Endogenous choice of decision variables

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Abstract

In this paper we allow the firms to choose their prices and quantities simultaneously. Quantities are produced in advance and their common sales price is determined by the market. Firms offer their "residual capacities" at their announced prices and the corresponding demand will be served to order. If all firms have small capacities, we obtain the Bertrand solution; while if at least one firm has a sufficiently large capacity, the Cournot outcome and a model of price leadership could emerge.

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1 Introduction

We consider a game in which the firms simultaneously choose their price and quantity decisions. This work complements Tasnádi (2006 and 2010) in which firms could choose their decision variable (price or quantity). In Tasnádi (2006) the firms chose between price and quantity, and second, they selected their prices or quantities with respect to their first-round decisions. Given the firms' first-stage decisions the price setters, if there were at least two of them, had to play mixed strategies like in the capacity-constrained Bertrand-Edgeworth game. Therefore, we had to limit ourselves to the case of symmetric capacities for which we found under reasonable assumptions the emergences of the Cournot game. In Chapter 5 of Tasnádi (2010) we considered the case in which the firms must choose their decision variables and their magnitudes simultaneously, by which approach we could avoid the problem of dealing with mixed strategies, and therefore, we could also investigate the case of asymmetric capacities. We found that for a large region of capacity constraints both the Cournot and the Forchheimer outcome could arise.

In contrast to the above mentioned two works, in which the firms had to make a binary choice, we allow in this paper the firms to make a smooth decision. The two extreme cases of

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setting zero quantity or setting a sufficiently large price would give us a purely price-setting firm and a purely quantity-setting firm, respectively. In particular, firms can produce a quantity for which the market determines price by equating supply and demand, and firms can choose the price for their additional production. In the interesting range of capacities we find the emergence of the Cournot and the Forchheimer outcome.

The endogenous choice of decision variables has been investigated in the literature for homogeneous good oligopoly markets by Dastidar (1996) and Qin and Stuart (1997). Dastidar (1996) considered a two-stage duopoly game in which the firms choose their decision variables in stage 1 and the magnitude of the selected decision variable in stage 2. In case of two quantity setters a Cournot duopoly game is played in stage 2, in case of two price setters a Bertrand duopoly game is played in stage 2, while in case of one price-setter and one quantity setter we have that the quantity setter takes the price set by the price setter as given. Dastidar (1996) finds that a mixed equilibrium never occurs, the Cournot game always emerges as an equilibrium, while the Bertrand game may emerge as the outcome of the two-stage game. Qin and Stuart (1997) formulated an oligopoly game in which some firms set their quantities and the remaining firms set their prices. They showed that if the firms are free to choose their decision variable, then both the Bertrand and the Cournot outcome could emerge. Tasnádi (2006 and 2010) and this paper considers Bertrand-Edgeworth-type price-setting instead of Bertrand-type price-setting behavior. For more on the role of decision variables in the homogeneous good framework we refer to Friedman (1988).

The seminal paper allowing the duopolists to choose their decision variable, is due to Singh and Vives (1984) that investigates a heterogeneous goods two-stage duopoly market. Singh and Vives (1984) demonstrated the emergence of the Cournot game if goods are substitutes and the emergence of the Bertrand game if goods are complements. Klemperer and Meyer (1986) investigates a one-stage heterogeneous goods duopoly game and reports that a multiplicity of equilibria is possible. For more on the endogenous choice of the decision variable in the heterogeneous goods framework see, for instance, Szidarovszky and Molnár (1992), Tanaka (2001a and 2001b) and Reisinger and Ressner (2009), among others.

The remainder of the paper is organized as follows. Section 2 presents our framework. Section 3 contains our analysis. Finally, we conclude in Section 4.

2 Framework

Suppose that there are n firms on the market, where we shall denote the set of firms by $N = \{1, \ldots, n\}$. We assume that the firms have zero unit costs.[1] We shall denote by k_i the capacity constraint of firm i and by $K = \sum_{i=1}^{n} k_i$ the aggregate capacity of the firms. Let the capacity constraints be ordered decreasingly, that is $k_1 \geq \ldots \geq k_n > 0$. We summarize the assumptions imposed on the oligopolists' cost functions below:

Assumption 1. There are n firms on the market with zero unit costs and capacity constraints $k_1 \geq k_2 \geq \ldots \geq k_n > 0$.

The demand is given by function D on which we impose the following restrictions:

[1] Since the firms, which set also prices, basically compete in a residual production-to-order game (that is, production takes place after the firms' prices are revealed), the real assumption here is that the firms have identical unit costs.

