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Abstract

In this paper, we show that the capital asset pricing model can be derived from a three-period general equilibrium model. We show that our extended model yields a Pareto efficient outcome. This result indicates that the beta pricing formula could be applied in a long term model settings as well.

Keywords: general equilibrium, CAPM, intertemporal choice, Pareto efficiency

JEL Classification: D53, G12, D15

1 Introduction

The capital asset pricing model, routinely referred to as CAPM in the literature, accurately estimates the relationship between the risk and the expected return of an asset. Its foundations were established by Sharpe (1964); Lintner (1965); Mossin (1966). The CAPM model is in fact used for the estimation of expected returns of risky assets in equilibrium. The CAPM can be derived from a two-period general equilibrium model which provides a sound theoretical basis for one of the essential tools of modern portfolio management: the Return-Beta relationship.

In this paper, we extend the consumption-based capital asset pricing model to a three-period finance economy. This extension can potentially have remarkable effects on several other fields of application. For example, a minimum of three periods is both necessary for handling long term financial assets and adding time-inconsistent behaviour into the context of financial-economic modeling. We introduce the three-period intertemporal general equilibrium model with one asset and the consumption-based version of the popular CAPM model, Consumption Capital Asset Pricing Model (CCAPM).

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In Section 2 we introduce the three-period general equilibrium model and show that the resulting consumption plan is efficient if markets are complete and that the firs theorem of welfare economics remains fulfilled in the three-period model as well. Section 3 defines the CCAPM, which is followed by the derivation of the three-period CAPM in Section 4. As a foundation of our model, we use the well-known, two-period pricing equations described in the book by LeRoy and Werner (2001) which we frequently use as building blocks in this study.

2 The Three-Period Finance Economy

This section is dedicated to introduce the definitions and notations that are necessary elements for the dynamics of the model. The described structure is based on the one in the article by Habis and Herings (2011).

Let $t \in \{0,1,2\} = T$ denote the time periods. In each period t one event out of a finite set occurs. At every state $s \in \mathcal{S}$ we denote the date-event at period t by $s_t \in \mathcal{S}_t$, where the cardinality of $\mathcal{S}_t$ is S_t and $\mathcal{S} = \bigcup_t \mathcal{S}_t$ for all $t \in T$. For $t = 0$ we define $s_0 = 0$. Let s_t^+ be the set of successors of s_t for all $t = 0, 1$ and s_t^- the set of predecessors of s_t for all $t = 1, 2$. In each period there is a single, non-durable consumption good.

There are a finite number of agents $h \in H$ participating in the economy. Each agent h has initial endowments $(e^h_{s_t})_{s_t \in \{0\} \cup \mathcal{S}_1 \cup \mathcal{S}_2} \in \mathbb{R}^{(S_1+S_2+1)}$. Agents have preferences over consumption bundles $c^h_{s_t} \in \mathbb{R}^{(S_1+S_2+1)}$ where $s_t \in \mathcal{S}$. Each agent's preferences are represented by a von Neumann-Morgenstern utility function that is additively separable over time and at period 0 it is defined by

$$u^h(c^h) = v^h_0(c^h_0) + \delta_1 \sum_{s_1 \in \mathcal{S}_1} \rho_{s_1} v^h_{s_1}(c^h_{s_1}) + \delta_1 \delta_2 \sum_{s_1 \in \mathcal{S}_1} \rho_{s_1} \sum_{s_2 \in s_1^+} \rho_{s_2} v^h_{s_2}(c^h_{s_2}) \qquad (1)$$

where ρ_{s_1} denotes the probability of occurrence of event s_1 and ρ_{s_2} denotes the probability of occurrence of event s_2 given event s_1 has occurred, δ_t is a one-period discount factor and $v^h_{s_t}$ is a Bernoulli function.

We apply the following assumptions throughout the paper. We assume that $\rho_{s_t} > 0$ for all $s_t \in \mathcal{S}_t$ and $\sum_{s_1 \in \mathcal{S}_1} \rho_{s_1} = 1$, $\sum_{s_2 \in \mathcal{S}_2} \rho_{s_2} = 1$, $\delta_1, \delta_2 > 0$, the probabilities and discount factors are identical across agents, and that the Bernoulli utility function is strictly increasing. Furthermore $c^h \in X^h$ where $X^h \subset \mathbb{R}^{1+S_1+S_2}$ and X^h is the vector of consumption bundles for agent h.

The constraint of $\rho_{s_t} > 0$ means that the agents only take into account the future outcomes for which the objective probability of occurrence is positive, i.e. unlikely events do not affect their utility. A further simplifying assumption is that all agents apply the same discount factors and have no satiation point.

