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Uniqueness of Clearing Payment Matrices in Financial Networks

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Abstract. We study bankruptcy problems in financial networks in the presence of general bankruptcy laws. The set of clearing payment matrices is shown to be a lattice, which guarantees the existence of a greatest clearing payment and a least clearing payment. Multiplicity of clearing payment matrices is both a theoretical and a practical concern. We present a new condition for uniqueness that generalizes all the existing conditions proposed in the literature. Our condition depends on the decomposition of the financial network into strongly connected components. A strongly connected component that contains more than one agent is called a cycle, and the involved agents are called cyclical agents. If there is a cycle without successors, then one of the agents in such a cycle should have a strictly positive endowment. The division rule used by a cyclical agent with a strictly positive endowment should be positive monotonic, and the rule used by a cyclical agent with a zero endowment should be strictly monotonic. Because division rules involving priorities are not positive monotonic, uniqueness of the clearing payment matrix is a much bigger concern for such division rules than for proportional ones. As a final contribution of the paper, we exhibit the relationship between the uniqueness of clearing payment matrices and the continuity of bankruptcy rules, a property that is very much desired for stability of financial systems.

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Keywords: financial networks • systemic risk • bankruptcy rules • fixed points

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1. Introduction

Over the last decades, financial networks have become increasingly interconnected, and the size of mutual financial obligations has become ever larger, thereby making systemic risk a more and more relevant concern. The standard analysis of systemic risk is based on the seminal work by Eisenberg and Noe [11]. This model has resulted in a large and rich literature, either extending it (Cifuentes et al. [6], Rogers and Veraart [24], Schuldenzucker et al. [25], Shin [26]) or using it to relate the number and magnitude of defaults to the network topology (Acemoglu et al. [1], Cabrales et al. [2], Capponi et al. [3], Elliott et al. [12], Gai and Kapadia [15], Glasserman and Young [16]) or measuring systemic risk (Chen et al. [5], Demange [10]). The model has been fruitfully applied to the assessment of the systemic stability of financial systems; see, for instance, Elsinger et al. [13] for an application to the Austrian banking system. For overviews of this stream of the literature, we refer to the excellent surveys by Glasserman and Young [17] and Jackson and Pernoud [20].

In the Eisenberg and Noe [11] model, agents have endowments that include all the agents' tangible and intangible assets but exclude the claims and liabilities agents have toward the other agents. The mutual claims and liabilities between agents are given by a liability matrix. The asset value of an agent is obtained as the sum of the agent's endowments together with the payments received from the other agents who settle their liabilities. The equity of an agent is equal to the asset value minus the payments made to the other agents. It may well be that

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the asset value of an agent is not sufficient to cover all the agent's liabilities, in which case the agent has to default. Specific to the network setting is that the default of one agent has negative consequences for the asset value of other agents, which might result in those agents defaulting as well, a phenomenon known as contagion.

A clearing payment matrix describes how much the agents pay to each other. It needs to be consistent with the prevailing bankruptcy laws and should satisfy limited liability and priority of creditors. Limited liability imposes that the clearing payment matrix should not lead to negative equity for any of the agents. Priority of creditors is satisfied if the only circumstance under which an agent is allowed to default is when the agent has zero equity. Eisenberg and Noe [11] consider the case where the bankruptcy law prescribes that claimants should be paid an amount that is proportional to their claims and provide sufficient conditions that lead to a uniquely determined clearing payment matrix.

Although the principle of proportionality is important in actual bankruptcy law, Chatterjee and Eyigungor [4], Flores-Szwagrzak [14], and Moulin [23] argue priority to be another important principle. Kaminski [21] explains that American bankruptcy law is a mixed lexicographic-proportional system. In this paper, we study the uniqueness of clearing payment matrices under general bankruptcy laws. We represent bankruptcy laws by division rules, which describe how insolvent agents should make payments to their claimants as a function of their asset value.

We present a system of equations with the property that a payment matrix is a clearing payment matrix if and only if it is a solution to the system of equations. This system is then used to establish that the set of clearing payment matrices has a lattice structure, which implies that there always exists a greatest clearing payment matrix and a least clearing payment matrix. Our research question is to find conditions such that the clearing payment matrix is unique, which implies that the greatest and least clearing payment matrices coincide.

The uniqueness of clearing payment matrices is of great theoretical and practical concern. Without uniqueness of the clearing payment matrix, bankruptcy laws together with the principles of limited liability and priority of creditors are not sufficient to pin down the payments that agents should make to each other. As argued in Jackson and Pernoud [19], the existence of multiple clearing payment matrices contributes to the fragility of the financial system as a consequence of endogenous uncertainty. Pessimistic beliefs about the payments to be received from other agents can become self-fulfilling and lead to a credit freeze, where agents stop making payments to each other. To make things worse as argued in Csóka and Herings [8], under a large class of decentralized clearing processes, payments are made according to the least clearing payment matrix.

The uniqueness of the clearing payment matrix turns out to depend on the type of monotonicity of the division rules. Strict monotonicity requires that an increase in asset value of a defaulting agent leads to strictly higher payments to all claimants with a nonzero claim. Positive monotonicity weakens strict monotonicity and requires that an increase in asset value of a defaulting agent leads to strictly higher payments to all claimants receiving a strictly positive payment at the original asset value. Proportional division rules are strictly monotonic, and constrained equal losses division rules are positive monotonic; however, priority division rules and constrained equal awards division rules are not even positive monotonic.

Our sufficient condition for uniqueness of the clearing payment matrix depends on the decomposition of the liability matrix into strongly connected components. A strongly connected component is a maximal set of agents such that for any two distinct agents in the set, there is a chain of liabilities from one agent to the other (i.e., any two distinct agents in a strongly connected component are successors of each other). A strongly connected component that consists of more than one agent is called a cycle. An agent that is part of a cycle is called a cyclical agent.

We demonstrate that the clearing payment matrix is unique if the following three properties are satisfied. First, every cycle without successors contains at least one agent with a strictly positive endowment. Second, any cyclical agent with a strictly positive endowment uses a positive monotonic division rule. Third, any cyclical agent with a zero endowment uses a strictly monotonic division rule. Notice that our sufficient condition for uniqueness only puts assumptions on the cyclical agents.

Our sufficient condition for uniqueness generalizes existing conditions that are found in the literature, even for the case where all agents use proportional division rules. Eisenberg and Noe [11] assume proportional division rules and require the financial network to be regular; every agent has at least one successor with a strictly positive endowment. Koster [22] generalizes this finding and shows that regularity is a sufficient condition for uniqueness of the clearing payment matrix when all agents use strictly monotonic division rules. Regularity implies that in every cycle without successors, there is at least one agent with a strictly positive endowment. The other two properties required in our uniqueness condition are trivially satisfied when all agents use proportional or more generally, strictly monotonic division rules.

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Glasserman and Young [16] also study the case of proportional division rules and obtain uniqueness by assuming the existence of an outside sector, which is such that every agent has direct or indirect liabilities to an agent in the outside sector. We show how the outside sector can be represented by a single agent in the network, who has no liabilities to other agents in the network. It then follows that there are no cycles in the network without successors. Because moreover, proportional division rules are strictly monotonic, our uniqueness condition is trivially satisfied.

Groote Schaarsberg et al. [18] call a financial network hierarchical if the liability matrix can be transformed into an upper triangular matrix. They show that the clearing payment matrix is unique in hierarchical financial networks without further assumptions on the division rules. Because a financial network is hierarchical if and only if there are no cycles, their condition for uniqueness is subsumed by ours.

Our uniqueness condition is quite weak when all agents use proportional division rules because then, it only requires that in cycles without successors, at least one agent has a strictly positive endowment. On the contrary, when agents use priority division rules, we present an example with a multiplicity of clearing payment matrices even when all agents have strictly positive endowments.

We conclude the paper by examining a class of bankruptcy rules, which are defined by associating with each financial network its greatest clearing payment matrix. Without continuity, an arbitrarily small change in the endowments or the liabilities can lead to a complete disruption of the payment matrix, which poses a threat to the stability of the financial system. We show that uniqueness of clearing payment matrices is a sufficient condition for continuity of such bankruptcy rules. We demonstrate by means of an example how multiplicity of payment matrices leads to a lack of continuity. We also present an example to show that our sufficient condition is not necessary.

This paper is organized as follows. Section 2 presents the model of financial networks, the various forms of monotonicity of division rules, and the definition of a clearing payment matrix. Section 3 is devoted to the lattice structure of the set of clearing payment matrices. Section 4 presents an example to show that under priority division rules, multiplicity of payment matrices can occur even when all endowments are strictly positive. This section also presents the main result of the paper: the sufficient condition for clearing payment matrices to be unique. Section 5 discusses the relation to other conditions for uniqueness that are found in the literature. We examine the connection between the uniqueness of clearing payment matrices and continuity of bankruptcy rules in Section 6. Finally, Section 7 presents the conclusion. All proofs except those related to Section 5 are relegated to Section 7.

2. Financial Networks

A financial network N is a quadruple (I, z, L, d) with the following interpretation. The set of *agents* in the financial network is given by the finite set I. The vector $z \in \mathbb{R}_+^I$ represents the nonnegative *endowments* of the agents, which include all the agents' tangible and intangible assets but exclude the claims and liabilities agents have toward each other. The *liability matrix* $L \in \mathbb{R}_+^{I \times I}$ describes the mutual claims of the agents. Its entry L_{ij} is the liability of agent i toward agent j or equivalently, the claim of agent j on agent i. In general, it can occur that agent i has a liability toward agent j and vice versa, so it may happen that simultaneously $L_{ij} > 0$ and $L_{ji} > 0$. We make the normalizing assumption that $L_{ii} = 0$. The total liabilities of agent $i \in I$ are denoted by $\overline{L}_i = \sum_{j \in I} L_{ij}$.

