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# Utility-refined and budget-refined $\varepsilon$-competitive equilibria

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ABSTRACT

In finitely additive exchange economies the usual definition of competitive equilibrium can result in an empty equilibrium set, and therefore one has to consider notions of $\varepsilon$-competitive equilibria. In this paper we investigate the relationship between two notions of $\varepsilon$-competitive equilibria.



## 1. Introduction

The notion of competitive equilibrium plays a fundamental role in economics. In an exchange economy there exists a competitive equilibrium state if there is a price system at which each trader maximizes its utility level and the average endowment equals the average consumption. Arrow and Debreu (1954) have shown the existence of a competitive equilibrium if consumers have convex preferences.

The usual assumption that the traders behave as price takers seems to be unrealistic in case of only finitely many traders. Aumann (1964) gave an exact mathematical model for pure competition by taking a continuum of traders. In the same setting Aumann (1966) demonstrated the existence of a competitive equilibrium without imposing convexity on individual preferences.

Quite a number of researchers investigated the question whether in case of infinitely many agents assuming a continuum of traders is essential in order to demonstrate the existence of a competitive equilibrium. Weiss (1981) considered a finitely additive measure space above the set of traders, Brown and Robinson (1975) investigated nonstandard exchange economies, while Armstrong and Richter (1986) took coalitional preferences on a Boolean algebra. It has to be emphasized that these frameworks allow the set of traders to be countably infinite.

Recently, Khan et al. (2020) provided a counterexample to Weiss (1981) existence result by showing that as long as the measure space of traders is not countably additive, one can find an economy without competitive equilibria. Furthermore, they investigated the impact of relaxing $\sigma$-additivity to merely finite additivity. In contrast to previous works they assumed that the traders' types are given by their endowments and utility functions instead of their endowments and preferences. The latter point seems minor, but subtracting the same value $\varepsilon$ from each trader's utility function implies a kind of uniform measurement on a cardinal scale, whereas this is not the case if traders' preferences are given.

Tasnádi (2002) followed a different route, since subtracting the same $\varepsilon$ from each trader's budget set appears to be less problematic since money is measured naturally on a cardinal scale. In this paper we investigate the relationship between the $\varepsilon$-competitive equilibrium by Khan et al. (2020) and the $\varepsilon$-competitive equilibrium refining the budget set in the spirit of Tasnádi (2002).

## 2. The framework

We will be working in a Euclidean space $\mathbf{R}^d$; the dimensionality $d$ of the space represents the number of different commodities being traded in the market. The set of price vectors is $P = \mathbf{R}^d_+ \setminus \{\theta\}$. We use subscripts to denote coordinates. For any vectors $x, y \in \mathbf{R}^d$ we write $x \gg y$ if $x_i > y_i$ and $x \geq y$ if $x_i \geq y_i$ for all $i \in \{1, \ldots, d\}$. We write $x > y$ to mean $x \geq y$ but not $x = y$. Let $xy = \sum_{i=1}^d x_i y_i$ and $\|x\| = \max_{1 \leq i \leq d} |x_i|$. The symbol $\theta$ denotes the origin in $\mathbf{R}^d$. The *consumption set* is $\mathbf{R}^d_+ = \left\{x \in \mathbf{R}^d \mid x \geq \theta\right\}$ and the *set of utility functions* is $\mathcal{U} = \mathbf{R}^{\mathbf{R}^d_+}$.

We shall denote by $T$ the non-empty set of traders, by $\mathcal{T}$ an algebra above the set of traders, and by $\mu$ a non-atomic finitely additive measure above $(T, \mathcal{T})$. A measurable function $f : T \to \mathbf{R}^d$, such as that for each trader $t \in T$ we have $f(t) \in \mathbf{R}^d_+$, is called an *assignment*. We shall denote the set of assignments by $\mathcal{A}$.

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An *exchange economy* is a map $e : T \to \mathcal{U} \times \mathbf{R}^d_+$, where for any $t \in T$ let its utility function $u_t$ be the projection of $e(t)$ onto $\mathcal{U}$ and its endowment $\omega(t)$ the projection of $e(t)$ onto $\mathbf{R}^d_+$. We assume that $\omega$ is a measurable mapping. Then an *allocation* is an assignment $f : T \to \mathbf{R}^d_+$ for which $\int_T f(t)d\mu(t) = \int_T \omega(t)d\mu(t)$. $\mathcal{F}(e)$ stands for the set of allocations of exchange economy $e$. Let us call an ordered pair $(f, p) \in \mathcal{A} \times P$ a *state* of an economy $e$.