Assumption 2. The demand function D intersects the horizontal axis at quantity a and the vertical axis at price b. D is strictly decreasing and twice continuously differentiable on $(0, a)$; moreover, D is right-continuous at 0, left-continuous at b and $D(p) = 0$ for all $p \geq b$.

Clearly, a price-setting firm will not set its price above b. Let us denote by P the inverse demand function. Thus, $P(q) = D^{-1}(q)$ for $0 < q \leq a$, $P(0) = b$, and $P(q) = 0$ for $q > a$. In addition, we shall denote by p^c the market clearing price, i.e. $p^c = P(K)$.

The following technical assumption is needed to ensure the existence of an equilibrium in our model.

Assumption 3. The function $pD'(p) + D(p)$ is strictly decreasing on $(0, b)$.

We shall denote by $D_i^r(p) = (D(p) - (K - k_i))^+$ the residual demand curve of firm i and its inverse by $P_i^r(q)$. It can be easily verified that $P_i^r(q) = P(q + (K - k_i))$. Assuming efficient rationing, the function $\pi_i^r(p) = pD_i^r(p)$ will equal firm i's profit whenever it sets the highest price in the pure price-setting game and $p \geq p^c$. We ensure that every firm will be active in the market by the next assumption.

Assumption 4. We assume that $K - k_n < a$.

We shall denote by p_i^m the unique revenue maximizing price on the residual demand curve D_i^r and by q_i^m the unique revenue maximizing output on the inverse residual demand curve P_i^r, i.e. $p_i^m = \arg\max_{p \in [0,b]} pD_i^r(p)$ and $q_i^m = \arg\max_{q \in [0,a]} qP_i^r(q)$ for any $i \in N$. Of course, $q_i^m = D_i^r(p_i^m)$. Furthermore, it can be checked that $p_1^m \geq p_2^m \geq \ldots \geq p_n^m$ because of Assumptions 1-4. Let $\overline{\pi}_i = \pi_i^r(p_i^m)$. Clearly, p^c and p_i^m are well defined whenever Assumptions 1-4 are satisfied. Note that Assumption 4 also ensures that $p_i^m > 0$ and $\overline{\pi}_i > 0$.

Let $p_i \in [0, b]$ be the price and $q_i \in [0, k_i]$ be the quantity decision of firm $i \in N$. Hence, the price decisions and the quantity decisions are contained in vectors $\mathbf{p} \in [0, b]^n$ and $\mathbf{q} \in \times_{j=1}^n [0, k_j]$, respectively. If firm i sets a positive quantity q_i, then it will sell this amount at a price determined by the market. Furthermore, if firm i sets its price p_i below b, then it is willing to produce in addition an amount of at most $k_i - q_i$ units at price p_i, which we shall call the residual capacity of firm i. We to refer to firm i as a pure price setter if $p_i < b$ and $q_i = 0$ and as a pure quantity setter if $p_i = b$ and $q_i > 0$. Otherwise, a firm is a price and quantity setter at the same time. We denote by $J_{\mathbf{p},\mathbf{q}} = \{j \in N \mid q_j > 0\}$ the set of quantity-setting firms and by $I_{\mathbf{p},\mathbf{q}} = N \setminus J_{\mathbf{p},\mathbf{q}}$ the set of purely price-setting firms. For fixed decisions $(\mathbf{p}, \mathbf{q})$ the aggregate supply of the firms at price p is given by $S_{\mathbf{p},\mathbf{q}}(p) = \sum_{i \in N} q_j + \sum_{i \in N, p_i \leq p} (k_i - q_i)$.

We have to specify a demand-allocating mechanism and the price of the quantity-setting firms' product as a function of the price and quantity actions taken by the firms. We define the quantity-setting firms' sales price, denoted by $p^*(\mathbf{p}, \mathbf{q})$, to be the lowest price at which the demand is less or equal to the aggregate supply of the firms. Formally, $p^*(\mathbf{p}, \mathbf{q}) =$

$$\inf \{p \in [0, b] \mid D(p) \leq S_{\mathbf{p},\mathbf{q}}(p)\} = \min \{p \in [0, b] \mid D(p) \leq S_{\mathbf{p},\mathbf{q}}(p)\}.$$

We can write min instead of inf because $D(p) - S_{\mathbf{p},\mathbf{q}}(p)$ is a decreasing and right-continuous function in p. Observe that $p^*(\mathbf{p}, \mathbf{q})$ is even defined in case of $I_{\mathbf{p},\mathbf{q}} = N$.

