preference domain $\mathcal{D}$.

Hence, a subdomain is non-manipulable if staying within $\mathcal{D}$, individual manipulation is impossible. Note that non-manipulable subdomains exist since if $\mathcal{D}$ is 'disconnected' or 'sparse enough' so that none of the voters can change their preference relation individually in a way that for any given profile in $\mathcal{D}$ the resulting profile remains in $\mathcal{D}$, then $\mathcal{D}$ is a non-manipulable subdomain for any SCF. Maximal non-manipulable subdomains in which each agent faced the same restriction imposed on the set of its admissible preferences have been determined for the Borda count by Barbie et al. (2006) and for the plurality rule by Sanver (2009).

In order to prevent us from investigating 'disconnected' subdomains we introduce the following notion.

Definition 8 A domain $\mathcal{D} \subseteq \mathcal{P}^n$ is called *connected* if for any profiles $\succ, \succ' \in \mathcal{P}^n$ there exists a sequence of profiles $\succ^{(0)}, \succ^{(1)}, \ldots, \succ^{(l)} \in \mathcal{D}$ such that $\succ^{(0)} = \succ$, $\succ^{(l)} = \succ'$ and two subsequent profiles of the sequence differ only in the preference relation of one voter.

Informally, the connectedness of a domain means that from any profile of a domain any other profile of the same domain can be reached by a sequence of individual manipulations.

The most obvious index of manipulability, the so-called Nitzan-Kelly index (NKI) introduced by Nitzan (1985) and Kelly (1988), divides the number of manipulable profiles by the number of all profiles. We will consider two versions of the NKI: one assumes the impartial culture (IC), while the other one the anonymous impartial culture (AIC) of preference profiles. In the former case (IC) voters' preferences are chosen independently and based on the uniform distribution above the set of all profiles (i.e. each preference relation is assigned independently and equally likely to the voters), while in the latter case (AIC) only the number of occurrences of a preference relation in a preference profile matters and not their distribution between voters (i.e. each anonymous preference profile is equally likely).[3] In particular, label the different preferences from 1 to $m!$ and let $n_1, \ldots, n_{m!}$ be the respective number of preferences $\succ_1, \ldots, \succ_{m!}$ for a given profile. Then in case of anonymous profiles only these numbers matter, while the assignment of these preferences to the voters does not.

Henceforth, we shall denote by $M_a = \{i \in N \mid a = t(\succ_i)\}$ the set of agents with top alternative $a \in A$, where $t(\succ_i)$ stands for the top alternative of $\succ_i$. In addition, let $r_\tau(\succ, j) \in A$ be the jth ranked alternative by the plurality rule with a given tie-breaking rule τ and $s(\succ, a)$ be the number of top positions of alternative $a \in A$ in case of $\succ \in \mathcal{P}^n$, i.e. the plurality score of alternative a. Furthermore, let $p_\tau(\succ, j) = s(\succ, r_\tau(\succ, j))$ be the plurality score of the jth ranked alternative.

3 Individual manipulability

In this section we turn to the individual manipulability of plurality kth rules and we start with some general and simple observations. Let $a = r_\tau(\succ, k-1)$, $b = r_\tau(\succ, k)$ and $c = r_\tau(\succ, k+1)$. With a slight abuse of notation let a be undefined if $k = 1$ and c be undefined if $k = m$. Note that by individual manipulation the relative difference in plurality scores can change by at most 2. In particular, we have a successful manipulation of the plurality kth rule by moving the $k-1$th ranked alternative (in case of $k \geq 2$) into the kth position

1. If a leads by two and a voter having a as the top alternative reveals b as its top alternative assuming that τ gives priority to b over a for the respective profile,
2. If a leads by one and a voter having a as the top alternative reveals again b as its top alternative, however now we have to assume that τ should give priority to a over other potentially tied alternatives,
3. If a is tied with b, while c has a lower score, then the type of manipulation described in Point 2 still works,
4. If a, b, c are all tied and assuming that τ gives priority to a over lower ranked tied alternatives, a voter having alternative c as its top alternative and preferring a to b reveals b as its top alternative.

[3] For a detailed discussion of the IC vs. AIC assumptions see Eğecioğlu and Giritligil (2013).

Table 1 Moving a manipulation possibility downwards

$\succ_1$	$\succ_2$	$\succ_3$	$\succ_4$	$\succ_5$	$\succ_6$	$\succ_7$	$\succ_1$	$\succ_2$	$\succ_3$	$\succ_4$	$\succ_5$	$\succ_6$	$\succ_7$
a	a	a	b	b	c	c	c	a	a	c	b	c	c
b	d	b	d	a	d	b	b	d	b	d	a	d	b
c	b	d	a	d	a	d	a	b	d	a	d	a	d
d	d	c	c	c	b	a	d	d	c	b	c	b	a

Since we are moving in the above described cases the $k-1$th ranked alternative into a lower ranked winning position, we are speaking about *downwards manipulation*. More generally, downwards manipulation can be extended to the case when a $k-j$th ranked alternative is 'degraded' to the kth ranked alternative by benefiting the manipulating voter.

Now we turn to the question whether a successful manipulation of the plurality kth rule by moving the $k+1$th ranked alternative (in case of $k \leq m-1$) into the kth position is possible. Since the plurality score of b can only be decreased by a voter having b on top, which is not beneficial for that voter, this type of *upwards manipulation* has less cases than downwards manipulation. More specifically, we have the following three potentially successful upwards manipulation possibilities.

5. If b leads by one against c, then and a voter preferring c to b with another top ranked alternative reveals c as its top alternative (τ can be set appropriately depending on whether the third alternative is tied with b or not).
6. If a leads by at least one and b is tied with c, then a voter preferring c to b with another top ranked alternative reveals again c as its top alternative (in this case we have even less restrictions in choosing the tie-breaking rule for the manipulated profile).
7. If a, b and c are all tied, then a voter with top alternative a and preferring c to b reveals b as its top alternative (the tie-breaking rule should not change the ordering of alternatives potentially tied with c).

Upwards manipulation can be extended to the case when a $k+l$th ranked alternative is 'upgraded' to the kth ranked alternative by benefiting the manipulating voter.

First, we show that the plurality second rule is manipulable on at least as many profiles as the plurality rule. We motivate the main step of our proof by the example shown in Table 1 in which $m=4$ and $n=7$. The plurality rule is manipulable at the profile shown on the left-hand side of Table 1 by voter 7. In order to maintain that voter 7 manipulates by making b the winning alternative instead of a for the plurality 2nd rule, we make the top alternative c of voter 7 the plurality winner by switching the positions of a with c and b with c for the first voters having a and b on top, respectively. The resulting profile is shown on the right-hand side of Table 1 in which voter 7 can manipulate the plurality 2nd rule. The main point of this example is that we associate to the manipulable profile for the plurality rule another manipulable profile for the plurality 2nd rule in which the same voter manipulates in exactly the same way.

We shall denote by $\mathcal{D}_k$ and $\mathcal{D}_k^a$ the set of profiles and anonymous profiles, respectively, on which the plurality kth rule is individually manipulable.

