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UNIFIED APPROACH OF PRIMAL-DUAL INTERIOR-POINT ALGORITHMS FOR A NEW CLASS OF AET FUNCTIONS

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Abstract. We propose new short-step interior-point algorithms (IPAs) for solving $P_*(\kappa)$-linear complementarity problems (LCPs). In order to define the search directions we use the algebraic equivalent transformation technique (AET) of the system which characterizes the central path. A novelty of the paper is that we introduce a new class of AET functions. We present the complexity analysis of the IPAs that use this general class of functions in the AET technique. Furthermore, we also deal with a special case, namely $\varphi(t) = t^2 - t + \sqrt{t}$. This function differs from the ones used in the literature in the sense that it has inflection point. It does not belong to the class of concave functions determined by Haddou et al. [24]. Furthermore, the kernel function corresponding to this AET function is neither eligible nor self-regular kernel function. We prove that the IPAs using any member φ of the new class of AET functions have polynomial iteration complexity in the size of the problem, bit length of the integral data and in the parameter κ. Beside this, we also provide numerical results that show the efficiency of the introduced methods.
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1. Introduction

Linear complementarity problems have been extensively studied nowadays. Linear programming (LP) and linearly constrained (convex) quadratic programming (QP) problems are special cases of LCPs. Several applications of LCPs arise in different fields, such as engineering, computational mechanics, game theory, economics, see [11, 21]. It was shown that solvability of LCPs related to quitting games ensures the existence of different ε-equilibrium solutions, see [40]. Bimatrix games can be also formulated as LCPs, see [33]. The Arrow-Debreu competitive market equilibrium problem with linear and Leontief utility functions can be transformed to LCP [46]. For detailed study on LCPs see the books of Cottle et al. [11] and Kojima et al. [31]. In the book of Kojima et al. [31] the theory of interior-point algorithms for solving LCPs is highlighted.

LCPs belong to the class of NP-complete problems, see [10]. However, the properties of the problem's matrix have influence on the solvability of the LCPs. It is known that if the problem's matrix is skew-symmetric [39, 44, 45] or positive semidefinite [32], IPAs can find approximate solution of LCPs in polynomial time. Cottle, Pang, and Venkateswaran

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[12] introduced the class of sufficient matrices. The class of $P_*(\kappa)$-matrices was proposed by Kojima et al. [31]. If we consider the union of the sets $P_*(\kappa)$ for all nonnegative κ we obtain the class P_*, see [31]. Väliaho [41] proved that the class of P_*-matrices is equivalent to the class of sufficient matrices. In general, IPAs for solving $P_*(\kappa)$-LCPs have polynomial iteration complexity in the size of the problem, bit length of the integral data and the special parameter $\kappa \geq 0$. However, Klerk and E.-Nagy [20] showed that the handicap of the problem's matrix could be exponential in the bit length of the data. Furthermore, the complexity analyses of IPAs for $P_*(\kappa)$-LCPs depend on the special parameter κ. In spite of this fact, there are computational results in the literature for LCPs with matrices having exponential value κ, where the iteration numbers are much better than its predicted by the complexity results, see [15–17]. This means that it is worth trying to obtain better complexity results for such LCPs, as well.

An important aspect in the theory of IPAs is how we determine the search directions. Several approaches have been proposed in the literature. For example, there are methods that use barrier functions for defining search directions. Peng et al. [37] considered self-regular functions and in this way they reduced the theoretical complexity of long-step IPAs. Beside these, Bai et al. [9] introduced the class of eligible kernel functions. Lešaja and Roos [34] also analysed algorithms using eligible kernel functions. Furthermore, the AET technique for defining search directions in case of IPAs for LP was introduced by Darvay, see [13]. He applied a continuously differentiable, monotone increasing function $\varphi : (\xi^2, \infty) \to \mathbb{R}$, where $0 \leq \xi < 1$, on the modified nonlinear equation of the system defining the central path. In the literature, most of the IPAs do not use any transformation of the central path system, hence these IPAs refer to the case when $\varphi(t) = t$ in the AET technique. Darvay [13, 14] was the first who used the function $\varphi(t) = \sqrt{t}$ in the AET technique. In 2016, Darvay et al. [18] considered the case when $\varphi(t) = t - \sqrt{t}$ and they proposed small-update IPA for LP using this search direction. In [38], different IPAs have been presented for LP and sufficient LCPs using the AET technique. Kheirfam and Haghighi [30] introduced an IPA for $P_*(\kappa)$-LCPs which applies the function $\varphi(t) = \frac{\sqrt{t}}{2(1+\sqrt{t})}$ in the AET technique. Later on, Haddou et al. [24] proposed a family of concave functions. It should be mentioned that they used other type of transformation of the central path system. In a private communication, M. E.-Nagy and A. Varga [19] showed us the definition of a new class of AET functions for long-step IPAs. However, up to our knowledge, there are functions belonging to our class of AET functions, that are not members of the class of AET functions introduced by M. E.-Nagy and A. Varga [19]. IPAs using the AET approach for determining search directions have been also extended to LCPs, see [2–5, 7, 15, 17, 29, 35].

The aim of this paper is to introduce a new class of AET functions and to analyse IPAs for $P_*(\kappa)$-LCPs that are based on these new search directions. We analyse the new family of functions and compare to the class of concave functions given by Haddou et al. [24] and to other AET functions used in this approach. We also analyse the relationship of the kernel functions belonging to this new class of AET functions to the class of eligible kernel functions. We consider a special case belonging to this new class of AET functions, namely $\varphi(t) = t^2 - t + \sqrt{t}$. This function has inflection point, hence it does not belong to the class of concave functions proposed by Haddou et al. Moreover, the kernel function corresponding to this AET function is neither eligible nor self-regular kernel function. We present the complexity analysis of the new IPAs in the general case and after that we also consider the version when $\varphi(t) = t^2 - t + \sqrt{t}$. We prove that the IPAs using any member φ of the new class of AET functions have polynomial iteration complexity in the size of the problem, bit length of the integral data and in the parameter κ. We also provide numerical results in the special case when $\varphi(t) = t^2 - t + \sqrt{t}$ and we compare our method to IPAs using other AET functions. Up to our best knowledge, this is the first function belonging to this new class of AET functions, which has inflection point and for which the currently best known complexity results can be obtained.

The paper is organized in the following way. In Section 2 we present several results related to the theory of $P_*(\kappa)$-LCPs, the classical AET approach. We also introduce a new class of AET functions used in this paper. We compare the proposed family of functions to other techniques for determining search directions. Section 3 is devoted to the complexity analysis of the IPAs that are based on the new class of AET functions. Section 4 contains the numerical results related to the IPA using new search direction. Furthermore, in Section 5 some concluding remarks and further research topics are enumerated.

2. New class of AET functions for interior-point algorithms for solving $P_*(\kappa)$-linear complementarity problems

In the first part of this section we present some basic concepts related to the theory of $P_*(\kappa)$-LCPs and $P_*(\kappa)$-matrices.

2.1. **Linear complementarity problems and $P_*(\kappa)$-matrices.** The aim of the LCPs is to find vectors $\mathbf{x}, \mathbf{s} \in \mathbb{R}^n$, that satisfy the following constraints:

$$-M\mathbf{x} + \mathbf{s} = \mathbf{q},\ \mathbf{xs} = \mathbf{0},\ \mathbf{x}, \mathbf{s} \geq \mathbf{0}, \tag{LCP}$$

where $M \in \mathbb{R}^{n\times n}$, $\mathbf{q} \in \mathbb{R}^n$ and $\mathbf{xs}$ is the componentwise product of vectors $\mathbf{x}$ and $\mathbf{s}$. The feasible region, the interior and the solutions set of LCP are given as follows:

$$\begin{aligned} \mathcal{F} &:= \{(\mathbf{x}, \mathbf{s}) \in \mathbb{R}^n_\oplus \times \mathbb{R}^n_\oplus : -M\mathbf{x} + \mathbf{s} = \mathbf{q}\}, \\ \mathcal{F}^+ &:= \{(\mathbf{x}, \mathbf{s}) \in \mathbb{R}^n_+ \times \mathbb{R}^n_+ : -M\mathbf{x} + \mathbf{s} = \mathbf{q}\}, \\ \mathcal{F}^* &:= \{(\mathbf{x}, \mathbf{s}) \in \mathcal{F} : \mathbf{xs} = \mathbf{0}\}. \end{aligned}$$

Note that $\mathbb{R}^n_\oplus$ denotes the n-dimensional nonnegative orthant and $\mathbb{R}^n_+$ the positive orthant, respectively. Cottle et al. [12] introduced the class of *sufficient matrices*.

Definition 2.1. (Cottle et al. [12]) A matrix $M \in \mathbb{R}^{n\times n}$ is a *column sufficient matrix* if for all $\mathbf{x} \in \mathbb{R}^n$

$$X(M\mathbf{x}) \leq 0 \ \text{ implies } \ X(M\mathbf{x}) = 0,$$

where $X = diag(\mathbf{x})$. Analogously, matrix M is *row sufficient* if M^T is column sufficient. The matrix M is *sufficient* if it is both row and column sufficient.

Kojima et al. [31] proposed the notion of $P_*(\kappa)$-matrices.