At price level $p^*(\mathbf{p}, \mathbf{q})$ aggregate supply may exceed market demand. This case is illustrated in Figure 1 in which there are two price-setting firms and one quantity-setting

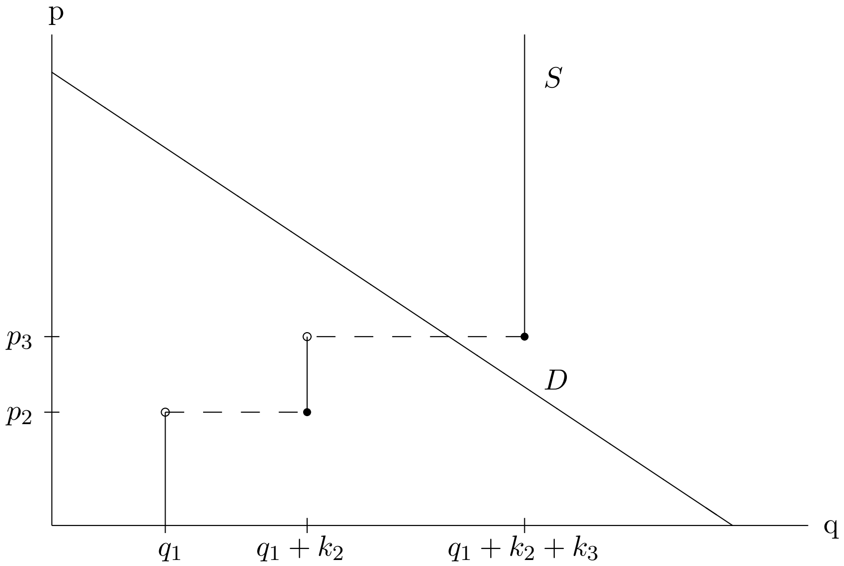

Figure 1: Quantity-setting firms product price

firm. In the situation presented in Figure 1 the sales price for the quantity-setting firm's product equals p_3 (where $p_1 = b$ and $q_2 = q_3 = 0$). In case of price ties, we assume for simplicity, that the demand is allocated first to the quantity-setting firms and afterwards the remaining demand is shared by the price-setting firms in proportion of their capacities. The first part of this assumption, i.e. that the demand is allocated first to the quantity-setting firms, expresses our intuition that price setters shall face quantity adjustment, while quantity setters shall face price adjustment. The demand satisfied by firm $j \in J_{\mathbf{p},\mathbf{q}}$ at price $p^*(\mathbf{p},\mathbf{q})$ is given by

$$\Delta_j^q(\mathbf{p},\mathbf{q}) = \begin{cases} q_j & \text{if } \; p^*(\mathbf{p},\mathbf{q}) > 0, \\ \min\left\{q_j, \frac{q_j}{\sum_{l=1}^n q_l} D(0)\right\} & \text{if } \; p^*(\mathbf{p},\mathbf{q}) = 0; \end{cases}$$

and therefore, its profit equals $\pi_j^q(\mathbf{p},\mathbf{q}) = p^*(\mathbf{p},\mathbf{q})\, q_j$ from producing quantity q_j. In addition, let $\pi_i^q(\mathbf{p},\mathbf{q}) = 0$ for any $i \in I_{\mathbf{p},\mathbf{q}}$. We define the demand satisfied by firm $i \in N$ resulting from its price-setting activity by

$$\Delta_i^p(\mathbf{p},\mathbf{q}) = \begin{cases} 0 & \text{if } \; p_i > p^*(\mathbf{p},\mathbf{q}) \text{ or } q_i = k_i, \\ \frac{k_i - q_i}{\sum_{p_l = p_i} k_l - q_l}\left(D(p_i) - \sum_{j=1}^n q_j - \sum_{p_l < p_i}(k_l - q_l)\right)^+ & \text{if } \; p_i = p^*(\mathbf{p},\mathbf{q}) \text{ and } q_i < k_i, \\ k_i - q_i & \text{if } \; p_i < p^*(\mathbf{p},\mathbf{q}) \text{ and } q_i < k_i. \end{cases}$$

Thus, firm $i \in N$ makes $\pi_i^p(\mathbf{p},\mathbf{q}) = p_i \Delta_i^p(\mathbf{p},\mathbf{q})$ profit from selling its residual capacity at price p_i. Finally, we shall define the profit of firm $i \in N$ by $\pi_i(\mathbf{p},\mathbf{q}) = \pi_i^p(\mathbf{p},\mathbf{q}) + \pi_i^q(\mathbf{p},\mathbf{q})$.