Proposition 1 *Assume that $m \geq 3$, $n \geq m+2$, and τ is a given tie-breaking rule. Considering individual manipulability, there exists a tie-breaking rule τ' such that the NKI of the plurality rule $f_{\tau}^{(1)}$ is smaller than or equal to the NKI of the plurality 2nd rule $f_{\tau'}^{(2)}$ on both the IC and the AIC, where in the latter case anonymous tie-breaking rules have to be assumed.*

Proof We will construct an injection $f : \mathcal{D}_1 \to \mathcal{D}_2$. Pick an arbitrary profile Π at which the plurality rule is manipulable.

We label the alternatives such that $|M_{a_1}| \geq |M_{a_2}| \geq \cdots \geq |M_{a_m}|$ and that in case of equalities they follow the ordering given by τ. Order the voters such that the $|M_{a_1}|$ voters with a_1 on the top come first, next the $|M_{a_2}|$ voters with a_2 on the top follow, and so forth. The ordering of voters having the same top alternative should be kept. We shall denote the reordered profile by Π_σ, where σ stands for the respective bijection.

Since the plurality rule is manipulable at Π_σ there exist $i, j \in A$ and a voter $q \in N$ such that either $|M_{a_1}| = |M_{a_i}|$ or $|M_{a_1}| = |M_{a_i}| + 1$, $q \in M_{a_j}$, and q prefers a_i to a_1 and can enforce outcome a_i by revealing a preference ordering with a_i as its top-ranked alternative. Select i, j, and q so that, following the respective lexicographic ordering, they are as small as possible. We exchange the preference ordering of voter q with that of the last voter having also a_j as the top alternative if this is not already the case. With a slight abuse of notation we shall denote the obtained permutation still by σ. Note that manipulability requires that Π_σ has at least three different top alternatives.

Our strategy will be to make alternative a_m the clear plurality winner and to maintain the ranking of the other alternatives based on their respective number of top positions such that voter q can make a_i the plurality 2nd rule winner by 'overtaking' a_1. We shall denote by l the smallest positive integer for which

- $l_k = \max\left\{|M_{a_k}| - l, 1\right\}$ for all $k = 1, \ldots, m-1$,
- $l_m = l_1 + \cdots + l_{k-1}$ and
- $|M_{a_1}| - l_1 < |M_{a_m}| + l_m$,

where $n \geq m+2$ is a sufficient condition to guarantee that the last inequality can be achieved independently from Π. Then we change the respective number of top positions of the alternatives to

$$|M_{a_1}| - l_1, \ldots, |M_{a_{m-1}}| - l_{m-1}, |M_{a_m}| + l_m,$$

where a_k and a_m should exchange their positions in the preference relations of the first l_k voters in M_{a_k} for all $k = 1, \ldots, m-1$. We shall denote the obtained profile by Π'_σ. In the special case of $|M_{a_1}| - l_1 = |M_{a_i}| - l_i$ we set the tie-breaking rule τ' at Π'_σ so that alternative a_1 is the most preferred one among the alternatives it is tied with. We emphasize that by our construction any alternative that was a top alternative of at least one voter in Π_σ remains also a top alternative of at least one voter in Π'_σ. The remaining at least one voter with the respective top alternative in Π'_σ serves as a separator so that Π_σ can be reconstructed from Π'_σ.

Thereafter, since σ is a bijection we have defined an injection $f : \mathcal{D}_1 \to \mathcal{D}_2$ and our result on the Nitzan-Kelly index on the IC follows. Finally, to arrive to the same statement on the AIC we just have to take into consideration that for anonymous SCFs

Table 2 Moving a manipulation possibility downwards

$\succ_1$	$\succ_2$	$\succ_3$	$\succ_4$	$\succ_5$	$\succ_6$	$\succ_7$	$\succ_1$	$\succ_2$	$\succ_3$	$\succ_4$	$\succ_5$	$\succ_6$	$\succ_7$
a	a	a	b	b	c	c	a	a	d	b	d	c	d
b	d	b	d	a	d	b	b	d	b	d	a	d	b
c	b	d	a	d	a	d	c	b	a	a	b	a	c
d	d	c	c	c	b	a	d	d	c	c	c	b	a

the AIC is basically related to a special ordering of preference relations. In particular, the ordering of profiles can be also obtained in two steps: first, order the profiles decreasingly based on their plurality scores, and second identical preferences should follow each other consecutively. □

Second, we show that the reverse plurality rule is manipulable on at least as many profiles as the plurality $m-1$th rule. We motivate the main step of our proof by the example shown in Table 2 in which $m = 4$ and $n = 7$. The plurality $m-1$th rule chooses alternative c for a tie-breaking rule preferring b to c at the profile shown on the left-hand side of Table 2. The profile is manipulable by the first five voters for a tie-breaking rule preferring b over c and c over a in the respective manipulated profiles. Let us pick the first voter with the $m-2$nd highest plurality score, that is voter 4. In order to maintain that voter 4 manipulates by making b the winning alternative instead of c for the reverse plurality rule, we make the top alternative b of voter 4 the plurality $m-1$th winner by switching the positions of a with d, b with d and c with d for the last voters having a, b and c on top, respectively. The resulting profile is shown on the right-hand side of Table 2 in which voter 4 can manipulate the plurality $m-1$th rule. Now voter 4 can manipulate the reverse plurality rule by revealing a preference relation having c on top. The main point is that voter 4 can manipulate the reverse plurality rule at the profile on the right-hand side of Table 2 in the same way as at the profile on the left-hand side of Table 2.

The proof of Proposition 2 will be quite similar to the proof of Proposition 1. However, in the previous proof we had upwards manipulation, whereas in the respective main part of the proof of our next proposition we will employ downwards manipulation.

Proposition 2 *Assume that $m \geq 3$, $n \geq m+2$, and τ is a given tie-breaking rule. Considering individual manipulability, there exists a tie-breaking rule τ' such that the plurality $m-1$th rule $f_\tau^{(m-1)}$ has not a larger NKI than the reverse-plurality rule $f_{\tau'}^{(m)}$ on both the IC and the AIC cases, where in the latter case anonymous tie-breaking rules have to be assumed.*

Proof We will construct an injection $f : \mathcal{D}_{m-1} \to \mathcal{D}_m$. Pick an arbitrary profile Π at which the plurality $m-1$th rule is manipulable.

We label the alternatives such that $|M_{a_1}| \geq |M_{a_2}| \geq \cdots \geq |M_{a_m}|$ and that in case of equalities they follow the ordering given by τ. Order the voters such that the $|M_{a_1}|$ voters with a_1 on the top come first, next the $|M_{a_2}|$ voters with a_2 on the top follow, and so forth. The ordering of voters having the same top alternative should be kept. We shall denote the reordered profile by Π_σ, where σ stands for the respective permutation.

Since the plurality $m-1$th rule is manipulable at Π_σ there exist $i, j \in A$ and a voter $q \in N$ such that $q \in M_{a_j}$, and q prefers a_i to a_{m-1} and can enforce outcome a_i. In particular, either $i = m$ (and then $j < m-1$) in case of upwards manipulation or $i < m-1$ in case of downwards manipulation. In the former case q reveals a preference ordering with a_i as its top alternative, while in the latter case q reveals a preference ordering with a_{m-1} as its top alternative. Select i, j, and q so that, following the respective lexicographic ordering, they are as small as possible, which implies the choice of downwards manipulation if both are possible. We exchange the preference ordering of voter q with that of the first voter having also a_j as the top alternative if this is not already the case. With a slight abuse of notation we will denote the obtained permutation still by σ.