Definition 2.2. (Kojima et al. [31]) Let $\kappa \geq 0$ be a nonnegative real number. A matrix $M \in \mathbb{R}^{n\times n}$ is a $P_*(\kappa)$-matrix if

$$(1+4\kappa)\sum_{i\in I_+(\mathbf{x})} x_i(Mx)_i + \sum_{i\in I_-(\mathbf{x})} x_i(Mx)_i \geq 0, \quad \forall \mathbf{x} \in \mathbb{R}^n, \tag{2.1}$$

where

$$I_+(\mathbf{x}) = \{1 \leq i \leq n : x_i(Mx)_i > 0\} \ \text{ and } \ I_-(\mathbf{x}) = \{1 \leq i \leq n : x_i(Mx)_i < 0\}.$$

A problem is called $P_*(\kappa)$-LCP if the problem's matrix of (LCP) is $P_*(\kappa)$-matrix. Throughout the paper we assume that $\mathcal{F}^+ \neq \emptyset$ and M is a $P_*(\kappa)$-matrix. Hence, we are dealing with $P_*(\kappa)$-LCPs. The handicap of M [41] is the smallest value of $\hat{\kappa}(M) \geq 0$ such that M is $P_*(\hat{\kappa}(M))$-matrix.

Definition 2.3. (Kojima et al. [31]) A matrix $M \in \mathbb{R}^{n\times n}$ is a P_*-*matrix* if it is a $P_*(\kappa)$-matrix for some $\kappa \geq 0$. Let $P_*(\kappa)$ denote the set of $P_*(\kappa)$-matrices. Analogously, we also use P_* to denote the set of all P_*-matrices, i.e.,

$$P_* = \bigcup_{\kappa\geq 0} P_*(\kappa).$$

Kojima et al. [31] proved that a P_*-matrix is column sufficient and Guu and Cottle [23] showed that it is row sufficient, too. This means that each P_*-matrix is sufficient. Väliaho [41] demonstrated the other inclusion, as well, proving that the class of P_*-matrices is the same as the class of sufficient matrices.

The *central path problem* in this case is

$$-M\mathbf{x}+\mathbf{s}=\mathbf{q} \qquad \mathbf{x},\mathbf{s}>\mathbf{0}, \qquad \mathbf{x}\mathbf{s}=\mu\mathbf{e}, \tag{CPP}$$

where $\mathbf{e}$ denotes the n-dimensional all-one vector and $\mu > 0$. Kojima et al. [31] proved that if M is a $P_*(\kappa)$-matrix, then the central path system has unique solution for every $\mu > 0$. In the following subsection we present the classical AET approach.

2.2. **Algebraic equivalent transformation technique.** In this subsection we present the AET technique in case of $P_*(\kappa)$-LCPs. Let $\varphi : (\bar{\xi},\infty) \to \mathbb{R}$, with $0 \leq \bar{\xi} < 1$, be a continuously differentiable and invertible function, such that $\varphi'(t) > 0$, $\forall t > \bar{\xi}$, see [13]. We use the notation $\varphi(\mathbf{x}) = [\varphi(x_1),\varphi(x_2)\ldots,\varphi(x_n)]^T$. System (CPP) can be written:

$$-M\mathbf{x}+\mathbf{s}=\mathbf{q} \qquad \mathbf{x},\mathbf{s}>\mathbf{0}, \qquad \varphi\left(\frac{\mathbf{x}\mathbf{s}}{\mu}\right)=\varphi(\mathbf{e}), \tag{CPP_φ}$$

Applying Newton's method we obtain the following system, see [16]:

$$\begin{aligned} -M\Delta\mathbf{x}+\Delta\mathbf{s} &= \mathbf{0}, \\ S\Delta\mathbf{x}+X\Delta\mathbf{s} &= \mathbf{a}_\varphi, \end{aligned} \tag{2.3}$$

where

$$\mathbf{a}_\varphi = \mu\frac{\varphi(\mathbf{e})-\varphi\left(\frac{\mathbf{x}\mathbf{s}}{\mu}\right)}{\varphi'\left(\frac{\mathbf{x}\mathbf{s}}{\mu}\right)}. \tag{2.4}$$

This system has unique solution, which follows from the following result.

Corollary 2.1. *(Kojima et al. [31]) Let $M \in \mathbb{R}^{n\times n}$ be a $P_*(\kappa)$-matrix, $\mathbf{x},\mathbf{s} \in \mathbb{R}^n_+$. Then, for all $\mathbf{a}_\varphi \in \mathbb{R}^n$ the system*

$$\begin{aligned} -M\Delta\mathbf{x}+\Delta\mathbf{s} &= \mathbf{0} \\ S\Delta\mathbf{x}+X\Delta\mathbf{s} &= \mathbf{a}_\varphi \end{aligned} \tag{2.5}$$

has a unique solution $(\Delta\mathbf{x},\Delta\mathbf{s})$, where X and S are the diagonal matrices obtained from the vectors $\mathbf{x}$ and $\mathbf{s}$.

Scaling plays important role in the theory of IPAs. Consider the following notations:

$$\mathbf{v}=\sqrt{\frac{\mathbf{x}\mathbf{s}}{\mu}}, \quad \mathbf{d} = \sqrt{\frac{\mathbf{x}}{\mathbf{s}}}, \quad \mathbf{d}_x=\frac{\mathbf{d}^{-1}\Delta\mathbf{x}}{\sqrt{\mu}}=\frac{\mathbf{v}\Delta\mathbf{x}}{\mathbf{x}}, \quad \mathbf{d}_s=\frac{\mathbf{d}\Delta\mathbf{s}}{\sqrt{\mu}}=\frac{\mathbf{v}\Delta\mathbf{s}}{\mathbf{s}}. \tag{2.6}$$

From these we obtain

$$\Delta\mathbf{x}=\frac{\mathbf{x}\mathbf{d}_x}{\mathbf{v}} \quad \text{and} \quad \Delta\mathbf{s}=\frac{\mathbf{s}\mathbf{d}_s}{\mathbf{v}}. \tag{2.7}$$

After substituting these into (2.3) we obtain the scaled system:

$$\begin{aligned} -\bar{M}\mathbf{d}_x + \mathbf{d}_s &= \mathbf{0}, \\ \mathbf{d}_x + \mathbf{d}_s &= \mathbf{p}_v, \end{aligned} \tag{2.8}$$

where $\bar{M} = DMD$, $D = diag(\mathbf{d})$ and

$$\mathbf{p}_v = \frac{\varphi(\mathbf{e}) - \varphi(\mathbf{v}^2)}{\mathbf{v}\varphi'(\mathbf{v}^2)}. \tag{2.9}$$

Table 1 contains the classical AET functions used in the literature and the corresponding vectors $\mathbf{a}_\varphi$ and $\mathbf{p}_v$.

φ	$\mathbf{a}_\varphi$	$\mathbf{p}_v$
$\varphi(t) = t$	$\mu\mathbf{e} - \mathbf{xs}$	$\mathbf{v}^{-1} - \mathbf{v}$
$\varphi(t) = \sqrt{t}$	$2(\sqrt{\mu\mathbf{xs}} - \mathbf{xs})$	$2(\mathbf{e} - \mathbf{v})$
$\varphi(t) = t - \sqrt{t}$	$\frac{\sqrt{\mu}\mathbf{xs}}{2\sqrt{\mathbf{xs}} - \sqrt{\mu}\mathbf{e}} - \mathbf{xs}$	$\frac{2(\mathbf{v} - \mathbf{v}^2)}{2\mathbf{v} - \mathbf{e}}$
$\varphi(t) = \frac{\sqrt{t}}{2(1+\sqrt{t})}$	$\sqrt{\frac{\mathbf{xs}}{\mu}}(\mu\mathbf{e} - \mathbf{xs})$	$\mathbf{e} - \mathbf{v}^2$

TABLE 1. AET functions used in the literature

Later on, Haddou et al. [24] proposed a family of smooth concave functions for monotone LCPs. However, it should be mentioned that they used other type of transformation of the central path system. They used functions $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ that satisfy the following conditions

H1. $\varphi(0) = 0$;
H2. $\varphi \in \mathcal{C}^3([0, +\infty))$;
H3. $\varphi'(t) > 0$, $\forall t \geq 0$;
H4. $\varphi''(t) \leq 0$, $\forall t \geq 0$;
H5. $\varphi'''(t) \geq 0$, $\forall t \geq 0$.

In the following subsection we introduce the new class of AET functions used in this paper.

2.3. **New class of AET functions.** We present the new class of AET functions which will be used in order to determine search directions.

Definition 2.4. Let $\varphi : (\xi, \infty) \to \mathbb{R}$ be a continuously differentiable, invertible function, such that $\varphi'(t) > 0$, $\forall t > \xi$, where $0 \leq \xi < 1$. All functions φ satisfying the conditions

AET1. $\exists\, c_1 \in \mathbb{R}_+$, such that

$$\left| \frac{\varphi(1) - \varphi(t^2)}{2t(1-t^2)\varphi'(t^2)} \right| \leq c_1,$$

for all $t > \xi$.

AET2. $\exists\, c_2 \in \mathbb{R}_+$, such that

$$\left| \frac{4t^2\varphi'(t^2)\left[(1-t^2)\varphi'(t^2) - \varphi(1) + \varphi(t^2)\right]}{(\varphi(1) - \varphi(t^2))^2} \right| \leq c_2,$$

for all $t > \xi$.

AET3. $\exists c_3 \in \mathbb{R}_+$ such that the inequality

$$\begin{aligned} 4t^2(\varphi(1)-\varphi(t^2))\varphi'(t^2) - c_3\left(\varphi(1)-\varphi(t^2)\right)^2 &\leq 4t^2(1-t^2)\left(\varphi'(t^2)\right)^2 \\ &\leq 4t^2(\varphi(1)-\varphi(t^2))\varphi'(t^2) + \left(\varphi(1)-\varphi(t^2)\right)^2 \end{aligned}$$

holds for all $t > \xi$,

belong to the new class of AET functions.

