International Journal of Solids and Structures 234-235 (2022) 111276

Contents lists available at ScienceDirect

International Journal of Solids and Structures

journal homepage: www.elsevier.com/locate/ijsolstr

Mono-unstable polyhedra with point masses have at least 8 vertices

Sándor Bozóki [a,b,*], Gábor Domokos [c,d], Flórián Kovács [c,e], Krisztina Regős [c]

[a] *Research Group of Operations Research and Decision Systems, Research Laboratory on Engineering and Management Intelligence, Institute for Computer Science and Control (SZTAKI), Eötvös Loránd Research Network (ELKH), Kende utca 13-17, Budapest, 1111, Hungary*
[b] *Department of Operations Research and Actuarial Sciences, Corvinus University of Budapest, Fővám tér 8, Budapest, 1093, Hungary*
[c] *MTA-BME Morphodynamics Research Group, Budapest University of Technology and Economics, Műegyetem rakpart 1-3., Budapest, 1111, Hungary*
[d] *Department of Mechanics, Materials and Structures, Budapest University of Technology and Economics, Műegyetem rakpart 1-3., Budapest, 1111, Hungary*
[e] *Department of Structural Mechanics, Budapest University of Technology and Economics, Műegyetem rakpart 1-3., Budapest, 1111, Hungary*

ARTICLE INFO

MSC:
52B10
77C20
52A38

Keywords:
Polyhedron
Static equilibrium
Monostatic polyhedron
Polynomial inequalities
Convex optimization

ABSTRACT

The monostatic property of convex polyhedra (i.e., the property of having just one stable or unstable static equilibrium point) has been in the focus of research ever since Conway and Guy (1969) published the proof of the existence of the first such object, followed by the constructions of Bezdek (2011) and Reshetov (2014). These examples establish $F \leq 14, V \leq 18$ as the respective *upper bounds* for the minimal number of faces and vertices for a homogeneous mono-stable polyhedron. By proving that no mono-stable homogeneous tetrahedron existed, Conway and Guy (1969) established for the same problem the lower bounds for the number of faces and vertices as $F, V \geq 5$ and the same lower bounds were also established for the mono-unstable case (Domokos et al., 2020b). It is also clear that the $F, V \geq 5$ bounds also apply for convex, homogeneous point sets with unit masses at each point (also called polyhedral 0-skeletons) and they are also valid for mono-monostatic polyhedra with exactly one stable and one unstable equilibrium point (both homogeneous and 0-skeletons). In this paper we draw on an unexpected source to extend the knowledge on mono-monostatic solids: we present an algorithm by which we improve the lower bound to $V \geq 8$ vertices on mono-*unstable* 0-skeletons. The problem is transformed into the (un)solvability of systems of polynomial inequalities, which is shown by convex optimization. Our algorithm appears to be less well suited to compute the lower bounds for mono-stability. We point out these difficulties in connection with the work of Dawson, Finbow and Mak (Dawson, 1985, Dawson et al., 1998, Dawson and Finbow, 2001) who explored the monostatic property of simplices in higher dimensions.

1. Introduction

1.1. Key definitions

We start by giving an informal definition of the most fundamental concepts. A rigid, convex body, supported on a horizontal surface under gravity, can be balanced in various positions. We refer to these positions as *static equilibria,* or, briefly, *equilibria*. Below we describe the generic version of these concepts in simple geometric terms.

We characterize the boundary of the rigid body by the scalar distance r, measured from the center of mass. Equilibria correspond to stationary points of r and in the generic case there exist three types of equilibria: stable equilibria correspond to minima, unstable equilibria to maxima, and saddle-type equilibria to saddle points. Later we will provide more formal definitions (Definitions 3 and 4) of unstable and stable equilibria on convex polyhedra. We also remark that our genericity condition is equivalent to requiring that r should be a Morse-function in two variables (Milnor, 1963).

We denote the respective numbers of stable, unstable and saddle-type equilibria by S, U and H and note that these numbers satisfy, based on the Poincaré–Hopf Theorem, $S + U - H = 2$. We say that the pair (S, U) defines the *equilibrium class* of the rigid body. We note that in the case of $S = U = 1$, the Poincaré–Hopf formula provides $H = 0$; however, we have $S, U > 0$ for all compact bodies. The latter condition follows from the fact that extrema of the potential energy function coincide with extrema of r. We are particularly interested in the case where either S, U or both have the minimal value of 1:

Definition 1. If $S = 1$ $(U = 1)$, then we call the body *mono-stable (mono-unstable)*, respectively.

* Corresponding author.
E-mail addresses: bozoki.sandor@sztaki.hu (S. Bozóki), domokos@iit.bme.hu (G. Domokos), kovacs.florian@emk.bme.hu (F. Kovács), regos.kriszti@gmail.com (K. Regős).

https://doi.org/10.1016/j.ijsolstr.2021.111276
Received 18 August 2021; Received in revised form 22 September 2021; Accepted 24 September 2021
Available online 7 October 2021

Definition 2. A rigid body is called *monostatic* if it is either mono-stable or mono-unstable, and it is called *mono-monostatic* if it is both mono-stable and mono-unstable.

We note that this terminology is different from the nomenclature used by Dawson (Dawson, 1985; Dawson et al., 1998; Dawson and Finbow, 2001) where the term *monostatic* is used to describe bodies which we call in the current paper *mono-stable*.

The minimal number of positions where a rigid body could be at rest have been investigated at least since Archimedes developed his famous design for ships (Nowacki, 2002). Having one stable position is of obvious advantage for ships; however, for rigid bodies under gravity, supported on a rigid surface this property, facilitating self-righting, could also be of advantage. In the current paper we investigate the monostatic property of polyhedra. Beyond having intriguing, purely mathematical aspects, polyhedra appear to have inspired a broad range of research at the interface of mechanics and geometry, which we briefly review next.

1.2. Polyhedra and mono-stability

Polyhedra have been receiving special attention in various fields of mechanics since the work of Cauchy (1813) on rigidity of polyhedral graphs. The development of adequate computational methods had a remarkable impact on the research of rigidity and mobility of bar-and-joint assemblies: the century-old conditions of Maxwell (1864) have been enhanced by symmetry-adapted approaches based on group theory (Guest and Fowler, 2005).

The rigidity and mobility analysis of highly symmetric polyhedral objects has not only been motivated by advanced methods (see the bifurcation analysis of motion by Kumar and Pellegrino (2000) or a tool for detecting finite mechanisms by symmetry arguments by Kangwai and Guest (1999)) but also by some discoveries in nanobiology. Virus capsids exhibiting a concerted rotational motion of polyhedral symmetry inspired a research on expandable polyhedra (Kovács et al., 2004). Regardless of symmetries, polyhedra often appear to be the best intuitive level of abstraction when modeling natural shapes. For example, the geometry of fragmentation has been described by using the theory of polyhedral convex mosaics (Domokos et al., 2020a).

The geometry associated with monostatic polyhedra appears to be a particularly intriguing subject: it is still unclear what are the minimal numbers F^S, V^S of faces and vertices (defining the combinatorial class (F, V)) for a convex, mono-stable polyhedron with homogeneous mass distribution. The first such object with $F = 19$ faces and $V = 34$ vertices (illustrated in Fig. 1) has been described by Conway and Guy in 1969. This construction was improved by Bezdek (2011) to $(F, V) = (18, 18)$ (see Fig. 1) and later by Reshetov (2014) to $(F, V) = (14, 24)$. These values define the best known *upper bounds*, so we have $F^S \leq 14, V^S \leq 18$. Even less is known about the lower bounds: the only result is due to Conway and Guy (1969) who proved that a homogeneous tetrahedron has at least two stable equilibria, so we have $F^S, V^S \geq 5$. The gaps between the upper and lower bounds (illustrated in Fig. 2) are substantial.