We distinguish between two different cases. In *Case A* we assume that the downwards manipulability (i.e. $i < m-1$) of the plurality $m-1$th rule is possible at Π_σ, which requires that $|M_{a_{m-1}}| \leq |M_{a_i}| \leq |M_{a_{m-1}}| + 2$. In *Case B* we assume that the downwards manipulability of the plurality $m-1$th rule is not possible at Π_σ (i.e. $i = m$), and therefore only upwards manipulabilty is feasible.

We start with *Case A*. Our strategy will be to ensure a lead of alternative a_m against alternative a_{m-1} in the number of top positions by at least 1 voter. Then voter q can make a_i the winner of the reverse-plurality rule by 'overtaking' a_{m-1}. We shall denote by l the smallest positive integer for which

- $l_k = \max\left\{|M_{a_k}| - l, 1\right\}$ for all $k = 1, \ldots, m-1$,
- $l_m = l_1 + \cdots + l_{k-1}$ and
- $|M_{a_{m-1}}| - l_{m-1} < |M_{a_m}| + l_m$.

Then we change the respective number of top positions of the alternatives to

$$|M_{a_1}| - l_1, \ldots, |M_{a_{m-1}}| - l_{m-1}, |M_{a_m}| + l_m,$$

where a_k and a_m should exchange their positions in the preference relations of the last l_k voters in M_{a_k} for all $k = 1, \ldots, m-1$. We shall denote the obtained profile by Π'_σ. In the special case of $|M_{a_i}| - l_i = |M_{a_{m-1}}| - l_{m-1}$ we set the tie-breaking rule τ' at Π'_σ so that alternative a_{m-1} is the least preferred one among the alternatives it is tied with. Note that downwards manipulation of the plurality $m-1$th rule is not possible at Π_σ if $|M_{a_1}| = |M_{a_2}| = cldots = |M_{a_m}|$. Taking into account that $n \geq m+2$, $|M_{a_1}| > |M_{a_m}|$ and $|M_{a_1}|$ could be reduced to one, we have $l \in \left\{1, \ldots, |M_{a_1}| - 1\right\}$, which in turn implies that l is well-defined. By the construction of Π'_σ all alternatives being at least twice on top in Π_σ remain at least once on top in Π'_σ. Since these alternatives are separated in Π'_σ by a certain number of preferences with a_m as their top alternatives it is straightforward to get back Π_σ from Π'_σ.

We continue with *Case B* in which only the upward manipulation of Π is possible. Then any voter in $M_{a_{m-1}}$ can manipulate the reverse-plurality rule by revealing a preference relation with a_m on top. Hence, $\Pi' = \Pi = f(\Pi)$ and the reverse-plurality rule is manipulable at Π.

Since only Case B produces profiles Π' in which manipulation can happen only between a_{m-1} and a_m, we can deduce from Π' whether it has been derived through either Case A or B. Therefore, the constructed f is a bijection.

To see that our statement is valid under the IC assumption we reorder the preferences by $(\sigma')^{-1}$ to obtain the desired profile Π'. Since for any two profiles in $\mathcal{D}_{m-1}$ belonging to the same anonymous class (i.e. associated with the same profile in $\mathcal{D}^a_{m-1}$) different permutations have to be applied in the beginning of the proof, we have defined an injection $f : \mathcal{D}_{m-1} \to \mathcal{D}_m$. □

For formulating an analogous statement for the SNTV rules it will be helpful to define the domain of profiles, which consists of the upwards manipulable domains. Therefore, we shall denote by $\mathcal{U}_k$ and $\mathcal{U}^a_k$ the set of profiles and anonymous profiles, respectively, on which the plurality kth rule is individually upwards manipulable.

Proposition 3 *Assume that $m \geq 3$, $n \geq m+2$ and τ is a given tie-breaking rule. Considering individual manipulability, there exists a tie-breaking rule τ' such that the plurality kth rule $f^{(k)}_\tau$, where $k = 2, \ldots, m-2$ has a smaller NKI on IC and AIC than the plurality $k+1$th rule $f^{(k+1)}_{\tau'}$, where in the latter case anonymous tie-breaking rules have to be assumed.*

Proof First, we construct an injection $f : \mathcal{U}_k \to \mathcal{U}_{k+1}$. Pick an arbitrary profile Π at which the plurality kth rule is upwards manipulable. This can be done in an analogous way to the proof of Proposition 1. Note that profiles like in Case B in the proof of Proposition 2 do not have to be treated separately.

Second, to extend f to $\mathcal{D}_k$ we have to deal with the profiles in $\mathcal{D}_k \setminus \mathcal{U}_k$. Note that these purely downwards manipulable profiles can be handled in an analogous way as Case A in the proof of Proposition 2.

Finally, whether the images of the profiles are obtained through either upwards or downwards manipulation can be easily recognized by checking were the transformed preferences can be found, that is either there are the first or last l_k preferences in the image for any $k = 1, \ldots, m-1$. □

Remark 1 The “antidictatorial rule”, which chooses the worst alternative of a fixed voter, is manipulable at each profile.

Remark 2 We have shown that individual manipulability, i.e. the number of manipulable profiles increases as we move from the plurality 1st rule to the plurality mth rule by varying k. We further show in the “Appendix” that all plurality kth rules, except the plurality 1st rule, are group manipulable. It is important to observe that the plurality 1st rule, like other standard scoring rules, satisfies the condition of positive or non-negative responsiveness.[4] This condition requires that if an alternative w is chosen at some preference profile and the position of w improves in the ranking of some voter while holding the ranking of all other alternatives fixed in this voter's preference relation, then w is also selected at the modified preference profile. It is easy to show that all plurality kth rules with $k > 1$ violate positive responsiveness. This condition means

[4] See Arrow (1963) (Page 25). We would like to thank an anonymous referee for pointing out the relationship between manipulability of plurality kth rules and the positive responsiveness property.

that if a chosen alternative w improves in an agent's preference but the preference ordering for all other pairs where both alternatives in the pair are different from w remains unchanged, then w must continue to be chosen in the modified profile. This monotonicity requirement, as is often observed in several allocation models, is critical for strategy-proofness.

Turning to the SNTV rule, it can be observed that its manipulability can be captured through upwards manipulations since a successful manipulation results in getting an alternative ranked outside of the top k alternatives (based on their plurality scores) into the set of top k winning alternatives. We shall denote the respective (upper) manipulable domains of the SNTV rule $g_\tau^{(k)}$ by $\mathcal{U}_k^s$.

Proposition 4 *Assume that $m \geq 3$, $n \geq m+2$ and τ is a given tie-preserving tie-breaking rule. Considering individual manipulability, the SNTV rule $g_\tau^{(k)}$, where $k = 1, \ldots, m-2$, has a smaller NKI on IC and AIC than the SNTV rule $g_\tau^{(k+1)}$, where in the AIC case anonymous tie-breaking rules have to be assumed.*

Proof We construct an injection $f : \mathcal{U}_k^s \to \mathcal{U}_{k+1}^s$. Pick an arbitrary profile $\succ$ at which the SNTV rule $g_\tau^{(k)}$ is manipulable. The heart of the proof is the proof of Proposition 1.