The determination of the payments to the agents takes place by means of division rules, $d = (d^i)_{i \in I}$. A division rule d^i describes which payments agent i makes to claimants in I as a function of the estate $E_i \in \mathbb{R}_+$ of agent i. More formally, the division rule of agent $i \in I$ is a function $d^i : \mathbb{R}_+ \to \mathbb{R}_+^I$ such that, for every $j \in I$, $d_j^i(E_i) \leq L_{ij}$ and $\sum_{j \in I} d_j^i(E_i) = \min\{E_i, \overline{L}_i\}$. Because $L_{ii} = 0$, it follows from the definition of a division rule that $d_i^i(E_i) = 0$. We assume that, for every $i \in I$, d^i is monotonic. More precisely, for every $j \in I$, for every $E_i, E_i' \in \mathbb{R}_+$ such that $E_i \leq E_i'$, it holds that $d_j^i(E_i) \leq d_j^i(E_i')$. It is well known that if d^i is monotonic, then it is continuous; see, for instance, Thomson [28].

Next, we define additional monotonicity conditions that will be useful in the analysis of the uniqueness of clearing payment matrices. The asset value at which agent i starts to make payments to an agent $j \in I$ with $L_{ij} > 0$ is denoted by $\underline{a}_{ij}$, so $d_j^i(E_i) = 0$ if $E_i \leq \underline{a}_{ij}$ and $d_j^i(E_i) > 0$ if $E_i > \underline{a}_{ij}$.

Definition 1. Let $N = (I, z, L, d)$ be a financial network. The division rule $d^i : \mathbb{R}_+ \to \mathbb{R}_+^I$ of agent $i \in I$ is strictly monotonic if, for every $j \in I$ such that $L_{ij} > 0$, for every $E_i, E_i' \in \mathbb{R}_+$ such that $0 \leq E_i < E_i' \leq \overline{L}_i$, it holds that $d_j^i(E_i) < d_j^i(E_i')$. It is *positive monotonic* if, for every $j \in I$ such that $L_{ij} > 0$, for every $E_i, E_i' \in \mathbb{R}_+$ such that $\underline{a}_{ij} \leq E_i < E_i' \leq \overline{L}_i$, it holds that $d_j^i(E_i) < d_j^i(E_i')$.

Strict monotonicity requires that an increase in the asset value of a defaulting agent leads to strictly higher payments to all claimants with a nonzero claim. The proportional rule is an example of strictly monotonic division

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rule. Positive monotonicity weakens strict monotonicity and requires that an increase in asset value of a defaulting agent leads to strictly higher payments to all claimants receiving a strictly positive payment at the original asset value. An example of a positive monotonic division rule is the constrained equal losses rule. The constrained equal awards rule is neither strictly monotonic nor positive monotonic. Formal definitions of these division rules follow later in this section.

An easy way to check the various monotonicity conditions is as follows. If we plot the payments chosen by the division rule for each claimant as a function of the estate, then strict monotonicity requires these plots to be strictly increasing, whereas positive monotonicity requires these plots to be strictly increasing as soon as they assign a strictly positive payment.

We now proceed by giving formal definitions of some important division rules: the proportional rule, the priority rule, the constrained equal awards rule, and the constrained equal losses rule.

The division rule d^i of agent $i \in I$ is equal to the *proportional rule* if it assigns to claimant $j \in I$ the amount

$$d^i_j(E_i) = \begin{cases} 0, & \text{if } L_{ij} = 0, \\ \min\left\{\dfrac{L_{ij}}{\overline{L}_i} E_i, L_{ij}\right\}, & \text{otherwise.} \end{cases}$$

Under the proportional division rule, the estate is divided in a proportional way over the claimants, up to the value of their claims.

The division rule d^i of agent $i \in I$ is equal to a *priority rule* if there exists a permutation $\pi : I \to \{1, \ldots, |I|\}$ such that the amount assigned to claimant $j \in I$ is equal to

$$d^i_j(E_i) = \max\left\{0, \min\left\{L_{ij}, E_i - \sum_{\{k \in I \mid \pi(k) < \pi(j)\}} L_{ik}\right\}\right\},$$

where $\{k \in I \mid \pi(k) < \pi(j)\}$ is the set of agents ranked before j according to π. Under the priority division rule, the claim of agent $j_1 = \pi^{-1}(1)$ is the first one to be paid; if there is any remaining estate (i.e., $E_i - L_{ij_1} > 0$), then the claim of agent $j_2 = \pi^{-1}(2)$ is paid next and so on.

Another example of a division rule is the constrained equal awards rule. If $E_i > \overline{L}_i$, then define $\lambda_i = \max_{j \in I} L_{ij}$. Otherwise, define $\lambda_i \in [0, \max_{j \in I} L_{ij}]$ as the unique solution to

$$\sum_{j \in I} \min\{L_{ij}, \lambda\} = E_i.$$

The *constrained equal awards rule* assigns to claimant $j \in I$ the amount

$$d^i_j(E_i) = \min\{L_{ij}, \lambda_i\}.$$

Under the constrained equal awards rule, all claimants get the same amount, up to the value of their claim.

The constrained equal losses division rule is the dual of the constrained equal awards rule and imposes that all claimants face the same loss, up to the value of their claim. If $E_i > \overline{L}_i$, then define $\mu_i = 0$. Otherwise, define $\mu_i \in [0, \max_{j \in I} L_{ij}]$ as the unique solution to

$$\sum_{j \in I} \max\{L_{ij} - \mu_i, 0\} = E_i.$$

The *constrained equal losses division rule* of agent i assigns to claimant $j \in I$ the amount

$$d^i_j(E_i) = \max\{L_{ij} - \mu_i, 0\}.$$

The analysis of financial networks is complicated because of the mutual liability structure and the contagion effects of default. The determination of the value of the estate E_i of agent $i \in I$ is endogenous as this value depends on the payments that are collected from the claims agent i has on the other agents.

Let $\mathcal{M}$ be the set of all matrices in $\mathbb{R}^{I \times I}_+$ with a zero diagonal. For $M \in \mathcal{M}$ and $i \in I$, let $M_i \in \mathbb{R}^I$ denote row i of M. For $M_i, M'_i \in \mathbb{R}^I$, we write $M_i < M'_i$ if $M_{ij} \leq M'_{ij}$ for all $j \in I$, and there is $j' \in I$ such that $M_{ij'} < M'_{ij'}$. Notice that the liability matrix L is an element of $\mathcal{M}$.

A *payment matrix* $P \in \mathcal{M}$ describes the mutual payments to be made by the agents: that is, P_{ij} is the monetary amount to be paid by agent $i \in I$ to agent $j \in I$. Under payment matrix $P \in \mathcal{M}$, the estate E_i of agent i is given by the *asset value* $a_i(P)$, defined as

$$a_i(P) = z_i + \sum_{j \in I} P_{ji}.$$

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Subtracting the payments as made by an agent from his asset value yields an agent's equity. More formally, the *equity* $e_i(P)$ of an agent $i \in I$ is given by

$$e_i(P) = a_i(P) - \sum_{j \in I} P_{ij} = z_i + \sum_{j \in I} (P_{ji} - P_{ij}).$$

It follows immediately from this expression that the sum over agents of their equities is the same as the sum over agents of their endowments.

We now extend the notions of priority of creditors and limited liability as defined for proportional division rules by Eisenberg and Noe [11] and feasibility for general division rules in a discrete setting by Csóka and Herings [8] to general division rules in the continuous setting studied here.

The image $\mathcal{F}_i$ of division rule d_i determines the feasible set of payments. More formally, we have

$$\mathcal{F}_i = d_i(\mathbb{R}_+) = \{d_i(E_i) \in \mathbb{R}^I_+ \mid E_i \in \mathbb{R}_+\}.$$

A payment matrix $P \in \mathcal{M}$ is feasible if for every $i \in I$, it holds that $P_i \in \mathcal{F}_i$. A payment matrix is feasible if every row i of the payment matrix belongs to the feasible set of payments of agent i, meaning that payments are made in accordance with the division rules. The set of all feasible payment matrices is denoted by $\mathcal{P}$, so

$$\mathcal{P} = \{P \in \mathcal{M} \mid \forall i \in I,\ P_i \in \mathcal{F}_i\}.$$

Definition 2. The matrix $P \in \mathcal{M}$ is a clearing payment matrix of the financial network $N = (I, z, L, d)$ if it satisfies the following three properties.

1. *Feasibility*. $P \in \mathcal{P}$.
2. *Limited liability*. For every $i \in I$, $e_i(P) \geq 0$.
3. *Priority of creditors*. For every $i \in I$, if $P_i < L_i$, then $e_i(P) = 0$.

Feasibility states that the payments are in accordance with the division rules. Limited liability requires that the total payments made by an agent do not exceed the asset value of the agent. Priority of creditors expresses that default is only allowed if equity is equal to zero. The analysis of Eisenberg and Noe [11] corresponds to the case where all agents use the proportional division rule. Csóka and Herings [8] address the general case for a discrete setup.

3. The Lattice of Clearing Payment Matrices

A first question is whether a clearing payment matrix always exists and if so, whether it is unique. To address these issues, we rewrite the conditions of a clearing payment matrix as the solution to a system of equations. Moreover, we show that the set of clearing payments matrices is a complete lattice. Csóka and Herings [8] have shown these results in a discrete setup, where all payments are integer multiples of some fixed unit of account. This section shows such results to remain true in the continuous setting. A final result in this section shows that all clearing payment matrices lead to the same value of equity, thereby slightly generalizing a result in Groote Schaarsberg et al. [18]. As demonstrated in Csóka and Herings [8], uniqueness of equity does not hold in the discrete setup.

The following theorem relates a clearing payment matrix to the solution of a particular system of equations.

Theorem 1. *The payment matrix $P \in \mathcal{M}$ is a clearing payment matrix of the financial network $N = (I, z, L, d)$ if and only if it solves the following system of equations*:

$$P_{ij} = d^i_j(a_i(P)), \quad i, j \in I.$$

Given a payment matrix P, one first computes the asset value $a_i(P)$ of agent $i \in I$, and then, one uses the division rule d^i to determine the payments to the other players. If these payments coincide with row i of the given payment matrix P for all $i \in I$, then P must be a clearing payment matrix and vice versa.