1.3. Generalizations

The above problem has many generalizations of which we list a few below and we also briefly summarize the related results:

1. Instead of looking at homogeneous polyhedra, one may look at *h-skeletons* which are polyhedra with mass uniformly distributed on their h-dimensional faces (see Fig. 3). If $h = 3$ then we have the original problem (homogeneous polyhedron). The importance of 2-skeletons as in Fig. 3(c) is shown by the work of Chen et al. (2018) on transformable polyhedra. The applications of 1-skeletons (sometimes called skeletal polyhedra) as in Fig. 3(b) include engineering and architectural structures (Chen et al., 2015; Luo et al., 2008). Leonardo da Vinci's illustration of twenty seven 1-skeletons appears in the book *Divina proportione* (Pacioli, 1509). A 0-skeleton is a homogeneous point set, with unit mass at each point (i.e., a polyhedron with unit masses at the vertices). We will denote the minimal face and vertex numbers for mono-stable h-skeletons by F_h^S, V_h^S, respectively. In the case of simplices, the center of mass of a homogeneous body and a 0-skeleton coincide as pointed out by Dawson and Finbow (2001) and also noted by Krantz et al. (2006), so Conway's result for the tetrahedron implies that for 0-skeletons we also have $F_0^S, V_0^S \geq 5$ and upper bounds are not known. For simplicity and consistency of notation, in the homogeneous case we will drop the lower index and use F^S, S^S, as before.
2. Instead of looking at mono-stable polyhedra one may look at mono-unstable polyhedra. For the homogeneous case, such polyhedra were first exhibited in Domokos et al. (2020b) with an example at $(F, V) = (18, 18)$. In the same paper, Conway's result for the nonexistence of a mono-stable tetrahedron was extended to the nonexistence of a mono-unstable tetrahedron, so for the minimal numbers F^U, V^U for the faces and vertices of a mono-unstable polyhedron we have $5 \leq F^U, V^U \leq 18$ and for 0-skeletons we have $5 \leq F_0^U, V_0^U$.
3. While our paper remains restricted to 3D problems, our algorithm is partially motivated by results for polytopes in higher dimensions. Dawson, Finbow and Mak (Dawson, 1985; Dawson et al., 1998; Dawson and Finbow, 2001) published a series of papers where they proved that in $d \geq 10$ dimensions mono-stable simplices exist, thus they proved for $d \geq 10$ that $F^S = F_0^S = d+1$ and they also proved that for $d < 9$ we have $F^S, F_0^S \geq d+2$
4. Instead of looking at mono-stable or mono-unstable objects one may look at homogenous mono-monostatic objects. Among piecewise smooth convex bodies the first such object, called Gömböc, was identified in 2007 (Várkonyi and Domokos). Little is known about the minimal number of faces and vertices for a polyhedron in this class. These lower bounds we will denote by F^*, V^*. Obviously, for homogeneous material distribution we have

$$F^* \geq F^S, F^U, \quad V^* \geq V^S, V^U, \tag{1}$$

however, despite a related prize (Domokos et al., 2020b), there are no other bounds known for either F^* or V^*.

1.4. Main result, strategy of the proof and outline of the paper

Our goal is to prove the statement claimed in the title of the paper:

Theorem 1. $V_0^U \geq 8$.

Using (1), it is immediately clear that from Theorem 1 we have

Lemma 1. $V_0^* \geq 8$

which also implies, via the Theorem of Steinitz (1906), $F_0^* \geq 6$. We will prove Theorem 1 in four steps:

1. In Section 2 we prove Theorem 2 which connects the vertex geometry (i.e., the collection of the V vectors pointing from the center of mass to the vertices) of a polytope, to the number U of its unstable equilibria. (We will also introduce the dual statement for faces and stable equilibria but we will not use the latter in the current paper for computations.)
2. In Section 3 we show that if $U = 1$ then Theorem 2 can be converted into a system of quadratic inequalities and we construct the general system for $d = 3$-dimensional 0-skeletons. The Theorem also admits to develop such systems for other cases, such as homogeneous polyhedra, or polytopes in other dimensions; however, in the current paper we only investigate 3 dimensional 0-skeletons.

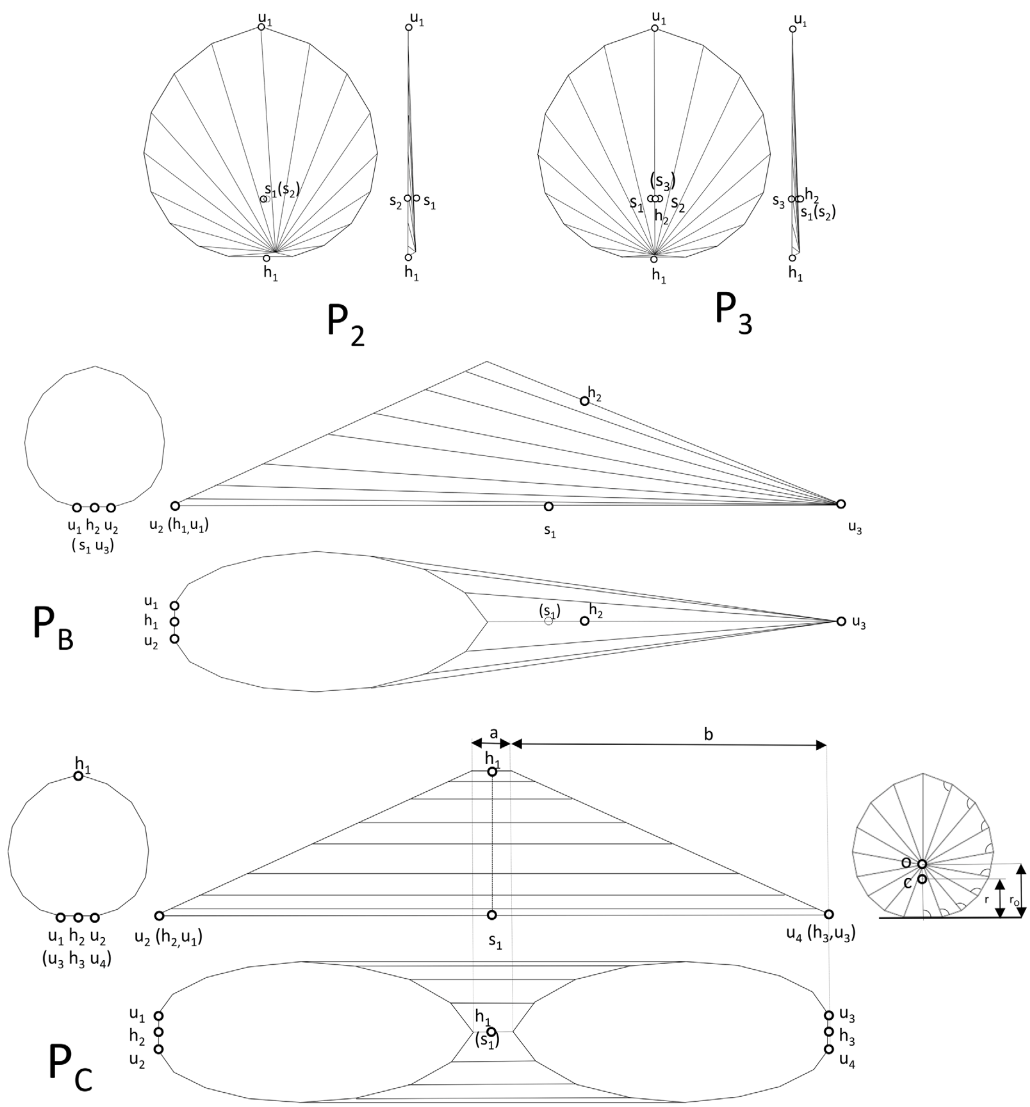

Fig. 1. Illustration of some monostatic polyhedra: Mono-unstable polyhedra P_2, P_3 (Domokos et al., 2020b) in classes $(F,V) = (18,18)$, $(S,U) = (1,2),(1,3)$, Bezdek's polyhedron (Bezdek, 2011), P_B in classes $(F,V) = (18,18),(S,U) = (1,3)$ and Conway's polyhedron (Conway and Guy, 1969) P_C in classes $(F,V) = (19,34),(S,U) = (1,4)$.

3. In Section 4 we develop and solve (i.e., prove the unsolvability of) this system for the tetrahedron, serving as an illustrative example of our method. Here we prove that $V^S \geq 5$ in a manner which is independent of the original proof of Conway.
4. In Section 5 we develop an optimization algorithm which can prove the unsolvability of the system of quadratic inequalities constructed in Section 3 by doing exact (rational) arithmetic and by using this code we prove Theorem 1.

The same algorithm could, in principle, also *solve* such a system (instead of proving its unsolvability). If the solution for $V = \bar{V}$ was convex and all systems with $V < \bar{V}$ were proven to be unsolvable then, in this case the algorithm would provide the *exact* value as $V_0^U = \bar{V}$. Since we did not find such a solution, Theorem 1 provides a lower bound for V_0^U. Analogous computations for other mass distributions (e.g., the homogeneous case) and other dimensions are, in principle, possible but have not been performed in this paper. Section 7 discusses why the dual case (i.e., the algorithm to compute F^S) is not pursued in the current paper and we establish the link between our formulae and those of Dawson (1985). We also draw some conclusions.