Let us introduce the following function: $f : (\xi, \infty) \to \mathbb{R}$:

$$f(t) = \frac{\varphi(1)-\varphi(t^2)}{t(\varphi'(t^2))}, \tag{2.10}$$

By using the function given in (2.10) we can give the definition of the new class of AET functions in the following way.

Proposition 2.1. *The conditions given in Definition 2.4 can be formulated in the following equivalent form:*

AETa. $\exists c_1 \in \mathbb{R}_+$, *such that* $g(t) = \frac{f(t)}{2(1-t^2)}$ *and* $|g(t)| \leq c_1$, *holds for all* $t > \xi$*;*

AETb. $\exists c_2 \in \mathbb{R}_+$, *such that* $h(t) = \frac{4(1-t^2-tf(t))}{f(t)^2} = \frac{1-2tg(t)}{(1-t^2)g(t)^2}$ *and* $|h(t)| \leq c_2$, *holds for all* $t > \xi$*;*

AETc. $\exists c_3 \in \mathbb{R}_+$ *such that the inequality*

$$tf(t) - c_3\frac{f(t)^2}{4} \leq 1-t^2 \leq tf(t) + \frac{f(t)^2}{4}$$

holds for all $t > \xi$*.*

Proof. Using the function given in (2.10) after some calculations we obtain that conditions AET1-3 can be formulated as the ones given in AETa-c. □

Remark 2.1. The values of the parameters c_1, c_2 and c_3 will have influence on the well-definedness of the algorithm. For this, we will give a relation between these parameters.

Table 2 contains examples for φ belonging to this new family of functions, the value of ξ and the values c_1, c_2 and c_3. The values given in Table 2 will be clear in the second part of the paper when we study the well-definedness of the algorithm. It should be mentioned that for the given values from Table 2 the introduced algorithms are well defined and the complexity analyses work. However, there are several other acceptable values for these parameters.

$\varphi(t)$	ξ	c_1	c_2	c_3
t	0.25	2	6	1
$\sqrt{t}$	0	2	6	1
$t-\sqrt{t}$	0.7	2	6	1
$t^2-t+\sqrt{t}$	0	2	8	8
$t^2+\sqrt{t}$	0	2	6	8

TABLE 2. Examples for φ belonging to the new class of AET function

Table 2 shows that most of the functions used in the literature from Table 1 belong to the new class of AET functions. However, it should be mentioned that the intervals on which the functions φ are defined play important role in this approach. For example, $\varphi(t)=t$ is only a member of this new class of AET functions if it is defined on a (ξ,∞) interval, where ξ is strictly positive. If ξ would be zero, then condition AET1 would not be satisfied for this function. Similar remark can be formulated in case of $\varphi(t)=t-\sqrt{t}$.

Remark 2.2. The functions $\varphi(t)=t^2-t+\sqrt{t}$ and $\varphi(t)=t^2+\sqrt{t}$ are new in this AET technique. Up to our best knowledge they are the first AET functions in the literature, that have inflection point.

We can compare our new class of AET functions to the class of concave functions proposed by Haddou et al. [24].

Example 2.1. Consider the function $\varphi(t)=\log(1+t)$, member of the class of concave functions introduced by Haddou et al. [24]. By using Definition 2.4, we can check that for this function condition AET3 is not satisfied.

In the following subsection we present the class of eligible kernel functions proposed in [9] and the relationship between the kernel function approach and the AET technique.

2.4. **Eligible kernel functions.** The determination of search directions in case of IPAs can be realized by using kernel functions.

Definition 2.5. (Bai et al. [9]) A function $\psi:\mathbb{R}_+\to\mathbb{R}_\oplus$ is called *kernel function* if it is twice continuously differentiable and if the following conditions hold:
K1. $\psi(1)=\psi'(1)=0$;
K2. $\psi''(t)>0$, for all $t>0$;
K3. $\lim_{t\downarrow 0}\psi(t)=\lim_{t\to\infty}\psi(t)=\infty$.

In some cases in the literature condition *K3.* of Definition 2.5 is used to define the notion of *coercive kernel function*, see [42, 43]. We can construct a barrier function $\Psi:\mathbb{R}^n_+\to\mathbb{R}$ in the following form:

$$\Psi(\mathbf{v})=\sum_{i=1}^{n}\psi(v_i),$$

where $\mathbf{v}\in\mathbb{R}^n_+$.

Peng et al. [37] modified the second equation of the scaled system to

$$\mathbf{d}_x+\mathbf{d}_s=-\nabla\Psi(\mathbf{v}).$$

Using this and the scaled system (2.8), we have

$$\mathbf{d}_x+\mathbf{d}_s=-\nabla\Psi(\mathbf{v})=\mathbf{p}_v=\frac{\varphi(\mathbf{e})-\varphi(\mathbf{v}^2)}{\mathbf{v}\varphi\prime(\mathbf{v}^2)}. \tag{2.11}$$

Hence, we can assign a corresponding kernel function to several functions φ appeared in the AET technique in the following way, see [1, 36]:

$$\psi(t)=\int_1^t\frac{\varphi(\bar\tau^2)-\varphi(1)}{\bar\tau\varphi'(\bar\tau^2)}d\bar\tau, \tag{2.12}$$

where the function ψ should satisfy the properties *K1.-K3.* of Lemma 2.5.

Peng et al. [37] considered self-regular functions and in this way they reduced the theoretical complexity of long-step IPAs. The definition of self-regular functions is given below.

Definition 2.6. (Peng et al. [37]) A function $\psi:(0,\infty)\to\mathbb{R},\ \psi\in\mathcal{C}^2$ is *self-regular* if it satisfies the conditions

SR1. $\psi(t)$ is strictly convex with respect to $t > 0$ and $\psi(t) = 0$ at its global minimal point $t = 1$, i.e. $\psi(1) = \psi'(1) = 0$. Further, there exist positive constants $\nu_2 \geq \nu_1 > 0$ and $p \geq 1$, $q \geq 1$ such that

$$\nu_1 \left(t^{p-1} + t^{-1-q}\right) \leq \psi''(t) \leq \nu_2 \left(t^{p-1} + t^{-1-q}\right), \; \forall t \in (0, \infty); \tag{2.13}$$

SR2. For any $t_1, t_2 > 0$,

$$\psi\left(t_1^r t_2^{1-r}\right) \leq r\psi(t_1) + (1-r)\psi(t_2), \; \forall r \in [0,1]. \tag{2.14}$$

The prototype self-regular kernel function is given by

$$\Upsilon_{p,q}(t) = \frac{t^{p+1}-1}{p(p+1)} + \frac{t^{1-q}-1}{q(q-1)} + \frac{p-q}{pq}(t-1), \tag{2.15}$$

where $p \geq 1$ and $q \geq 1$.
Bai et al [8] defined the class of *eligible kernel functions*.

Definition 2.7. (Bai et al. [8]) We call a kernel function *eligible kernel function* if it satisfies the following conditions:
EK1. $t\psi''(t) + \psi'(t) > 0$, $t < 1$;
EK2. $\psi'''(t) < 0$, $t > 0$;
EK3. $2\psi''(t)^2 - \psi'(t)\psi'''(t) > 0$, $t < 1$;
EK4. $\psi''(t)\psi'(\beta t) - \beta\psi'(t)\psi''(\beta t) > 0$, $t > 1$, $\beta > 1$.

Note that the class of eligible kernel functions contains some self-regular functions, as well as many non-self-regular functions as special cases, see [34]. However, it should be mentioned that all self-regular kernel functions $\Upsilon_{p,q}(t)$ with growth $p \leq 1$ belong to the class of eligible kernel functions.

Remark 2.3. Using (2.12) and (2.10) we have

$$\psi'(t) = -f(t). \tag{2.16}$$

From this and Proposition 2.1 we obtain that if we have a kernel function, then using AETa-c we can check whether the corresponding function(s) φ do(es) belong to the new class of AET without calculating the functions φ. Furthermore, if we have a function φ, then by using the conditions given in Proposition 2.1 we can check whether the conditions given in EK1-4 and K1-3 hold for the corresponding kernel function. Hence, we can compare functions from the class of new AET to corresponding kernel functions.

Example 2.2. Let $\psi(t) = \frac{1}{2}\left(t - \frac{1}{t}\right)^2$ be an eligible kernel function, see [34]. This is also a self-regular kernel function. Using (2.12) we have $f(t) = -\psi'(t) = \frac{1}{t^3} - t$. Note that condition AETb from Proposition 2.1 or AET2 from Definition 2.4 is not satisfied in this case, which means that the function(s) φ belonging to this eligible and self-regular kernel function is (are) not members of the new class of AET. Note that we do not have to calculate φ to check this.

Remark 2.4. The function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$, $\varphi(t) = t$ belongs to the class of concave functions proposed by Haddou et al. [24]. Furthermore, the kernel function corresponding to this function is eligible kernel function. It should be mentioned that $\varphi(t) = t$ belongs to the new class of AET functions if it is defined on (ξ, ∞), where $0 < \xi < 1$.

2.5. **Special case of the new class of AET.** Let us consider the special case mentioned in Subsection 2.3, $\varphi : (0, \infty) \to \mathbb{R}$.

$$\varphi(t) = t^2 - t + \sqrt{t}. \tag{2.17}$$

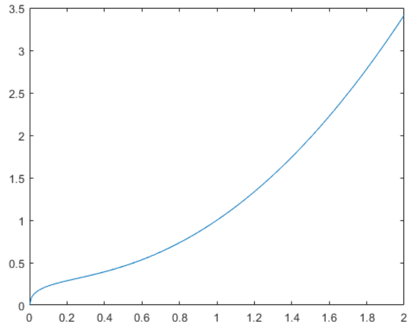

FIGURE 1. Graph of the function given in (2.17).