It is straightforward to verify that any solution to the system of equations in Theorem 1 satisfies feasibility, limited liability, and priority of creditors and so, must be a clearing payment matrix. For the other way around, if P is a clearing payment matrix and P_{ij} would fall short of $d_{ij}(a_i(P))$, then one would obtain a violation of priority of creditors. Limited liability can be used to show that P_{ij} cannot exceed $d_{ij}(a_i(P))$.

The system of equations we use to characterize clearing payment matrices is slightly different from the one used in Eisenberg and Noe [11] for proportional division rules and the one in Koster [22] for general division rules. In those two papers, each $|I|$-dimensional row of the payment matrix is represented by a one-dimensional parameter, like the fraction of total liabilities that are paid in Eisenberg and Noe [11].

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A *lattice* is a partially ordered nonempty set in which every pair of elements has a supremum and an infimum. A *complete lattice* is a lattice in which every nonempty subset has a supremum and an infimum. The partial order $\leq$ on $\mathcal{M}$ is defined in the usual way; for $P,\ P' \in \mathcal{M}$, it holds that $P \leq P'$ if and only if $P_{ij} \leq P'_{ij}$ for all $(i,j) \in I \times I$. The following result states that there always exists a least clearing payment matrix and a greatest clearing payment matrix.

Theorem 2. *The set of clearing payment matrices of the financial network* $N = (I,z,L,d)$ *is a complete lattice. In particular, there exists a least clearing payment matrix* P^- *and a greatest clearing payment matrix* P^+.

The literature has a number of results related to Theorem 2. Eisenberg and Noe [11] show that there is a least clearing payment matrix and a greatest clearing payment matrix when all agents use the proportional division rule. Csóka and Herings [8] prove the result of Theorem 2 for a discrete setup and do not allow for zero endowments. All these papers rely on Tarski's fixed point theorem for the proof.

Our next result, a modest generalization of a result by Groote Schaarsberg et al. [18] who assume that all agents use the same division rule, states that all clearing payment matrices lead to the same amount of equity.[1]

Theorem 3. *If* P *and* P' *are clearing payment matrices of a financial network* $N = (I,z,L,d)$, *then* $e(P) = e(P')$.

The result of Theorem 3 depends crucially on the fact that there is no smallest unit of account. Csóka and Herings [8] show that in discrete setups (for instance, resulting from the presence of a smallest unit of account), equity is in general not unique.

4. Uniqueness of Clearing Payment Matrices

Our next research question concerns the uniqueness of a clearing payment matrix.

Eisenberg and Noe [11, theorem 2] implies that in a financial network where all agents have strictly positive endowments and use proportional division rules, the clearing payment matrix is unique. Surprisingly, when replacing proportional division rules by priority division rules, the clearing payment matrix need not be unique as Example 1 demonstrates.

Example 1. Let $N = (I,z,L,d)$ be a financial network with four agents, $I = \{1,2,3,4\}$, all using the priority division rule corresponding to $\pi = (4,3,2,1)$. Tables 1 and 2 present the endowments, the liabilities, the least clearing payment matrix P^-, the greatest clearing payment matrix P^+, and the induced asset values and equities. The sets of defaulting agents are different in P^- and P^+. in P^-, agents 1, 2, and 3 default, whereas in P^+, only agent 3 is insolvent.

In Example 1, the solvent agent, agent 4, makes the same payments in P^- and P^+. All defaulting agents in P^-, agents 1, 2, and 3, make different payments in P^- and P^+. The next two propositions generalize these insights.

Proposition 1. *Let* $N = (I,z,L,d)$ *be a financial network, and let* P^- *and* P^+ *be the least and greatest clearing payment matrices of* N, *respectively. If* $i \in I$ *satisfies* $P_i^- < P_i^+$, *then* $e_i(P^-) = e_i(P^+) = 0$.

According to Proposition 1, if an agent makes different payments in two clearing payment matrices, then this agent has zero equity. This result follows immediately from the priority of creditors condition.

The next proposition shows how differences in payments by some agents propagate in the financial network. It provides conditions such that if there is a sequence of insolvent connected agents and the first agent pays more in the greatest than in the least clearing payment matrix to the second agent, then every agent in the sequence pays more in the greatest than in the least clearing payment matrix to the next agent.

For any two consecutive agents in the sequence, it is assumed that the former agent makes a strictly positive payment to the latter agent in the greatest clearing payment matrix. The agents in the sequence are also assumed to employ positive monotonic division rules. This will imply that when the asset value of an agent in the sequence is

Table 1. The clearing payment matrix P^- and its induced asset values and equities in Example 1 with priority division rules.

z	L				P^-				$a(P^-)$	$e(P^-)$
1	0	8	0	0	0	2	0	0	2	0
1	0	0	10	0	0	0	4	0	4	0
1	7	0	0	5	0	0	0	5	5	0
1	1	1	0	0	1	1	0	0	6	4

Table 2. The clearing payment matrix P^+ and its induced asset values and equities in Example 1 with priority division rules.

z	L				P^+				$a(P^+)$	$e(P^+)$
1	0	8	0	0	0	8	0	0	8	0
1	0	0	10	0	0	0	10	0	10	0
1	7	0	0	5	6	0	0	5	11	0
1	1	1	0	0	1	1	0	0	6	4

strictly below the asset value at the greatest clearing payment matrix, then strictly less is paid to the next agent in the sequence, implying that the asset value of that agent is strictly below the asset value at the greatest clearing payment matrix. Lower asset values then propagate across the sequence and lead to lower payments.

Proposition 2. *Let $N=(I,z,L,d)$ be a financial network; P^- and P^+ be the least and greatest clearing payment matrices of N, respectively; and $i_1\in I$ be such that $P^-_{i_1}<P^+_{i_1}$. Let $(i_1,\ldots,i_{k'})$ be a sequence of agents in I such that, for every $k=1,\ldots,k'-1$,*

$$d^{i_k} \text{ is positive monotonic},$$
$$e_{i_k}(P^+)=0,$$
$$P^+_{i_k i_{k+1}}>0.$$

Then, for every $k=1,\ldots,k'-1$, it holds that $P^-_{i_k i_{k+1}}<P^+_{i_k i_{k+1}}$.

Proposition 2 makes clear that if an agent makes different payments in two clearing payment matrices and this agent makes directly or indirectly a strictly positive payment to some other defaulting agents, then there is also multiplicity of payments by these other defaulting agents, at least when division rules are positive monotonic. The proof is by induction and establishes that along the sequence of agents, the asset value at the least payment matrix is strictly below the asset value at the greatest clearing payment matrix.

A sequence of $k'\geq 2$ distinct agents $(i_1,\ldots,i_{k'})$ is a *directed path* in a matrix $M\in\mathcal{M}$ if, for every $k\in\{1,\ldots,k'-1\}$, $M_{i_k i_{k+1}}>0$. Agent $j\in I$ is connected to agent $i\in I$ in M if there is a directed path $(i_1,\ldots,i_{k'})$ in M such that $i_1=i$ and $i_{k'}=j$.

Let $N=(I,z,L,d)$ be a financial network. A set of agents $S\subset I$ is said to be a *strongly connected component* in L if any two distinct agents in S are connected to each other in L and the set S is maximal with regard to this property.

For every $i\in I$, let $O(i)$ denote the strongly connected component in L to which i belongs. The collection $\mathcal{O}=\{O(i)\mid i\in I\}$ is a partition of I. We construct the directed graph $(\mathcal{O},D)$ by defining

$$D=\{(O,O')\in\mathcal{O}\times\mathcal{O}\mid \exists i\in O,\ \exists j\in O',\ L_{ij}>0\},$$

and so, for two distinct elements $O,O'\in\mathcal{O}$, there is an arc from O to O' if there is $i\in O$ and $j\in O'$ such that $L_{ij}>0$. The successors of $O\in\mathcal{O}$ in the directed graph $(\mathcal{O},D)$ are given by the strongly connected components that are connected to O in $(\mathcal{O},D)$. The directed graph $(\mathcal{O},D)$ has no cycles. We can, therefore, order the sets in $\mathcal{O}$ and write $\mathcal{O}=\{O_1,\ldots,O_R\}$, where $(O_r,O_{r'})\in D$ implies $r<r'$. In general, this order is not uniquely determined.

An agent $i\in I$ is said to be a cyclical agent and the set $O(i)$ is said to be a cycle if $O(i)$ consists of at least two elements. Agent i is cyclical if and only if there is a directed path of agents in L starting at agent i such that the last agent on the path has a strictly positive liability toward agent i. The set of all cyclical agents is denoted by C.

We illustrate the directed graph $(\mathcal{O},D)$ in Example 2.

Example 2. Consider a financial network with 13 agents $I=\{1,2,\ldots,13\}$. In Figure 1, if agent i has a strictly positive liability to agent j, then we draw an arc from i to j.

The collection $\mathcal{O}=\{O_1,O_2,O_3,O_4,O_5\}$ consists of the strongly connected components in L, where $O_1=\{1\}$, $O_2=\{2,3,4,5,6\}$, $O_3=\{7,8,12\}$, $O_4=\{9\}$, and $O_5=\{10,11,13\}$. The arcs between the strongly connected components are given by $D=\{(O_1,O_3),(O_2,O_4),(O_2,O_5)\}$. The successor of O_1 is O_3, and the successors of O_2 are O_4 and O_5. The directed graph $(\mathcal{O},D)$ has no cycles. The sets O_2, O_3, and O_5 are cycles. The set of cyclical agents is equal to $C=O_2\cup O_3\cup O_5=\{2,3,4,5,6,7,8,10,11,12,13\}$. For $r\in\{1,2,3,4,5\}$, agents in O_r only have strictly positive liabilities to agents in $O_{r'}$ with $r'\geq r$. In this example, we could have chosen a different order on the collection of strongly connected components. For instance, we could have ordered $O=\{9\}$ before $O'=\{7,8,12\}$.