2. The geometric construction

2.1. Problem statement

Let P be a d-dimensional convex polytope with F faces $(f_i, i = 1,2,\ldots,F)$ and V vertices $(v_i, i = 1,2,\ldots,V)$.

We identify the faces by the *face vectors* $\mathbf{q}_i$ orthogonal to face f_i with $|\mathbf{q}_i|$ being equal to the orthogonal distance of the face f_i from the center of mass o. We also note that this definition of face vector differs from that given by Dawson (1985), where the magnitude of face vectors is proportional to the area of the given face. In particular, for a simplex, Dawson's vectors have ratios reciprocal to those here. We will introduce Dawson's face vectors in Eq. (30) in Section 6 and discuss their relationship to our construction.

The vertices are identified by *vertex vectors* $\mathbf{r}_i$, $(i = 1,2,\ldots,V)$ with origin at the center of mass o and we assume P to be generic in the sense that if $i \neq j$ then $|\mathbf{r}_i| \neq |\mathbf{r}_j|$.

Definition 3. We call Z_i the (ith) *vertex-orthoplane* of P if Z_i contains the ith vertex p_i and it is orthogonal to the line $[o, p_i]$. We say that p_i is an (unstable) equilibrium point of P if the vertex-orthoplane Z_i intersects P only in the vertex p_i. More precisely: we speak about a

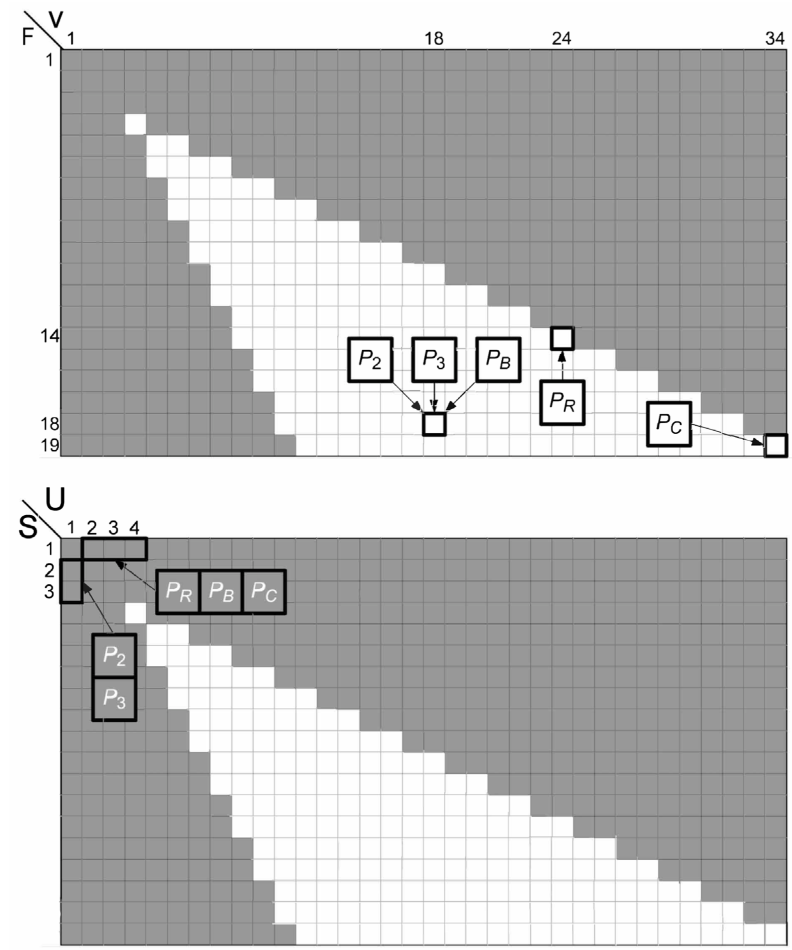

Fig. 2. Location of some monostatic polyhedra (including the ones shown in Fig. 1) on the $[F, V]$ (face and vertex number) grid and $[S, U]$ (number of stable and unstable equilibria) grid: Conway's polyhedron (Conway and Guy, 1969) P_C in classes $(F, V) = (19, 34), (S, U) = (1, 4)$, Bezdek's polyhedron (Bezdek, 2011), P_B in classes $(F, V) = (18, 18), (S, U) = (1, 3)$, Reshetov's polyhedron (Reshetov, 2014), P_R in classes $(F, V) = (14, 24), (S, U) = (1, 2)$. Mono-unstable polyhedra P_2, P_3 in classes $(F, V) = (18, 18)$, $(S, U) = (1, 2), (1, 3)$. White squares correspond on the $[F, V]$ grid to combinatorial classes where, according to Steinitz (1906, 1922) we find polyhedra, on the $[S, U]$ grid the same squares correspond to equilibrium classes where we find polyhedra with $S = F, U = V$ (Domokos et al., 2020b).

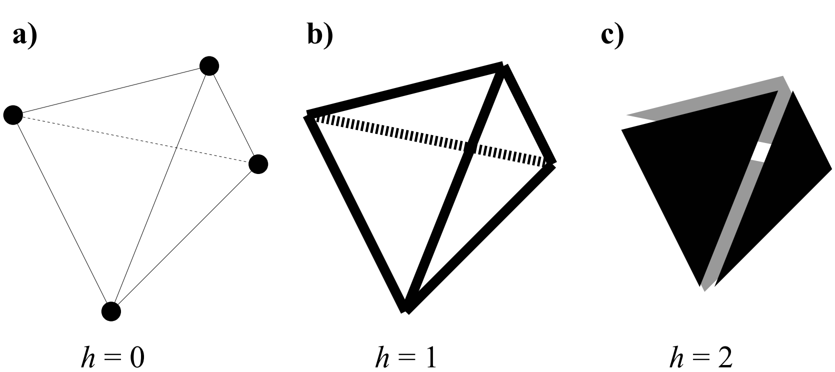

Fig. 3. Illustration of homogeneous material distributions for polyhedra: (a) 0-skeleton, (b) 1-skeleton, (c) 2-skeleton. The case $h = 3$, i.e., uniform mass distribution over the entire volume is not shown for simple graphical reasons.

nondegenerate equilibrium point p_i if $Z_i \cap P = p_i$, and the equilibrium is degenerate if Z_i also contains at least one edge of P. The term 'unstable equilibrium' refers to a nondegenerate one henceforth unless stated otherwise. We denote the number of unstable equilibrium points of P by $U(P)$.

We remark that we have $U(P) \leq V$, since unstable equilibria can be carried only by a vertex. Our main result connects the vertex geometry (i.e., the set of position vectors $\mathbf{r}_i$) of P with the number $U(P)$ of its unstable equilibria via the following formula:

Theorem 2. *Let P be a convex polytope. Then*

$$U(P) = \sum_{i=1}^{V} \left\lfloor \frac{1}{2} + \frac{\sum_{j=1}^{V} \operatorname{sign}\left((\mathbf{r}_i - \mathbf{r}_j)\mathbf{r}_i\right)}{2(V-1)} \right\rfloor . \tag{2}$$

We also phrase an analogous, though more restricted result for stable equilibria.

Definition 4. We say that face f_i carries a (nondegenerate) stable equilibrium point of P if the orthogonal projection o_i of the center of mass o onto the plane of f_i is contained in the interior of f_i. The equilibrium is degenerate if $o_i \in \mathrm{bd}(f_i)$ but the term 'stable equilibrium' refers to a nondegenerate one henceforth unless stated otherwise. We denote the number of stable equilibrium points of P by $S(P)$ and we have $S(P) \leq F$. We call W_i the (ith) *face-orthoplane* of P if W_i contains the orthogonal projection o_i of the center of mass onto the face f_i and it is orthogonal to the line $[o, o_i]$.

Theorem 3. *Let P be a convex polytope. Then*

$$S(P) = \sum_{i=1}^{F} \left[\frac{1}{2} + \frac{\sum_{j=1}^{F} \mathrm{sign}\left((\mathbf{q}_j - \mathbf{q}_i)\mathbf{q}_j \right)}{2(F-1)} \right]. \tag{3}$$

Next we prove Theorems 2 and 3. In Section 6 we will show that in the case of homogeneous simplices the statement of Theorem 3 follows from a result of Dawson (1985).

2.2. Proof of Theorem 2

Let p_i and p_j denote two vertices with respective position vectors $\mathbf{r}_i, \mathbf{r}_j$ and let o denote the center of mass of the polytope P. Below we define a special, asymmetric relationship between these vertices:

Definition 5. We say that vertex p_i is in the *shadow* of vertex p_j if and only if

$$(\mathbf{r}_i - \mathbf{r}_j)\mathbf{r}_i < 0. \tag{4}$$

Remark 1. Condition (4) implies that if p_i is in the shadow of p_j then the vertex-orthoplane Z_i will intersect P not only at the vertex p_i, so p_i cannot be an equilibrium point.