The graph of the function given in (2.17) is shown in Figure 2.17.

It is interesting that this function has inflection point. Up to our best knowledge this is the first AET function in the literature which has inflection point.

Using (2.4) and (2.9) we can calculate the corresponding $\mathbf{a}_\varphi$ and $\mathbf{p}_v$ in this case:

$$\mathbf{a}_\varphi = \mu \frac{\mathbf{e} - \frac{\mathbf{x}^2\mathbf{s}^2}{\mu^2} + \frac{\mathbf{xs}}{\mu} - \sqrt{\frac{\mathbf{xs}}{\mu}}}{2\frac{\mathbf{xs}}{\mu} - \mathbf{e} + \frac{1}{2}\sqrt{\frac{\mu}{\mathbf{xs}}}} = \frac{2\mu^2\sqrt{\mathbf{xs}} - 2\mathbf{x}^2\mathbf{s}^2\sqrt{\mathbf{xs}} + 2\mu\mathbf{xs}\sqrt{\mathbf{xs}} - 2\mu\mathbf{xs}\sqrt{\mu}}{4\mathbf{xs}\sqrt{\mathbf{xs}} - 2\mu\sqrt{\mathbf{xs}} + \mu\sqrt{\mu}} \tag{2.18}$$

and

$$\mathbf{p}_v = \frac{\mathbf{e} - \mathbf{v}^4 + \mathbf{v}^2 - \mathbf{v}}{2\mathbf{v}^3 - \mathbf{v} + \frac{1}{2}\mathbf{e}} = \frac{2(\mathbf{e} - \mathbf{v})(\mathbf{e} + \mathbf{v}^2 + \mathbf{v}^3)}{4\mathbf{v}^3 - 2\mathbf{v} + \mathbf{e}}. \tag{2.19}$$

Hence, in our case

$$f(t) = \frac{2(1 - t^4 + t^2 - t)}{4t^3 - 2t + 1}, \tag{2.20}$$

and using (2.16), in our case when φ is the function given in (2.17) we have

$$f'(t) = \frac{-8t^6 + 4t^4 + 8t^3 - 28t^2 + 4t + 2}{(4t^3 - 2t + 1)^2} = -\psi''(t). \tag{2.21}$$

$$f''(t) = \frac{4(8t^6 - 36t^5 + 108t^4 - 20t^3 - 6t^2 - 12t + 3)}{(4t^3 - 2t + 1)^3} = -\psi'''(t). \tag{2.22}$$

Note that in case of kernel functions we have $\psi''(t) \geq 0$. However, using (2.16) and (2.21) we can conclude that the kernel function corresponding to the function φ given in (2.17) is not a kernel function in the sense, that ψ is not convex for all $t > 0$. From the same reason we get that the kernel function belonging to the function φ is neither self-regular, because in case of self-regular functions (2.13) should be satisfied. However, (2.21) takes positive and negative values as well. In case of eligible kernel functions $\psi'''(t) < 0$, hence we should have $f''(t) > 0$, for $t > 0$. However, the expression given in (2.22) is not positive for all $t > 0$. The kernel function corresponding to the function φ given in (2.17) does not belong to the class of eligible kernel functions.

This function φ does not belong neither to the class of Haddou's class of concave functions, because $\varphi''(t) = 2 - \frac{1}{4t\sqrt{t}}$ is not negative for all $t \geq 0$.

In the following subsection we present short-step IPAs for solving $P_*(\kappa)$-LCPs, that use this new class of functions in the AET technique to determine search directions.

2.6. **New interior-point algorithms for solving $P_*(\kappa)$-linear complementarity problems.** Firstly, we deal with the determination of the search directions. For this, we consider system (2.3), where $\mathbf{a}_\varphi$ depends on the function φ used, which is member of the class new AET functions.

As special case, when $\varphi(t) = t^2 - t + \sqrt{t}$, $\mathbf{a}_\varphi$ will be the expression given in (2.18). We define the centrality measure $\delta : \mathbb{R}^n_+ \times \mathbb{R}^n_+ \times \mathbb{R}_+ \to \mathbb{R} \cup \{\infty\}$ as

$$\delta(\mathbf{x}, \mathbf{s}, \mu) := \delta(\mathbf{v}) = \frac{\|\mathbf{p}_v\|}{2}, \tag{2.23}$$

where $\|\cdot\|$ denotes the standard Euclidean norm, and $\mathbf{p}_v$ is given in (2.9). Consider the τ-neighbourhood of a fixed point of the central path as

$$\mathcal{N}_2(\tau, \mu) := \{(\mathbf{x}, \mathbf{s}) \in \mathcal{F}^+ : \delta(\mathbf{x}, \mathbf{s}, \mu) \leq \tau\}, \tag{2.24}$$

where $\delta(\mathbf{x}, \mathbf{s}, \mu)$ is given in (2.23), $\mu > 0$ is fixed and τ is a threshold parameter.

In Algorithm 2.1 we define a whole class of IPAs for solving $P_*(\kappa)$-LCPs, which is based on the new class of AET functions.

Algorithm 2.1 : IPAs for $P_*(\kappa)$-LCPs based on a new class of AET functions

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Let $\epsilon > 0$ be the accuracy parameter, $0 < \theta < 1$ the update parameter and τ the proximity parameter. Furthermore, a known upper bound κ of the handicap $\hat{\kappa}(M)$ is given. Assume that for $(\mathbf{x}^0, \mathbf{s}^0)$ the $\left(\mathbf{x}^0\right)^T \mathbf{s}^0 = n\mu^0$, $\mu^0 > 0$ holds such that $\delta(\mathbf{x}^0, \mathbf{s}^0, \mu^0) \leq \tau$.

begin

  $k := 0$;

  while $\left(\mathbf{x}^k\right)^T \mathbf{s}^k > \epsilon$ **do**

  begin

   (determination of search directions)

   compute $(\Delta x^k, \Delta s^k)$ from (2.3) with φ belonging to the new class of AET functions;

   let $\mathbf{x}^k := \mathbf{x}^k + \Delta x^k$ and $\mathbf{s}^k := \mathbf{s}^k + \Delta s^k$;

   (update of the parameter μ)

   $\mu^{k+1} := (1 - \theta)\, \mu^k$;

   k:=k+1;

  end

end

Remark 2.5. The default values of the parameters τ and θ will be given later in the complexity analysis of the algorithm.

Remark 2.6. If we consider the special case $\varphi(t) = t^2 - t + \sqrt{t}$, then the default value of θ is $\frac{1}{(200+100\kappa)\sqrt{n}}$ and the default value of τ is $\tau = \frac{1}{32+16\kappa}$.

Remark 2.7. Note that using the function from (2.17) the proximity measure from (2.23) will be

$$\delta(\mathbf{x}, \mathbf{s}, \mu) = \frac{1}{2} \left\| \frac{\mathbf{e} - \mathbf{v}^4 + \mathbf{v}^2 - \mathbf{v}}{2\mathbf{v}^3 - \mathbf{v} + \frac{1}{2}\mathbf{e}} \right\|. \tag{2.25}$$

In the following section we present the complexity analysis of the IPAs using the new class of AET functions defined by conditions AET1-3 of Definition 2.4.

3. Complexity analysis

The first lemma shows the strict feasibility of the full-Newton step.

Lemma 3.1. *Let* $(\boldsymbol{x}, \boldsymbol{s}) \in \mathcal{F}^+$ *be given, such that* $\delta(\boldsymbol{x}, \boldsymbol{s}; \mu) \leq \frac{1}{\sqrt{1+4\kappa}}$. *For any function satisfying AET3 of Definition 2.4, we have that* $(\mathbf{x}^+, \mathbf{s}^+) \in \mathcal{F}^+$.

Proof. It should be mentioned that only the right hand side of AET3 should be satisfied in this lemma, namely:

$$4t^2(\varphi(1)-\varphi(t^2))\varphi'(t^2)+\left(\varphi(1)-\varphi(t^2)\right)^2 \geq 4t^2(1-t^2)\left(\varphi'(t^2)\right)^2, \quad t>\xi. \tag{3.1}$$

We have

$$\begin{aligned} \mathbf{x}(\alpha)\mathbf{s}(\alpha) &= (\mathbf{x}+\alpha\Delta\mathbf{x})(\mathbf{s}+\alpha\Delta\mathbf{s}) = \mathbf{x}\mathbf{s}+\alpha(\mathbf{s}\Delta\mathbf{x}+\mathbf{x}\Delta\mathbf{s})+\alpha^2\Delta\mathbf{x}\Delta\mathbf{s} \\ &= \mu\mathbf{v}^2+\mu\mathbf{v}\alpha(\mathbf{d}_x+\mathbf{d}_s)+\alpha^2\mu\mathbf{d}_x\mathbf{d}_s \\ &= \mu((1-\alpha)\mathbf{v}^2+\alpha(\mathbf{v}^2+\mathbf{v}\mathbf{p}_v+\alpha\mathbf{d}_x\mathbf{d}_s). \end{aligned} \tag{3.2}$$

Our aim is to show that $\mu\alpha(\mathbf{v}^2+\mathbf{v}\mathbf{p}_v+\alpha\mathbf{d}_x\mathbf{d}_s) \geq 0$. We have $\mathbf{p}_v = \mathbf{d}_x+\mathbf{d}_s$ and using the notations $\mathbf{q}_v := \mathbf{d}_x-\mathbf{d}_s$ it can be seen that

$$\mathbf{d}_x\mathbf{d}_s = \frac{\mathbf{p}_v^2-\mathbf{q}_v^2}{4}. \tag{3.3}$$