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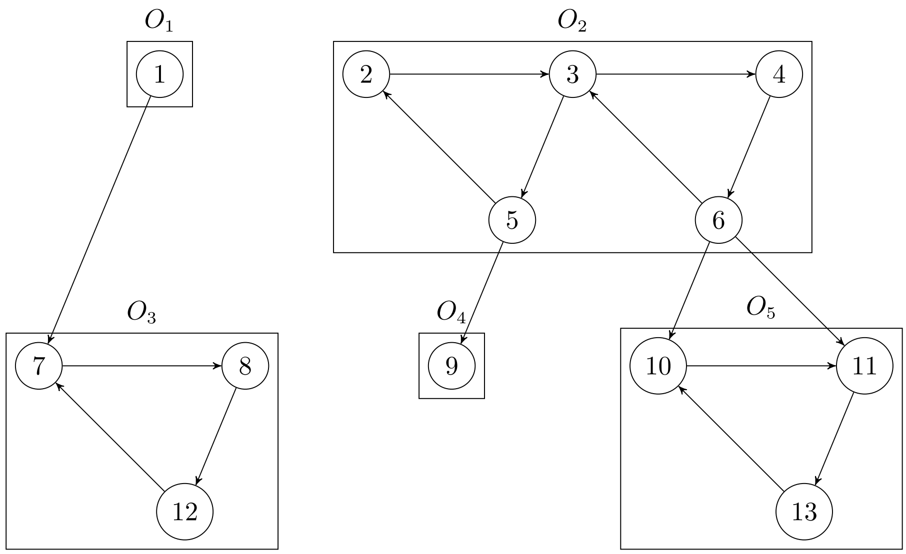
Figure 1. Agents, liabilities, and the ordered sets $\mathcal{O}$ in Example 2.



The next result presents a sufficient condition for the clearing payment matrix to be unique. This condition only involves the cyclical agents.

Theorem 4. *Let $N = (I, z, L, d)$ be a financial network such that*
1. *if $O \in \mathcal{O}$ is a cycle and has no successors, then $\sum_{i \in O} z_i > 0$;*
2. *for every $i \in C$ with $z_i > 0$, d^i is positive monotonic; and*
3. *for every $i \in C$ with $z_i = 0$, d^i is strictly monotonic.*

Then, N has a unique clearing payment matrix.

The sufficient conditions for the clearing payment matrix to be unique in Theorem 4 are that (1) at least one agent in a cycle without successors has a strictly positive endowment, (2) the division rules of cyclical agents with strictly positive endowments are positive monotonic, and (3) the division rules of cyclical agents with zero endowments are strictly monotonic. A detailed comparison of how these conditions relate to other sufficient conditions for uniqueness as found in the literature is made in Section 5.

Our sufficient condition shows that with strictly monotonic division rules, uniqueness of the clearing payment matrix is to be expected. The only condition that is needed in this case is that the aggregate endowments in a cycle without successors are strictly positive, which is very mild. Our sufficient condition is also not very restrictive for positive monotonic division rules. In that case, it requires that cyclical agents have strictly positive endowments. On the other hand, when division rules are monotonic but not positive monotonic, then our sufficient condition requires that there are no cyclical agents. This is definitely restrictive. As illustrated by Example 1 for priority division rules, it is possible to have multiple clearing payment matrices, even when all endowments are strictly positive. Example 3 presents a similar example for constrained equal awards rules. We obtain the policy implication that strictly monotonic division rules like proportional or positive monotonic division rules like constrained equal losses are preferable to monotonic division rules in financial networks as they avoid multiplicity of clearing payment matrices and thereby, as is argued in Section 6, potential discontinuities in the financial system.

The proof of Theorem 4 proceeds in three steps. First, the agents are partitioned into the strongly connected components of the liability matrix. In the first step, it is shown that payments made by the cyclical agents are uniquely determined. The first step distinguishes the case where the agents in a particular cycle have strictly positive aggregate endowments or receive a strictly positive amount from agents outside the cycle and the case where these agents all have zero endowments and do not receive any payments from agents outside the cycle. In the former case, the proof of this step relies heavily on Proposition 2 and shows that the agents in the cycle are connected in the greatest clearing payment matrix to an agent with strictly positive equity. The latter agent fully pays its liabilities, and from Proposition 1 on the uniqueness of equity across all clearing payment matrices, it should have uniquely determined incoming payments. This is then shown to imply uniqueness of payments

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made by all agents in the cycle. When the endowments of all agents in the cycle are equal to zero, then the assumptions of the theorem imply that all the agents have strictly monotonic division rules and that the cycle has a successor. Nonuniqueness of the payments of the agents in the cycle would imply a strictly positive payment in the greatest clearing payment matrix to an agent that belongs to a successor of the cycle, which leads to a contradiction. Steps 2 and 3 in the proof use induction on the tree that is formed by the partition of the agents in strongly connected components. Step 2 considers the base case for the payments made by agents in O_1, and Step 3 contains the induction step. If O_1 is a cycle, then uniqueness of the payments made by the corresponding agents follows from Step 1. If O_1 is a singleton, then uniqueness of the payments by the corresponding agent follows easily from limited liability and priority of creditors. The induction in Step 3 essentially repeats these arguments.

5. Relation to Other Uniqueness Conditions in the Literature

The literature has found a number of conditions to obtain a unique clearing payment matrix. In this section, we show how earlier findings by Csóka and Herings [7], Eisenberg and Noe [11], Glasserman and Young [16], Groote Schaarsberg et al. [18], and Koster [22] are special cases of the condition presented in Theorem 4.

We start with the uniqueness result derived in Eisenberg and Noe [11]. Eisenberg and Noe [11] restrict attention to proportional division rules. As mentioned before, such rules are strictly monotonic. To explain their conditions, we need the definitions of risk orbit and regular financial network.

Given a financial network $N=(I,z,L,d)$, the set consisting of agent $i \in I$ together with all agents $j \in I$ that are connected to agent i in L is denoted by $O^+(i)$. Eisenberg and Noe [11] refer to $O^+(i)$ as the risk orbit of agent i. The set $O^+(i)$ is equal to the union of $O(i)$ and the sets in $\mathcal{O}$ that are successors of $O(i)$ in $(\mathcal{O},D)$. The financial network $N=(I,z,L,d)$ is said to be *regular* if, for every $i \in I$, $\sum_{j \in O^+(i)} z_j > 0$.[2] Regularity rules out the situation where agent i has a zero endowment and none of the agents to which agent i has direct or indirect liabilities have a strictly positive endowment. A sufficient condition for regularity is that all agents have strictly positive endowments. Another sufficient condition for regularity is that one agent has a strictly positive endowment and $\mathcal{O}$ consists of a single set, so there is a single strongly connected component.

Proposition 3 (Eisenberg and Noe [11]). *Let $N=(I,z,L,d)$ be a regular financial network such that all agents use the proportional division rule. Then, N has a unique clearing payment matrix.*

Proof. If $O \in \mathcal{O}$ is a cycle and has no successors, then for every $i \in O$, $O^+(i) = O$. Then, regularity implies that $\sum_{i \in O} z_i > 0$, so condition (1) of Theorem 4 is satisfied.

Because the proportional division rule is strictly monotonic, conditions (2) and (3) of Theorem 4 are satisfied. □

Glasserman and Young [16] extend the Eisenberg and Noe [11] model by allowing agents to have liabilities to nodes outside the network. More precisely, $b_i \geq 0$ corresponds to the total liabilities of agent $i \in I$ to nodes outside the network. Glasserman and Young [16] have the following condition for uniqueness of the clearing payment matrix. Every agent has strictly positive direct or indirect liabilities to nodes outside the network. Using our notation, for every agent $i \in I$, there is $j \in O^+(i)$ such that $b_j > 0$.

We continue by embedding the Glasserman and Young [16] model into the Eisenberg and Noe [11] framework and next, reformulate their uniqueness condition. To do so, we represent the nodes outside the network by a single agent 0. The agents inside the network are represented by the set $\{1,\dots,n\}$. We assume, without loss of generality, that $z_0 = 0$. Agent 0 has no liabilities to nodes in the network, and so, for every $i \in \{1,\dots,n\}$, $L_{0i} = 0$. The liabilities of agents in the network to nodes outside the network are now replaced by liabilities to agent 0; so, for every $i \in \{1,\dots,n\}$, $L_{i0} = b_i$. We obtain a financial network $N=(I,z,L,d)$ by taking $I = \{0,1,\dots,n\}$ and for every $i \in I$, d^i equal to the proportional division rule. The uniqueness condition of Glasserman and Young [16] can now be formulated as the requirement that, for every $i \in I$, $0 \in O^+(i)$. Notice that $O^+(0) = \{0\}$ by construction.

Proposition 4 (Glasserman and Young [16]). *Let $N=(I,z,L,d)$ be a financial network such that $I=\{0,1,\dots,n\}$, $z_0=0$, for every $i \in I$, $L_{0i}=0$, and all agents use the proportional division rule. If, for every $i \in I$, $0 \in O^+(i)$, then N has a unique clearing payment matrix.*

Proof. It clearly holds that $\{0\} \in \mathcal{O}$, and by assumption, $\{0\}$ is a successor of every $O \in \mathcal{O} \setminus \{\{0\}\}$. As there are no cycles without successors, condition (1) of Theorem 4 is, therefore, trivially satisfied.

Because the proportional division rule is strictly monotonic, conditions (2) and (3) of Theorem 4 are satisfied. □

The conditions of Propositions 3 and 4 are quite different. Proposition 3 makes sure that there are strictly positive endowments among every agent and its successors, whereas Proposition 4 makes no assumption regarding the endowments but assumes strictly positive liabilities to a designated agent, representing the outside sector.