Lemma 2. *If vertex p_j shadows vertex p_i, then*

$$|\mathbf{r}_i| < |\mathbf{r}_j|. \tag{5}$$

Proof. By Definition 5, $\mathbf{r}_i^2 < \mathbf{r}_i\mathbf{r}_j$, but $0 < (\mathbf{r}_i - \mathbf{r}_j)^2 = \mathbf{r}_i^2 - 2\mathbf{r}_i\mathbf{r}_j + \mathbf{r}_j^2 < -\mathbf{r}_i^2 + \mathbf{r}_j^2$.

Lemma 3. *P has an unstable equilibrium point at the vertex p_i if and only if the p_i is not in the shadow of any other vertex p_j ($j \neq i$).*

Proof. The vertex-orthoplane Z_i defines two half-spaces. If the vertex p_i is not in the shadow of any other vertex then the center of mass o and all other vertices will be contained in the same half-space. This implies that the vertex-orthoplane Z_i does not intersect P except at p_i, so p_i is an unstable equilibrium point of P.

Remark 2. There are three possible shadowing relationships between two vertices p_i and p_j:

1. Vertex p_i is shadowing vertex p_j.
2. Vertex p_j is shadowing vertex p_i.
3. Vertex p_i and vertex p_j are not shadowing each other.

Definition 6. The elements $s_{i,j}$ of the $V \times V$ *vertex shadowing matrix* $\mathbf{S}(P)$ are defined as follows: If $i \neq j$ then $s_{i,j} = -1$ if p_j is shadowing p_i and $s_{i,j} = 1$ if p_j is not shadowing p_i. If $i = j$ then $s_{i,i} = 0$. Based on Definition 5 we have

$$s_{i,j} = \mathrm{sign}\left((\mathbf{r}_i - \mathbf{r}_j)\mathbf{r}_i \right). \tag{6}$$

Now we conclude the proof of Theorem 2.

Proof. Based on Definition 6, for the absolute value of the row-sums of the shadowing matrix $\mathbf{S}(P)$ we can write

$$\left| \sum_{j=1}^{V} s_{i,j} \right| \leq V - 1 \tag{7}$$

with equality if and only if p_i is an unstable equilibrium point of P. According to the formula (2) of Theorem 2 in this case we add one to the value of $U(P)$.

2.3. Proof of Theorem 3

First we define *shadowing* for stable equilibria analogously to Definition 5:

Definition 7. We say that face f_i is in the *shadow* of face f_j if and only if

$$(\mathbf{q}_j - \mathbf{q}_i)\mathbf{q}_j < 0. \tag{8}$$

Remark 3. Condition (8) implies that the ith face-orthoplane W_i will intersect P, so o_i cannot be an equilibrium point.

Lemma 4. *If face f_j shadows face f_i, then*

$$|\mathbf{q}_j| < |\mathbf{q}_i|. \tag{9}$$

Proof. By Definition 7, $\mathbf{q}_j^2 < \mathbf{q}_i\mathbf{q}_j$, but $0 < (\mathbf{q}_j - \mathbf{q}_i)^2 = \mathbf{q}_j^2 - 2\mathbf{q}_j\mathbf{q}_i + \mathbf{q}_i^2 < -\mathbf{q}_j^2 + \mathbf{q}_i^2$.

Lemma 5. *P has a stable equilibrium point on the face f_i if and only if the face f_i is not in the shadow of any other face f_j ($j \neq i$).*

Proof. If f_i is not in the shadow of any other face then P will *not* tip to any other face if positioned on f_i, so f_i carries a stable equilibrium point.

Definition 8. The elements $\bar{s}_{i,j}$ of the $F \times F$ *face shadowing matrix* $\bar{\mathbf{S}}(P)$ are defined as follows: If $i \neq j$ then $\bar{s}_{i,j} = -1$ if f_j is shadowing f_i and $\bar{s}_{i,j} = 1$ if f_j is not shadowing f_i. If $i = j$ then $\bar{s}_{i,i} = 0$. Based on Definition 7 we have

$$\bar{s}_{i,j} = \mathrm{sign}\left((\mathbf{q}_j - \mathbf{q}_i)\mathbf{q}_j \right). \tag{10}$$

Now we conclude the proof of Theorem 3.

Proof. Based on Definition 8, for the absolute value of the row-sums of the shadowing matrix $\bar{\mathbf{S}}(P)$ we can write

$$\left| \sum_{j=1}^{F} s_{i,j} \right| \leq F - 1 \tag{11}$$

with equality if and only if f_i is a stable equilibrium point of P. According to the formula (3) of Theorem 3 in this case we add one to the value of $S(P)$.

3. The Algorithm

3.1. Problem statement

Formula (2) of Theorem 2 is, by default, capable to compute the shadowing matrix $\mathbf{S}(P)$ of a given polytope P, based on position vectors $\mathbf{r}_i$, ($i = 1, \ldots, V$).

However, one can also use this formula *to find* monostatic polytopes with $U(P) = 1$ unstable equilibrium point.

3.2. Notation and definition of equations

Definition 9. Let us denote the vertex at largest distance from the center of mass by p_1.

Lemma 6. *Vertex p_1 always carries an unstable equilibrium.*

Proof. All vertices p_i with $i > 1$ are on the same side of the vertex-orthoplane at p_1, so none of the vertices p_i, $i > 1$ may shadow p_1. Via Definition 5 this provides the proof.

Lemma 7. *Let P be a mono-unstable polytope ($U(P) = 1$). Then, for any $1 < i \le V$ there exists at least one $0 < j \le V$ such that $s_{i,j} = -1$.*

Proof. We prove Lemma 7 by contradiction. If the statement of the lemma is false then we have for some $i^* > 1$ a row of the shadowing matrix with positive entries (except for the main diagonal). This means that vertex p_{i^*} is not shadowed by any other vertex, i.e., it is an equilibrium (in addition to the equilibrium located at p_1). However, having $U(P) > 1$ equilibria contradicts the statement of the lemma.

3.3. Material distribution

Since polytopes have multi-level geometric structure, material homogeneity may be interpreted in various ways. In Section 1.3 we introduced h-skeletons where material is uniform on the h-faces of the polyhedron. Fig. 3(a)–(c) illustrate h-skeletons for 3-dimensional polyhedra. While Theorems 2 and 3 are valid irrespective of material distribution, if we want to utilize these results to find monostatic polytopes (as mentioned in Section 3.1), the difficulty of the resulting algebraic system will strongly depend on the material distribution. In this paper we concentrate on 0-skeletons which yield the simplest balance equations. We also mention that computing for the $h = 3$ homogeneous distribution in $d = 3$ dimensions is identical to the problem of 0-skeletons if we only investigate simplices; however, the equilibria of that class of polyhedra are well understood. The dual case, corresponding to Theorem 3 is more challenging: none of the $h = i$ $(i = 0, 1, \dots, d)$ homogeneous skeletons yields convenient algebraic equations. Nevertheless, as we will point out in Section 6, one may define a somewhat artificial distribution for which computations appear to be straightforward.

3.4. Control parameters

The problem is defined by 3 integer parameters: d is the dimension of the polytope, $0 \le h \le d$ defines the material distribution and $V > d$ provides the number of vertices. For brevity, we will characterize the problem with the triplet (d, h, V). If V is not specified, then the triplet will refer to the problem of finding the smallest value of V for which a monostatic polyhedron exists.

Now we regard the following set of equations:

$$s_{i,j} = -1, \quad i = 2, 3, \dots, V, \quad j \in \{1, 2, \dots, i-1\} \tag{12}$$

(note the condition $i < j$ that will be justified below) and proceed by

Definition 10. The (d, h, V)*-expansion* of (12) is constructed in the following manner:

1. We construct *systems*, each consisting of $V - 1$ inequalities with index i running from 2 to V, defining $(V - 1)!$ systems by using admissible permutations of j.
2. Each system is supplemented by d moment equations guaranteeing that the center of mass o of the h-homogeneous polytope is at the origin.

So the (d, h, V)-expansion of (12) in d dimensions will consist of $(V-1)!$ systems, each having $V - 1$ inequalities and d equations to which, for brevity, we will henceforth refer as a *system of inequalities* or *system*.