3 The analysis

In this Section we determine those conditions under which an equilibrium in pure strategies exists. We will proceed step by step in order to obtain a better understanding of the equilibrium behavior of our oligopoly game. The following Lemma states that in a pure-strategy equilibrium the purely price-setting firms must set the same prices.

Lemma 1. *Under Assumptions 1-4, if* $(\mathbf{p}, \mathbf{q})$ *is a pure-strategy equilibrium, then* $p_i = p_j$ *for any* $i, j \in I_{\mathbf{p},\mathbf{q}}$.

Proof. The statement is obviously true if $|I_{\mathbf{p},\mathbf{q}}| \leq 1$. Thus, we have only to consider the case of $|I_{\mathbf{p},\mathbf{q}}| > 1$. Let firm j be one of the firms setting the lowest price; that is, $p_j \leq p_i$ for all $i \in I_{\mathbf{p},\mathbf{q}}$. Suppose that $p_j < p_i$ holds for a firm $i \in I_{\mathbf{p},\mathbf{q}}$. If $\Delta_i^p(\mathbf{p}, \mathbf{q}) > 0$, then firm j can increase its profit by setting its price arbitrarily close to but below p_i. If $\Delta_i^p(\mathbf{p}, \mathbf{q}) = 0$, then $\pi_i(\mathbf{p}, \mathbf{q}) = 0$. But firm i can make positive profit, for instance, by switching its price to $P\left(\frac{1}{2}(K - k_i + a)\right)$ so that it faces a positive demand because of Assumption 4. Thus, firm i would deviate, and therefore, $p_j < p_i$ cannot be the case. □

Next, we prove that in a pure-strategy equilibrium every purely price-setting firm's price must be equal to the sales price of the quantity-setting firms.

Lemma 2. *Let Assumptions 1-4 be satisfied and* $(\mathbf{p}, \mathbf{q})$ *be a pure-strategy equilibrium. If* $|J_{\mathbf{p},\mathbf{q}}| > 0$, *then* $p_i = p^*(\mathbf{p}, \mathbf{q})$ *for any* $i \in I_{\mathbf{p},\mathbf{q}}$.

Proof. Clearly, the Lemma holds if $|I_{\mathbf{p},\mathbf{q}}| = 0$. Therefore, in what follows we can assume that $|I_{\mathbf{p},\mathbf{q}}| > 0$. Suppose that $p_i \neq p^*(\mathbf{p}, \mathbf{q})$ for some $i \in I_{\mathbf{p},\mathbf{q}}$. Recall that any firm $i \in I_{\mathbf{p},\mathbf{q}}$ setting its price below $p^*(\mathbf{p}, \mathbf{q})$ can sell its entire capacity k_i. Moreover, observe that $p^*(\mathbf{p}, \mathbf{q})$ will not change if prices lower than $p^*(\mathbf{p}, \mathbf{q})$ change as long as they remain lower than $p^*(\mathbf{p}, \mathbf{q})$. Thus, if $p_i < p^*(\mathbf{p}, \mathbf{q})$, firm i can increase its profit by setting a price slightly below $p^*(\mathbf{p}, \mathbf{q})$, since it is still selling k_i but at a higher price. If $p_i > p^*(\mathbf{p}, \mathbf{q})$, then firm i does not sell anything and makes zero profit. But firm i can achieve positive profit by making a sufficiently large price reduction because of Assumption 4; a contradiction. □

Furthermore, if the equilibrium market price is larger than the market-clearing price and there is at least one purely price-setting firm in the market, then in a pure-strategy equilibrium the quantity-setting firms produce at their capacity limits, and thus there are no "partially" price-setting firms in the market.

Lemma 3. *Let Assumptions 1-4 be satisfied and* $(\mathbf{p}, \mathbf{q})$ *be a pure-strategy equilibrium. If* $|J_{\mathbf{p},\mathbf{q}}| > 0$, $|I_{\mathbf{p},\mathbf{q}}| > 0$ *and* $p^*(\mathbf{p}, \mathbf{q}) > p^c$, *then we must have* $q_j = k_j$ *for all* $j \in J_{\mathbf{p},\mathbf{q}}$ *in any pure-strategy equilibrium.*

Proof. From Lemma 2 we know that in a pure-strategy equilibrium $p_i = p^*(\mathbf{p}, \mathbf{q})$ must hold for all $i \in I_{\mathbf{p},\mathbf{q}}$. Moreover, every firm's profit must be positive because of Assumption 4, which implies $\Delta_i^p(\mathbf{p}, \mathbf{q}) > 0$ for any $i \in I_{\mathbf{p},\mathbf{q}}$ and $\Delta_j^q(\mathbf{p}, \mathbf{q}) > 0$ for any $j \in J_{\mathbf{p},\mathbf{q}}$.