Let $\delta' > \delta > 0$ be fixed reals. We assume that the utility functions of the traders satisfy the following conditions:

- $\delta$-*monotonicity*: for all $t \in \mathcal{T}$ and all $x, y \in \mathbf{R}^d_+$ we have $x > y \Rightarrow u_t(x) > u_t(y) + \delta\|x - y\|$,
- $\delta'$-*monotonicity*: for all $t \in \mathcal{T}$ and all $x, y \in \mathbf{R}^d_+$ we have $x > y \Rightarrow u_t(x) < u_t(y) + \delta'\|x - y\|$,
- *continuity*: for all $t \in T$ the utility functions $u_t$ are continuous,
- *measurability*: for all $f, g \in \mathcal{A}$ we have $\{t \in T \mid u_t(f(t)) > u_t(g(t))\} \in \mathcal{T}$.

In contrast to most of the results on economies with infinitely many traders Khan et al. (2020) assume that traders' preferences are given by their utility functions $(u_t)_{t\in T}$, which makes a difference when defining their notion of $\varepsilon$-competitive equilibrium since $\varepsilon$ is uniform above the set of traders and is subtracted from all utility functions in their definition. However, usually employing different measurements of scale should not effect the equilibrium. Therefore, to be in line with this stronger requirement, when defining our other type of $\varepsilon$-competitive equilibrium, which is related to relaxing the condition on the budget constraints, we need the stronger notions of $\delta$-monotonicity and $\delta'$-monotonicity, where the latter is not needed in the proof of Proposition 1. If one focuses on traders' preferences, our two monotonicity assumptions can be assured by appropriate monotonic transformations of the traders' utility functions.

The first notion of $\varepsilon$-competitive equilibrium investigated in Khan et al. (2020), which we call $\varepsilon$-utility competitive equilibrium, relaxes the requirement on the optimality of the utility level. For an $\varepsilon > 0$ we shall denote by

$$U_t^\varepsilon = \{x \in B_t(p) \mid \forall y \in B_t(p) : u_t(x) \geq u_t(y) - \varepsilon\}$$

the $\varepsilon$-optimal bundles of traders $t \in T$, where

$$B_t(p) = \left\{x \in \mathbf{R}^d_+ \mid px \leq p\omega(t)\right\}.$$

**Definition 1.** For an $e = (u, \omega)$ and $\varepsilon > 0$ the state $(f, p)$ is an $\varepsilon$-utility competitive equilibrium if $p \in \mathbf{R}^d_+ \setminus \{\theta\}$, $f \in \mathcal{F}(e)$, $f(t) \in B_t(p)$ for all $t$ and there exists a $T_\varepsilon \in \mathcal{T}$ such that:

1. $\mu(T_\varepsilon) \leq \varepsilon$ and
2. $f(t) \in U_t^\varepsilon$ for any $t \in T_\varepsilon^c$.

We shall denote the set of such states by $W_\varepsilon^u(e)$.

We turn to the second definition of $\varepsilon$-competitive equilibrium, which we call $\varepsilon$-budget competitive equilibrium and which relaxes the optimal consumer-choice by neglecting other choices within the $\varepsilon$ neighborhood of the budget constraint. For any $\varepsilon > 0$ let

$$B_t^\varepsilon = \left\{x \in \mathbf{R}^d_+ \mid px \leq p\left(\omega(t) - \mathbf{1}\varepsilon\right)\right\} \text{ and}$$
$$C_t^\varepsilon = \left\{x \in \mathbf{R}^d_+ \mid px > p\left(\omega(t) - \mathbf{1}\varepsilon\right)\right\}$$

the $\varepsilon$-budget set of trader $t \in T$ and its complement, respectively, where $\mathbf{1} = (1, \ldots, 1) \in \mathbf{R}^d_+$.