There are J_{s_t} assets at each $s_t \in \{0\} \cup \mathcal{S}_1$. The set of assets at event s_t is $\mathcal{J}_{s_t}$. Each asset j pays (random) dividends $d_{s_{t+1},j}$ at date-events $s_{t+1} \in s_t^+$. We denote the vector of dividends by $d_{s_t} = (d_{s_t,1}, \ldots, d_{s_t,J_{s_t^-}})$ where $s_t \in \mathcal{S}_1 \cup \mathcal{S}_2$, and the pay-off matrices by $A_{s_t} = (d_1, \ldots, d_{J_{s_t}}) \in \mathbb{R}^{|s_t^+| \times J_{s_t}}$ where $s_t \in \{0\} \cup \mathcal{S}_1$. The price of asset j at date-events $s_t \in \{0\} \cup \mathcal{S}_1$ is $q_{s_t,j} \in \mathbb{R}$. We denote the vector of asset prices by $q_{s_t} = (q_{s_t,1}, \ldots, q_{s_t,J_{s_t}})$, and the collection of prices over date-events by $q = (q_{s_t})_{s_t \in \{0\} \cup \mathcal{S}_1}$. We assume that assets are in zero net supply. At date-event $s_t \in \{0\} \cup \mathcal{S}_1$ agent h chooses a portfolio-holding $\theta^h_{s_t} = (\theta^h_{s_t,1}, \theta^h_{s_t,2}, \ldots, \theta^h_{s_t,J_{s_t}}) \in \mathbb{R}^{J_{s_t}}$.

The finance economy $\mathcal{E} = ((u^h, e^h)_{h=1,\ldots,H}; (A_{s_t})_{s_t \in \{0\} \cup \mathcal{S}_1})$ is defined by the agents' utility functions and endowments, and the pay-off matrices.

A *competitive equilibrium* for an economy $\mathcal{E}$ is a collection of portfolio-holdings $\theta^* = (\theta^{1*}, \theta^{2*}, \ldots, \theta^{H*}) \in \mathbb{R}^{H \times J \times (S_1+1)}$, consumption $c^* = (c^{1*}, c^{2*}, \ldots, c^{H*}) \in \mathbb{R}^{H \times (S_1+S_2+1)}$ and asset prices $q^* \in \mathbb{R}^{J \times (S_1+1)}$ that satisfy the following conditions:

1. For $h = 1, \ldots, H$,

$$(c^{h*}, \theta^{h*}) \in \arg \max_{c^h \in X^h, \theta^h \in \mathbb{R}^{J \times (S_1+1)}} u^h(c^h) \tag{2}$$

$$\begin{aligned} \text{s. t. } c^h_0 + q_0 \theta^h_0 &= e^h_0, \\ c^h_{s_1} + q_{s_1} \theta^h_{s_1} &= e^h_{s_1} + d_{s_1} \theta^h_0, \quad \text{for } s_1 \in \mathcal{S}_1, \\ c^h_{s_2} &= e^h_{s_2} + d_{s_2} \theta^h_{s_2^-}, \quad \text{for } s_2 \in \mathcal{S}_2, \end{aligned}$$

2.

$$\sum_{h=1}^{H} \theta^{h*} = 0, \tag{3}$$

3.

$$\sum_{h=1}^{H} c^{h*} = \sum_{h=1}^{H} e^h. \tag{4}$$

Note that the third condition is always satisfied when the first and the second are.

If Assumption 2 is met (i.e. agents have strictly increasing utility functions) equilibrium prices exclude arbitrage opportunities in the following sense.

Asset prices q are *arbitrage-free* if there is no $\theta^h = (\theta^h_{s_t})_{s_t \in \{0\} \cup \mathcal{S}_1}$ such that

$$\begin{aligned} q_0 \theta^h_0 &\leq 0, & (5) \\ \forall s_t \in \mathcal{S}_1 \cup \mathcal{S}_2 : q_{s_t} \theta^h_{s_t} &\leq A_{s_t^-} \theta^h_{s_t^-}, & (6) \end{aligned}$$

with at least one strict inequality.

Markets are *complete* if for every income stream $y \in \mathbb{R}^{S_1+S_2}$ there exists a portfolio plan $(\theta^h_{s_t})_{s_t \in \{0\} \cup \mathcal{S}_1}$ such that

$$\begin{aligned} \forall s_1 \in \mathcal{S}_1 &: \quad d_{s_1} \theta^h_0 - q_{s_1} \theta^h_{s_1} = y_{s_1}; \\ \forall s_2 \in \mathcal{S}_2 &: \quad d_{s_2} \theta^h_{s_2^-} = y_{s_2}. \end{aligned}$$

That is, for each date-event $s_t \in \{0\} \cup \mathcal{S}_1$ and arbitrary payoffs in immediate successors of s_t, there exists a portfolio that generates those payoffs. Such a portfolio exists if and only if A_{s_t} has rank $|s_t^+|$, which is stated in the following proposition:

Markets are complete if and only if for every $s_t \in \{0\} \cup \mathcal{S}_1$ the following condition is met

$$rank(A_{s_t}) = |s_t^+|. \tag{7}$$

Proof. The proof is given in (Habis and Herings, 2011). □

If there are no arbitrage opportunities on the financial markets and the markets are complete, then there exists a unique, strictly positive state price vector $(\pi_{s_t})_{s_t \in \{0\} \cup \mathcal{S}_1} \in \mathbb{R}^{S_1+1}$ such that

$$q_{s_t} = \pi_{s_t}^\top \cdot A_{s_t}. \tag{8}$$

Proof. The proof is given in (Magill and Quinzii, 1996). □

The following additional assumptions will be made throughout this section: We assume that

1. asset 1 is risk free, so $d_{s_t,1} = 1 \; \forall s_t \in \mathcal{S}_1 \cup \mathcal{S}_2$, and its return is $R^f = 1/q_{s_t,1}$,
2. and $\{c^h \in X^h | u^h(c^h) \geq u^h(e^h)\} \subset \operatorname{int}(X^h)$, which prevents the solution of the agent's maximization problem form occurring at the boundary of the consumption set.