The three differences from the proof of Proposition 1 are the following ones:

- The situation of ties can be handled more easily since it does not matter whether the alternative a_i entering the top k or $k+1$ alternatives in plurality scores by the manipulation of voter q will just make it into the winning set or even better, and therefore assuming tie-preserving tie-breaking rules without modifying τ to τ' does the job.
- It still does not matter that $|M(a_m)|$ will be ranked highest since it cannot loose its top position by a single-voter manipulation.
- The definition of $\succ_i^D$ reduces for $\succ$ the comparison of sets $g_\tau^{(k)}(\succ) = \{a_1, \ldots, a_k\}$ and $g_\tau^{(k)}(\succ_j', \succ_{-j}) = \{a_1, \ldots, a_{l-1}, a_i, a_{l+1}, \ldots, a_{k-1}\}$ to the comparison of alternatives a_i and a_k by $\succ_i$, which is done in the proofs of Propositions 1 and 3.

□

Note that the SNTV rule $g_\tau^{(m)}$ selects the set of all alternatives, and therefore a statement analogous to Proposition 2 does not hold true.

4 Non-manipulable subdomains

In this section we determine non-manipulable domains for the plurality kth rules, where we have to treat the plurality, the plurality kth with $1 < k < m$ and the reverse plurality rules separately.

Lemma 1 *A non-manipulable domain for the plurality rule for any tie-breaking rule is given by*

$$\mathcal{N}_1 = \left\{ \succ \in \mathcal{P}^n \mid p_\tau(\succ, 1) - p_\tau(\succ, 2) \geq 1 \right\}. \tag{1}$$

Furthermore, $\mathcal{N}_1$ is a maximal non-manipulable connected subdomain for the plurality rule for any tie-breaking rule.

Proof Clearly, a voter ranking the plurality winner on top has no incentive to misrepresent its preference relation, while voters with other top alternatives can individually only increase the votes of their lower ranked alternatives by the plurality rule by one. Therefore, for any profile in $\mathcal{N}_1$ any feasible manipulation has to result in an equal number of votes for the two leading alternatives based on their plurality scores in the manipulated profile, which is impossible without leaving $\mathcal{N}_1$, and therefore $\mathcal{N}_1$ is non-manipulable.

We prove that $\mathcal{N}_1$ is a maximal connected non-manipulable domain (for any tie-breaking rule). Note that $\mathcal{N}_1$ contains all unanimous profiles, that is all profiles in which all voters have the same top alternative. Pick an arbitrary profile $\succ$ such that $p_\tau(\succ, 1) = p_\tau(\succ, 2) > p_\tau(\succ, 3) \geq 0$.[5] Let $a = r_\tau(\succ, 1)$ and $b = r_\tau(\succ, 2)$. By the definition of $\mathcal{N}_1$ there exists a profile $\succ' \in \mathcal{N}_1$ such that $\succ_{-i} = \succ'_{-i}$ and $a \succ'_i b$ for a voter $i \in N$ who has top alternative a in $\succ_i$ and another top alternative c in $\succ'_i$. If b and c get the same plurality scores at $\succ'$, assume that $b\tau(\succ')c$. Then voter i can manipulate profile $\succ'$ by revealing $\succ_i$ instead of $\succ'_i$, which implies that $\mathcal{N}_1 \cup \{\succ\}$ is manipulable. Hence, we can conclude that $\mathcal{N}_1$ is a maximal connected non-manipulable domain. □

It is worth mentioning that $\mathcal{N}_1$ determines an 'improper' decomposition by letting $\mathcal{N}_2 = \ldots = \mathcal{N}_m = \emptyset$.

A non-manipulable domain can be given easily in an analogous way to Lemma 1, which is also maximal in the sense that the differences in plurality scores cannot be reduced further; however, it still can be extended by some profiles without violating non-manipulability.

Lemma 2 *A non-manipulable domain for the plurality kth rule, where $k = 2, \ldots, m-1$, for any tie-breaking rule is given by*

$$\mathcal{N}_k^u = \left\{ \succ \in \mathcal{P}^n \mid p_\tau(\succ, k-1) - 2 \geq p_\tau(\succ, k) \geq p_\tau(\succ, k+1) + 1 \right\}. \tag{2}$$

Let $L \in \mathcal{P}$ be a given linear ordering of alternatives. Then

$$\begin{aligned} \mathcal{N}_k = \Big\{ \succ \in \mathcal{P}^n \mid\ & p_\tau(\succ, k-1) - 1 = p_\tau(\succ, k) \geq p_\tau(\succ, k+1) + 1 \text{ and } \forall i \in A : \\ & p_\tau(\succ, i) = p_\tau(\succ, k-1) \Rightarrow r_\tau(\succ, i)\ L\ r_\tau(\succ, k) \Big\} \cup \mathcal{N}_k^u \end{aligned} \tag{3}$$

is a maximal non-manipulable connected subdomain for the plurality kth rule for any tie-breaking rule.[6]

Proof Taking downwards and upwards manipulations into account, a lead by at least 3 votes by the $k-1$th ranked alternative over the kth ranked alternative and a lead

[5] Let us remark that since we are looking for a maximal connected non-manipulable domain containing $\mathcal{N}_1$ more than two alternatives cannot have the highest plurality scores in a profile obtained by a manipulation of a profile in $\mathcal{N}_1$.

[6] The precondition of the implication of 3 in definitely true for $i = k-1$ and also holds for other alternatives $a_{k-l}, \ldots, a_{k-2}$ possibly tied with a_{k-1}.

by at least 2 votes by the kth ranked alternative over the $k+1$th alternative makes manipulations impossible. However, since by individual misrepresentation we cannot leave $\mathcal{N}_k^u$ the differences for both downwards and upwards manipulations can be reduced by 1 as formulated in (2), and therefore $\mathcal{N}_k^u$ is indeed non-manipulable. Note that the differences in plurality scores in (2) cannot be decreased 'uniformly' (i.e. for all profiles) since reducing 2 to 1 in the first inequality would make it possible to switch a lead by one vote of the $k-1$th alternative over the kth alternative into the opposite direction without leaving the restricted subdomain and reducing 1 to 0 in the second inequality obviously opens the door for manipulation. This is also the reason why we can extend $\mathcal{N}_k^u$ (where superscript u stands for uniform) to $\mathcal{N}_k$ without allowing space for manipulation.