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Groote Schaarsberg et al. [18] provide a sufficient condition for uniqueness of the clearing payment matrix for the case with general division rules. A financial network is called hierarchical if, by reordering the agents, the matrix L can be transformed into an upper triangular matrix (i.e., a matrix with zeros below the diagonal).

Proposition 5 (Groote Schaarsberg et al. [18]). *Let $N=(I,z,L,d)$ be a hierarchical financial network.*[3] *Then, N has a unique clearing payment matrix.*

Proof. If the matrix L can be written in an upper triangular form, then are no cycles and no cyclical agents, so the conditions of Theorem 4 are trivially satisfied. □

Even when attention is restricted to proportional division rules, the conditions of Proposition 5 are independent from those of Propositions 3 and 4. Unlike Propositions 3 and 4, cycles are not allowed. On the other hand, Proposition 5 makes no assumptions on endowments and does not require that all agents have a strictly positive liability to a designated agent. The proposition also applies to division rules, which are not proportional.

Koster [22] extends the regularity condition of Eisenberg and Noe [11] from proportional division rules to strictly monotonic division rules.

Proposition 6 (Koster [22]). *Let $N=(I,z,L,d)$ be a regular financial network such that all agents use a strictly monotonic division rule.*[4] *Then, N has a unique clearing payment matrix.*

Proof. If $O\in\mathcal{O}$ is a cycle and has no successors, then for every $i\in O$, $O^+(i)=O$. Then, regularity implies that $\sum_{i\in O} z_i>0$, so condition (1) of Theorem 4 is satisfied.

Because division rules are assumed to be strictly monotonic, conditions (2) and (3) of Theorem 4 are satisfied. □

Proposition 6 shows that the conditions of Proposition 3 can be extended from proportional division rules to arbitrary strictly monotonic division rules. The conditions in Proposition 6 are clearly independent from those of Proposition 5.

As we noted before, the constrained equal losses division rule does not satisfy strict monotonicity. Our generalization to positive monotonicity in Theorem 4 guarantees that the clearing payment matrix is unique when all agents use the constrained equal losses division rule and endowments of cyclical agents are all strictly positive, a result that was also stated in the working paper by Csóka and Herings [7, theorem 7.8]. This result is in stark contrast with the case of constrained equal awards division rules as will be demonstrated in the next example. This is surprising because both division rules can be considered as each other's dual and share many common features; see Thomson [28]. The example has the same primitives as Example 1, except that the priority division rules are replaced by constrained equal awards rules.

Example 3. Let $N=(I,z,L,d)$ be a financial network with three agents $I=\{1,2,3,4\}$, all using constrained equal awards rules. Tables 3 and 4 present the endowments, the liabilities, the least clearing payment matrix P^-, the greatest clearing payment matrix P^+, and the induced asset values and equities. Because constrained equal awards division rules are not positive monotonic, positivity of the endowments is not sufficient to guarantee uniqueness of the clearing payment matrix.

6. Bankruptcy Rules

We consider a class of bankruptcy rules, which are defined by assigning to each financial network a clearing payment matrix and in case of multiplicity of clearing payment matrices, the greatest one. Continuity is a very desirable property of such bankruptcy rules. Without continuity, an arbitrarily small change in the endowments or the liabilities can lead to a complete disruption of the payment matrix, clearly being very detrimental to the stability of the financial system. In this section, we show that uniqueness of the clearing matrix is sufficient to obtain

Table 3. The clearing payment matrix P^+ and its induced asset values and equities in Example 3 with constrained equal awards rules.

z	L				P^-				$a(P^-)$	$e(P^-)$
1	0	8	0	0	0	7	0	0	7	0
1	0	0	10	0	0	0	9	0	9	0
1	7	0	0	5	5	0	0	5	10	0
1	1	1	0	0	1	1	0	0	6	4

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Table 4. The clearing payment matrix P^- and its induced asset values and equities in Example 3 with constrained equal awards rules.

z	L				P^+				$a(P^+)$	$e(P^+)$
1	0	8	0	0	0	8	0	0	8	0
1	0	0	10	0	0	0	10	0	10	0
1	7	0	0	5	6	0	0	5	11	0
1	1	1	0	0	1	1	0	0	6	4

continuity of the bankruptcy rule. When there is multiplicity of payment matrices, we have examples showing that there is no selection of them that makes the bankruptcy rule continuous. We also present an example with multiplicity of clearing payment matrices but a continuous bankruptcy rule, which shows that uniqueness of the clearing payment matrix is not a necessary condition to obtain a continuous bankruptcy rule.

In this section, we fix the set of agents I and the division rules d, but we allow the endowments z and the liability matrix L to vary. To avoid confusion, we make the dependence of the division rule on the vector of liabilities explicit and write $d^i(E, L_i)$ for the division rule of agent $i \in I$. We also make the dependence on endowments of the asset value and equity explicit and write $a_i(z, P)$ and $e_i(z, P)$ for the asset value and equity of agent $i \in I$, respectively.

We assume that, for every $i \in I$, d^i is continuous in L_i, an assumption that is satisfied for all commonly used division rules. The set of financial networks, denoted by $\mathcal{N}$, consists of all pairs of vectors of endowments in $\mathbb{R}^I_+$ and liability matrices in $\mathcal{M}$, so

$$\mathcal{N} = \mathbb{R}^I_+ \times \mathcal{M}.$$

We next define the bankruptcy rule $b : \mathcal{N} \to \mathcal{M}$ by associating the greatest clearing payment matrix to the financial network $N = (z, L) \in \mathcal{N}$.

Definition 3. Let I and d be given. The *bankruptcy rule* $b : \mathcal{N} \to \mathcal{M}$ is given by

$$b(z, L) = P^+, \quad (z, L) \in \mathcal{N},$$

where P^+ is the greatest clearing payment matrix for the financial network (z, L).

In general, each agent could use a different division rule. However, if a bankruptcy rule is based on the same division rule for each agent, we associate the name of the division rule to the bankruptcy rule. For instance, if all agents use the constrained equal awards division rule, then we call the resulting bankruptcy rule the constrained equal awards bankruptcy rule.

We endow $\mathcal{N}$ with the standard topology, based on the Euclidean topology for endowments and liabilities. Continuity is an attractive property of a bankruptcy rule because it implies that small changes in the financial network induce small changes in the resulting payment matrix. Continuity is also used as an axiom in Csóka and Herings [9] in order to axiomatize the proportional bankruptcy rule. The next result relates the uniqueness of the clearing payment matrix to the continuity of the bankruptcy rule b and states that b is continuous at any financial network with a unique clearing payment matrix. The proof is based on a standard limit argument.

Theorem 5. *Let I and d be given. Let $(\bar{z}, \bar{L}) \in \mathcal{N}$ be such that the financial network $\overline{N} = (\bar{z}, \bar{L})$ has a unique clearing payment matrix. Then, b is continuous at $\overline{N}$.*

Consider the proportional bankruptcy rule on the domain of financial networks (z, L) where the first condition of Theorem 4 is satisfied or the constrained equal losses bankruptcy rule on the domain of financial networks (z, L) where all cyclical agents have strictly positive endowments. By Theorem 5, these bankruptcy rules are continuous. Such a result was also presented in the working paper by Csóka and Herings [7] for the special case where all endowments are strictly positive. They also present the following example to show that a similar result does not hold for the constrained equal awards bankruptcy rule.

Example 4. Let $N = (z, L)$ be a financial network with three agents, $I = \{1, 2, 3\}$, all using the constrained equal awards rule. Table 5 presents the endowments, the liabilities, the payment matrix P^+ resulting from the constrained equal awards bankruptcy rule, and the induced asset values and equities. Agents are all able to pay their liabilities, although agents 1 and 2 end up with zero equity.

Now, for $\varepsilon > 0$, consider the financial network $N^\varepsilon = (z, L^\varepsilon)$ as displayed in Table 6, where the liabilities of both agents 1 and 2 to agent 3 have gone up by ε.

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Table 5. The payment matrix, asset values, and equities resulting from constrained equal awards rules in Example 4 for the financial network $N = (z, L)$.

z	L			P^+			$a(z,P^+)$	$e(z,P^+)$
1	0	2	1	0	2	1	3	0
1	2	0	1	2	0	1	3	0
1	0	0	0	0	0	0	3	3

Because constrained equal awards require the same payments from agent 1 to agents 2 and 3, up to their claims, agent 1 can pay at most one unit to both agents. The same is true for the payments of agent 2 to agents 1 and 3. Under these payments, agents 1 and 2 end up with zero equity and default partially on all their liabilities. We have that

$$\lim_{\varepsilon\downarrow 0} b(z^\varepsilon, L^\varepsilon) = \lim_{\varepsilon\downarrow 0} P^\varepsilon = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \neq \begin{bmatrix} 0 & 2 & 1 \\ 2 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = P^+ = b(z, L),$$

so although the financial networks N^ε converge to N when ε tends to zero, the corresponding payment matrices do not converge.

The lack of continuity of b in Example 4 is not resolved by making another selection from the set of clearing payment matrices. The financial network N has many clearing payment matrices compatible with constrained equal awards rules. The greatest clearing payment matrix is equal to P^+, and the least clearing payment matrix is equal to P^-, given by

$$P^- = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

The following example shows that an alternative definition of the constrained equal awards bankruptcy rule that selects the least clearing payment matrix would not solve the lack of continuity.

Example 5. For $\varepsilon > 0$, consider the financial network $N^\varepsilon = (z^\varepsilon, L)$ as displayed in Table 7.

The payment matrix P^ε is the unique clearing payment matrix in the financial network N^ε under constrained equal awards rules. The financial networks N^ε tend to the financial network N of Example 4 as ε goes to zero. The payment matrices P^ε are all equal to $b(z, L)$. Selecting the least clearing payment matrix for N under constrained equal awards rules instead of the greatest clearing payment matrix $b(z, L)$, or in fact selecting any clearing payment matrix for N different from the greatest clearing payment matrix, would then lead to a violation of continuity in this example.