We use $E_{s_t}(c_{s_t^+})$ to denote the expectation of $c_{s_t^+}$ conditional on date-event s_t, so $E_{s_t}(c_{s_t^+}) = \sum_{s_{t+1} \in s_t^+} \rho_{s_t} c_{s_t}$.

2.1 Efficiency

According to the *First Welfare Theorem* the complete-markets equilibria provide Pareto-efficient consumption allocations. An allocation is Pareto-optimal if it is impossible to reallocate the total endowment so as to make some agents better off without making any agent worse off. Specifically, an allocation c^h is Pareto-optimal if there does not exist an alternative allocation $\bar{c}^h$ which is feasible,

$$\sum_{h=1}^{H} \bar{c}^h = \sum_{h=1}^{H} e^h, \tag{9}$$

weakly preferred by every agent,

$$u^h(\bar{c}^h) \geq u^h(c^h), \tag{10}$$

and strictly preferred by at least one agent, so that (10) holds with strict inequality for at least one agent.

(First Welfare Theorem) Let (θ^*, c^*, q^*) be a competitive equilibrium for $\mathcal{E}$. If asset markets are complete, then c^* is Pareto-optimal.

Proof. The proof can be obtained by contradiction. Suppose that c^{*h} is the complete-market equilibrium consumption allocation, and that there is a feasible allocation $\tilde{c}^h$ such that $u^h(\tilde{c}^h) \geq u^h(c^{*h})$ for every h, with strict inequality for some h.

Using the framework of Definition 2, the consumption plan c^{*h} maximizes utility $u^h(c^h)$ subject to the budget constraints

$$
\begin{aligned}
c_0^{*h} &= e_0^h - \pi_0 d_{s_1} \theta_0^h & (11)\\
c_{s_1}^{*h} &= e_{s_1}^h + d_{s_1}\theta_0^h - \pi_{s_1} d_{s_2} \theta_{s_1}^h & (12)\\
c_{s_2}^{*h} &= e_{s_2}^h + d_{s_2}\theta_{s_1}^h, & (13)
\end{aligned}
$$

where π_{s_t} is the unique state price vector associated with $q_{s_t}^*$. Note that π_{s_t} is strictly positive.

Multiplying equation (13) by π_{s_1} and adding the result to equation (12), we obtain

$$c_{s_1}^{*h} + \pi_{s_1} c_{s_2}^{*h} = e_{s_1}^h + \pi_{s_1} e_{s_2}^h + d_{s_1}\theta_0^h. \tag{14}$$

Multiplying equation (14) by π_0 and adding the result to equation ((11)), we obtain

$$c_0^{*h} + \pi_0 c_{s_1}^{*h} + \pi_0 \pi_{s_1} c_{s_2}^{*h} = e_0^h + \pi_0 e_{s_1}^h + \pi_0 \pi_{s_1} e_{s_2}^h, \tag{15}$$

thus the budget constraints of the original utility-maximization problem in (2) are equivalent to equation (15). Consequently, the optimal consumption plan c^{*h} maximizes $u^h(c^h)$ subject to equation (15).

Since $u^h(c^h)$ is strictly increasing, we have

$$\tilde{c}_0^h + \pi_0 \tilde{c}_{s_1}^h + \pi_0 \pi_{s_1} \tilde{c}_{s_2}^h \geq c_0^{*h} + \pi_0 c_{s_1}^{*h} + \pi_0 \pi_{s_1} c_{s_2}^{*h} \tag{16}$$

for every h, with strict inequality for some h, who are strictly better off with $\tilde{c}^h$ than with c^{*h}. Summing over all agents and applying equation (15), we obtain

$$\sum_{h=1}^{H} \tilde{c}_0^h + \sum_{h=1}^{h=H} \pi_0 \tilde{c}_{s_1}^h + \sum_{h=1}^{H} \pi_0 \pi_{s_1} \tilde{c}_{s_2}^h > e_0 + \pi_0 e_{s_1} + \pi_0 \pi_{s_1} e_{s_2}, \tag{17}$$

which contradicts the assumption that consumption allocation $\tilde{c}^h$ is feasible. □

Proving this proposition is a new development, and it is a crucial requirement for deriving the three-period model and finding a Pareto-efficient result at the same time.