We prove that $\mathcal{N}_k$ is a maximal non-manipulable connected domain (for any tie-breaking rule). First, pick an arbitrary profile $\succ$ such that either $p_\tau(\succ, k-1) - 2 \geq p_\tau(\succ, k) = p_\tau(\succ, k+1)$, or $p_\tau(\succ, k-1) - 1 = p_\tau(\succ, k) = p_\tau(\succ, k+1)$ and $r_\tau(\succ, k-1)\ L\, r_\tau(\succ, k)$. Let $a = r_\tau(\succ, k), b = r_\tau(\succ, k+1)$ and $c = r_\tau(\succ, k-1)$. By the definition of $\mathcal{N}_k^u$ there exists a profile $\succ' \in \mathcal{N}_k^u$ such that $\succ_{-i} = \succ'_{-i}$ and $a \succ'_i b$ for a voter $i \in N$ who has top alternative a in $\succ_i$ and top alternative c in $\succ'_i$. Then voter i can manipulate profile $\succ'$ by revealing $\succ_i$ instead of $\succ'_i$, which implies that $\mathcal{N}_k \cup \{\succ\}$ is (upwards) manipulable.

Second, pick an arbitrary profile $\succ$ such that $p_\tau(\succ, k-1) - 1 = p_\tau(\succ, k) \geq p_\tau(\succ, k+1) + 1$ and $\succ \notin \mathcal{N}_k$. Let $a = r_\tau(\succ, k-1)$, $b = r_\tau(\succ, k)$ and $c = r_\tau(\succ, k+1)$. Choose a voter i with top alternative a in $\succ$. Since $\succ \notin \mathcal{N}_k$ the profile $\succ' = (\succ'_i, \succ_{-i})$ in which a voter i has top alternative b in $\succ'_i$ is in $\mathcal{N}_k$. Hence, i can manipulate $\succ$ by revealing $\succ'_i$.

Third, pick an arbitrary profile $\succ$ such that $p_\tau(\succ, k-1) = p_\tau(\succ, k) > p_\tau(\succ, k+1) + 1$. Let $a = r_\tau(\succ, k-1)$, $b = r_\tau(\succ, k)$ and $c = r_\tau(\succ, k+1)$. Now choose a voter $i \in N$ with top alternative a. Clearly, voter i can manipulate $\succ$ by revealing a preference relation $\succ'_i$ with top alternative b. In addition, $\succ' = (\succ'_i, \succ_{-i}) \in \mathcal{N}_k$ since b leads by two votes over a and a still leads by at least one vote over c in $\succ'$.

Forth, pick an arbitrary profile $\succ$ such that $p_\tau(\succ, k-1) = p_\tau(\succ, k) = p_\tau(\succ, k+1) + 1$. Let $a = r_\tau(\succ, k-1)$, $b = r_\tau(\succ, k)$ and $c = r_\tau(\succ, k+1)$. Then to assure that $\mathcal{N}_k \cup \{\succ\}$ remains connected $\succ$ has to be obtained by a misrepresentation of voter i such that for profile $\succ' = (\succ'_i, \succ_{-i})$ either (i) $s(\succ', b) = s(\succ', a) + 1 \geq s(\succ', c) + 2$ if bLa or (ii) $s(\succ', a) = s(\succ', b) + 1 \geq s(\succ', c) + 2$ if aLb. In case (i) b has to be the top alternative of $\succ'_i$, while it is not the top alternative of $\succ_i$. Therefore, i can manipulate $\succ'$ by revealing $\succ_i$. In case (ii) considering another tie-breaking rule τ' that switches the positions of a and b at profile $\succ$ leads to case (i).

Hence, we can conclude that $\mathcal{N}_k$ is a maximal connected non-manipulable domain. □

Lemma 3 *A non-manipulable domain for the reverse plurality rule for any tie-breaking rule is given by*

$$\mathcal{N}_m^u = \left\{ \succ \in \mathcal{P}^n \mid p_\tau(\succ, m-1) - 2 \geq p_\tau(\succ, m) \right\}.$$

Let $L \in \mathcal{P}$ be a given linear ordering of alternatives. Then

$$\mathcal{N}_m = \{\succ \in \mathcal{P}^n \mid p_\tau(\succ, m-1) - 1 = p_\tau(\succ, m) \text{ and } \forall i \in A: \\ p_\tau(\succ, i) = p_\tau(\succ, m-1) \Rightarrow r_\tau(\succ, i)\ L\ r_\tau(\succ, m)\} \cup \mathcal{N}_m^u \quad (4)$$

is a maximal non-manipulable connected subdomain for the reverse plurality rule for any tie-breaking rule.

Proof The proof can be done by repeating some parts of the proof of Lemma 2 related to downwards manipulation for $k = m$ since upwards manipulation is impossible. □

We formulate the following statement concerning the manipulability of the plurality kth rule and that of the SNTV rule with k winners.

Lemma 4 *Let us restrict ourselves to tie-order preserving tie-breaking rules. Then if the plurality kth rule $f_\tau^{(k)}$ is upwards manipulable, then the SNTV rule $g_\tau^{(k)}$ is manipulable. The opposite direction may fail in case of multiple ties from the $k-j$th to the $k+l$th positions, where $j, l \geq 1$. However, if the SNTV rule $g_\tau^{(k)}$ is manipulable at a profile, then there exists a tie-order preserving tie-breaking rule τ' such that both $f_{\tau'}^{(k)}$ and $g_{\tau'}^{(k)}$ are manipulable at that profile.*

Proof For a given profile $\succ$ let $a_i = f_\tau^{(i)}(\succ)$ for all $i = 1, \ldots, m$. If the plurality kth rule is upwards manipulable at profile $\succ$, then there exists a voter i such that $a_j \succ_i a_k$, $j > k$, $\succ_i' \in \mathcal{P}$ and $f_\tau^{(k)}(\succ_i', \succ_{-i}) = a_j$, which in turn implies that $g_\tau^{(k)}(\succ_i', \succ_{-i}) \succ_i^L g_\tau^{(k)}(\succ)$ since the alternatives $a_1, \ldots, a_{k-1}$ remain the top $k-1$ alternatives at $\succ' = (\succ_i', \succ_{-i})$ and $\{a_j\} \succ_i^L \{a_k\}$.

A manipulation by voter i of the SNTV rule for the given tie-breaking rule τ at profile $\succ$ by revealing $\succ_i'$ could move an alternative from the $k+j$th position to the $k-l$th position without an improvement at the kth position for the manipulator, and thus without a manipulation of the plurality kth rule. However, then either alternatives a_{k-1} and a_k are tied or not in both $\succ$ and $\succ'$. In the former case, since there is no improvement at the kth position (i.e. $a_k \succ_i a_{k-1}$)[7], there exists a tie-breaking rule τ', which reorders the tied alternatives a_{k-1} and a_k at $\succ$ (i.e. prior to manipulation), while in the latter case a_{k-l} will be tied with a_{k-1} after the manipulation and let τ' reorder these two alternatives at $(\succ_i', \succ_{-i})$ (i.e. after manipulation). □

We also determine a maximal non-manipulable connected domain for the SNTV rule.

Lemma 5 *A non-manipulable domain for the SNTV rule choosing k winners, where $k = 2, \ldots, m-1$, for any tie-breaking rule is given by*

$$\overline{\mathcal{N}}_k = \{\succ \in \mathcal{P}^n \mid p_\tau(\succ, k) \geq p_\tau(\succ, k+1) + 1\}. \quad (5)$$

Furthermore, $\overline{\mathcal{N}}_k$ is a maximal non-manipulable connected domain for the SNTV rule choosing k winners for any tie-breaking rule.