We conclude this section with an example that shows the converse of Theorem 5 to be false. It is possible that a bankruptcy rule is continuous at a particular financial network $\overline{N}$, whereas the financial network $\overline{N}$ has multiple clearing payment matrices.

Example 6. Let $I = \{1, 2\}$. Because each agent has only one creditor, the choice of the division rules is irrelevant. Let $\overline{N} = (\overline{z}, \overline{L})$ be the financial network where agents have zero endowments and owe each other one unit. Table 8 presents the endowments, the liabilities, the least clearing payment matrix P^-, and the greatest clearing payment matrix P^+. Because according to our definition, a bankruptcy rule selects the greatest clearing payment matrix, it holds that

$$b(\overline{N}) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Table 6. The payment matrix, asset values, and equities resulting from constrained equal awards rules in Example 4 for the financial network $N^\varepsilon = (z, L^\varepsilon)$.

z	L^ε			P^ε			$a(z,P^\varepsilon)$	$e(z,P^\varepsilon)$
1	0	2	$1+\varepsilon$	0	1	1	2	0
1	2	0	$1+\varepsilon$	1	0	1	2	0
1	0	0	0	0	0	0	3	3

Table 7. The payment matrix, asset values, and equities resulting from the constrained equal awards rules in Example 5 for the financial network $N^{\varepsilon} = (z^{\varepsilon}, L)$.

z^{ε}	L			P^{ε}			$a(z^{\varepsilon}, P^{\varepsilon})$	$e(z^{\varepsilon}, P^{\varepsilon})$
$1+\varepsilon$	0	2	1	0	2	1	$3+\varepsilon$	ε
$1+\varepsilon$	2	0	1	2	0	1	$3+\varepsilon$	ε
1	0	0	0	0	0	0	3	3

The next lemma, proved in Section 7, asserts that the bankruptcy rule b is continuous, so in particular, it is continuous at $\overline{N}$.

Lemma 1. *Let* $I = \{1, 2\}$. *The bankruptcy rule* $b : \mathcal{N} \to \mathcal{M}$ *is continuous.*

7. Conclusion

We consider financial networks where agents are linked to each other with financial contracts. In case of bankruptcy, the payments to other agents are determined by bankruptcy law. Although the literature has almost exclusively focused on proportional division rules, in bankruptcy law certain claims are often given priority over other claims. We, therefore, allow for general division rules. The set of clearing payment matrices can be shown to be a lattice, so there always exists a least clearing payment matrix and a greatest clearing payment matrix. Because these clearing payment matrices may not all coincide, we are interested in conditions that guarantee uniqueness.

Uniqueness of clearing payment matrices depends heavily on the structural properties of the network of financial liabilities, which can be represented as a directed graph. We partition the network of financial liabilities into strongly connected components. A strongly connected component that consists of more than one agent is called a cycle. An agent that is part of such a component is said to be cyclical.

We provide the following sufficient condition for the clearing payment matrix to be unique. (1) For every cycle without successors, at least one agent has a strictly positive endowment. (2) The division rules of cyclical agents with strictly positive endowments are positive monotonic. (3) The division rules of cyclical agents with zero endowments are strictly monotonic.

Positive monotonicity is a new condition. It requires that if a defaulting agent makes a strictly positive payment to another agent and the asset value of the former agent increases, then the payment to the latter agent increases as well. The well-known constrained equal losses division rule is positive monotonic, but its dual, the constrained equal awards division rule, is not. The proportional division rule is strictly monotonic and therefore, positive monotonic. Priority-based division rules are not positive monotonic.

Our sufficient condition for uniqueness is easily seen to imply several uniqueness conditions that have been provided before in the literature, which mostly considers proportional division rules. One case is where all agents have strictly positive endowments. Another is where there are no cyclical agents or all agents have a strictly positive liability to the outside sector. We show how the outside sector can be represented by an additional agent without liabilities to the other agents. We demonstrate that priority-based division rules as well as constrained equal awards division rules may lead to a multiplicity of clearing payment matrices, even when all endowments are strictly positive.

We define a class of bankruptcy rules by assigning the greatest clearing payment matrix to a financial network. We show that uniqueness of clearing payment matrices is a sufficient condition for the desirable property of continuity of such bankruptcy rules. We show by means of an example that bankruptcy rules that are based on constrained equal awards division rules violate continuity. The violation of continuity is not repaired when defining

Table 8. The payment matrix, asset values, and equities resulting from constrained equal awards rules in Example 6 for the financial network $\overline{N} = (\overline{z}, \overline{L})$.

z	L		P^{-}		P^{+}	
0	0	1	0	0	0	1
0	1	0	0	0	1	0

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bankruptcy rules alternatively by assigning the least clearing payment matrix, or in fact any clearing payment matrix different from the greatest clearing payment matrix, to a financial network. These results also highlight once more that the network aspect leads to different structural properties. In the absence of network aspects, monotonicity of the division rules is sufficient for continuity, whereas in financial networks, even stronger monotonicity requirements like positive or strict monotonicity are not sufficient to guarantee continuity of the bankruptcy rule.

Proof of Theorem 1. (⇒)

Let $P \in \mathcal{M}$ be a clearing payment matrix of the financial network N. Consider some $i \in I$. We define $P'_i = d^i(a_i(P))$. Because $P_i \in \mathcal{F}_i$ and d^i is monotonic, it holds that (a) $P_i < P'_i$, (b) $P_i = P'_i$, or (c) $P_i > P'_i$.

Case a. $P_i < P'_i$.

We have that

$$e_i(P) = a_i(P) - \sum_{j \in I} P_{ij} > a_i(P) - \sum_{j \in I} P'_{ij} = a_i(P) - \sum_{j \in I} d^i_j(a_i(P)) \geq 0,$$

where the weak inequality follows from the definition of a division rule. From $P_i < P'_i \in \mathcal{F}_i$, it follows that $P_i < L_i$. Because $e_i(P) > 0$, P does not satisfy priority of creditors. We conclude that case (a) cannot occur.

Case c. $P_i > P'_i$.

Let $E_i \in [0, \overline{L}_i]$ be such that $P_i = d^i(E_i)$. From $d^i(E_i) = P_i > P'_i = d^i(a_i(P))$ and the monotonicity of d^i, it follows that $a_i(P) < E_i$. We have that

$$E_i = \sum_{j \in I} d^i_j(E_i) = \sum_{j \in I} P_{ij} = a_i(P) - e_i(P) \leq a_i(P),$$

where the inequality follows because P satisfies limited liability. This contradicts our earlier conclusion that $a_i(P) < E_i$. It follows that case (c) cannot occur.

We have derived that case (b) holds, so $P_i = P'_i = d^i(a_i(P))$.

(⇐)

Let $P \in \mathcal{M}$ be a solution to the system of equations

$$P_{ij} = d^i_j(a_i(P)), \quad i, j \in I.$$

1. Feasibility. It holds that $P \in \mathcal{P}$ because for every $i \in I$, $P_i = d^i(a_i(P))$.
2. Limited liability. For every $i \in I$, we have that

$$e_i(P) = a_i(P) - \sum_{j \in I} P_{ij} = a_i(P) - \sum_{j \in I} d^i_j(a_i(P)) \geq a_i(P) - a_i(P) = 0,$$

where the weak inequality follows from the definition of a division rule.

3. Priority of creditors. Assume $i \in I$ is such that $P_i < L_i$. We have that

$$\sum_{j \in I} P_{ij} = \sum_{j \in I} d^i_j(a_i(P)) = \min\{a_i(P), \overline{L}_i\} = a_i(P),$$

where the second equality follows from the definition of a division rule and the third equality because $P_i < L_i$, so $\sum_{j \in I} P_{ij} = \sum_{j \in I} d^i_j(a_i(P)) < \sum_{j \in I} L_{ij} = \overline{L}_i$. We have that

$$e_i(P) = a_i(P) - \sum_{j \in I} P_{ij} = 0. \quad \square$$

Proof of Theorem 2. Let $\varphi : \mathcal{P} \to \mathcal{P}$ be defined by

$$\varphi_{ij}(P) = d^i_j(a_i(P)), \quad P \in \mathcal{P}, \; i, j \in I.$$

It follows from Theorem 1 that P^* is a clearing payment matrix if and only if P^* is a fixed point of φ.

The set $\mathcal{P}$ is clearly a complete lattice.

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We show that φ is monotonic. Let $P, P' \in \mathcal{P}$ be such that $P \le P'$. For every $i \in I$, it holds that

$$\varphi_i(P) = d^i(a_i(P)) = d^i\left(z_i + \sum_{j\in I} P_{ji}\right) \le d^i\left(z_i + \sum_{j\in I} P'_{ji}\right) = d^i(a_i(P')) = \varphi_i(P'),$$

where the inequality follows the fact that d^i is monotonic.

By Tarski's fixed point theorem (Tarski [27]), the set of fixed points of φ is a complete lattice with respect to $\le$. It follows that the set of fixed points has a least element and a greatest element. □

Proof of Theorem 3. Let P^+ be the greatest clearing payment matrix, which exists because of Theorem 2, and let P be an arbitrary clearing payment matrix of N. Let $i \in I$. If $e_i(P) > 0$, then

$$0 < e_i(P) = a_i(P) - \overline{L}_i \le a_i(P^+) - \overline{L}_i \le e_i(P^+).$$

If $e_i(P) = 0$, then $e_i(P^+) \ge e_i(P)$ because P^+ satisfies limited liability.