When markets are incomplete, equilibrium consumption allocations are in general not Pareto-optimal and the First Welfare Theorem typically fails, since agents may not be able to implement the trades required to attain the optimal allocation. Equilibrium consumption allocations, however, can be optimal in a restricted sense. We turn now to a less ambitious notion of efficiency: are markets performing well in the sense that it is impossible to improve social welfare by using the asset market?

If we consider efficiency as a program carried out by a social planner with certain objectives we can distinguish myopic and forward-looking planners.

Based on the results above, we can assume that the mentioned theorems can be proved in such constrained cases as well but that is the subject of future research.

In this section, we got familiarized with the model's system and formalized the environment. Before we arrive at the applications, let us brush up on the CCAPM model definitions.

3 The Consumption Capital Asset Pricing Model

First, we shortly run through the most relevant aspects of the Capital Asset Pricing Model based on the relevant section of Bodie, Kane, and Marcus (2011). Then, we move on to introduce the Consumption Capital Asset Pricing Model using the definitions from the same book as source.

As we also said this in the Introduction, the CAPM estimates the relationship between the risk and the expected return of an asset.

The model assumes that the utility of an asset is dependent exclusively on the expected return, and the covariance of returns of the asset. The risk premium on the market portfolio can be given as a function of its risk and the risk aversion of the representative investor:

$$E(r_M) - R_f = \overline{A}\sigma_M^2 \tag{18}$$

where σ_M^2 is the variance of the market portfolio, $\overline{A}$ is the coefficient of the average risk-aversion, and R_f is the risk-free rate.

The risk premium of the individual assets is proportional to the risk premium of the market portfolio and its beta coefficient. The beta describes the relationship between the individual asset's return and the the market portfolio's return:

$$\beta_j = \frac{Cov(r_j, r_M)}{\sigma_M^2}, \tag{19}$$

Thus the risk premium in case of individual assets is:

$$E(r_j) - R_f = \frac{Cov(r_j, r_M)}{\sigma_M^2}[E(r_M - R_f] = \beta_j[E(r_M) - R_f]. \tag{20}$$

which is the most popular expression of the CAPM: the expected return - beta relationship.

As it holds true for individual assets, the equation holds for any linear combinations of these assets. This relationship can be understood as a risk-reward equation. The beta of the asset accurately describes the risk because it is proportional to the risk the asset contributes to the risk of the optimal portfolio with.

The graphical representation of this expected return - beta relationship is the security-market line, or SML.

Let us now move on to the Consumption Capital Asset Pricing Model (CCAPM), where the CAPM is centered around consumption, first introduced by Rubinstein (1976), Lucas (1978), and Breeden (1979).

We examine a life-long consumption plan, where the agents, in each period, need to decide about the division of their wealth between today's consumption and the investments and savings that ensure the consumption of the future periods They reach the optimum if the marginal utility coming from spending an additional unit of wealth today equals the marginal utility coming from the expected future consumption that is financed using this same unit of wealth.

The future wealth can increase as a result of wage income and the return of the units of wealth invested in the optimal complete portfolio.

A financial asset is more risky in terms of consumption if it has a positive covariance with the increase in consumption. In other words, its payoff is higher when the consumption is already high, and lower when the consumption is relatively constrained.[1] As a result, the optimal risk premium is higher for those assets that show higher positive covariance with the increase in consumption. Based on this observation, we can describe the risk premium of an asset as function of the risk of consumption:

$$E(R_j) = \beta_{jC}(E(rc) - R_f), \tag{21}$$

where the portfolio C can be translated as a *consumption-tracking portfolio*, which is the portfolio which correlates positively to the greatest extent with the increase in consumption.

The β_{jC} can be interpreted as the coefficient of the regression line where we explain R_j return premium of asset j using the return premium of the consumption-tracking portfolio as the explanatory variable.

With the previously defined risk-free rate R_f, we define the risk premium that is independent from the uncertainty of consumption as $(E(rc) - R_f)$ which is also determined using the return premium of the consumption-tracking portfolio.

This is very similar to the traditional CAPM: the consumption-tracking portfolio plays the role of the market portfolio in the CAPM. However, opposing the original CAPM theory, the beta of the consumption capital asset pricing model is not necessarily 1, in fact it is entirely realistic and empirically observed that this beta can be greater than 1. This means that the linear relationship between the market risk premium and the consumption portfolio can be written as

$$E(R_M) = \alpha_M + \beta_{MC}E(R_C) + \epsilon_M \tag{22}$$

where α_M and ϵ_M ensures the possibility of empirical deviations from the exact model defined by equation (21), and that β_{MC} is not necessarily 1.

The CCAPM is attractive, as it compactly expresses the idea of *consumption hedging* and the potential changes in the investment opportunities. Furthermore, it integrates this in the parameter of the distribution of returns in a one-factor model setup.

[1] We also note this when we discuss the three-period model later.

As a summary, we define the CCAPM below in a format that fits the purposes of this study. The Consumption Capital Asset Pricing Model (CCAPM) is a version of the Capital Asset Pricing Model where the expected return premium of the market portfolio is replaced by the return premium of the consumption-tracking portfolio. This model establishes a relationship between the investors' sensitivity to the changes in consumption and the risk of the assets.