[7] Note that the manipulation of the SNTV rule by voter i cannot result in a decrease of the plurality scores of alternatives $a_1, \ldots, a_k$.

Proof One has to employ the parts of the proof of Lemma 2 related to upwards manipulation. □

From the proof of Lemma 2 it also follows that if we restrict ourselves to upwards manipulations of the plurality kth rule, then we also obtain $\overline{\mathcal{N}}_k$ as the maximal upwards non-manipulable domain for the plurality kth rule.

5 A decomposition with a corollary

Observe that any SCF $f : \mathcal{P}^n \to A$ can be decomposed into plurality kth rules, that is there exists $\mathcal{X}_k \subseteq \mathcal{P}^n$ ($k = 1, \ldots, m$) such that

- $\cup_{k=1}^{m} \mathcal{X}_k = \mathcal{P}^n$,
- $\mathcal{X}_k \cap \mathcal{X}_l = \emptyset$ if $k \neq l$, and
- $f_{|\mathcal{X}_k} = f^{(k)}_{|\mathcal{X}_k}$.

We illustrate a decomposition by the following example.

Example 1 Let $m = 4$, $n = 3$ and the tie-breaking rule τ equaling the constant tie-breaking rule given by the fixed ordering $a\tau b\tau c\tau d$. Let f_τ^B be the Borda count with tie-breaking rule τ.

Then, for instance, for profile $\Pi =$

$\succ_1$	$\succ_2$	$\succ_3$
a	a	b
b	b	c
c	c	d
d	d	a

we have $f_\tau^B = f_\tau^{(2)}$, that is on the above profile Π the Borda winner equals with the choice of the plurality 2nd rule. Clearly, on unanimous profiles the Borda count equals the plurality rule. We do not list a partition of the 46656 profiles for $m = 4$ and $n = 3$ determining the profiles on which the Borda count is equal to a given plurality kth rule.

From the Gibbard–Satterthwaite theorem (1973/75) we know that only the dictatorial SFCs are non-manipulable and onto in case of at least three alternatives. Therefore, our aim with the above described decomposition is to obtain a maximal non-manipulable subdomain of $\mathcal{P}^n$. Plurality kth rules are especially helpful since the room for manipulation by an individual is extremely small since it can just increase the received votes of an alternative by just one and decrease the received votes of an other alternative by just one, thus keeping our analysis as simple as possible.

In order to arrive to a non-manipulable subdomain we have to find disjoint subdomains $\mathcal{N}_1, \mathcal{N}_2, \ldots, \mathcal{N}_m$ on which the plurality first, the plurality second, and so on rules will be employed. Our strategy will be to assure that manipulation within any $\mathcal{N}_k$ becomes impossible and none of the voters can move by an individual misrepresentation a profile from $\mathcal{N}_i$ to another $\mathcal{N}_j$. Note that while $\mathcal{N}_i$ is connected, their union will

Table 3 A decomposition for $m = 3$ and $n = 15$

Rank	$\mathcal{N}_1'$						$\mathcal{N}_2'$		$\mathcal{N}_3'$		
1^{st}	13	12	11	11	10	9	9	8	8	7	7
2^{nd}	2	3	4	3	4	4	6	7	5	6	5
3^{rd}	0	0	0	1	1	2	0	0	2	2	3

be disconnected. For simplicity we start with the case in which the non-manipulability on $\mathcal{N} = \mathcal{N}_1 \cup \ldots \cup \mathcal{N}_m \subset \mathcal{P}^n$ has to hold for any tie-breaking rule.

From Lemmas 1–3 it follows that $\mathcal{N}' = \cap_{i=1}^m \mathcal{N}_m$ is non-manipulable for any plurality kth rule, which in itself is a trivial observation; however, it just does not allow manipulation for those subdomains of $\mathcal{N}'$ on which the plurality kth rule is employed, but admits manipulations by moving from a profile in $\mathcal{N}_i$ to a profile in $\mathcal{N}_j$. Therefore, a slightly more sophisticated construction to arrive to a partition of a subdomain of $\cup_{i=1}^m \mathcal{N}_m$ is needed to employ several plurality kth rules at the same time with a non-manipulable subdomain.

From the lessons of Lemmas 1–3 we construct a non-manipulable subdomain, with appropriately defined subdomains $\mathcal{N}_i'$ on which the plurality kth rules will be employed, respectively.[8] Clearly, if we would take the non-manipulable domain from Lemma 1, we could not introduce the other plurality kth rules into our decomposition. We know that a safe difference in plurality scores for the plurality rule is 1 downwards, while for the other plurality kth rules 2 and 1 upwards and downwards, respectively. If we relax our assumption on the plurality rule and require a larger difference in the plurality scores of the top two alternatives, then we will give room for the other plurality kth rules. This is the reason for the relaxed condition $p_\tau(\succ, 1) - p_\tau(\succ, 2) \geq 5$ in the definition of $\mathcal{N}_1'$ in the upcoming Proposition 5. We can see similar conditions in the definitions of $\mathcal{N}_k'$ for all $k = 2, \ldots, l$, i.e. $p_\tau(\succ, k) - p_\tau(\succ, k+1) \geq 6$. The other inequalities guarantee that $\mathcal{N}_i'$ and $\mathcal{N}_j'$ are disjoint if $i \neq j$.

We present a motivating example in Table 3 before stating Proposition 5. The columns of Table 3 express the distribution of the 15 voters according to the numbers of first, second and third ranked alternatives. Each column gathers many preferences. In addition, the underlying preferences are grouped into the three possible non-manipulable subdomains. As it can be seen, the first ranked alternatives by the profiles in $\mathcal{N}_1'$ have a large lead over their second ranked alternatives, in particular, they have a lead of at least five top positions. The difference in the number of top positions between the second and third ranked alternatives lie between 2 and 4. From this we can immediately see that manipulation within $\mathcal{N}_1'$ is not possible. Turning to $\mathcal{N}_2'$, we can see differences in the number of top positions of at least six between the second and third alternatives, whereas the differences in top positions between the first and second ranked alternatives lie between 1 and 3. It follows that $\mathcal{N}_2'$ is non-manipulable. Looking at $\mathcal{N}_3'$, the differences in the number of top positions between the first and second ranked and between the second and third ranked alternatives lie between 1 and 3 and between 2 and 4, respectively. Obviously, $\mathcal{N}_3'$ is non-manipulable. Finally, it is

[8] It is worthwhile to mention that there are multiple ways for coming up with a decomposition utilizing all plurality kth rules.

impossible to move from a preference relation in $\mathcal{N}_i'$ to a preference relation in $\mathcal{N}_j'$, where $i \neq j \in \{1, 2, 3\}$, by a single-voter manipulation.