Suppose that $q_j < k_j$ for a firm $j \in J_{\mathbf{p},\mathbf{q}}$. Then we have to distinguish the following three cases: (i) $p_j < p^*(\mathbf{p}, \mathbf{q})$, (ii) $p_j = p^*(\mathbf{p}, \mathbf{q})$ and (iii) $p_j > p^*(\mathbf{p}, \mathbf{q})$. In case (i) firm j can increase its profit by increasing its output to k_j because this will not decrease $p^*(\mathbf{p}, \mathbf{q})$, since its previously sold $k_j - q_j$ units can be now sold at price $p^*(\mathbf{p}, \mathbf{q})$. In case (ii) a purely price-setting firm could capture market from j by unilaterally undercutting price p_j.[2] Finally, in case (iii) firm j selling a positive amount at price p_j would be in contradiction with the definition of $p^*(\mathbf{p}, \mathbf{q})$. Hence, quantity-setting firm j could increase its profits by increasing its quantity since this would just result in a decrease of sales for the purely price-setting firms. We obtained in any of the three cases a contradiction, and therefore we must have $q_j = k_j$ in any pure-strategy equilibrium with an equilibrium price larger than the market-clearing price. □

[2] At this point we are employing $p^*(\mathbf{p}, \mathbf{q}) > p^c$.

Now we are ready to prove the main propositions considering our oligopoly game. The next proposition establishes that the Bertrand solution is the only possible Nash equilibrium candidate in the presence of at least two purely price-setting firms.

Proposition 1. *Let Assumptions 1-4 be satisfied and* $(\mathbf{p}, \mathbf{q})$ *be a pure-strategy equilibrium. If* $|I_{\mathbf{p},\mathbf{q}}| \geq 2$*, then the only possible pure-strategy Nash equilibrium candidate is* $q_j = k_j$ *or* $p_j = p^c$ *for all* $j \in J_{\mathbf{p},\mathbf{q}}$ *and* $p_i = p^c$ *for all* $i \in I_{\mathbf{p},\mathbf{q}}$*.*

Proof. Suppose that $|J_{\mathbf{p},\mathbf{q}}| = 0$. By Lemma 1 we know that in a possible equilibrium every purely price-setting firm sets the same price. But, if that price level exceeds p^c, then their sales will be less than their capacity level, and therefore, any of them can gain from undercutting its opponents.

Now, we consider case $|J_{\mathbf{p},\mathbf{q}}| > 0$. By Lemma 2 we must have $p_i = p^* (\mathbf{p}, \mathbf{q})$ for any $i \in I_{\mathbf{p},\mathbf{q}}$. Of course, $p^c \leq p^* (\mathbf{p}, \mathbf{q})$. Suppose that $p^c < p^* (\mathbf{p}, \mathbf{q})$. Then we know by Lemma 3 that $q_j = k_j$ for all $j \in J_{\mathbf{p},\mathbf{q}}$. But if $p^c < p^* (\mathbf{p}, \mathbf{q})$, then any price-setting firm $i \in I_{\mathbf{p},\mathbf{q}}$ will sell less than k_i products. Therefore, by just unilaterally undercutting price $p^* (\mathbf{p}, \mathbf{q})$, any price-setting firm can increase its sales radically, and thus, increase its profits. Hence, in an equilibrium we must have $p^c = p^* (\mathbf{p}, \mathbf{q})$ and only strategy profiles specified by Proposition 1 remain. □

If $k_i \leq q_i^m$, then we will say that firm $i \in N$ has *scarce capacity*, while otherwise we will say that firm $i \in N$ has *sufficient capacity*. Note that a firm with scarce capacity will be eager to produce at its capacity limit. We shall denote by H the set of those firms having only scarce capacity, i.e. $H = \{i \in N \mid k_i \leq q_i^m\}$. It can be verified that condition $k_i \leq q_i^m$ is equivalent to $p^c \geq p_i^m$. Thus, $H = \{i \in N \mid h \leq i \leq n\}$ for some $h \in \{1, \ldots, n+1\}$ since the sequence p_i^m is nonincreasing.