3.5. Possible scenarios

Solving the $(V - 1)!$ systems of $V - 1$ inequalities and d equations for the Vd scalar coordinates $r_{i,k}$, $(i = 1, 2, \dots, V,\ k = 1, 2, \dots, d)$ of the position vectors $\mathbf{r}_i$ may yield two types of result:

1. If *none* of the $(V - 1)!$ systems has a solution, that implies the existence of a row for $i > 1$ in the shadowing matrix with *only positive entries*. This is the necessary and sufficient condition for P *not* to be monostatic, so it proves the nonexistence of d-dimensional monostatic polytopes P with V vertices.
2. If *any* of the $(V-1)!$ systems has a solution, that implies that for $i > 1$, in *each row* of the shadowing matrix there is a negative element. According to Lemma 7, this is the necessary and sufficient condition for P to be monostatic. However, we remark that our algorithm does not include the condition that the solution should be convex, so this has to be checked additionally.

Remark 4. The algorithm is only testing the lower triangle of the shadowing matrix, characterized by $i > j$. It is easy to see that this is sufficient: If the (d, h, V) problem has a solution, this means that there exists a monostatic, d-dimensional, h-homogeneous polytope with V vertices. Based on Lemma 7, the shadowing matrix $\mathbf{S}(P)$ of a monostatic polytope P has the property that for $i > 1$, in each row there will be at least one negative element $s_{i,j} = -1$. If all vertices $p_i\,(i = 2, \dots, V)$ are labeled such that $|\mathbf{r}_1| \ge |\mathbf{r}_2| \ge \cdots \ge |\mathbf{r}_V|$, no negative entries in the upper triangle of $\mathbf{S}(P)$ can occur by Lemma 2. In other words, p_1 always carries an unstable equilibrium, p_2 is shadowed by p_1, p_3 is shadowed either by p_1 or p_2, etc., which finally yields $(V-1)!$ possible shadowing systems.

3.6. Initial conditions

In d dimensions a rigid body has d translational degrees of freedom and $\binom{d}{2}$ rotational degrees of freedom. By fixing the origin at $\mathbf{r} = 0$ the translations are eliminated. In addition, we fix $n = \binom{d}{2} + 1$ coordinates to eliminate the rotations and also to specify one element of a family of similar polytopes.

In $d = 2$ dimensions we have $n = 2$, so, in addition to fixing the origin, we fix the coordinates of one point. In $d = 3$ dimensions we have $n = 4$, so, in addition to fixing the origin, we fix all 3 coordinates of the first point and one additional coordinate of the second point. In $d = 4$ dimensions we have $n = 7$, so, in addition to fixing the origin, we fix all 4 coordinates of the first point and 3 coordinates of the second point.

3.7. General system of inequalities for 0-skeletons in 3 dimensions, i.e., the $(d, h, V) = (3, 0, V)$ problem

Eq. (12) is equivalent to a system of inequalities which can be written as

$$\sum_{k=1}^{d} r_{i,k}^2 \le \sum_{k=1}^{d} r_{i,k} r_{j,k}, \quad (i = 2, \dots, V; j = 1, \dots, i-1) \tag{13}$$

expressing that the ith point is shadowed by the jth point.

If we consider the $h = 0$ problem then the balance equations for the center of mass can be written as:

$$\sum_{i=1}^{V} r_{i,k} = 0, \quad k = 1, 2, \dots, d. \tag{14}$$

We remark that in the case of a simplex, the balance equations for the $h = 0$ and $h = d$ cases coincide. Using the considerations of Section 3.6, we fix $n = \binom{3}{2} + 1 = 4$ coordinates:

$$\begin{aligned} r_{1,1} &= 1 \\ r_{1,2} &= 0 \\ r_{1,3} &= 0 \\ r_{2,3} &= 0, \end{aligned} \tag{15}$$

which, by Definition 9 implies

$$|r_{i,k}| < 1, \quad i > 1, \quad k = 1, 2, \ldots, d. \tag{16}$$

Henceforth, it is our goal to prove the unsolvability the system (13)–(15) for various values of V.

4. Tetrahedron: the $(d, h, V) = (3, 3, 4)$ and $(d, h, V) = (3, 0, 4)$ problems

Lemma 8. *There exists no mono-unstable tetrahedron.*

Proof.

Remark 5. The statement of the lemma is equivalent to the claim that the $(3, 0, 4)$-expansion of (13) has no solution. We will prove the latter. We also mention that the $(d, H, V) = (3, 0, 4)$ problem is illustrated in Fig. 3(a).

Next, using Definition 10 we construct the $(3, 0, 4)$-expansion of (13) in two steps.

In the first step, after substituting the initial conditions (15) we obtain the following $V - 1 = 3$ groups of inequalities:

$$r_{2,1}^2 + r_{2,2}^2 \le r_{2,1} \tag{17}$$

$$\left\{ \begin{array}{lcl} r_{3,1}^2 + r_{3,2}^2 + r_{3,3}^2 & \le & r_{3,1} \\ r_{3,1}^2 + r_{3,2}^2 + r_{3,3}^2 & \le & r_{2,1} r_{3,1} + r_{2,2} r_{3,2} \end{array} \right\} \tag{18}$$

$$\left\{ \begin{array}{lcl} r_{4,1}^2 + r_{4,2}^2 + r_{4,3}^2 & \le & r_{4,1} \\ r_{4,1}^2 + r_{4,2}^2 + r_{4,3}^2 & \le & r_{2,1} r_{4,1} + r_{2,2} r_{4,2} \\ r_{4,1}^2 + r_{4,2}^2 + r_{4,3}^2 & \le & r_{3,1} r_{4,1} + r_{3,2} r_{4,2} + r_{3,3} r_{4,3} \end{array} \right\} \tag{19}$$

In the second step, we have $1 \times 2 \times 3 = 6$ possibilities to pick one inequality each from the three groups (17)–(19) and complementing these with the $d = 3$ substitution of (14) yields 6 systems, each consisting of 3 equations and 3 inequalities. Here the moment balance equations for the $h = 0$ and $h = 3$ problem coincide since the tetrahedron is a simplex, so the above system describes both problems. Here follows the expanded system:

$$\left\{ \begin{array}{llcl} (a) & r_{2,1}^2 + r_{2,2}^2 & \le & r_{2,1} \\ (b) & r_{3,1}^2 + r_{3,2}^2 + r_{3,3}^2 & \le & r_{3,1} \\ (c) & r_{4,1}^2 + r_{4,2}^2 + r_{4,3}^2 & \le & r_{4,1} \\ (d) & 1 + r_{2,1} + r_{3,1} + r_{4,1} & = & 0 \\ (e) & r_{2,2} + r_{3,2} + r_{4,2} & = & 0 \\ (f) & r_{3,3} + r_{4,3} & = & 0 \end{array} \right\} \tag{20}$$

$$\left\{ \begin{array}{llcl} (a) & r_{2,1}^2 + r_{2,2}^2 & \le & r_{2,1} \\ (b) & r_{3,1}^2 + r_{3,2}^2 + r_{3,3}^2 & \le & r_{3,1} \\ (c) & r_{4,1}^2 + r_{4,2}^2 + r_{4,3}^2 & \le & r_{2,1} r_{4,1} + r_{2,2} r_{4,2} \\ (d) & 1 + r_{2,1} + r_{3,1} + r_{4,1} & = & 0 \\ (e) & r_{2,2} + r_{3,2} + r_{4,2} & = & 0 \\ (f) & r_{3,3} + r_{4,3} & = & 0 \end{array} \right\} \tag{21}$$

$$\left\{ \begin{array}{llcl} (a) & r_{2,1}^2 + r_{2,2}^2 & \le & r_{2,1} \\ (b) & r_{3,1}^2 + r_{3,2}^2 + r_{3,3}^2 & \le & r_{3,1} \\ (c) & r_{4,1}^2 + r_{4,2}^2 + r_{4,3}^2 & \le & r_{3,1} r_{4,1} + r_{3,2} r_{4,2} + r_{3,3} r_{4,3} \\ (d) & 1 + r_{2,1} + r_{3,1} + r_{4,1} & = & 0 \\ (e) & r_{2,2} + r_{3,2} + r_{4,2} & = & 0 \\ (f) & r_{3,3} + r_{4,3} & = & 0 \end{array} \right\} \tag{22}$$

$$\left\{ \begin{array}{llcl} (a) & r_{2,1}^2 + r_{2,2}^2 & \le & r_{2,1} \\ (b) & r_{3,1}^2 + r_{3,2}^2 + r_{3,3}^2 & \le & r_{2,1} r_{3,1} + r_{2,2} r_{3,2} \\ (c) & r_{4,1}^2 + r_{4,2}^2 + r_{4,3}^2 & \le & r_{4,1} \\ (d) & 1 + r_{2,1} + r_{3,1} + r_{4,1} & = & 0 \\ (e) & r_{2,2} + r_{3,2} + r_{4,2} & = & 0 \\ (f) & r_{3,3} + r_{4,3} & = & 0 \end{array} \right\} \tag{23}$$