**Definition 2.** For an $e = (u, \omega)$ and an $\varepsilon > 0$ the state $(f, p)$ is an $\varepsilon$-*budget competitive equilibrium* if $p \in \mathbf{R}^d_+ \setminus \{\theta\}$, $f \in \mathcal{F}(e)$, $f(t) \in B_t(p)$ for all $t$ and there exists a $T_\varepsilon \in \mathcal{T}$ such that:

1. $\mu(T_\varepsilon) \leq \varepsilon$ and
2. for all $t \in T_\varepsilon^c$ and for all $g \in \mathcal{A}$ the relation $u_t(g(t)) > u_t(f(t))$ implies $p \cdot g(t) > p \cdot (\omega(t) - \mathbf{1}\varepsilon)$.

We shall denote the set of such states by $W_\varepsilon^b(e)$.

## 3. The relationship between the two approaches

Looking at the two definitions of $\varepsilon$-competitive equilibrium, we can see that there is only a difference in points 2 of Definitions 1 and 2. Therefore, first we compare the respective conditions imposed on each trader separately.

**Lemma 3.1.** *For any $t \in T$, any $p \in P$ and any $\kappa > 0$ there exists an $\varepsilon > 0$ such that*

$$\begin{aligned}&\{x \in B_t(p) \mid \forall y \in B_t(p) : u_t(x) \geq u_t(y) - \varepsilon\}\\ &\subseteq \left\{x \in \mathbf{R}^d_+ \mid px > p\left(\omega(t) - \mathbf{1}\kappa\right)\right\}.\end{aligned} \tag{3.1}$$

**Proof.** We are done if $U_t^\varepsilon = \emptyset$. Assume that $U_t^\varepsilon \neq \emptyset$, which implies that $B_t(p)$ is bounded by $\delta$-monotonicity. Then we prove the statement by contradiction. Suppose that there exists a $t \in T$ and a $\kappa > 0$ such that for all $\varepsilon > 0$ we can find an

$$x \in U_t^\varepsilon \setminus C_t^\kappa \Leftrightarrow x \in U_t^\varepsilon \cap B_t^\kappa. \tag{3.2}$$

Let $y^* \in B_t(p)$ be a utility maximizing bundle and let $u^* = u_t(y^*)$. Pick a monotonically decreasing null sequence $(\varepsilon^{(i)})_{i=1}^\infty$. Then by (3.2) there exists a sequence of bundles $(x^{(i)})_{i=1}^\infty$ within $U_t^{\varepsilon^{(i)}} \cap B_t^\kappa$ converging to $x^*$ such that $\lim_{i\to\infty} u_t(x^{(i)}) = u_t(x^*) = u^*$ by the continuity of $u_t$. Note that a higher utility level than $u^*$ can be attained in the 'northeast direction' from $x^*$ at the budget constraint by the monotonicity of $u_t$, which is in contradiction with the definition of $u^*$. □

For a fixed trader $t \in T$, a fixed price $p \in P$ and a given $\kappa > 0$ we shall denote by $\varepsilon_{t,p}(\kappa)$ the supremum of $\varepsilon$s satisfying (3.1). Observe that

$$U_t^{\varepsilon_{t,p}(\kappa)} \cap \left\{x \in \mathbf{R}^d_+ \mid px = p\left(\omega(t) - \mathbf{1}\kappa\right)\right\} \neq \emptyset$$

by the continuity and monotonicity of the utility functions.

**Lemma 3.2.** *If $p \gg \theta$, then $\varepsilon_{t,p}(\kappa)$ decreases as $\kappa$ decreases and $\lim_{\kappa\to 0} \varepsilon_{t,p}(\kappa) = 0$ for any $t \in T$.*

**Proof.** Let $t \in T$ be a given trader and $p \in P$ be a given price, hence we omit the subscripts $t$ and $p$ of $\varepsilon$.

(i) Since if $\kappa > \kappa'$ and a pair $(\kappa', \varepsilon)$ satisfies (3.1), then the pair $(\kappa, \varepsilon)$ also satisfies (3.1), the first statement of the lemma follows.