4 The Three-Period CAPM

In this section, we prove that the β pricing formula, that relates the return of a risky asset to the return of the market portfolio can also be derived in the introduced three-period finance general equilibrium model.

Though many publications has tackled the possibility of deriving the CAPM in different environments (such as missing conditions or differing model environments) this perspective is a unique one as the capital asset pricing equation has not been derived in a three-period model previously. Though it is a topic of future research but this result also means that the CAPM could be used for asset pricing in long term models with long-lived assets as well.

First, we define the utility function of the rational agents (h) as follows:

$$u^h(c^h) = v_0^h(c_0^h) + \delta_1 \sum_{s_1 \in S_1} \rho_{s_1} v_{s_1}^h(c_{s_1}^h) + \delta_1\delta_2 \sum_{s_1 \in S_1} \rho_{s_1} \sum_{s_2 \in S_1^+} \rho_{s_2} v_{s_2}^h(c_{s_2}^h). \quad (23)$$

Agent h maximizes this utility subject to her constraints on endowments, income and even costs which were formalized in Definition 2. Since markets are complete, it follows from Proposition 2, that there exists a unique and strictly positive state price vector π_{s_t}. The asset price vector $q_{s_t} = \pi_{s_t}^T \cdot A_{s_t}$ then follows from the agents' optimization problem:

$$\mathcal{L}^h = u^h(c^h) - \lambda_0^h(c_0^h - e_0^h + q_0\theta_0^h) - \lambda_{s_1}^h(c_{s_1}^h + q_{s_1}\theta_{s_1}^h - e_{s_1}^h - d_{s_1}\theta_0^h) - \lambda_{s_2}^h(c_{s_2}^h - e_{s_2}^h - d_{s_2}\theta_{s_2^-}^h), \quad (24)$$

where $\lambda_{s_t}^h$ denote the Lagrange-multipliers. The first-order conditions, which are necessary and sufficient for (c^{h*}, θ^{h*}) to be a solution, are that there exist $\lambda^{h*} \in \mathbb{R}_{++}^{1+S_1+S_2}$ such that

$$\nabla\mathcal{L}^h(c^{h*}, \theta^{h*}, \lambda^{h*}) = 0, \quad (25)$$

which is equivalent to

$$\nabla u^h(c^{h*}) = \lambda^{h*}, \quad \text{and} \quad (26)$$

$$-q_{s_t}\lambda_{s_t} + d_{s_t^+}\lambda_{s_t^+} = 0, \forall s_t \in \{0\} \cup \mathcal{S}_1. \quad (27)$$

The partial derivatives by $(c_0^h, c_{s_1}^h, c_{s_2}^h, \theta_0^h, \theta_{s_1}^h)$ can be seen in Appendix A.1. Solving this system of equations for q_{s_t}:

$$q_{s_t} = A_{s_t} \frac{\lambda_{s_t^+}^h}{\lambda_{s_t}^h}, \text{s.t.} \lambda_{s_t}^h \neq 0 \quad (28)$$

then we substitute with the respective values of the λ^h multipliers and get

$$q_{s_t} = A_{s_t} \frac{\delta_{t+1} \sum_{s_{t+} \in S_t^+} \rho_{s_t^+} \partial v^h_{s_t^+}(c^h_{s_t^+})/\partial c^h_{s_t^+}}{\partial v^h_{s_t}(c^h_{s_t})/\partial c^h_{s_t}}. \tag{29}$$

It becomes apparent that what we get is the *marginal rate of substitution* (MRS) between the consumption levels of the different periods. Equation (29) means that for each $s_t \in \{0\} \cup \mathcal{S}_1$ date-event, an agent h invests in j assets, such that the marginal cost of each additional $q_{s_t,j}$ unit equals its marginal utility, which is in fact the present value of the future dividends of agent h.

By the definition of the expected value described in Section 2, we substitute the respective part of equation (29) and we get[2]

$$q_{s_t} = \frac{\delta_{t+1} E_{s_t}[\partial_{c_{s_t^+}} v^h_{s_t^+}(c^{h*}) A_{s_t}]}{\partial_{c_{s_t}} v^h_{s_t}(c^{h*})} = E(\mathrm{MRS}^h_{s_t} A_{s_t}), \quad \text{for all } s_t \in \{0\} \cup \mathcal{S}_1, \tag{30}$$

where $v_{s_t^+} = (v_{s_t+1})_{s_{t+1} \in s_t^+}$ and we can see the MRS between the consumption levels of period t and of all states belonging to the period t^+.