In the following proposition we establish that the Bertrand solution is the unique Nash equilibrium solution in pure strategies of any mixed oligopoly game if every firm has scarce capacity.

Proposition 2. *Under Assumptions 1-4, if* $H = N$ *and* $(\mathbf{p}, \mathbf{q})$ *is an equilibrium in pure strategies, then it must be payoff and price equivalent with the Bertrand solution, i.e. the equilibrium price equals the market-clearing price and all firms sell their entire capacity.*

Proof. We can verify that $q_j = k_j$ or $p_j = p^c$ for any $j \in J_{\mathbf{p},\mathbf{q}}$ and $p_i = p^c$ for any $i \in I_{\mathbf{p},\mathbf{q}}$ is a Nash equilibrium because of Assumptions 2, 3 and $k_i \leq q_i^m$ for all $i \in N$.

Assume that $|I_{\mathbf{p},\mathbf{q}}| = 0$. Then the partial price setter selecting the highest price above p^c would reduce its price because of $p_i^m \geq p^c$. In addition, if there are more partial price setters choosing the same highest price above p^c, then each of them could benefit from a slight price reduction. Hence, only profiles specified in Proposition 2 can be equilibrium profiles.

We turn to the analysis of case $I_{\mathbf{p},\mathbf{q}} = \{i\}$. Suppose that $p^c < p^* (\mathbf{p}, \mathbf{q})$. Then by Lemma 3 each quantity-setting firm $j \in J_{\mathbf{p},\mathbf{q}}$ sets $q_j = k_j$ and by Lemma 2 the purely price-setting firm i should set price $p_i = p^* (\mathbf{p}, \mathbf{q})$. Since by $p^c \geq p_i^m$ firm i has an incentive for a price reduction $p^c < p^* (\mathbf{p}, \mathbf{q})$ cannot be the case in an equilibrium. Again, only profiles specified in Proposition 2 can be equilibrium profiles.

Finally, if $|I_{\mathbf{p},\mathbf{q}}| \geq 2$, then Proposition 1 yields the desired result. □

Let us denote by p_i^d the smallest price satisfying $p_i^d \min\{k_i, D\left(p_i^d\right)\} = p_i^m D_i^r\left(p_i^m\right)$. Suppose that firm i has sufficient capacity (implying $p_i^d < p_i^m$), then firm i is indifferent to whether serving residual demand at price level p_i^m or selling its entire capacity at the lower price level p_i^d. We will need the following Lemma established for the two firm case by Deneckere and Kovenock (1992) and for the n firm case by (Tasnádi 2010, Lemma 7).

Lemma 4. *Suppose that firm i and j have both sufficient capacity and that Assumptions 1-4 are satisfied. If $i < j$, then $p_i^d \geq p_j^d$. In addition, if $k_i > k_j$, then $p_i^d > p_j^d$.*

Next, we investigate the case of only one purely price-setting firm. Because of Proposition 2 we may assume in what follows that there is a firm having sufficient capacity.

Proposition 3. *Under Assumptions 1-4, if $(\mathbf{p}, \mathbf{q})$ is an equilibrium in pure strategies such that $I_{\mathbf{p},\mathbf{q}} = \{i\} \subset N \setminus H = \{1, \ldots, h-1\}$ then the equilibrium is given by*

$$\forall j \in J_{\mathbf{p},\mathbf{q}} : q_j = k_j \text{ and } p_i = p_i^m = \arg\max_{p \in [0,b]} pD_i^r(p). \tag{1}$$

In addition, the above type of equilibrium exists if and only if $p_1^d \leq p_i^m$.