Using the definition of $\mathbf{p}_v$ given in (2.9) we obtain that (3.1) is equivalent with

$$\mathbf{v}\mathbf{p}_v+\frac{\mathbf{p}_v^2}{4} \geq \mathbf{e}-\mathbf{v}^2, \tag{3.4}$$

which leads to

$$\mathbf{v}^2+\mathbf{v}\mathbf{p}_v \geq \mathbf{e}-\frac{\mathbf{p}_v^2}{4}. \tag{3.5}$$

Using (3.2), (3.3) and (3.5) we have

$$\begin{aligned} \frac{\mathbf{x}(\alpha)\mathbf{s}(\alpha)}{\mu} &= (1-\alpha)\mathbf{v}^2+\alpha\left(\mathbf{v}^2+\mathbf{v}\mathbf{p}_v+\alpha\frac{\mathbf{p}_v^2}{4}-\alpha\frac{\mathbf{q}_v^2}{4}\right) \\ &\geq (1-\alpha)\mathbf{v}^2+\alpha\left(\mathbf{e}-(1-\alpha)\frac{\mathbf{p}_v^2}{4}-\alpha\frac{\mathbf{q}_v^2}{4}\right). \end{aligned}$$

We have $\dfrac{\mathbf{x}(\alpha)\mathbf{s}(\alpha)}{\mu} \geq 0$, if $\left\|(1-\alpha)\dfrac{\mathbf{p}_v^2}{4}-\alpha\dfrac{\mathbf{q}_v^2}{4}\right\|_\infty \leq 1$ holds.
From [16] we obtain

$$\|\mathbf{q}_v\| \leq \sqrt{1+4\kappa}\,\|\mathbf{p}_v\| = 2\sqrt{1+4\kappa}\delta. \tag{3.6}$$

Then, we have

$$\begin{aligned} \left\|(1-\alpha)\frac{\mathbf{p}_v^2}{4}-\alpha\frac{\mathbf{q}_v^2}{4}\right\|_\infty &\leq (1-\alpha)\frac{\|\mathbf{p}_v\|^2}{4}+\alpha\frac{\|\mathbf{q}_v\|^2}{4} \\ &\leq (1-\alpha)\frac{\|\mathbf{p}_v\|^2}{4}+\alpha(1+4\kappa)\frac{\|\mathbf{p}_v\|^2}{4} \\ &= (1+4\alpha\kappa)\delta^2 \leq (1+4\kappa)\delta^2. \end{aligned}$$

This means that $\left\|(1-\alpha)\frac{\mathbf{p}_v^2}{4}-\alpha\frac{\mathbf{q}_v^2}{4}\right\|_\infty \leq 1$ holds if we have $\delta \leq \frac{1}{\sqrt{1+4\kappa}}$. In this way the lemma is proven.

□

In the following lemma we analyse the conditions under which the Newton process is quadratically convergent.

Lemma 3.2. *Suppose that* $\delta(\mathbf{x},\mathbf{s};\mu) \leq \frac{1}{\sqrt{1+4\kappa}}$. *Let* $(\boldsymbol{x},\boldsymbol{s}) \in \mathcal{F}^+$ *and* $\bar{\mathbf{v}} = \sqrt{\frac{\boldsymbol{x}^+\boldsymbol{s}^+}{\mu}}$ *be given. For any function* φ *satisfying AET1 and AET2 of Definition 2.4, we can say that after a primal-dual Newton barrier step we have:*

$$\delta(\boldsymbol{x}^+,\boldsymbol{s}^+;\mu) \leq c_1(c_2+2+4\kappa)\delta(\boldsymbol{x},\boldsymbol{s};\mu)^2,$$

where $c_1, c_2 \in \mathbb{R}_+$.

Proof. It should be noted that in the proof we will use the form AETa and AETb of Proposition 2.1. Using Proposition 2.1 and (2.23) we get

$$\delta(\mathbf{x}^+,\mathbf{s}^+;\mu) := \frac{\|\mathbf{p}_{\bar{\mathbf{v}}}\|}{2} = \left\|(\mathbf{e}-\bar{\mathbf{v}}^2)g(\bar{\mathbf{v}})\right\|, \tag{3.7}$$

where the function g is given in Proposition (2.1). Using the assumption that $\exists c_1 \in \mathbb{R}_+$, for which $|g(t)| \leq c_1$, we obtain

$$\delta(\mathbf{x}^+,\mathbf{s}^+;\mu) \leq c_1\left\|\mathbf{e}-\bar{\mathbf{v}}^2\right\|. \tag{3.8}$$

We know from (3.2) that

$$\left\|\mathbf{e}-\bar{\mathbf{v}}^2\right\| = \left\|\mathbf{e}-\frac{\mathbf{x}^+\mathbf{s}^+}{\mu}\right\| = \left\|\mathbf{e}-\mathbf{v}^2-\mathbf{v}\mathbf{p}_v-\frac{\mathbf{p}_v^2}{4}+\frac{\mathbf{q}_v^2}{4}\right\|. \tag{3.9}$$

Using condition AETb of Proposition 2.1 we have

$$\mathbf{e}-\mathbf{v}^2-\mathbf{v}\mathbf{p}_v \;=\; h(\mathbf{v})\frac{\mathbf{p}_v^2}{4}. \tag{3.10}$$

Using (3.6), (3.9) and (3.10) we obtain

$$\left\|\mathbf{e}-\bar{\mathbf{v}}^2\right\| \;\leq\; \left\|\mathbf{e}-\mathbf{v}^2-\mathbf{v}\mathbf{p}_v\right\|+\left\|\frac{\mathbf{p}_v^2}{4}\right\|+\left\|\frac{\mathbf{q}_v^2}{4}\right\| \leq (2+c_2+4\kappa)\delta^2. \tag{3.11}$$

From (3.8) and (3.11) we have

$$\delta(\mathbf{x}^+,\mathbf{s}^+;\mu) \leq c_1(c_2+2+4\kappa)\delta(\mathbf{x},\mathbf{s};\mu)^2,$$

which proves the lemma.

□

In the next lemma we investigate the effect of the full-Newton step on the duality gap.

Lemma 3.3. *Let* $\delta := \delta(\boldsymbol{x},\boldsymbol{s};\mu)$ *and suppose that the vectors* $\boldsymbol{x}^+$ *and* $\boldsymbol{s}^+$ *are obtained using a full-Newton step, thus* $\boldsymbol{x}^+ = \boldsymbol{x}+\Delta\boldsymbol{x}$ *and* $\boldsymbol{s}^+ = \boldsymbol{s}+\Delta\boldsymbol{s}$. *For any function* φ *satisfying AET3 of Definition 2.4 with* $c_3 \in \mathbb{R}_+$, *we have that*

$$\left(\boldsymbol{x}^+\right)^T\boldsymbol{s}^+ \leq \mu(n+(c_3+1)\delta^2).$$

Proof. Note that only the left hand side of AET3 will be used in this proof, namely

$$4t^2(\varphi(1)-\varphi(t^2))\varphi'(t^2)-c_3\left(\varphi(1)-\varphi(t^2)\right)^2 \le 4t^2(1-t^2)\left(\varphi'(t^2)\right)^2, \quad t>\xi. \tag{3.12}$$

Using the definition of $\mathbf{p}_v$ in (2.9) we get that (3.12) is equivalent with

$$\mathbf{v}\mathbf{p}_v - c_3\frac{\mathbf{p}_v^2}{4} \le \mathbf{e}-\mathbf{v}^2. \tag{3.13}$$

From (3.2) and (3.13) we get

$$\frac{1}{\mu}\mathbf{x}^+\mathbf{s}^+ = \mathbf{v}^2+\mathbf{v}\mathbf{p}_v+\mathbf{d}_x\mathbf{d}_x \le \mathbf{e}+\frac{c_3}{4}\mathbf{p}_v^2+\mathbf{d}_x\mathbf{d}_s.$$

After some calculations we have

$$\begin{aligned}\left(\mathbf{x}^+\right)^T\mathbf{s}^+ &= \mathbf{e}^T(\mathbf{x}^+\mathbf{s}^+) \le \mu(\mathbf{e}^T\mathbf{e}+\frac{c_3}{4}\mathbf{e}^T\mathbf{p}_v^2+\mathbf{e}^T\mathbf{d}_x\mathbf{d}_s)\\ &= \mu(n+\frac{c_3}{4}\|\mathbf{p}_v\|^2+\mathbf{d}_x^T\mathbf{d}_s)\\ &\le \mu(n+c_3\delta^2+\delta^2) = \mu(n+(c_3+1)\delta^2).\end{aligned}$$

The last inequality holds, since

$$\mathbf{d}_x^T\mathbf{d}_s = \frac{\|\mathbf{d}_x+\mathbf{d}_s\|^2-\|\mathbf{d}_x-\mathbf{d}_s\|^2}{4} \le \frac{\|\mathbf{d}_x+\mathbf{d}_s\|^2}{4} = \frac{\|\mathbf{p}_v\|^2}{4}.$$

□

The next lemma examines the effect which a Newton step followed by an update of the parameter μ has on the proximity measure.