Proposition 5 *Let $l < m$ and*

$$\begin{aligned}
\mathcal{N}_1' = \{\succ \in \mathcal{P}^n \mid & \, p_\tau(\succ, 1) - p_\tau(\succ, 2) \geq 5 \\
& 4 \geq p_\tau(\succ, i) - p_\tau(\succ, i+1) \geq 2 \textit{ for all } i = 2, \ldots l\}, \\
\mathcal{N}_k' = \{\succ \in \mathcal{P}^n \mid & \, 3 \geq p_\tau(\succ, 1) - p_\tau(\succ, 2) \geq 1 \\
& p_\tau(\succ, k) - p_\tau(\succ, k+1) \geq 6 \\
& 4 \geq p_\tau(\succ, i) - p_\tau(\succ, i+1) \geq 2 \textit{ for all } i = 2, \ldots l \textit{ and } i \neq k\}, \\
\mathcal{N}_{l+1}' = \{\succ \in \mathcal{P}^n \mid & \, 3 \geq p_\tau(\succ, 1) - p_\tau(\succ, 2) \geq 1 \\
& 4 \geq p_\tau(\succ, i) - p_\tau(\succ, i+1) \geq 2 \textit{ for all } i = 2, \ldots l\},
\end{aligned}$$

where $k = 2, \ldots, l$. Assuming that the plurality kth rule is employed on $\mathcal{N}_k'$, the subdomain $\mathcal{N} = \cup_{i=1}^{l+1} \mathcal{N}_i$ is non-manipulable for any tie-breaking rule for the just defined SCF on $\mathcal{N}$.

Proof Lemmas 1–3 imply that the subdomains $\mathcal{N}_k'$ are non-manipulable and the construction of $\mathcal{N}_1', \ldots, \mathcal{N}_{l+1}'$ assures that changing the preference relation of a single voter does not allow to us to move a profile from $\mathcal{N}_i'$ to another $\mathcal{N}_j'$. □

Our next proposition utilizes our decomposition and the derived non-manipulable domains for plurality kth rules.

Proposition 6 *Let $\mathcal{X}_1, \ldots, \mathcal{X}_m$ be a decomposition into plurality kth rules of SCF $f : \mathcal{P}^n \to A$ and let $\mathcal{M}_k = \mathcal{X}_k \cap \mathcal{N}_k'$ for any $k = 1, \ldots, m$. Then $\mathcal{M} = \cup_{i=1}^m \mathcal{M}_i$ is a non-manipulable domain for f.*

Though the statement of Proposition 6 is straightforward, we think that its result is surprising since it means that for a subdomain of preference profiles for which the plurality scores of the alternatives cannot be too close the respective SCF is non-manipulable. Hence, based on a simple statement on plurality scores and the decomposition of an arbitrary SCF we can arrive to a non-manipulable domain of that SCF.

6 Concluding remarks

In this paper we have introduced plurality kth rules and investigated their individual and group manipulability (the latter in the "Appendix"). For larger k the plurality kth rule is manipulable on more profiles. Any social choice function can be expressed as a 'combination' of plurality kth rules, which employed for providing non-manipulable domains for any social choice function. We established connections between the plurality kth rules and the SNTV rule.

Investigating Borda kth rules or more generally scoring kth methods, could be a direction of possible future extensions of this work. Capturing the manipulability of plurality k rules is definitely simpler than, for instance, obtaining similar results for Borda kth rules. The growing literature on multi-winner elections serves as a motivation for these type of extensions.

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Appendix: Group manipulability

Concerning group manipulability, we obtain an even more negative result since for any plurality kth rule, with the exception of the plurality rule, there exists a tie-breaking rule such that it is group manipulable at any profile. However, if we require group manipulability for any tie-breaking rule, then the reverse-plurality rule remains the only one that can be manipulated at any profile.

We define below the group manipulability of SCFs. Consider a preference profile $(\succ_1, \ldots, \succ_n) \in \mathcal{P}^n$.

Definition 9 An SCF $f : \mathcal{P}^n \to A$ is *group manipulable by a non-empty set of voters* $M \subseteq N$ at $(\succ_1, \ldots, \succ_n) \in \mathcal{P}^n$ if for all $i \in M$,

$$\exists \succ_i' \in \mathcal{P} \text{ such that } f(\succ_M', \succ_{-M}) \succ_i f(\succ_M, \succ_{-M}).$$

An SCF $f : \mathcal{P}^n \to A$ is *group manipulable at* $(\succ_1, \ldots, \succ_n) \in \mathcal{P}^n$ if there exists a non-empty set of voters $M \subseteq N$ who can manipulate f.

First, we start with a result showing that the reverse-plurality rule is extremely vulnerable to group manipulation.

Proposition 7 *There exists an anonymous tie-breaking rule for which the reverse-plurality rule is group manipulable at any profile with* $n \geq 2$.

Proof Pick a profile $(\succ_1, \ldots, \succ_n) \in \mathcal{P}^n$ and an ordering $a_1, a_2, \ldots, a_m$ of the alternatives such that $|M_{a_1}| \leq |M_{a_2}| \leq \cdots \leq |M_{a_m}|$. Choose an $a_i \neq a_1$ for which $|M_{a_i}| > 0$ and then a tie-breaking τ in which a_1 has the highest priority and a_i the second highest priority. Then the reverse-plurality rule f_τ^* selects alternative a_1 and the group M_{a_i} of voters can manipulate by all of them switching to preferences with a_1 on the top. $\square$

From the proof of Proposition 7 we can see that the tie-breaking rule has to be set in a specific way for being able to group manipulate at profile Π if and only if there are at least two alternatives receiving zero vote. The next proposition shows that we need exactly at least a certain number of voters such that we do not need to care about the tie-breaking rule.

Proposition 8 *The reverse-plurality rule is group manipulable for any tie-breaking rule at any profile if and only if for odd m we have* $n > (m-1)^2/4$ *and for even m we have* $n > (m^2 - 2m)/4$.

Proof Pick a profile $(\succ_1, \dots, \succ_n) \in \mathcal{P}^n$ and an ordering $a_1, a_2, \dots, a_m$ of the alternatives such that $|M_{a_1}| \le |M_{a_2}| \le \cdots \le |M_{a_m}|$.

Observe in the proof of Proposition 7 we needed to set a tie-breaking rule because it might be the case that in profile $(\succ_1, \dots, \succ_n) \in \mathcal{P}^n$ at least two alternatives are not a top alternative of any voter and then for any other profile the proof would work for any tie-breaking rule. Hence, we can restrict ourselves to profiles for which $|M_{a_1}| = |M_{a_2}| = 0$. Let $k = \max\{j \in \{1, \dots, m\} \mid |M_{a_j}| = 0\}$. Note that if $|M_{a_l}| \ge k$, then the voters in M_{a_l} can manipulate by choosing alternatives $a_1, \dots, a_k$ as the set of their top alternatives. Suppose that $(\succ_1, \dots, \succ_n) \in \mathcal{P}^n$ is non-manipulable in the described way, then we must have $n \le (m-k)(k-1)$ since all of the alternatives being a top alternative of somebody can be at most $k-1$ times on the top. Expression $(m-k)(k-1)$ is maximized in k at $k = (m+1)/2$ if m is odd, and at $k = m/2$ or $k = m/2 + 1$ if m is even. Then we must have $n \le n_o^* = (m-1)^2/4$ and $n \le n_e^* = (m^2 - 2m)/4$, respectively, to make a manipulation of the described type infeasible.