$$\left\{ \begin{array}{llcl} (a) & r_{2,1}^2 + r_{2,2}^2 & \le & r_{2,1} \\ (b) & r_{3,1}^2 + r_{3,2}^2 + r_{3,3}^2 & \le & r_{2,1} r_{3,1} + r_{2,2} r_{3,2} \\ (c) & r_{4,1}^2 + r_{4,2}^2 + r_{4,3}^2 & \le & r_{2,1} r_{4,1} + r_{2,2} r_{4,2} \\ (d) & 1 + r_{2,1} + r_{3,1} + r_{4,1} & = & 0 \\ (e) & r_{2,2} + r_{3,2} + r_{4,2} & = & 0 \\ (f) & r_{3,3} + r_{4,3} & = & 0 \end{array} \right\} \tag{24}$$

$$\left\{ \begin{array}{llcl} (a) & r_{2,1}^2 + r_{2,2}^2 & \le & r_{2,1} \\ (b) & r_{3,1}^2 + r_{3,2}^2 + r_{3,3}^2 & \le & r_{2,1} r_{3,1} + r_{2,2} r_{3,2} \\ (c) & r_{4,1}^2 + r_{4,2}^2 + r_{4,3}^2 & \le & r_{3,1} r_{4,1} + r_{3,2} r_{4,2} + r_{3,3} r_{4,3} \\ (d) & 1 + r_{2,1} + r_{3,1} + r_{4,1} & = & 0 \\ (e) & r_{2,2} + r_{3,2} + r_{4,2} & = & 0 \\ (f) & r_{3,3} + r_{4,3} & = & 0 \end{array} \right\} \tag{25}$$

Next we show that none of the systems (20)–(25) has a solution.

4.0.1. Systems (20)–(22)

From part (a) of either of (20)–(22) we have $r_{2,1} \ge 0$, and so (d) with (16) imply both $r_{3,1} \le 0, r_{4,1} \le 0$. This, however, contradicts (b), so systems (20)–(22) have no solution.

4.0.2. System (23)

The proof is essentially identical to the proof for systems (20)–(22) but the contradiction is found at (c) instead of (b).

4.0.3. System (24)

From (24)(a) we have $r_{2,1} \ge 0$, and this implies, via (24)(d) that $r_{3,1} < 0, r_{4,1} < 0$. Substituting this into (24)(b) and (c) we get $r_{2,2} r_{3,2} > 0, r_{2,2} r_{4,2} > 0$, respectively implying that $r_{2,2}, r_{3,2}$ and $r_{4,2}$ have the same sign. This, however, contradicts (24)(e), so system (24) has no solution.

4.0.4. System (25)

The proof is similar to the proof of system (24). From (25)(a) we have $r_{2,1} \ge 0$, and this implies, via (25)(d) that $r_{3,1} < 0, r_{4,1} < 0$. This yields, via (25)(b) $r_{2,2} r_{3,2} > 0$ and further, via (25)(e) $r_{3,2} r_{4,2} < 0$. (25)(f) yields $r_{3,3} r_{4,3} < 0$. Since we already showed that $r_{2,1} r_{4,1} < 0$, substituting into (25)(c) yields a contradiction, so system (25) has no solution.

5. Computing the lower bound for V_0^U: the $(d, h, V) = (3, 0, \{5, 6, 7\})$ problems

Our goal is to prove Theorem 1, to which we now give an equivalent, more detailed formulations as

Corollary 1. *The $(d, h, V) = (3, 0, \{5, 6, 7\})$ problems have no solution.*

We prove Corollary 1 (and thus also Theorem 1 and Lemma 1) by applying the algorithm laid out in Section 3.

Proof. We show that the system (13)–(14) has no solution for $d = 3, h = 0, V \le 7$. The case of $V = 4$ was proven by elementary considerations in Section 4. The case $V = 5$ can also be proven in a similar way; however, only 7 of the 24 systems have such short proofs, the others require more steps: the total length is 22 (hand written) pages. The reason of that it is not included in this paper is that we have found another, optimization-based method which proves the unsolvability for $V = 6$ and $V = 7$, too.

Departing from (14), $r_{V,k}$ can be expressed as

$$r_{V,k} = -\sum_{i=1}^{V-1} r_{i,k}, \quad k = 1,2,3. \tag{26}$$

The substitutions (26) convert the system of inequalities (13) and Eqs. (14) into $V-1$ inequalities:

$$\sum_{k=1}^{3} r_{i,k}^2 - \sum_{k=1}^{3} r_{i,k} r_{j,k} \le 0, \quad i = 2,\dots,V-1, \tag{27}$$

$$\sum_{k=1}^{3}\left(-\sum_{i=1}^{V-1} r_{i,k}\right)^2 - \sum_{k=1}^{3}\left(-\sum_{i=1}^{V-1} r_{i,k}\right) r_{j,k} \le 0, \tag{28}$$

where $j \in \{1,\dots,i-1\}$, resulting in $(V-1)!$ systems.

According to (15), $r_{1,1} = 1, r_{1,2}, r_{1,3}, r_{2,3} = 0$, and since $r_{V,1}$, $r_{V,2}$ and $r_{V,3}$ were eliminated in (26), the number of free variables $r_{i,k}$ is $3V-7$. Let us define the parametric function $f : \mathbb{R}^{3V-7} \to \mathbb{R}$ as the weighted sum of the left hand sides of the inequalities (27)–(28) above

$$f = \sum_{i=2}^{V-1} c_i \left(\sum_{k=1}^{3} r_{i,k}^2 - \sum_{k=1}^{3} r_{i,k} r_{j,k}\right) + c_V \left[\sum_{k=1}^{3}\left(-\sum_{i=1}^{V-1} r_{i,k}\right)^2 - \sum_{k=1}^{3}\left(-\sum_{i=1}^{V-1} r_{i,k}\right) r_{j,k}\right], \tag{29}$$

where the coefficients $c_i, i = 2,\dots,V$ are arbitrary positive numbers.

Let us fix the values $j \in \{1,\dots,i-1\}$ arbitrarily. If there exist positive coefficients $c_i, i = 2,\dots,V$ such that f is positive everywhere, then the unsolvability of system (27)–(28) follows: assume for contradiction that it has a solution, and substitute this solution in (29), yielding a non-positive value of f, which is not possible once f is positive.

How to find coefficients and how to check the positivity of f? Positive polynomials appeared in Hilbert's 17th problem (Marshall, 2008; Prestel and Delzell, 2001) and have many applications; however, the research on sufficiently general and computationally feasible algorithms includes a lot of unsolved problems (Lasserre, 2010; Henrion and Garulli, 2005).

However, in our case, surprisingly enough a simple randomized search for the coefficients works, at least up to $V = 7$ vertices. We benefit from that if f is convex and quadratic, then its minimum value can be calculated simply. The positivity of f can be proved by showing its convexity and finding a positive minimum. Since f is a multi-variate polynomial of degree two, the first-order conditions of minimality yield a system of linear equations. In order to check convexity, the second-order condition is the positive definiteness of Hf, the Hessian of f. Note that Hf depends on coefficients c_i only, following again from that f is quadratic.

The algorithm for proving unsolvability of the system of inequalities (27)–(28) is summarized below.

```
for all systems of inequalities
step 1. generate random positive integer values of c_2,...,c_n
step 2. if Hf is positive definite
     then if min f > 0,
         then print('this system is unsolvable')
             (go to the next system in the for cycle)
         else go to step 1
     else go to step 1
end for
```

Tables S1-S4 in the Supplementary material include the coefficients and the minimum value of function f for all systems written for $V = 4,5,6,7$ vertices, respectively. Since the minimum values are positive, all systems are unsolvable.

The computational approach above, implemented in Maple, does not include numerical errors because all calculations deal with integer and rational numbers, resulting in exact rational numbers, too.

Remark 6. In the case of $V = 8$ vertices, there are some systems of inequalities (e.g., system (27)–(28) written for the choice of $j = i-1$ for all $i = 2,\dots,V-1$), where we could not prove that f is positive with appropriate coefficients as all the random trials led to negative minima, and at the same time at least one inequality of the system (27)–(28) was violated — neither unsolvability of all of these systems, or solvability of at least one of these systems follows.