Lemma 3.4. *Suppose that* $\delta(\mathbf{x},\mathbf{s};\mu) \le \frac{1}{\sqrt{1+4\kappa}}$. *Let* $\boldsymbol{v}^+ = \sqrt{\frac{\boldsymbol{x}^+\boldsymbol{s}^+}{\mu^+}}$, *where* $\mu^+ = (1-\theta)\mu$ *and let* $\eta = \sqrt{1-\theta}$. *For any function* φ *satisfying AET1 and AET2 of Definition 2.4 with* $c_1, c_2 \in \mathbb{R}_+$, *after a primal-dual Newton step we have:*

$$\delta(\boldsymbol{x}^+,\boldsymbol{s}^+;\mu^+) \le \frac{c_1}{\eta^2}\left(\theta\sqrt{n}+((c_2+2)+4\kappa)\delta^2\right).$$

Proof. In the proof we will use the form AETa and AETb of Proposition 2.1. Using the definition of the proximity measure given in (2.9) we have

$$\delta(\mathbf{x}^+,\mathbf{s}^+;\mu^+) = \left\|(\mathbf{e}-(\mathbf{v}^+)^2)g(\mathbf{v}^+)\right\|.$$

From condition AETa of Proposition 2.1 , we get

$$\delta(\mathbf{x}^+,\mathbf{s}^+;\mu^+) \le c_1\left\|\mathbf{e}-(\mathbf{v}^+)^2\right\|. \tag{3.14}$$

Furthermore,

$$\begin{aligned}\left\|\mathbf{e}-(\mathbf{v}^+)^2\right\| &= \left\|\mathbf{e}-\frac{1}{\eta^2}\frac{\mathbf{x}^+\mathbf{s}^+}{\mu}\right\| = \left\|\mathbf{e}-\frac{1}{\eta^2}\left(\mathbf{v}^2+\mathbf{v}\mathbf{p}_v+\frac{\mathbf{p}_v^2}{4}-\frac{\mathbf{q}_v^2}{4}\right)\right\|\\ &= \frac{1}{\eta^2}\left\|\eta^2\mathbf{e}-\left(\mathbf{v}^2+\mathbf{v}\mathbf{p}_v+\frac{\mathbf{p}_v^2}{4}-\frac{\mathbf{q}_v^2}{4}\right)\right\|\\ &= \frac{1}{\eta^2}\left\|-\theta\mathbf{e}+\mathbf{e}-\mathbf{v}^2-\mathbf{v}\mathbf{p}_v-\frac{\mathbf{p}_v^2}{4}+\frac{\mathbf{q}_v^2}{4}\right\|.\end{aligned} \tag{3.15}$$

From (3.10), (3.14), (3.15) and condition AETb of Proposition 2.1 we obtain

$$\begin{aligned}\delta(\mathbf{x}^+,\mathbf{s}^+;\mu^+) &\leq \frac{c_1}{\eta^2}\left(\|\theta\mathbf{e}\|+\left\|\mathbf{e}-\mathbf{v}^2-\mathbf{v}\mathbf{p}_v\right\|+\left\|\frac{\mathbf{p}_v^2}{4}\right\|+\left\|\frac{\mathbf{q}_v^2}{4}\right\|\right)\\ &\leq \frac{c_1}{\eta^2}\left(\theta\sqrt{n}+((c_2+2)+4\kappa)\delta^2\right). \qquad (3.16)\end{aligned}$$

□

In the following lemma we set the values of the parameters θ and τ and we prove that for these values the IPAs using the new class of AET functions are well defined.

Lemma 3.5. *Let $\varphi:(\xi,\infty)\to\mathbb{R}$ satisfying AET1-3 of Definition 2.4. Consider $n\geq 1$, $\theta=\frac{2}{25c_2(2+\kappa)\sqrt{n}}$ and $\tau=\frac{1}{2c_2(2+\kappa)}$. If $c_1<\frac{100c_2-4}{41c_2+50}$ and $c_2>\frac{1}{2}$, then we have*

$$\delta(\boldsymbol{x}^+,\boldsymbol{s}^+;\mu^+)<\tau,$$

hence the IPAs defined in Algorithm 2.1 are well defined.

Proof. We suppose that $\tau=\frac{1}{2c_2(2+\kappa)}$ and $c_2>\frac{1}{2}$. From here we have $\frac{1}{2c_2(2+\kappa)}<\frac{1}{2+\kappa}<\frac{1}{\sqrt{1+4\kappa}}$. Using this and the assumption that AET1-3 of Definition 2.4 are satisfied, from Lemma 3.1 we get that $(\mathbf{x}^+,\mathbf{s}^+)\in\mathcal{F}^+$.

Using Lemma 3.4 we have

$$\delta(\mathbf{x}^+,\mathbf{s}^+;\mu^+) \leq \frac{c_1}{\eta^2}\left(\theta\sqrt{n}+((2+c_2)+4\kappa)\delta^2\right). \qquad (3.17)$$

Considering $\theta=\frac{2}{25c_2(2+\kappa)\sqrt{n}}$ we get

$$\theta\sqrt{n}=\frac{2}{25c_2(2+\kappa)}. \qquad (3.18)$$

Moreover, using $n\geq 1$, $\kappa\geq 0$ we have

$$\frac{1}{1-\theta}\leq\frac{1}{1-\frac{2}{50c_2}}=\frac{25c_2}{25c_2-1}. \qquad (3.19)$$

Substituting the value of τ in (3.17) and using $\kappa\geq 0$, (3.18) and (3.19), we get

$$\begin{aligned}\delta(\mathbf{x}^+,\mathbf{s}^+;\mu^+) &\leq c_1\cdot\frac{25c_2}{25c_2-1}\left(\frac{2}{25c_2(2+\kappa)}+(c_2-6+8+4\kappa)\frac{1}{(2c_2(2+\kappa))^2}\right)\\ &= c_1\cdot\frac{25c_2}{25c_2-1}\left(\frac{4}{25(2c_2(2+\kappa))}+\frac{c_2-6}{4c_2^2(2+\kappa)^2}+\frac{4(2+\kappa)}{4c_2^2(2+\kappa)^2}\right)=\\ &\leq c_1\cdot\frac{25c_2}{25c_2-1}\frac{1}{2c_2(2+\kappa)}\left(\frac{4}{25}+\frac{c_2-6}{4c_2}+\frac{2}{c_2}\right)\\ &= c_1\frac{25c_2}{25c_2-1}\frac{41c_2+50}{100c_2}\frac{1}{2c_2(2+\kappa)}=\frac{c_1(41c_2+50)}{100c_2-4}\tau. \qquad (3.20)\end{aligned}$$

The obtained result should be less than τ, hence using $c_2>\frac{1}{2}>\frac{1}{25}$, we get

$$c_1<\frac{100c_2-4}{41c_2+50}, \qquad (3.21)$$

which gives the result. □

Remark 3.1. It should be mentioned that the parameters c_1 and c_2 of the functions given in Table 2 satisfy the condition $c_2>\frac{1}{2}$ and $c_1<\frac{100c_2-4}{41c_2+50}$.

The following lemma gives upper bound on the number of iterations.

Lemma 3.6. *Consider* $\varphi : (\xi, \infty) \to \mathbb{R}$ *satisfying AET1-3 of Definition 2.4. Let* $n \geq 1$, $\theta = \frac{2}{25c_2(2+\kappa)\sqrt{n}}$, $\tau = \frac{1}{2c_2(2+\kappa)}$, $c_1 < \frac{100c_2-4}{41c_2+50}$ *and* $c_2 > \frac{1}{2}$ *and* $c_3 < 16c_2^2 - 1$. *We assume that the pair* $(\boldsymbol{x}^0, \boldsymbol{s}^0)$ *is strictly feasible,* $\mu^0 = \frac{(\boldsymbol{x}^0)^T \boldsymbol{s}^0}{n}$ *and* $\delta(\boldsymbol{x}^0, \boldsymbol{s}^0; \mu^0) < \tau$. *Let* $\boldsymbol{x}^k$ *and* $\boldsymbol{s}^k$ *be the two vectors obtained by the algorithms given in Algorithm 2.1 after* k *iterations. Then, for*

$$k \geq \left\lceil \frac{1}{\theta} \log \frac{\mu^0(n+1)}{\varepsilon} \right\rceil$$

we have $(\boldsymbol{x}^k)^T \boldsymbol{s}^k < \varepsilon$.

Proof. From Lemma 3.3 and $c_3 < 16c_2^2 - 1$ we have

$$\begin{aligned} \left(\mathbf{x}^k\right)^T \mathbf{s}^k &\leq \mu^k \left(n + (c_3+1) \cdot \frac{1}{(2c_2(2+\kappa))^2} \right) \\ &= (1-\theta)^k \mu^0 \left(n + (c_3+1) \cdot \frac{1}{(2c_2(2+\kappa))^2} \right) < (1-\theta)^k \mu^0 (n+1). \end{aligned} \tag{3.22}$$

The condition $(\mathbf{x}^k)^T \mathbf{s}^k < \varepsilon$ holds if

$$(1-\theta)^k \mu^0 (n+1) < \varepsilon. \tag{3.23}$$

Taking the logarithm of both sides of (3.23) we have

$$k \log(1-\theta) + \log\left(\mu^0 (n+1)\right) < \log \varepsilon.$$

Using that $-\log(1-\theta) \geq \theta$ finally we get

$$k\theta \geq \log\left(\mu^0 (n+1)\right) - \log \varepsilon = \log \frac{\mu^0 (n+1)}{\varepsilon},$$

which proves the lemma. □

Remark 3.2. Condition $c_3 < 16c_2^2 - 1$ is satisfied for all functions given in Table 2.