Proof. First, we demonstrate that if $p_1^d \leq p_i^m$, then the strategy profile given by equation (1) is an equilibrium. Suppose that a quantity-setting firm chooses a strategy (p_j, q_j) different from (b, k_j). Clearly, firm j would reduce its profits by selling its residual capacity at a price $p_j < p_i^m$. If $p_j > p_i^m$, then it can sell $q_j' = \left(D(p) - q_j - \sum_{l \neq j} k_l\right)^+$ units at price p_j, which means that firm j faces its residual demand curve. Hence, firm j would be better off by selling k_j units at price p_i^m because of $p_j^d \leq p_1^d \leq p_i^m$. For quantity-setting firm j the case of $p_j = p_i^m$ remains to be investigated. In the latter case a unilateral output decrease from k_i to at most q_i^m will not change the sales price for the quantity-setting firms' product, but only increase the purely price-setting firm's sales. Moreover, if firm j reduces its output below q_i^m, then the sales price for the quantity-setting firms' product will equal to $P_j^r\left(q_j + \sum_{l \neq j} k_l\right)$. Thus, a unilateral output decrease by any firm $j \in J_{\mathbf{p},\mathbf{q}}$ with $j > i$ will inevitably lead to a decrease in its own profit level because of $p_j^m \leq p_i^m$ and Assumptions 2 and 4. Furthermore, an output reduction by firm $j \in J_{\mathbf{p},\mathbf{q}}$ with $j < i$ will increase its profits if and only if $p_j^d > p_i^m$, because then a decrease in output to q_j^m yields $p_j^d k_j$ profits, which is greater than $p_i^m k_j$. Regarding that the sequence p_j^d is nonincreasing, we have shown that the quantity-setting firms will not deviate from $q_j = k_j$. It can be easily checked that firm i will not deviate from price p_i^m. Hence, equation (1) determines a Nash equilibrium profile, which determines the set of equilibrium profiles in the presence of one purely price-setting firm because of Lemmas 2 and 3.

Second, we establish that if $p_1^d > p_i^m$, then there is a lack of Nash equilibrium with one purely price-setting firm. We already know by Lemmas 2 and 3 that in an equilibrium $q_j = k_j$ for all $j \in J_{\mathbf{p},\mathbf{q}}$ and $p_i = p^*(\mathbf{p}, \mathbf{q})$ must hold. Therefore, price-setting firm i sets price p_i^m and sells q_i^m amount of product. This means that $p_i = p^c(\mathbf{p}, \mathbf{q})$ must be equal to p_i^m in a Nash equilibrium. But, then firm 1 will unilaterally decrease its outputs, because $p_1^m q_1^m = p_1^d k_1 > p_i^m k_1$ and we conclude that a Nash equilibrium does not exist. □

Checking the proof of Proposition 3, we obtain the following Corollary.

Corollary 1. *Let Assumptions 1-4 and $p_1^d \leq p_i^m$ be satisfied, and let $i \in N \setminus H = \{1, \ldots, h-1\}$. Then*

$$\forall j \in N \setminus \{i\} : q_j = k_j \ , \ q_i \in [0, q_i^m] \text{ and } p_i = p_i^m = \arg\max_{p \in [0,b]} pD_i^r(p) \tag{2}$$

determines all equilibria with one partial or pure price-setting firm.

We have to emphasize that by Proposition 3 our oligopoly game, with a price-setting firm i having sufficient capacity and fulfilling condition $p_1^d \leq p_i^m$, yields an implementation of Forchheimer's model of dominant firm price leadership, because the price-setting firm sets its price by maximizing profits with respect to its residual demand curve and the sales price for the remaining firms' product equals that price. Let us remark that, to act as a price leader, a firm does not have to possess the largest capacity on the market so far the requirements of Proposition 3 are satisfied. For more on price leadership we refer to Ono (1982), Deneckere and Kovenock (1992), van Damme and Hurkens (2004), Tasnádi (2004) or Yano and Komatsubara (2006).

We still have to consider the Cournot game, that is the case in which every firm behaves as a pure quantity setter. The existence of a Nash equilibrium in the Cournot game has been investigated extensively in the literature (see for instance Szidarovszky and Yakowitz, 1977; Novshek, 1985; Amir, 1996; Forgó, 1996). Regarding our assumptions, the results known to us cannot be applied directly to demonstrate existence in our model. Particularly, we assume that the function $pD(p)$ is strictly concave, which does not even imply that $qP(q)$ is concave. To verify this consider demand function $D(p) = 1 - \frac{4}{3}p^{3/4}$, which satisfies Assumption 2 and 3, but for which $qP(q)$ is convex in the interval $(6/7, 1)$. Moreover, it can be verified that the concavity of $pD(p)$ does not even imply the log-concavity of $P(q)$. Hence, even Amir's (1996) existence theorem cannot be applied. However, we will prove the existence of a Nash equilibrium by applying Debreu's (1952) existence theorem.

Proposition 4. *Under Assumptions 1-4, the Cournot game has an equilibrium in pure strategies.*

Proof. Firm i's strategy set $[0, k_i]$ is compact and its payoff function $q_i \; P\left(\sum_{j\in N} q_j\right)$ is continuous. In order to apply Debreu's (1952) existence theorem we still have to show that the firms' payoff functions are quasiconcave in their own decision variable. We will demonstrate that $\Pi_i(q_i, Q_{-i}) = q_i P(q_i + Q_{-i})$ is single peaked in q_i for any fixed value of $Q_{-i} \in [0, K - k_i]$, which in turn implies quasiconcavity.