(ii) Since $\varepsilon(\kappa)$ decreases as $\kappa$ decreases $\lim_{\kappa\to 0} \varepsilon(\kappa)$ exists. Suppose that $\lim_{\kappa\to 0} \varepsilon(\kappa) = \varepsilon^* > 0$. Let again $y^* \in B_t(p)$ be a utility maximizing bundle and let $u^* = u_t(y^*)$. Then by the monotonicity of $u_t$ the set $U_t^{\varepsilon^*} \cap B_t(p)$ has a positive hyper-volume to the northeast, which is in contradiction with $\kappa \to 0$ and the definition of $\varepsilon(\kappa)$. □

**Proposition 1.** *If $W_\varepsilon^u(e) \neq \emptyset$ for all $\varepsilon > 0$, then $W_\kappa^b(e) \neq \emptyset$ for all $\kappa > 0$.*

**Proof.** Let $\Delta = \{p \in P \mid \mathbf{1}p = 1 \text{ and } p \gg \theta\}$. Pick $\kappa > 0$ arbitrarily and let

$$\varepsilon^* = \inf\left\{\varepsilon_{t,p}(\kappa) \mid t \in T \text{ and } p \in \Delta\right\}.$$

We claim that $\varepsilon^* > 0$. Suppose that $\varepsilon^* = 0$. Then for any $p \in \Delta$ and any $t \in T$ there exists an $x_t \in U_t^{\varepsilon_{t,p}(\kappa)} \cap \left\{x \in \mathbf{R}^d_+ \mid px = p\left(\omega(t) - \mathbf{1}\kappa\right)\right\}$. Note that $\|p\| \leq 1$ and let $y_t = x_t + e_i\kappa \in B_t(p)$, where $e_i$ denotes the $i$th unit vector. By $\delta$-monotonicity we have

$$u_t(y_t) - u_t(x_t) > \delta|y_t - x_t| = \delta\kappa \tag{3.3}$$

and by $x_t, y_t \in U_t^{\varepsilon_{t,p}(\kappa)} \cap B_t(p)$ we have

$$u_t(y_t) - u_t(x_t) \le \varepsilon_{t,p}(\kappa), \tag{3.4}$$

and therefore,

$$0 < \delta\kappa < \varepsilon_{t,p}(\kappa), \tag{3.5}$$

which in turn will be violated by infinitely many $(t,p) \in T \times \Delta$ by our indirect assumption $\varepsilon^* = 0$, yielding a contradiction.

Select an $(f,p) \in W^u_{\varepsilon^*}(e)$, where $p \gg \theta$ by $\delta$-monotonicity. Hence, $\mu(T_{\varepsilon^*}) \le \varepsilon^*$ and $f(t) \in U_t^{\varepsilon^*}$ for all $t \in T^c_{\varepsilon^*}$.

First, if $\varepsilon^* \le \kappa$, then for any $t \in T^c_{\varepsilon^*}$ and any $g \in \mathcal{A}$ we must have that $u_t(g(t)) > u_t(f(t))$ implies $p \cdot g(t) > p \cdot (\omega(t) - \mathbf{1}\kappa)$ because of $\varepsilon^* \le \varepsilon_{t,p}(\kappa)$. Thus, $(f,p) \in W^b_\kappa$.

Second, if $\varepsilon^* > \kappa$, then from Lemma 3.2 it follows that one can start with a sufficiently small positive $\kappa' < \kappa$ so that the respective $\varepsilon'$ associated with $\kappa'$ will be at most as large as $\kappa$, and therefore from an $(f,p) \in W^u_{\varepsilon'}(e)$ we can get $(f,p) \in W^b_\kappa$. □

Now we turn to the converse statement. Trivially, the following lemma holds true since for any given $\varepsilon > 0$ a sufficiently large $\kappa > 0$ does the job.

**Lemma 3.3.** *For any $t \in T$, any $p \in P$ and any $\varepsilon > 0$ there exists a $\kappa > 0$ such that*

$$\begin{aligned} &\{x \in B_t(p) \mid \forall y \in B_t(p) : u_t(x) \ge u_t(y) - \varepsilon\} \\ &\subseteq \left\{x \in \mathbf{R}^d_+ \mid px > p\left(\omega(t) - \mathbf{1}\kappa\right)\right\}. \end{aligned} \tag{3.6}$$

For a fixed trader $t \in T$ and a given $\varepsilon > 0$ and $p \in P$ we shall denote by $\kappa_{t,p}(\varepsilon)$ the infimum of $\kappa$s satisfying (3.6).