Equation (30) asserts that each agent h invests in each asset j at each date-event $s_t \in \{0\} \cup \mathcal{S}_1$ in such a way that the marginal cost of an additional unit of the security $q_{s_t,j}$ is equal to its marginal benefit, the present value for agent h of its future stream of dividends. Although the $MRS^h_{s_t}$ of each agent can be different as a result of the shape of the utility function (e.g. based on their attitude towards risk), they cannot disagree on asset prices in equilibrium. If one projects the individual $MRS^h_{s_t}$s onto the marketed subspace $\langle A_{s_t} \rangle$ one obtains a unique pricing vector, given that $q_{s_t} = \pi^\top_{s_t} \cdot A_{s_t}$ which is the one defined in (30). For asset prices q_{s_t} we define the one-period return $r_{s_t^+, \theta_{s_t}}$ for a portfolio θ_{s_t}, with $q_{s_t}\theta_{s_t} \neq 0$, by

$$r_{s_t^+, \theta_{s_t}} = \frac{A_{s_t} \theta^h_{s_t}}{q_{s_t} \theta^h_{s_t}}. \tag{31}$$

This reflects the general definition of returns: we divide the pay-offs of the securities in the portfolio by their price. We will furthermore use the usual formula of the covariance:

$$E(yz) = cov(y,z) + E(y)E(z) \tag{32}$$

to rewrite equation (30) in the following manner:

$$1 = \frac{\delta_{t+1} E_{s_t}[\partial_{c_{s_t^+}} v^h_{s_t^+}(c^{h*}) r_{s_t^+, \theta_{s_t}}]}{\partial_{c_{s_t}} v^h_{s_t}(c^{h*})}, \tag{33}$$

[2]For the sake of clearer notation, we will substitute the traditional notation ($\frac{\partial f(x)}{\partial x}$) of the partial derivative of any function $f(x)$ with respect to x variable by simply writing $\partial_x f(x)$.

then using $cov_{s_t}(x_{s_t^+}, y_{s_t^+})$ to denote the conditional covariance between two variables and the above definitions we get

$$1 = \frac{\delta_{t+1} E_{s_t}[r_{s_t^+, \theta_{s_t}}] E_{s_t}[\partial_{c_{s_t^+}} v^h_{s_t^+}(c^{h*})]}{\partial_{c_{s_t}} v^h_{s_t}(c^{h*})} + \frac{\delta_{t+1} cov_{s_t}(\partial_{c_{s_t^+}} v^h_{s_t^+}(c^{h*}), r_{s_t^+, \theta_{s_t}})}{\partial_{c_{s_t}} v^h_{s_t}(c^{h*})}. \quad (34)$$

Rearranging this yields the equation of the one-period expected return

$$E_{s_t}[r_{s_t^+, \theta_{s_t}}] = \frac{\partial_{c_{s_t}} v^h_{s_t}(c^{h*})}{\delta_{t+1} E_{s_t}[\partial_{c_{s_t^+}} v^h_{s_t^+}(c^{h*})]} - \frac{cov_{s_t}(\partial_{c_{s_t^+}} v^h_{s_t^+}(c^{h*}), r_{s_t^+, \theta_{s_t}})}{E_{s_t}[\partial_{c_{s_t^+}} v^h_{s_t^+}(c^{h*})]} \quad (35)$$

where the expression

$$R^f_{s_t} = \frac{\partial_{c_{s_t}} v^h_{s_t}(c^{h*})}{\delta_{t+1} E_{s_t}[\partial_{c_{s_t^+}} v^h_{s_t^+}(c^{h*})]} \quad (36)$$

is the return of the one-period risk-free asset[3]. Plugging this into equation (35) we retrieve the consumption-based capital asset pricing formula

$$E_{s_t}[r_{s_t^+, \theta_{s_t}}] = R^f_{s_t} - \delta_{t+1} R^f_{s_t} \frac{cov_{s_t}(\partial_{c_{s_t^+}} v^h_{s_t^+}(c^{h*}), r_{s_t^+, \theta_{s_t}})}{\partial_{c_{s_t}} v^h_{s_t}(c^{h*})}. \quad (37)$$

This equation shows that for each asset the risk premium (which is the difference between the expected return of the risky assets and the risk-free rate) is proportional to the covariance between its return rate and the marginal rate of substitution between the date-events of s_t and s_t^+ (with a negative proportionality constant).

To be precise, $\partial_{c_{s_t^+}} v^h_{s_t^+}(c^{h*}) / \partial_{c_{s_t}} v^h_{s_t}(c^{h*})$ in equation (37), is not the marginal rate of substitution between the state-dependent consumptions of date-events s_t^+ and s_t, as the probabilities are missing. Similarly, we will refer to the marginal utility of consumption by the notion $\partial_{c_{s_t^+}} v^h_{s_t^+}(c^{h*})$, although the probabilities are missing here as well. There is no reason to be held up by this terminological imprecision, as we are not diverting from the conventional methodology of the literature, see LeRoy and Werner (2001). For a strictly risk-averting decision maker, $\partial_{c_{s_t^+}} v^h_{s_t^+}(c^{h*})$ is a negative function of the consumption in s_t^+. Thus, the security, that has a high pay-off when the consumption is high, and has a low pay-off when the consumption is low as well, has a greater expected return than the risk-free security. Let us now, in contrast, consider a security, that has a high pay-off when the consumption is low, and has a low pay-off when the consumption is high. Following the above concept, such a security would have an expected return which is less than that of the risk-free asset. Such securities

[3]The definition of the risk-free asset is the one described in LeRoy and Werner (2001) as $R^f_{s_t} = \frac{1}{\sum_{s_t \in \{0\} \cup \mathcal{S}_1 \cup \mathcal{S}_2} q_{s_t}}$ which, in equilibrium, is equivalent with $R^f_{s_t}$ in our equations.

can then be used to decrease the risk of consumption for the decision makers. If the covariance of an asset's return and the MRS is zero, the asset has the same expected return as the risk-free asset.