The lower bound for the number of minimal vertices, that a three-dimensional mono-unstable 0-skeleton must have, has now been improved to 8. Upper bounds may come when, for a given V, a system of polynomial inequalities (27)–(28) has a solution, and the corresponding polyhedron is convex (see Section 3.5).

6. The dual problem: search for mono-stable polyhedra

So far we demonstrated how Theorem 2 can be converted into an algorithm to establish the lower bound for the number of vertices of a mono-unstable polyhedron. To illustrate the algorithm we computed the case of 0-skeletons and found the lower bound $V_0^U \ge 8$ (Theorem 1).

Theorem 3 is the dual of Theorem 2, and it could, in principle, serve as the basis of a dual algorithm, searching for mono-stable polyhedra. However, as we will explain below, the only mass-distribution where this computation would be of comparable difficulty as 0-skeletons for mono-unstable polyhedra is physically rather counter-intuitive. We will call polyhedra with this highly special mass distribution *dual 0-skeletons* (see Definition 11) and we will show that their static balance properties run against intuition. In particular, we will show (Proposition 2) that for dual 0-skeletons of simplices, Dawson's tipping condition (derived for homogeneous simplices) is *reversed*. Since dual 0-skeletons are the only mass distribution where the search for stable equilibria would result in equations analogous to the ones examined in this paper for unstable equilibria, our algorithm appears to have limited practical applications for finding mono-stable polyhedra.

To better understand the background, we point out the connection of our work to Dawson's research on mono-stable simplices.

6.1. Dawson's projection criterion

Starting on a proof by Conway for the non-existence of a mono-stable tetrahedron, Dawson (1985) investigated the existence of mono-stable simplices in d dimensions. This research was continued in Dawson et al. (1998), Dawson and Finbow (2001) and ultimately led to the proof that for $d < 9$ no mono-stable simplex exists and for $d > 9$ there exist mono-stable simplices. The $d = 9$ case is not yet resolved. The problem of the existence of mono-stable simplices is closely related to our problem and below we will point out the main connections as well as the main differences.

The key idea in Dawson's arguments is what he calls the *projection criterion*:

$$|\mathbf{x_i}| < |\mathbf{x_j}| \cos\theta_{ij}, \tag{30}$$

where $\mathbf{x}_i$ is orthogonal to face f_i of the simplex s and $|\mathbf{x}_i|$ measures the area of f_i and θ_{ij} is the angle between $\mathbf{x}_i$ and $\mathbf{x}_j$. If the projection criterion (30) holds, then a $h = 3$- (or $h = 0$)-homogeneous simplex, if placed on face f_i, will tip over to face f_j, so we will call (30) the *tipping condition*, see also in Heppes (1967).

6.2. Equivalence between the shadowing criterion and the projection criterion

Proposition 1. *In the case of homogeneous simplices and 0-skeletal simplices, Eq. (8) is equivalent to Dawson's tipping condition (30).*

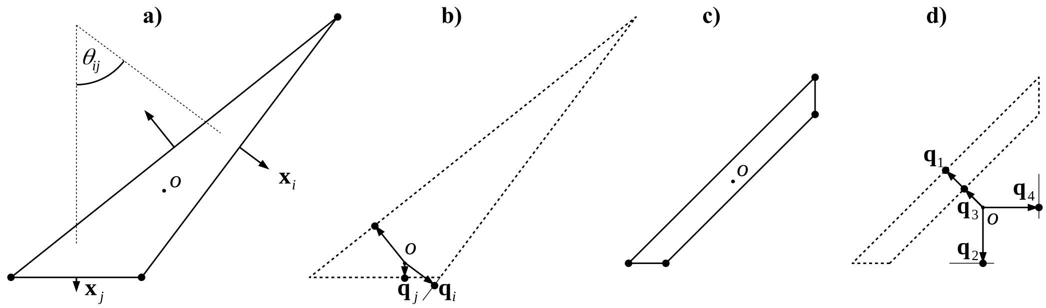

Fig. 4. Illustration of 0-skeletons and dual 0-skeletons in 2D. (a) 0-skeleton of triangle, (b) uniquely defined dual 0-skeleton of the same triangle (c) 0-skeleton of a quadrangle (d) possible dual 0-skeleton of the same quadrangle. Large black dots mark unit masses.

Proof. First we show that

$$|\mathbf{x}_i||\mathbf{q}_i| = d \cdot \frac{\mathrm{Vol}(s)}{d+1}, \tag{31}$$

where Vol(s) denotes the volume of the simplex s.

Let s be a d-dimensional simplex with center of mass o, let f_i be a $(d-1)$-dimensional face of s, let the center of mass of f_i be denoted by o_i and let v_i be the vertex opposite f_i. It is known (Krantz et al., 2006) that $|v_i, o|/|o, o_i| = d$ (where $|a, b|$ denotes the length of the line segment $\overline{ab}$).

Let s_i be a d-dimensional simplex defined by the following $d+1$ vertices: face f_i defines d vertices and we add o as the $(d+1)$st vertex. $|\mathbf{q}_i|$ is the height of s_i orthogonal to f_i. From this it follows that $\mathrm{Vol}(s)/\mathrm{Vol}(s_i) = d+1$ for $i = 1, 2, \dots, d+1$. Since the left hand side of (31) is constant, (30) implies that

$$|\mathbf{q}_j| - |\mathbf{q}_i| \cos\theta_{ij} < 0, \tag{32}$$

and this yields (8) via

$$|\mathbf{q}_j|^2 - |\mathbf{q}_j||\mathbf{q}_i| \cos\theta_{ij} < 0. \tag{33}$$

6.3. Dual 0-skeletons

As noted in Proposition 1, Dawson's tipping condition (30) only applies if the polyhedron is either a homogeneous simplex or a 0-skeleton of a simplex and it is only equivalent to Eq. (8) under the same condition. Now we show that there exists a material distribution for which Dawson's tipping condition is *reversed*: if (30) holds then the simplex will tip from face f_j to face f_i.

Definition 11. We will call a 3-dimensional polytope with F faces a *dual 0-skeleton* if $\sum \mathbf{q}_i = 0$, with the vectors $\mathbf{q}_i$, $(i = 1, 2, \dots, F)$ defined in Section 2.1.

The balance equations for the center of mass of a dual 0-skeleton can be written as

$$\sum_{i=1}^{F} q_{i,k} = 0, \quad k = 1, 2, \dots, d, \tag{34}$$

which is analogous to Eq. (14). Based on Theorem 3 and the balance Eqs. (34) we can construct the dual version of the algorithm presented in Section 3.

The name for dual 0-skeletons is motivated by their role in the case of simplices where the dual 0-skeleton is uniquely defined and corresponds to a center of mass in the interior of the simplex. In fact, it is straightforward to find the center of mass o for the dual 0-skeleton in a simplex: let S be a d-dimensional simplex and we regard the vectors $\mathbf{x}_i$ $(i = 0, 1, \dots, d+1)$ introduced in Eq. (30). Assume all $\mathbf{x}_i$ have identical origin o and we denote the endpoints of the vectors by X_i. Now we regard the set of planes p_i each of which is normal to the corresponding vector $\mathbf{x}_i$ at the endpoint X_i. It is easy to see that the simplex S' defined by these planes will be similar to S, the set of vectors $\mathbf{q}'_i$ for S' can be defined as $\mathbf{q}'_i = \mathbf{x}_i$. If we place unit masses at the points X_i then this mass distribution defines the dual 0-skeleton of S'. Dawson proved that a $h = 3$ homogeneous (or $h = 0$ homogeneous 0-skeleton) simplex can tip from face i to face j if and only if his tipping condition (30) is true. In the case of a dual 0-skeleton simplex Dawson's tipping condition is *reversed*: it implies the opposite, i.e., that it would tip from face j to face i.

Proposition 2. *For the dual 0-skeleton of any d-dimensional simplex S the tipping condition* (30) *is reversed.*

Proof. The construction scheme described at Definition 11 proves that the center of mass o for the dual-0-skeleton of a d-dimensional simplex S is an interior point of S and $\mathbf{q}_i = \alpha\mathbf{x}_i$ with $\alpha > 0$ holds for any pair $(\mathbf{x}_i, \mathbf{q}_i)$. Expressing the face shadowing condition (8) in terms of $\mathbf{x}_i$ yields $(\mathbf{x}_j - \mathbf{x}_i)\mathbf{x}_j < 0$, which can be obtained from (30) by interchanging subscripts i and j.

In the case of d-dimensional polytopes with $F > d+1$ faces the center of mass corresponding to a dual 0-skeleton may not be in the interior of the polytope. Fig. 4 illustrates the 0-skeleton and the dual 0-skeleton of a triangle and also shows a quadrangle where the center of mass for the dual skeleton is not contained in the interior.