**Lemma 3.4.** *If $p \gg \theta$, then $\kappa_{t,p}(\varepsilon)$ decreases as $\varepsilon$ decreases and $\lim_{\varepsilon\to 0} \kappa_{t,p}(\varepsilon) = 0$ for any $t \in T$.*

**Proof.** Let $p \in P$ and $t \in T$ be given, hence we omit the subscripts $t$ and $p$ of $\kappa$.

(i) Since if $\varepsilon > \varepsilon'$ and a pair $(\kappa, \varepsilon)$ satisfies (3.6), then the pair $(\kappa, \varepsilon')$ also satisfies (3.6), the first statement of the lemma follows.

(ii) Since $\kappa(\varepsilon)$ decreases as $\varepsilon$ decreases $\lim_{\varepsilon\to 0}\kappa(\varepsilon)$ exists. Suppose that $\lim_{\varepsilon\to 0}\kappa(\varepsilon) = \kappa^* > 0$. Let again $y^* \in B_t(p)$ be a utility maximizing bundle, which has to lie at the boundary of $B_t(p)$ by the monotonicity of $u_t$, and let $u^* = u_t(y^*)$. Then the set $U_t^\varepsilon \cap C_t^{\kappa^*} \subseteq U_t^\varepsilon \cap C_t^{\kappa(\varepsilon)}$ has a positive hyper-volume, which is even bounded from below by a positive value, for any $\varepsilon > 0$, and therefore we can achieve utility $u^*$ in the interior of $B_t(p)$ contradicting the monotonicity of $u_t$. □

**Proposition 2.** *If $W^b_\kappa(e) \ne \emptyset$ for all $\kappa > 0$, then $W^u_\varepsilon(e) \ne \emptyset$ for all $\varepsilon > 0$.*

**Proof.** Let $\Delta = \{p \in P \mid \mathbf{1}p = 1 \text{ and } p \gg \theta\}$. Pick $\varepsilon > 0$ arbitrarily and let $\kappa^* = \inf\left\{\kappa_{t,p}(\varepsilon) \mid t \in T \text{ and } p \in \Delta\right\}$.

We claim that $\kappa^* > 0$. Suppose that $\kappa^* = 0$. Pick $p \in \Delta$, $t \in T$, $y_t$ a utility-maximizing bundle within $B_t(p)$ and $x_t \in U_t^\varepsilon \cap \left\{x \in \mathbf{R}^d_+ \mid px \ge p\left(\omega(t) - \mathbf{1}\kappa_{t,p}(\varepsilon)\right)\right\}$ to the 'south-east' of $y_t$ so that $u_t(y_t) - u_t(x_t) = \varepsilon$. By $\delta'$-monotonicity we have

$$\varepsilon = u_t(y_t) - u_t(x_t) < \delta'\|y_t - x_t\| \le \delta'\kappa_{t,p}(\varepsilon), \tag{3.7}$$

which cannot be the case since for infinitely many $(t,p) \in T \times P$ the right-hand side will be less than $\varepsilon$ by $\kappa^* = 0$; a contradiction.

Select an $(f,p) \in W^b_{\kappa^*}(e)$, where $p \gg \theta$ by $\delta$-monotonicity. Then for all $t \in T^c_{\kappa^*}$ we have that $u_t(g(t)) > u_t(f(t))$ implies $g(t) \in C_t^{\kappa^*}$. Since $U_t^\varepsilon$ intersects $B_t^{\kappa_{t,p}(\varepsilon)}$ and the indifference curve going through $f(t)$ lies in the closure of $C_t^{\kappa^*}$ it follows that $f(t) \in U_t^\varepsilon$.

First, if $\varepsilon \ge \kappa^*$, then we are done and $(f,p) \in W^u_\varepsilon$. Second, if $\varepsilon < \kappa^*$, then from Lemma 3.4 it follows that one can start with a sufficiently small positive $\varepsilon' < \varepsilon$ so that the respective $\kappa^*$ associated with $\varepsilon'$ will be at most as large as $\varepsilon$ so that we arrive to $W^u_\varepsilon \ne \emptyset$. □

## 4. Concluding remarks

From Proposition 1 it follows that whenever for all $\varepsilon > 0$ the $\varepsilon$-competitive equilibria defined by Khan et al. (2020) are nonempty the $\varepsilon$-competitive equilibria introduced in this paper are also nonempty for all $\varepsilon > 0$. Khan et al. (2020) provide conditions on the existence of their $\varepsilon$-competitive equilibrium.

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