Theorem 3.1. *Let* $\varphi : (\xi, \infty) \to \mathbb{R}$ *satisfying AET1-3 of Definition 2.4. Consider* $n \geq 1$, $\theta = \frac{2}{25c_2(2+\kappa)\sqrt{n}}$ *and* $\tau = \frac{1}{2c_2(2+\kappa)}$. *If* $c_1 < \frac{100c_2-4}{41c_2+50}$, $c_2 > \frac{1}{2}$ *and* $c_3 < 16c_2^2 - 1$, *then the IPAs given in Algorithm 2.1 require no more than*

$$\mathcal{O}\left((2+\kappa)\sqrt{n} \log \frac{(n+1)\mu^0}{\epsilon} \right)$$

interior-point iterations.

In the following subsection we analyse the special case when $\varphi(t) = t^2 - t + \sqrt{t}$, which is the first AET function which has inflection point.

3.1. **Special case.** The first step of this research was to define and analyse an IPA using a new type of function in the AET technique, which has inflection point. From this analysis we built up the new class of AET functions. Hence, in this subsection, we summarize the corollaries of the lemmas presented in the previous part in the special case when $\varphi(t) = t^2 - t + \sqrt{t}$.

Corollary 3.1. *Let* $(\boldsymbol{x}, \boldsymbol{s}) \in \mathcal{F}^+$ *be given, such that* $\delta(\boldsymbol{x}, \boldsymbol{s}; \mu) < \frac{1}{\sqrt{1+4\kappa}}$. *In case of* $\varphi(t) = t^2 - t + \sqrt{t}$ *we have that* $(\mathbf{x}^+, \mathbf{s}^+) \in \mathcal{F}^+$.

Corollary 3.2. *Let* $(\boldsymbol{x}, \boldsymbol{s}) \in \mathcal{F}^+$ *and* $\bar{\mathbf{v}} = \sqrt{\frac{\boldsymbol{x}^+ \boldsymbol{s}^+}{\mu}}$ *be given, such that* $\delta(\boldsymbol{x}, \boldsymbol{s}; \mu) \leq \frac{1}{\sqrt{1+4\kappa}}$. *In case of* $\varphi(t) = t^2 - t + \sqrt{t}$ *we have that after a primal-dual Newton barrier step:*

$$\delta(\boldsymbol{x}^+, \boldsymbol{s}^+; \mu) \leq 2(10+4\kappa)\delta(\boldsymbol{x}, \boldsymbol{s}; \mu)^2.$$

Proof. In the proof we use that in case of $\varphi(t) = t^2 - t + \sqrt{t}$

$$g(t) := \frac{1+t^2+t^3}{(4t^3-2t+1)(t+1)} \tag{3.24}$$

function, for which we have $|g(t)| \leq 2$. This means that $c_1 = 2$ in this case.

Furthermore,

$$h(t) = \frac{(4t^3-2t+1)(-2t^3-4t^2-2t+1)}{(1+t^2+t^3)^2}. \tag{3.25}$$

It can be shown that $|h(t)| \leq 8$, for all $t > 0$, hence $c_2 = 8$.
Using Lemma 3.2 we have

$$\delta(\mathbf{x}^+, \mathbf{s}^+; \mu) \leq 2(10+4\kappa)\delta(\mathbf{x}, \mathbf{s}; \mu)^2.$$

□

Corollary 3.3. *Let* $\delta := \delta(\boldsymbol{x}, \boldsymbol{s}; \mu)$ *and suppose that the vectors* $\boldsymbol{x}^+$ *and* $\boldsymbol{s}^+$ *are obtained using a full-Newton step, thus* $\boldsymbol{x}^+ = \boldsymbol{x} + \Delta\boldsymbol{x}$ *and* $\boldsymbol{s}^+ = \boldsymbol{s} + \Delta\boldsymbol{s}$. *In case of* $\varphi(t) = t^2 - t + \sqrt{t}$ *we have*

$$\left(\boldsymbol{x}^+\right)^T \boldsymbol{s}^+ \leq \mu(n + 9\delta^2).$$

Corollary 3.4. *Let* $\boldsymbol{v}^+ = \sqrt{\frac{\boldsymbol{x}^+ \boldsymbol{s}^+}{\mu^+}}$, *where* $\mu^+ = (1-\theta)\mu$ *and let* $\eta = \sqrt{1-\theta}$. *Suppose that* $\delta(\mathbf{x}, \mathbf{s}; \mu) \leq \frac{1}{\sqrt{1+4\kappa}}$. *In case of* $\varphi(t) = t^2 - t + \sqrt{t}$ *used in the AET technique, after a primal-dual Newton step we have:*

$$\delta(\boldsymbol{x}^+, \boldsymbol{s}^+; \mu^+) \leq \frac{2}{\eta^2}\left(\theta\sqrt{n} + (10+4\kappa)\delta^2\right).$$

Proof. From Corollary 3.2 we get that in case of $\varphi(t) = t^2 - t + \sqrt{t}$, $c_1 = 2$ and $c_2 = 8$. Substitution of these values in Lemma 3.4 gives the desired result. □

Corollary 3.5. *Let* $n \geq 1$, $\theta = \frac{1}{(200+100\kappa)\sqrt{n}}$ *and* $\tau = \frac{1}{32+16\kappa}$. *Then, we have*

$$\delta(\boldsymbol{x}^+, \boldsymbol{s}^+; \mu^+) < \tau,$$

hence the algorithm is well defined.

Corollary 3.6. *We assume that the pair* $(\boldsymbol{x}^0, \boldsymbol{s}^0)$ *is strictly feasible,* $\mu^0 = \frac{(\boldsymbol{x}^0)^T \boldsymbol{s}^0}{n}$ *and* $\delta(\boldsymbol{x}^0, \boldsymbol{s}^0; \mu^0) < \tau := \frac{1}{32+16\kappa}$. *Let* $\boldsymbol{x}^k$ *and* $\boldsymbol{s}^k$ *be the two vectors obtained by the algorithm given in Algorithm 2.1 after* k *iterations. Then, for*

$$k \geq \left\lceil \frac{1}{\theta} \log \frac{\mu^0(n+1)}{\varepsilon} \right\rceil$$

we have $(\boldsymbol{x}^k)^T \boldsymbol{s}^k < \varepsilon$.

Corollary 3.7. *Let* $n \geq 1$, $\theta = \frac{1}{(200+100\kappa)\sqrt{n}}$ *and* $\tau = \frac{1}{32+16\kappa}$. *Then, Algorithm 2.1 requires no more than*

$$\mathcal{O}\left((2+\kappa)\sqrt{n}\log\frac{(n+1)\mu^0}{\epsilon}\right)$$

interior-point iterations.

4. Numerical results

We implemented a variant of the proposed PD IPAs in the C++ programming language. About the implementation there is a detailed explanation in [28]. We did all computations on a desktop computer with Intel quad-core 2.6 GHz processor and 16 GB RAM. Due to the fact that in many cases we do not have information about the value of κ, we used Algorithm 2.1 in our implementation. We set the values $\theta = 0.999$ and $\epsilon = 10^{-5}$.

Moreover, it should be mentioned that most of the numerical results related to $P_*(\kappa)$-LCPs are related to problems where the value of κ is zero, that lead to LP problems. Gurtuna et al. [22] and Asadi et al. [6] provided numerical results related to $P_*(\kappa)$-LCPs having positive handicap, by considering 2×2 or 3×3 matrices. They also analysed block diagonal matrices formed by the aformentioned ones. Darvay et al. [16] presented numerical results where they solved $P_*(\kappa)$-problems with matrices having positive κ generated by Illés and Morapitiye [25].

De Klerk and E.-Nagy [20] proved that the handicap of the matrix can be exponential in the size of the problem. They considered the following matrix which was proposed by Csizmadia:

$$M = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ -1 & 1 & 0 & \cdots & 0 \\ -1 & -1 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & -1 & \cdots & 1 \end{pmatrix}, \tag{4.1}$$

and they proved that $\hat{\kappa}(M) \geq 2^{2n-8} - 0.25$. However, we obtained promising results in this case as well.

We generated 10-10 $(\bar{\mathbf{x}}, \bar{\mathbf{s}})$ starting point pairs for each size of (4.1) matrix from three different intervals, $[0,1]^n$, $[0,10]^n$ and $[0,100]^n$ for coordinates $\bar{\mathbf{x}}$ and $\bar{\mathbf{s}}$, respectively. Given the $P_*(\kappa)$-property of matrix M, we generated test problems with vector $\bar{\mathbf{q}}$ given by

$$\bar{\mathbf{q}} = -M\bar{\mathbf{x}} + \bar{\mathbf{s}}.$$

The computational results for problems generated by the points $(\bar{\mathbf{x}}, \bar{\mathbf{s}})$ for interval $[0,1]^n$ can be found in Table 3, for $[0,10]^n$ in Table 4 and for $[0,100]^n$ in Table 5. However, in case of 400×400 matrices there were several problems, where numerical errors appeared. In case of interval $[0,1]^n$ there were two, in case of interval $[0,100]^n$ there was one example, where the duality gap stopped decreasing after several steps. This phenomenon can be explained by the relatively high local κ value, which was computed as

$$\kappa(\Delta\mathbf{x}) = -\frac{1}{4} \frac{\Delta\mathbf{x}^T M \Delta\mathbf{x}}{\sum_{i \in \mathcal{I}_+(\Delta\mathbf{x})} \Delta\mathbf{x}_i (M\Delta\mathbf{x})_i}.$$

This caused after several steps the value α to be zero due to some numerical errors. In case of each used function φ this problem appeared after the same iteration number. It can be seen from Tables 3, 4, 5 that if $\bar{\mathbf{x}}$ and $\bar{\mathbf{s}}$ are from the same interval, the size of the interval does not significantly change the iteration number or the CPU time.