Summarizing, we may say that the utilization of Theorem 3 as the basis of an algorithm to find mono-stable polyhedra is computationally feasible only if we investigate dual 0-skeletons; however, the latter (except for the case of simplices) may not be physically relevant, so the dual algorithm, based on Theorem 3 does not appear to be of practical interest. Since the attention was previously focused on mono-stable polyhedra (and the mono-unstable case was not considered), this observation may explain why this algorithm was not investigated earlier.

7. Concluding remarks

In this paper we improved the previously known (Conway and Guy, 1969; Domokos et al., 2020b) lower bound $V_0^U \geq 5$ on the minimal number of vertices for a convex, mono-unstable 0-skeleton to $V_0^U \geq 8$. This result also implies the same lower bound for the minimal number of vertices for a convex mono-monostatic 0-skeleton, so we also proved $V_0^* \geq 8$.

On one hand, we think that these lower bounds are not yet close to the actual minimal values. On the other hand, although no mathematical evidence exists, intuitively it looks plausible that the same lower bounds are valid for V^U, V^*, i.e., for homogeneous polyhedra.

The algorithm presented in this paper is, in principle, also able to compute V^U; however, we did not yet attempt to implement the balance equations for this case. Also, the same algorithm is (again, in principle) capable to compute higher dimensional problems. Dawson (1985) investigated the minimal dimension in which a simplex may be mono-stable and in our notation his result can be written for $d = 10$ dimensions as $V_0^S = 11$.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Acknowledgments

The authors are grateful to the anonymous reviewers and the editor of this journal for their constructive comments and suggestions.

Support of the NKFIH Hungarian Research Fund grant 134199 and of the NKFIH Fund TKP2021 BME-NVA and TKP2020 BME-NCS, carried out at the Budapest University of Technology and Economics, is kindly acknowledged. The research of S. Bozóki was supported by the Hungarian National Research, Development and Innovation Office (NKFIH) under Grant NKFIA ED_18-2-2018-0006.

Appendix A. Supplementary data

Supplementary material related to this article can be found online at https://doi.org/10.1016/j.ijsolstr.2021.111276. Tables S1-S4 in the Supplementary material include coefficients $c_2, c_3, \dots, c_V$, and the minimum value of function f, found by the algorithm presented in Section 5, for all systems written for $V = 4, 5, 6, 7$ vertices, respectively.

References

Bezdek, A., 2011. On stability of polyhedra. In: Workshop on Discrete Geometry, Sep 13-16 2011, Fields Institute, Canada. pp. 2490–2491.

Cauchy, A., 1813. Recherche sur les polyèdres — premier mémoire. J. de L'école Polytech. 9, 66–86.

Chen, Y., Sareh, P., Feng, J., 2015. Effective insights into the geometric stability of symmetric skeletal structures under symmetric variations. Int. J. Solids Struct. 69–70, 277–290. http://dx.doi.org/10.1016/j.ijsolstr.2015.05.023.

Chen, Y., Yang, F., You, Z., 2018. Transformation of polyhedrons. Int. J. Solids Struct. 138, 193–204. http://dx.doi.org/10.1016/j.ijsolstr.2018.01.012.

Conway, J.H., Guy, R.K., 1969. Stability of polyhedra. SIAM Rev. 11 (1), 78–82. http://dx.doi.org/10.1137/1011014.

Dawson, R., 1985. Monostatic simplexes. Amer. Math. Monthly 92 (8), 541–546. http://dx.doi.org/10.1080/00029890.1985.11971676.

Dawson, R., Finbow, W., 2001. Monostatic simplexes III. Geom. Dedicata 84, 101–113. http://dx.doi.org/10.1023/A:1010339220243.

Dawson, R., Finbow, W., Mak, P., 1998. Monostatic simplexes II. Geom. Dedicata 70, 209–219. http://dx.doi.org/10.1023/A:1004941706441.

Domokos, G., Jerolmack, D.J., Kun, F., Török, J., 2020a. Plato's Cube and the natural geometry of fragmentation. Proc. Natl. Acad. Sci. 117 (31), 18178–18185. http://dx.doi.org/10.1073/pnas.2001037117.

Domokos, G., Kovács, F., Lángi, Z., Regős, K., Varga, P.T., 2020b. Balancing polyhedra. Ars Math. Contemp. 19 (1), 95–124. http://dx.doi.org/10.26493/1855-3974.2120.085.

Guest, S.D., Fowler, P.W., 2005. A symmetry-extended mobility rule. Mech. Mach. Theory 40, 1002–1014. http://dx.doi.org/10.1016/j.mechmachtheory.2004.12.017.

Henrion, D., Garulli, A. (Eds.), 2005. Positive polynomials in control. In: Lecture Notes in Control and Information Sciences, vol. 312, Springer-Verlag Berlin Heidelberg.

Heppes, A., 1967. A double tipping tetrahedron. SIAM Rev. 9 (3), 599–600. http://dx.doi.org/10.1137/1008076.

Kangwai, R.D., Guest, S.D., 1999. Detection of finite mechanisms in symmetric structures. Int. J. Solids Struct. 36 (36), 5507–5527. http://dx.doi.org/10.1016/S0020-7683(98)00234-0.

Kovács, F., Tarnai, T., Fowler, P.W., Guest, S.D., 2004. A class of expandable polyhedral structures. Int. J. Solids Struct. 41 (3), 1119–1137. http://dx.doi.org/10.1016/j.ijsolstr.2003.09.046.

Krantz, S.G., McCarthy, J.E., Parks, H.R., 2006. Geometric characterizations of centroids of simplices. J. Math. Anal. Appl. 316 (1), 87–109. http://dx.doi.org/10.1016/j.jmaa.2005.04.046.

Kumar, P., Pellegrino, S., 2000. Computation of kinematic paths and bifurcation points. Int. J. Solids Struct. 37 (46), 7003–7027. http://dx.doi.org/10.1016/S0020-7683(99)00327-3.

Lasserre, J.B., 2010. Moments, Positive Polynomials and their Applications. In: Imperial College Press Optimization Series, vol. 1, Imperial College Press.

Luo, Y., Yu, Y., Liu, J., 2008. A retractable structure based on Bricard linkages and rotating rings of tetrahedra. Int. J. Solids Struct. 45 (2), 620–630. http://dx.doi.org/10.1016/j.ijsolstr.2007.08.018.

Marshall, M., 2008. Positive Polynomials and Sums of Squares. In: Mathematical Surveys and Monographs, vol. 146, The American Mathematical Society.

Maxwell, J.C., 1864. On the calculation of the equilibrium and stiffness of frames. Phil. Mag. 27, 294–299. http://dx.doi.org/10.1080/14786446408643668.

Milnor, J., 1963. Morse Theory. In: Based on lecture notes by M. Spivak and R. Wells. Annals of Mathematics Studies, No. 51, Princeton University Press, Princeton, N.J., p. vi+153, 0163331.

Nowacki, H., 2002. Archimedes and ship stability. In: Kaklis, P.D., Papanikolaou, A., Spyrou, K. (Eds.), Passenger Ship Design, Construction, Operation and Safety: Euroconference; Knossos Royal Village, Anissaras, Crete, Greece, October 15–17, 2001. National Technical Univ. of Athens, Department of Naval Architecture and Marine Engineering, Athens, pp. 335–360.

Pacioli, L., 1509. Divina Proportione. Paganini, Venice, URL https://ia801600.us.archive.org/23/items/divinaproportion00paci/divinaproportion00paci.pdf.

Prestel, A., Delzell, C., 2001. Positive Polynomials – from Hilbert's 17th Problem To Real Algebra. In: Springer Monographs in Mathematics, Springer-Verlag, Berlin Heidelberg.

Reshetov, A., 2014. A unistable polyhedron with 14 faces. Internat. J. Comput. Geom. Appl. 24 (1), 39–59. http://dx.doi.org/10.1142/S0218195914500022.

Steinitz, E., 1906. Über Die Eulerschen Polyederrelationen. Archiv Für Mathematik Und Physik 11, 86–88.

Steinitz, E., 1922. Polyeder und raumeinteilungen. In: Encyklopädie Der Mathematischen Wissenschaften, Dritter Band: Geometrie. vol. 3, pp. 1–139.

Várkonyi, P.L., Domokos, G., 2006. Static equilibria of rigid bodies: Dice, pebbles and the Poincaré-Hopf theorem. J. Nonlinear Sci. 16, 255–281. http://dx.doi.org/10.1007/s00332-005-0691-8.