Pick an arbitrary value for Q_{-i} from interval $[0, K - k_i]$. Let us define the function $F : [0, a - Q_{-i}] \to [0, c]$ by $F(q) = P(q + Q_{-i})$, where $c = P(Q_{-i})$. We shall denote by G the inverse function of F. It can be checked that $G(p) = D(p) - Q_{-i}$. Let $\Pi_i^*(p) = p(D(p) - Q_{-i})^+$. Of course, $\Pi_i^*(0) = 0$, $\Pi_i^*(p) = 0$ for any $p \geq c$, and Π_i^* is strictly concave in $(0, c)$. Hence, it has a unique maximum, denoted by $p^* \in (0, c)$, which can be determined by the following equation:

$$\frac{d}{dp}\Pi_i^*(p) = G(p) + pG'(p) = 0 \tag{3}$$

The first-order condition corresponding to problem $\max_{q_i} \Pi_i(q_i, Q_{-i})$ is

$$\frac{d}{dq}\Pi_i(q, Q_{-i}) = F(q) + qF'(q) = 0. \tag{4}$$

We check that $q^* = G(p^*)$ satisfies equation (4):

$$\frac{d}{dq}\Pi_i(q^*, Q_{-i}) = F(q^*) + q^*F'(q^*) = p^* + G(p^*)\frac{1}{G'(p^*)} = 0,$$

where the last equality holds clearly because p^* is a solution of equation (3) and $G'(p^*) \neq 0$. Furthermore, q^* is the unique solution of equation (4) since otherwise equation (3) will not have a unique solution. Finally, output level q^* corresponds to a maximum since $\Pi_i(q^*, Q_{-i}) > 0$ and $\Pi_i(0, Q_{-i}) = \Pi_i(a - Q_{-i}, Q_{-i}) = 0$. □

The next theorem summarizes our results concerning our oligopoly game:

Theorem 1. *Under Assumptions 1-4 the following statements hold true concerning our oligopoly game:*

1. *If* $q_i^m \geq k_i$ *for all* $i \in N$*, then any Nash equilibrium yields the Bertrand solution.*

2. *If* $i < h$ *and* $p_1^d \leq p_i^m$*, then the second type of equilibria result in a model of price leadership.*

3. *If there exists an* $i \in N$ *such that* $q_i^m < k_i$*, then third type of equilibria are given by the Cournot solutions.*

4. *Another type of Nash equilibrium does not exist.*

Proof. Points 1 and 2 follow from Proposition 2 and Corollary 1.

Next, we show point 3, that is (p, y_i) for all $i \in N$ is a Nash equilibrium, where $\mathbf{y}$ denotes a Cournot solution and $p \geq p^* = P\left(\sum_{i=1}^n y_i\right)$. Suppose that firm $i \in N$ considers to become a pure or partial price setter, and thus switches to another strategy $(p_i, q_i) \neq (p, y_i)$. Since in any case firm i faces residual demand curve $\left(D(p_i) - q_i - \sum_{j \neq i} y_j\right)^+$ as a partial $(q_i > 0)$ or pure $(q_i = 0)$ price setter it can be easily verified that (p, y_i) is a best response to the other firms strategies.

Finally, point 4 follows from our established facts that the presence of at least one pure price setter implies that there cannot be a partial price setter (Lemma 3), scarce capacities have to result in the Bertrand solution (Proposition 2), the presence of a pure or partial price setter has to result in a kind of price leadership (Corollary 1) and in the absence of a pure and partial price setter we obtain a Cournot outcome. □

4 Concluding remarks

Let us remark that if $p_1^d > p_{h-1}^m$, then the only price leadership equilibrium that emerges yields an implementation in Nash equilibrium of the classical dominant firm model of price leadership.

If every firm has scarce capacity, where by scarce we mean that the unconstrained profit-maximizing output with respect to its residual demand curve exceeds its capacity constraint, then any firm will produce at its capacity limit. Thus, the Bertrand solution arises. We want to highlight that if at least one firm does not have scarce capacity, then either the Cournot game or price leadership emerges. Therefore, if some additional assumptions are satisfied, we have also given a game-theoretic foundation of Forchheimer's model of dominant-firm price leadership (see Scherer and Ross, 1990).

In a follow up research we would like to single out either the Cournot solution or the dominant firm model of price leadership if both of these two types of equilibria exist. Investigating their stability properties, might help us in solving the problem of multiple equilibria.

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