We have that $e(P) \le e(P^+)$. From $\sum_{i\in I} e_i(P) = \sum_{i\in I} e_i(P^+) = \sum_{i\in I} z_i$ and $e(P) \le e(P^+)$, it follows that $e(P) = e(P^+)$. □

Proof of Proposition 1. Let $i \in I$ be such that $e_i(P^+) > 0$. By Theorem 3, it holds that $e_i(P^-) = e_i(P^+) > 0$. Feasibility and priority of creditors now imply that $P_i^- = P_i^+ = L_i$. Therefore, if $i \in I$ is such that $P_i^- < P_i^+$, then $e_i(P^+) = 0$, and by Theorem 3, $e_i(P^-) = 0$. □

Proof of Proposition 2. Let $E^-, E^+ \in \mathbb{R}^I_+$ be such that, for every $i \in I$, $E_i^- \le E_i^+ \le \overline{L}_i$, and

$$P_i^- = d^i(E_i^-),$$
$$P_i^+ = d^i(E_i^+).$$

For every $k = 1, \ldots, k' - 1$, it holds by Theorem 3 that $e_{i_k}(P^-) = e_{i_k}(P^+)$, and because d^{i_k} is positive monotonic and $P^+_{i_k i_{k+1}} > 0$, it holds that $E^+_{i_k} > \underline{a}_{i_k i_{k+1}}$.

We now show by induction that

$$E^-_{i_k} < E^+_{i_k}, \quad k = 1, \ldots, k' - 1. \tag{1}$$

For $k = 1$, $E^-_{i_1} < E^+_{i_1}$ follows from $P^-_{i_1} < P^+_{i_1}$.
Assume that, for some $k \le k' - 2$, it holds that $E^-_{i_k} < E^+_{i_k}$. We show that $E^-_{i_{k+1}} < E^+_{i_{k+1}}$.
Because $e_{i_{k+1}}(P^-) = e_{i_{k+1}}(P^+) = 0$, it holds that

$$\sum_{j\in I} d_j^{i_{k+1}}(E^-_{k+1}) = z_{i_{k+1}} + \sum_{j\in I} d^j_{i_{k+1}}(E_j^-), \tag{2}$$
$$\sum_{j\in I} d_j^{i_{k+1}}(E^+_{k+1}) = z_{i_{k+1}} + \sum_{j\in I} d^j_{i_{k+1}}(E_j^+). \tag{3}$$

We argue that the right-hand side of (3) is greater than that of (2). Because $E^- \le E^+$, we have that

$$d^j_{i_{k+1}}(E_j^+) \ge d^j_{i_{k+1}}(E_j^-), \quad j \in I.$$

It holds by the induction hypothesis, positive monotonicity of d^{i_k}, and $E^+_{i_k} > \underline{a}_{i_k i_{k+1}}$ that

$$d^{i_k}_{i_{k+1}}(E^+_{i_k}) > d^{i_k}_{i_{k+1}}(E^-_{i_k}).$$

The left-hand side of (3) is then also greater than that of (2), so

$$\sum_{j\in I} d_j^{i_{k+1}}(E^+_{k+1}) > \sum_{j\in I} d_j^{i_{k+1}}(E^-_{k+1}),$$

implying that

$$E^-_{i_{k+1}} < E^+_{i_{k+1}}.$$

It now follows that, for every $k = 1, \ldots, k' - 1$, $P^-_{i_k i_{k+1}} = d^{i_k}_{i_{k+1}}(E^-_{i_k}) < d^{i_k}_{i_{k+1}}(E^+_{i_k}) = P^+_{i_k i_{k+1}}$. □

Proof of Theorem 4. By Theorem 2, N has a least clearing payment matrix P^- and a greatest clearing payment matrix P^+. Let $E^-, E^+ \in \mathbb{R}^I_+$ be such that, for every $i \in I$, $E_i^- \le E_i^+ \le \overline{L}_i$, and

$$P_i^- = d^i(E_i^-),$$
$$P_i^+ = d^i(E_i^+).$$

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By Theorem 3, it holds that

$$e_i(P^-) = e_i(P^+), \quad i \in I. \tag{4}$$

Let $\mathcal{O} = \{O_1, \ldots, O_R\}$ be the partition of strongly connected components of L, where $(O_r, O_{r'}) \in D$ implies $r < r'$.

In Step 1, we show equality of the rows in P^- and P^+ that correspond to cyclical agents. Steps 2 and 3 use induction to complete the proof. Step 2 considers the base case for rows in O_1, and Step 3 contains the induction step.

Step 1. For every $i \in C$, $P_i^- = P_i^+$.
Suppose there is $i_1 \in C$ such that

$$E_{i_1}^- < E_{i_1}^+. \tag{5}$$

We consider all the endowments and incoming payments of the agents in the cycle $O(i_1)$ and distinguish two cases.

Case 1. $\sum_{j \in O(i_1)} z_j + \sum_{i \in I \setminus O(i_1)} \sum_{j \in O(i_1)} P_{ij}^+ > 0$.
The set of agents S consisting of i_1 together with the agents to which i_1 makes direct or indirect payments when the payment matrix is P^+ is defined as

$$S = \{i_1\} \cup \{j \in I \mid j \text{ is connected to } i_1 \text{ in } P^+\}.$$

Suppose $\sum_{j \in S} e_j(P^+) = 0$. It follows that the agents in S cannot have strictly positive endowments or receive strictly positive payments from agents outside S as otherwise, these resources would end up somewhere in S, so $\sum_{j \in S} z_j + \sum_{i \in I \setminus S} \sum_{j \in S} P_{ij}^+ = 0$. We have that, for every $j \in S$, $z_j = 0$, and for every $j \in S \cap O(i_1)$, d^j is strictly monotonic. Because $E_{i_1}^+ > 0$, the strict monotonicity of d^j for every $j \in S \cap O(i_1)$ implies that $O(i_1) \subset S$. We have that

$$0 = \sum_{j \in S} z_j + \sum_{i \in I \setminus S} \sum_{j \in S} P_{ij}^+ \geq \sum_{j \in O(i_1)} z_j + \sum_{i \in I \setminus S} \sum_{j \in O(i_1)} P_{ij}^+ = \sum_{j \in O(i_1)} z_j + \sum_{i \in I \setminus O(i_1)} \sum_{j \in O(i_1)} P_{ij}^+ > 0,$$

where the second equality follows from the fact that members of $S \setminus O(i_1)$ have no liabilities to agents in $O(i_1)$ and the strict inequality follows from the assumption of Case 1. We have obtained a contradiction. Consequently, it follows that $\sum_{j \in S} e_j(P^+) > 0$.

Subcase 1.1. $\sum_{j \in S \cap O(i_1)} e_j(P^+) > 0$.
It holds by Proposition 1 that $e_{i_1}(P^+) = 0$. Because all agents in S are connected to i_1 in P^+, there is a finite sequence of agents $(i_1, \ldots, i_{k'})$ in $S \cap O(i_1)$ such that

$$\begin{aligned} P_{i_k i_{k+1}}^+ &= d_{i_{k+1}}^{i_k}(E_{i_k}^+) > 0, & k &= 1, \ldots, k'-1, \\ e_{i_k}(P^+) &= 0, & k &= 1, \ldots, k'-1, \\ e_{i_{k'}}(P^+) &> 0. \end{aligned}$$

It follows from Proposition 2 that $E_{i_{k'-1}}^- < E_{i_{k'-1}}^+$. It, therefore, holds that

$$\begin{aligned} e_{i_{k'}}(P^+) - e_{i_{k'}}(P^-) &= \sum_{j \in I} d_{i_{k'}}^j(E_j^+) - \sum_{j \in I} d_j^{i_{k'}}(E_{k'}^+) - \sum_{j \in I} d_{i_{k'}}^j(E_j^-) + \sum_{j \in I} d_j^{i_{k'}}(E_{k'}^-) \\ &= \sum_{j \in I} d_{i_{k'}}^j(E_j^+) - \sum_{j \in I} d_{i_{k'}}^j(E_j^-) \\ &> 0, \end{aligned}$$

where the second equality follows from $E_{i_{k'}}^- = E_{i_{k'}}^+ = \overline{L}_{i_{k'}}$ and the inequality follows from $E^- \leq E^+$, $E_{i_{k'-1}}^- < E_{i_{k'-1}}^+$, positive monotonicity of $d^{i_{k'-1}}$, and $E_{i_{k'-1}}^+ > \underline{a}_{i_{k'-1} i_{k'}}$. We have obtained a contradiction to (4). Consequently, it follows that $E_{i_1}^- = E_{i_1}^+$, so $P_{i_1}^- = P_{i_1}^+ = d^{i_1}(E_{i_1}^+)$.

Subcase 1.2. $\sum_{j \in S \cap O(i_1)} e_j(P^+) = 0$.
Because $\sum_{j \in S} e_j(P^+) > 0$, it holds that $S \setminus O(i_1) \neq \emptyset$. Let $i' \in O(i_1)$ and $j' \in S \setminus O(i_1)$ be such that $P_{i'j'}^+ > 0$. Now, there is a finite path of agents $(i_1, \ldots, i_{k'})$ in S such that $i_{k'-1} = i'$, $i_{k'} = j'$,

$$\begin{aligned} P_{i_k i_{k+1}}^+ &= d_{i_{k+1}}^{i_k}(E_{i_k}^+) > 0, & k &= 1, \ldots, k'-1, \\ e_{i_k}(P^+) &= 0, & k &= 1, \ldots, k'-1. \end{aligned}$$

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Because $i_{k'-1} \in O(i_1)$, it follows that $i_1, \ldots, i_{k'-1} \in O(i_1)$. We can use Proposition 2 to conclude that $E^-_{i_{k'-1}} < E^+_{i_{k'-1}}$, and therefore, $P^-_{i'j'} < P^+_{i'j'}$.