On the other hand, we generated another examples as well with the matrix (4.1) with different sizes. We generated 10-10 examples with random starting points $\bar{\mathbf{x}} \in [0,1]^n$ and $\mathbf{s} \in [9,11]^n$, $\bar{\mathbf{x}} \in [9,11]^n$ and $\bar{\mathbf{s}} \in [0,1]^n$, $\bar{\mathbf{x}} \in [0,100]^n$ and $\bar{\mathbf{s}} \in [9900, 11000]^n$, $\bar{\mathbf{x}} \in [9900, 11000]^n$ and $\bar{\mathbf{s}} \in [0,100]^n$. The obtained results can be seen in Tables 6, 7, 8 and 9, respectively. All tables contain the averages of maximum local κ, as well.

It can be seen from Tables 6, 7, 8 and 9 that if the values of $\bar{\mathbf{x}}$ are significantly smaller than the values of $\bar{\mathbf{s}}$, the problem can be solved relatively easier with less iteration numbers, even in case of large-size matrix (4.1). However, if the values of $\bar{\mathbf{s}}$ are smaller, the problem becomes even harder than in the case when the starting points are the all one vectors. This phenomenon can be explained by the difference of the local κ which is relatively small in the case when the values of $\bar{\mathbf{x}}$ are smaller and relatively high in the other case. The detailed explanation of the connection between the local κ and the change of duality gap and the step length can be found in [28].

n	$\varphi(t)=t$			$\varphi(t)=\sqrt{t}$			$\varphi(t)=t^2-t+\sqrt{t}$		
	Nr. of Iter.	CPU (s)	max local κ	Nr. of Iter.	CPU (s)	max local κ	Nr. of Iter.	CPU (s)	max local κ
10	13.5	0.1016	252.5210	14.0	0.1014	248.8704	23.3	0.2700	253.01779
100	69.6	13.1915	$2.8965\cdot 10^{34}$	70.3	11.7055	$2.8905\cdot 10^{34}$	81.2	15.2739	$2.8969\cdot 10^{34}$
400	266.75	896.5609	$8.3266\cdot 10^{137}$	267.25	763.6711	$8.1513\cdot 10^{137}$	279.0	809.6871	$8.3419\cdot 10^{137}$

TABLE 3. Numerical results with $\theta = 0.999$ for $P_*(\kappa)$-LCPs with matrix given in (4.1) with $\bar{\mathbf{x}}, \bar{\mathbf{s}} \in [0,1]^n$

n	$\varphi(t)=t$			$\varphi(t)=\sqrt{t}$			$\varphi(t)=t^2-t+\sqrt{t}$		
	Nr. of Iter.	CPU (s)	max local κ	Nr. of Iter.	CPU (s)	max local κ	Nr. of Iter.	CPU (s)	max local κ
10	14.2	0.0987	211.8509	15.9	0.1142	208.7034	29.6	0.2755	212.3371
100	74.6	14.2480	$4.8624\cdot 10^{33}$	76.6	13.1046	$4.8457\cdot 10^{33}$	91.8	16.2140	$4.8651\cdot 10^{33}$
400	253.6	716.8654	$4.5225\cdot 10^{137}$	256.1	735.9447	$4.5266\cdot 10^{137}$	272.7	876.3592	$4.5242\cdot 10^{137}$

TABLE 4. Numerical results with $\theta = 0.999$ for $P_*(\kappa)$-LCPs with matrix given in (4.1) with $\bar{\mathbf{x}}, \bar{\mathbf{s}} \in [0,10]^n$

n	$\varphi(t)=t$			$\varphi(t)=\sqrt{t}$			$\varphi(t)=t^2-t+\sqrt{t}$		
	Nr. of Iter.	CPU (s)	max local κ	Nr. of Iter.	CPU (s)	max local κ	Nr. of Iter.	CPU (s)	max local κ
10	14.7	0.1042	211.8509	17.9	0.1249	208.7034	36.5	0.3014	212.3371
100	73.6	13.0567	$3.0703\cdot 10^{33}$	77.3	12.5555	$3.0473\cdot 10^{33}$	96.4	17.8456	$3.0713\cdot 10^{33}$
400	263.0	796.7204	$1.8622\cdot 10^{137}$	266.89	745.1423	$1.8564\cdot 10^{137}$	288.22	934.7279	$1.8625\cdot 10^{137}$

TABLE 5. Numerical results with $\theta = 0.999$ for $P_*(\kappa)$-LCPs with matrix given in (4.1) with $\bar{\mathbf{x}}, \bar{\mathbf{s}} \in [0,100]^n$

n	$\varphi(t)=t$			$\varphi(t)=\sqrt{t}$			$\varphi(t)=t^2-t+\sqrt{t}$		
	Nr. of Iter.	CPU (s)	max local κ	Nr. of Iter.	CPU (s)	max local κ	Nr. of Iter.	CPU (s)	max local κ
10	6.0	0.0452	7.2784	8.0	0.0630	7.0675	22.6	0.2046	7.1097
100	9.5	1.9026	327003.0236	11.9	2.1738	321470.2586	27.0	5.7808	327269.0307
400	18.0	42.7345	$9.0870\cdot 10^{17}$	20.6	47.9526	$9.0672\cdot 10^{17}$	36.1	85.7836	$9.0888\cdot 10^{17}$

TABLE 6. Numerical results with $\theta = 0.999$ for $P_*(\kappa)$-LCPs with matrix given in (4.1) with $\bar{\mathbf{x}} \in [0,1]^n$, $\bar{\mathbf{s}} \in [9,11]^n$

n	$\varphi(t)=t$			$\varphi(t)=\sqrt{t}$			$\varphi(t)=t^2-t+\sqrt{t}$		
	Nr. of Iter.	CPU (s)	max local κ	Nr. of Iter.	CPU (s)	max local κ	Nr. of Iter.	CPU (s)	max local κ
10	22.3	0.1739	682.0966	24.7	0.1901	684.2502	36.3	0.3449	682.0185
100	274.2	52.4393	$1.9735\cdot 10^{55}$	276.6	51.1744	$1.9805\cdot 10^{55}$	287.9	58.6141	$1.9732\cdot 10^{55}$
400	-	-	-	-	-	-	-	-	-

TABLE 7. Numerical results with $\theta = 0.999$ for $P_*(\kappa)$-LCPs with matrix given in (4.1) with $\bar{\mathbf{x}} \in [9,11]^n$, $\bar{\mathbf{s}} \in [0,1]^n$

In the following section some concluding remarks are presented.

n	$\varphi(t)=t$			$\varphi(t)=\sqrt{t}$			$\varphi(t)=t^2-t+\sqrt{t}$		
	Nr. of Iter.	CPU (s)	max local κ	Nr. of Iter.	CPU (s)	max local κ	Nr. of Iter.	CPU (s)	max local κ
10	6.0	0.0414	9.0557	12.4	0.0900	8.5673	39.5	0.3124	6.2562
100	8.0	1.5736	1742.3699	14.0	2.5820	1731.8622	42.0	8.4011	1724.6807
400	9.3	23.5005	133784.0036	16.0	39.2535	134356.1961	44.9	103.3628	132590.5281

Table 8. Numerical results with $\theta = 0.999$ for $P_*(\kappa)$-LCPs with matrix given in (4.1) with $\bar{\mathbf{x}} \in [0,100]^n$, $\bar{\mathbf{s}} \in [9900, 11000]^n$

n	$\varphi(t)=t$			$\varphi(t)=\sqrt{t}$			$\varphi(t)=t^2-t+\sqrt{t}$		
	Nr. of Iter.	CPU (s)	max local κ	Nr. of Iter.	CPU (s)	max local κ	Nr. of Iter.	CPU (s)	max local κ
10	15.0	0.1106	108.8984	22.2	0.1655	109.9658	46.3	0.3665	109.1237
100	569.0	109.1256	$1.2750\cdot 10^{56}$	577.6	119.9684	$1.2778\cdot 10^{56}$	600.9	129.3901	$1.2748\cdot 10^{56}$
400	-	-	-	-	-	-	-	-	-

Table 9. Numerical results with $\theta = 0.999$ for $P_*(\kappa)$-LCPs with matrix given in (4.1) with $\bar{\mathbf{x}} \in [9900, 11000]^n$, $\bar{\mathbf{s}} \in [0,100]^n$

5. Conclusions

In this paper we proposed new IPAs for solving $P_*(\kappa)$-LCPs. The novelty of the paper is that we introduced a new class of AET functions in order to determine the search directions. We proved that the IPAs using any member φ of this new class of AET functions have polynomial iteration complexity in the size of the problem, bit length of the integral data and in the parameter κ. We also provided numerical results that show the efficiency of the IPA in the special case when $\varphi(t) = t^2 - t + \sqrt{t}$. The new class of functions defined by conditions AET1-3 of Definition 2.4 differs from the existing classes. We showed that $\varphi(t) = t^2 - t + \sqrt{t}$ belonging to our new class is not member of the existing classes of AET functions. The kernel function corresponding to this AET function φ is neither self-regular, nor eligible kernel function, see Subsection 2.5. We also provided example for eligible and self-regular kernel function for which the corresponding AET function is not member of our new class, see Example 2.2. As further research it would worth analysing the system of differential inequalities given in AET1-3. Another interesting research topic would be to extend the presented IPAs in a similar way that Illés et al. did in [26, 27], for general LCPs. It would be good to collect general LCP test problems in order to make the algorithms testable in practice.

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