Based on equation (37) the risk premium of a security is solely dependent on the covariance between its return and the MRS between the date-events s_t and s_t^+. This covariance can be understood as the degree of risk of the security, which has two significant features. Firstly, it can only be used if the economy is in the state of equilibrium. Secondly, this covariance-measure provides not just a partial but a complete ordering of the risk of returns.

If the marginal rate of substitution is constant, the consumption-based asset pricing equation defined in equation (37) gives a *fair price*. The MRS can be deterministic in two cases: if the consumption of the agent is deterministic as well, and if the agent is risk-indifferent.

In order to illustrate further details of the optimization process of the agents, in the next assumption, we will show the $v_{s_t}^h$ utility function which is quadratic with respect to the $t+1$ period consumption.

Let $X^h = \mathbb{R}^{1+S_1+S_2}$ and $v_{s_t}^h(c_{s_t}^h) = \xi_t c_{s_t}^h - \frac{1}{2}\alpha_t (c_{s_t}^h)^2$ be a quadratic utility-function.

Substituting this into Equation (37) we get

$$E_{s_t}[r_{s_t^+,\theta_{s_t}}] = R_{s_t}^f - \delta_{t+1} R_{s_t}^f \frac{cov_{s_t}(\xi_{t+1} - \alpha_{t+1} c_{s_{t+}}^h, r_{s_t^+,\theta_{s_t}})}{\xi_t - \alpha_t c_{s_t}^h}, \tag{38}$$

then it follows that the expected return of an arbitrary asset j is

$$E_{s_t}[r_{s_t^+,j}] = R_{s_t}^f + \frac{\delta_{t+1}\alpha_{t+1} R_{s_t}^f}{\xi_t - \alpha_t c_{s_t}^h} cov_{s_t}(c_{s_t^+}^h, r_{s_t^+,j}). \tag{39}$$

In a *securities market economy* the aggregated endowment is in the *asset span* which means it can be attained from the pay-offs of portfolio of some securities. This portfolio is the market portfolio with its return denoted by $r_{s_t^+}^M$. Equation (39) holds for returns of portfolios as well. In particular it holds for the market return $r_{s_t^+}^M$ so that

$$E_{s_t}[r_{s_t^+}^M] = R_{s_t}^f + \frac{\delta_{t+1}\alpha_{t+1} R_{s_t}^f}{\xi_t - \alpha_t c_{s_t}^h} cov_{s_t}(c_{s_t^+}^h, r_{s_t^+}^M). \tag{40}$$

Dividing Equation (39) by (40) after subtracting $R_{s_t}^f$ from both and thus eliminating the term $\frac{\delta_{t+1}\alpha_{t+1} R_{s_t}^f}{\xi_t - \alpha_t c_{s_t}^h}$ one obtains

$$\frac{E_{s_t}[r_{s_t^+,j}] - R_{s_t}^f}{E_{s_t}[r_{s_t^+}^M] - R_{s_t}^f} = \frac{cov_{s_t}(c_{s_t^+}, r_{s_t^+,j})}{cov_{s_t}(c_{s_t^+}, r_{s_t^+}^M)} \tag{41}$$

where, as we assume, the market risk premium is nonzero.

If equilibrium consumptions lie in the span of the market return and the risk-free return, then $c^h_{s_t^+}$ and $r^M_{s_t^+}$ are perfectly correlated. Accordingly $c^h_{s_t^+}$ can be replaced by $\varphi r^M_{s_t^+}$-vel. Finally, for a portfolio $\theta^h_{s_t} \in \mathbb{R}^{J_{s_t}}$ we define $\beta_{\theta_{s_t}}$-t

$$\beta_{\theta_{s_t}} = \frac{cov_{s_t}(r^M_{s_t^+}, r_{s_t^+,\theta})}{var(r^M_{s_t^+})}. \tag{42}$$

This $\beta_{\theta_{s_t}}$ will be the consumption beta of the CCAPM, mentioned in Section 3, which reflects how the risk of a security is related to the risk of the market portfolio.