We find that

$$\begin{aligned}\sum_{j\in S\setminus O(i_1)} e_j(P^+) &= \sum_{j\in S\setminus O(i_1)} z_j + \sum_{i\in O(i_1)} \sum_{j\in S\setminus O(i_1)} P^+_{ij} + \sum_{i\in I\setminus S} \sum_{j\in S\setminus O(i_1)} P^+_{ij} \\ &> \sum_{j\in S\setminus O(i_1)} z_j + \sum_{i\in O(i_1)} \sum_{j\in S\setminus O(i_1)} P^-_{ij} + \sum_{i\in I\setminus S} \sum_{j\in S\setminus O(i_1)} P^-_{ij} \\ &= \sum_{j\in S\setminus O(i_1)} e_j(P^-),\end{aligned}$$

a contradiction to Theorem 3. Consequently, it follows that $E^-_{i_1} = E^+_{i_1}$, so $P^-_{i_1} = P^+_{i_1} = d^{i_1}(E^+_{i_1})$.

Case 2. $\sum_{j\in O(i_1)} z_j + \sum_{i\in I\setminus O(i_1)} \sum_{j\in O(i_1)} P^+_{ij} = 0$.
It follows that $\sum_{j\in O(i_1)} z_j = 0$. By the first assumption of the theorem, there is $O_r \in \mathcal{O}$ such that $(O(i_1), O_r) \in D$. It holds by the third assumption of the theorem that, for every $i \in O(i_1)$, d^i is strictly monotonic. Because $P^+_{i_1} > 0$, we find that, for every $i \in O(i_1)$, $a_i(P^+) > 0$, so in particular, there is a strictly positive payment to an agent in O_r. The amount $\sum_{j\in O(i_1)} z_j + \sum_{i\in I\setminus O(i_1)} \sum_{j\in O(i_1)} P^+_{ij}$ must at least be equal to this strictly positive payment, leading to a contradiction. Consequently, it follows that $E^-_{i_1} = E^+_{i_1}$, so $P^-_{i_1} = P^+_{i_1} = d^{i_1}(E^+_{i_1})$.

Step 2. For every $i \in O_1$, $P^-_i = P^+_i$.

Case 1. O_1 is a singleton.
Let $i \in I$ be such that $O_1 = \{i\}$. If $z_i \geq \overline{L}_i$, then feasibility and priority of creditors imply that $E^-_i = E^+_i = \overline{L}_i$. If $z_i < \overline{L}_i$, then limited liability and priority of creditors imply that $e_i(P^-) = e_i(P^+) = 0$, so $E^-_i = E^+_i = z_i$. It follows in both cases that $P^-_i = P^+_i = d^i(E^+_i)$.

Case 2. O_1 is a cycle.
This case follows from Step 1.

Step 3. Assume for some $r' < R$, for every $r \leq r'$, for every $i \in O_r$, $P^-_i = P^+_i$. Then, for every $i \in O_{r'+1}$, $P^-_i = P^+_i$.

Case 1. $O_{r'+1}$ is a singleton.
Let $j \in I$ be such that $O_{r'+1} = \{j\}$. We have that $a_j(P^-) = a_j(P^+) = z_j + \sum_{r=1}^{r'} \sum_{i\in O_r} P^+_{ij}$. If $a_j(P^-) = a_j(P^+) \geq \overline{L}_j$, then feasibility and priority of creditors imply that $E^-_j = E^+_j = \overline{L}_j$. If $a_j(P^-) = a_j(P^+) < \overline{L}_j$, then limited liability and priority of creditors imply that $e_j(P^-) = e_j(P^+) = 0$, so $E^-_j = E^+_j = a_j(P^+)$. It follows in both cases that $P^-_i = P^+_i = d^i(E^+_i)$.

Case 2. $O_{r'+1}$ is a cycle.
This case follows from Step 1. □

Proof of Theorem 5. Let $(N^n)_{n\in\mathbb{N}} = (z^n, L^n)_{n\in\mathbb{N}}$ be a sequence of financial networks in $\mathcal{N}$ that converges to the financial network $\overline{N} = (\overline{z}, \overline{L})$. We have to show that the sequence of payment matrices $(P^n)_{n\in\mathbb{N}}$ defined by $P^n = b(z^n, L^n)$ converges to the payment matrix $b(\overline{z}, \overline{L})$.

The convergence of the sequence $(L^n)_{n\in\mathbb{N}}$ implies that it is bounded, which in turn, implies that the sequence $(P^n)_{n\in\mathbb{N}}$ is bounded, so it has a convergent subsequence $(P^{n^m})_{m\in\mathbb{N}}$, with limit, say, $\overline{P} \in \mathcal{M}$. For every $m \in \mathbb{N}$, it holds by Theorem 1 that

$$P^{n^m}_{ij} = d^i_j(a_i(z^{n^m}, P^{n^m}), L_i), \quad i, j \in I.$$

We have that

$$\overline{P}_{ij} = \lim_{m\to\infty} P^{n^m}_{ij} = \lim_{m\to\infty} d^i_j(a_i(z^{n^m}, P^{n^m}), L_i) = d^i_j(a_i(\overline{z}, \overline{P}), L_i), \quad i, j \in I,$$

where the third equality uses that d^i_j and a_i are continuous. It follows by Theorem 1 that $\overline{P}$ is a clearing payment matrix of the financial network $\overline{N}$. Because $\overline{N}$ has a unique clearing payment matrix by assumption, $\overline{P}$ is also the greatest clearing payment matrix and therefore, equal to $b(\overline{z}, \overline{L})$ by definition of b. We have shown that any convergent subsequence of the bounded sequence $(P^n)_{n\in\mathbb{N}}$ converges to $b(\overline{z}, \overline{L})$, and it follows that the sequence $(P^n)_{n\in\mathbb{N}}$ itself converges to $b(\overline{z}, \overline{L})$. □

Proof of Lemma 1. We first characterize the greatest clearing payment matrix for every financial network in $\mathcal{N}$. Consider any financial network $N = (z, L) \in \mathcal{N}$. We distinguish four cases.

Case 1. $z_1 + L_{21} - L_{12} \geq 0$ and $z_2 + L_{12} - L_{21} \geq 0$.
It is feasible for both agents to fully pay their liabilities, which then obviously results in the greatest clearing payment matrix. We have that

$$P_{12} = L_{12},$$
$$P_{21} = L_{21}.$$

Case 2. $z_1 + L_{21} - L_{12} \geq 0$ and $z_2 + L_{12} - L_{21} < 0$.
In this case, agent 2 ends up bankrupt in any clearing payment matrix. Because at least one agent is solvent in the greatest clearing payment matrix, agent 1 does not default in the greatest clearing payment matrix. It holds that

$$P_{12} = L_{12},$$
$$P_{21} = z_2 + L_{12}.$$

Case 3. $z_1 + L_{21} - L_{12} < 0$ and $z_2 + L_{12} - L_{21} \geq 0$.
Exploiting symmetry, it follows from the analysis in Case 2 that

$$P_{12} = z_1 + L_{21},$$
$$P_{21} = L_{21}.$$

Case 4. $z_1 + L_{21} - L_{12} < 0$ and $z_2 + L_{12} - L_{21} < 0$.
This case cannot occur because it would require that

$$0 > z_1 + L_{21} - L_{12} + z_2 + L_{12} - L_{21} = z_1 + z_2,$$

whereas both z_1 and z_2 are nonnegative.

Next, consider a converging sequence of financial networks $(z^n, L^n)_{n\in\mathbb{N}} \in \mathcal{N}$ with limit $(z, L) \in \mathcal{N}$ and suppose, to obtain a contradiction, that $(b(z^n, L^n))_{n\in\mathbb{N}}$ does not converge to $b(z, L)$. By passing to a subsequence, we can assume that there is one case out of the cases distinguished such that all financial networks in the sequence satisfy the conditions of that particular case and that the sequence $(b(z^n, L^n))_{n\in\mathbb{N}}$ has a limit different from $b(z, L)$. If all the financial networks in the sequence satisfy the conditions of Case 1, then the fact that the set of such networks is closed implies that the limit (z, L) satisfies the conditions of Case 1, and continuity of b at (z, L) follows from the expressions for the payment matrix in that case. If all the financial networks in the sequence satisfy the conditions of Case 2 and the limit (z, L) satisfies the conditions of Case 2 as well, then continuity of b at (z, L) follows immediately from the expressions for the payment matrix corresponding to Case 2. If the limit (z, L) does not satisfy the conditions of Case 2, then it must satisfy the conditions of Case 1, and we have $z_1 + L_{21} - L_{12} \geq 0$ and $z_2 + L_{12} - L_{21} = 0$. It follows that

$$\lim_{n\to\infty} b_{12}(z^n, L^n) = \lim_{n\to\infty} L^n_{12} = L_{12} = b_{12}(z, L),$$
$$\lim_{n\to\infty} b_{21}(z^n, L^n) = \lim_{n\to\infty} z^n_2 + L^n_{12} = z_2 + L_{12} = L_{21} = b_{21}(z, L).$$

The analysis for the situation where all the financial networks in the sequence satisfy the conditions of Case 3 follows by symmetry. We obtain that the limit of the sequence $(b(z^n, L^n))_{n\in\mathbb{N}}$ is equal to $b(z, L)$, which leads to a contradiction. Consequently, b is continuous at every financial network in $\mathcal{N}$. □

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Endnotes

[1] Groote Schaarsberg et al. [18] define a division rule as a function of the estate and the vector of liabilities, so they treat the vector of liabilities as a variable.

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[2] The formal definition of a risk orbit in Eisenberg and Noe [11] does not make clear whether agent *i* itself is included in its risk orbit when $O(i)$ is not a cycle. The informal discussion later on in the paper suggests that agent *i* is included. Indeed, Eisenberg and Noe [11] assert that the assumption that every agent has a strictly positive endowment is a sufficient condition for regularity, which would not be true without the inclusion of agent *i* in its risk orbit. Because the condition of regularity is weaker when agent *i* is included in the risk orbit and all the proofs in Eisenberg and Noe [11] remain valid in this case, we assume here that agent *i* is included in its risk orbit.

[3] Groote Schaarsberg et al. [18] require additionally that all agents use the same division rule, where a division rule is a function of the estate and the vector of liabilities.

[4] Koster [22] requires additionally that all agents use the same division rule, where a division rule is a function of the estate and the vector of liabilities.

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