Finally, we show that the bounds on n are tight, that is for any profile with at most n_o^* or at most n_e^* voters there exist profiles and tie-breaking rules such that manipulation is not possible. Let k be the value determined in the above paragraph separately for odd and even m. Then pick a profile in which $0 < |M_{a_l}| \le k-1$ for any $l \in \{k+1, \dots, m\}$ and in any of the preferences with a_l on the top the alternatives $a_1, \dots, a_k$ are ranked above the top alternatives of the other voters ($\{a_{k+1}, \dots, a_m\} \setminus \{a_l\}$), and pick the tie-breaking rule preferring alternatives having smaller indices to alternatives having larger indices. Clearly, any group of voters M_{a_l}, where $l > k$, cannot manipulate without cooperating with voters from other groups. However, such cooperations are not viable based on our tie-breaking rule because each of them prefer the first k alternatives to the top alternatives of the other ones. □

We can see from Proposition 8 that the price for maintaining the group manipulability of the reverse-plurality rule at any profile for any fixed tie-breaking rule is quite low since the required minimal number of voters is fairly small in the number of alternatives and is usually satisfied in real-life voting situations.

The result of Proposition 7 can be extended to the plurality kth rules with the exception of the plurality rule.

Proposition 9 *For any* $k = 2, \dots, m-1$ *there exists an anonymous tie-breaking rule for which the plurality kth rule is group manipulable at any profile with* $n \ge 2$.

Proof Pick a profile $\Pi = (\succ_1, \dots, \succ_n) \in \mathcal{P}^n$ and pick an ordering $a_1, a_2, \dots, a_m$ of the alternatives such that $|M_{a_1}| \ge |M_{a_2}| \ge \cdots \ge |M_{a_m}|$. Choose $\tau(\Pi)$ in a way that makes a_k the kth ranked alternative in profile Π. Let $\Delta = |M_{a_1}| - |M_{a_k}|$.

If $\Delta > 0$, then construct profile Π' from Π by changing the preference relations of Δ voters from M_{a_1} to preference relations with a_k on the top. Select $\tau(\Pi')$ such that a_1 becomes the kth ranked alternative in profile Π'. Then the respective voters can group manipulate.

If $\Delta = 0$, then to obtain profile Π' from Π just change the preference relation of one voter in $M(a_1)$ to another preference relation still having a_1 on the top, but ordering the other alternatives differently. Select $\tau(\Pi')$ such that a_1 becomes the kth ranked alternative in profile Π'. Then the respective voter can manipulate. It is worthwhile mentioning that these changes can be done consistently by considering profiles in which $|M_{a_1}| = \cdots = |M_{a_k}|$ and taking an appropriate circular ordering of these profiles. □

Our next proposition shows that surprisingly the selection of a tie-breaking rule in Proposition 9 plays a central role since we cannot maintain group manipulability at any profile for any tie-breaking rule even if we have sufficiently many voters for the plurality kth rules with the exceptions of the reverse-plurality rule. From another point of view an analogous statement to Proposition 8 does not hold for the plurality kth rules, where $1 \leq k < m$.

Proposition 10 *For any $k = 1, \ldots, m-1$ there exists a tie-breaking rule and there exist profiles on which the plurality kth rule is not group manipulable.*

Proof For any $k = 2, \ldots, m-1$ we provide a tie-breaking rule and a profile at which the plurality kth rule is not group manipulable. We consider profiles satisfying $|M_{a_1}| \geq |M_{a_2}| \geq \cdots \geq |M_{a_{k+1}}| \geq 0 = |M_{a_{k+2}}| = \cdots = |M_{a_m}|$ and $|M_{a_1}|$ almost equals $|M_{a_{k+1}}|$ by which we mean that $|M_{a_1}| - |M_{a_{k+1}}| \leq 1$. In case of possible ties the tie-breaking rule has to select a_k as the plurality kth winner. We restrict our set of investigated profiles even further by requiring that a_k should be the second ranked alternative in the orderings of voters not in M_{a_k}. For these type of profiles a group of voters manipulates only if it can make the top alternative of its voters the plurality kth winner, which implies that voters with different top alternatives do not cooperate. If at least two voters in M_{a_i} reveal another top alternative, then there will be at least k alternatives receiving more votes than a_i and the misrepresentation will not be beneficial. If only one voter in M_{a_i} reveals a different top alternative, then a_i can have the kth most vote only in a tie. If in this case the tie-breaking rule is set in a way that alternative a_i cannot win, then the plurality kth rule is not group manipulable at this profile.

For the plurality rule consider, for instance, unanimous profiles (i.e $N = M_{a_1}$), which are not group manipulable even for any tie-breaking rule. □

References

Arrow KA (1963) Social choice and individual values. Wiley, New York

Aleskerov F, Kurbanov E (1999) Degree of manipulability of social choice procedures. In: Alkan A et al (eds) Current trends in economics. Springer, Berlin

Barbie M, Puppe C, Tasnádi A (2006) Non-manipulable domains for the borda count. Econ Theory 27:411–430

Bednay D, Moskalenko A, Tasnádi A (2017) Does avoiding bad voting rules lead to the good ones? Oper Res Lett 45:448–451

Eğecioğlu Ö, Giritligil AE (2013) The impartial, anonymous, and neutral culture model: a probability model for sampling public preference structures. J Math Soc 37:203–222

Elkind E, Faliszewski P, Slinko A (2015) Distance rationalization of voting rules. Soc Choice Welfare 45:345–377

Elkind E, Faliszewski P, Skowron P, Slinko A (2017) Multiwinner voting: a new challenge for social choice theory. In: Endriss U (ed) Trends in computational social choice. AI Access Foundation

Faliszewski P, Skowron P, Slinko A, Talmon N (2018) Multiwinner analogues of the plurality rule: axiomatic and algorithmic views. Soc Choice Welfare 51:513–550

Gibbard A (1973) Manipulation of voting schemes: a general result. Econometrica 41:587–601

Lang J, Xia L (2016) Voting in combinatorial domains. In: Brandt F, Conitzer V, Endriss U, Lang J, Procaccia AD (eds) Handbook of computational social choice. Cambridge University Press, pp 197–222

Kelly J (1988) Minimal manipulability and local strategy-proofness. Soc Choice Welfare 5:81–85

Kilgour M (2018) Multi-winner voting. Estudios de Economía Aplicada 36:167–180

Maus S, Peters H, Storcken T (2007a) Anonymous voting and minimal manipulability. J Econ Theory 135:533–544

Maus S, Peters H, Storcken T (2007b) Minimally manipulable anonymous social choice functions. Math Soc Sci 53:239–254

Nitzan S (1985) The vulnerability of point-voting schemes to preference variation and strategic manipulation. Public Choice 47:349–370

Nitzan S (1981) Some measures of closeness to unanimity and their implications. Theo Decis 13:129–138

Richelson JT (1978) A characterization result for the plurality rule. J Econ Theory 19:548–550

Roberts FS (1991) Characterizations of the plurality function. Math Soc Sci 21:101–127

Sanver MR (2009) Strategy-proofness of the plurality rule over restricted domains. Econ Theory 39:461–430

Satterthwaite M (1975) Strategy-proofness and Arrow's conditions: existence and correspondence theorems for voting procedures and social welfare functions. J Econ Theory 10:187–217