Then the following CAPM-pricing formula holds for each $\theta^h_{s_t} \in \mathbb{R}^{J_{s_t}}$ thus

$$E_{s_t}[r_{s_t^+,\theta}] - R^f_{s_t} = \beta_{\theta_{s_t}}(E_{s_t}[r^M_{s_t^+}] - R^f_{s_t}); \tag{43}$$

which is, in fact, the formula of the *security market line*:

$$E_{s_t}[r_{s_t^+,\theta}] = R^f_{s_t} + \beta_{\theta_{s_t}}(E_{s_t}[r^M_{s_t^+}] - R^f_{s_t}). \tag{44}$$

As it is also stated in LeRoy and Werner (2001), the assumption, that the equilibrium consumption choice is in the span of the market return and risk-free return is trivial in a *representative-agent economy*. This is because the optimal consumption of each agent in the economy is equal to the per capita pay-off of the market portfolio. Since we assumed that all agents has the same quadratic utility function this holds true for the economy defined in this paper.

Hence, we have proven that the CCAPM formula can be derived from a three-period utility maximization model; in other words, we extended the results of the widely known two-period model to three periods. This is significant as a stand-alone result but it can also provide a basis for numerous future research topics which require a multi-period model. One such case is the analysis of long term securities or the long term efficiency of incomplete markets.

A Appendix

A.1 Partial derivatives

The partial derivatives of the Lagrangean function, all solved when they equal to zero:

$$\frac{\partial \mathcal{L}^h}{\partial c_0^h} = \frac{\partial v_0^h(c_0^h)}{\partial c_0^h} - \lambda_0^h = 0,$$

$$\frac{\partial \mathcal{L}^h}{\partial c_{s_1}^h} = \frac{\delta_1 \sum_{s_1 \in S_1} \rho_{s_1} \partial v_{s_1}^h(c_{s_1}^h)}{\partial c_{s_1}^h} - \lambda_{s_1}^h = 0,$$

$$\frac{\partial \mathcal{L}^h}{\partial c_{s_2}^h} = \frac{\delta_1 \delta_2 \sum_{s_1 \in S_1} \rho_{s_1} \sum_{s_2 \in S_1^+} \rho_{s_2} \partial v_{s_2}^h(c_{s_2}^h)}{\partial c_{s_2}^h} - \lambda_{s_2}^h = 0,$$

$$\frac{\partial \mathcal{L}^h}{\partial \theta_0^h} = -\lambda_0^h q_0 + d_{s_1} \lambda_{s_1}^h = 0,$$

$$\frac{\partial \mathcal{L}^h}{\partial \theta_{s_1}^h} = -\lambda_{s_1}^h q_{s_1} + d_{s_2} \lambda_{s_2}^h = 0.$$

The derivatives with respect to the consumption variables are equivalent to the

$$\Delta u^h(c^{h*}) = \lambda^{h*} \tag{45}$$

matrix equation which means that, in $t = 0$, the Lagrange multipliers are equal to the partial derivatives of the utility function with respect to the consumption variables in the respective date-event.

The partial derivatives with respect to the portfolio-holdings is as follows:

$$-q_{s_t} \lambda_{s_t}^h + A_{s_t} \lambda_{s_t^+}^h = 0, \forall s_t \in \{0\} \cup \mathcal{S}_1$$

References

BODIE, Z., A. KANE, AND A. J. MARCUS (2011): *Investments - 9th ed.*chap. The Capital Asset Pricing Model, pp. 280–317. Douglas Reiner, McGraw-Hill/Irwin, New York.

BREEDEN, D. (1979): "An Intertemporal Asset Pricing Model with Stochastic Consumption and Investment Opportunities," *Journal of Financial Economics*, 7, 265–296., doi: https://doi.org/10.1016/0304-405X(79)90016-3.

HABIS, H., AND J.-J. HERINGS (2011): "Core concepts for incomplete market economies," *Journal of Mathematical Economics*, 47(5), 595–609, doi: https://doi.org/10.1016/j.jmateco.2011.07.005.

LEROY, S. F., AND J. WERNER (2001): *Principles of Financial Economics*pp. 135–145. Cambridge University Press, Cambridge and New York, doi: https://doi.org/10.1017/cbo9780511753787.

LINTNER, J. (1965): "The Valuation of Risk Assets and the Selection of Risky Investments in Stock Portfolios and Capital Budgets," *Review of Economics and Statistics*, 47(1), 13–37., doi: https://doi.org/10.2307/1924119.

LUCAS, R. (1978): "Asset Prices in an Exchange Economy," *Econometrica*, 46, 1429–1445., doi: https://doi.org/10.2307/1913837.

MAGILL, M. J. M., AND M. QUINZII (1996): *Theory of Incomplete Markets*. MIT Press, Cambridge, Massachusetts.

MOSSIN, J. (1966): "Equilibrium in a Capital Asset Market," *Econometrica*, 34(4), 768–783., doi: https://doi.org/10.2307/1910098.

RUBINSTEIN, M. (1976): "The Valuation of Uncertain Income Streams and the Pricing of Options," *Bell Journal of Economics and Management Science*, 7, 407–425., doi: https://doi.org/10.2307/3003264.

SHARPE, W. (1964): "Capital Asset Prices: A Theory of Market Equilibrium Under The Conditions Of Risk," *The Journal of Finance*, 19(3), 425–442., doi: https://doi.org/10.2307/2977928.