Stable project allocation under distributional constraints

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Abstract: In a two-sided matching market when agents on both sides have preferences the stability of the solution is typically the most important requirement. However, we may also face some distributional constraints with regard to the minimum number of assignees or the distribution of the assignees according to their types. These two kind of requirements can be challenging to reconcile in practice. Our research is motivated by two real applications, a project allocation problem and a workshop assignment problem, both involving some distributional constraints. We used integer programming techniques to find reasonably good solutions with regard to the stability and the distributional constraints. Our approach can be useful in a variety of different applications, such as resident allocation with lower quotas, controlled school choice or college admissions with affirmative action.

Keywords: stable matching, two-sided markets, project allocation, linear programming, multi-criteria decision making

1 Introduction

Centralised matching scheme has been used since 1952 in the US to allocate junior doctors to hospitals [29]. Later, the same technology has been used in school choice programs in large cities, such as New York [3] and Boston [4]. Similar schemes have been established in Europe for university admissions and school choice as well. For instance, in Hungary both the secondary school and the higher education admission schemes are organised nationwide, see [9] and [10], respectively. In the above mentioned applications it is common that the preferences of the applicants and the rankings of the parties on the other side are collected by a central coordinator and a so-called stable allocation is computed based on the matching algorithm of Gale and Shapley [19]. Two-sided matching markets, and the above applications

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in particular, have been extensively studied in the last decades, see [32] and [26] for overviews from game theoretical and computational aspects, respectively.

This paper is motivated by two applications at the Corvinus University of Budapest. In the first application the task is to allocate students to projects in such a way that the number of students allocated to each project is between a lower and an upper quota. This is a natural requirement present in many applications, such as the Japanese resident allocation scheme [22, 23, 20]. Furthermore, there are also separate lower bounds on the number of foreign students assigned to each company. In the second application the goal is to assign students to companies for solving case studies in a conference, and here again some distributional constraints are imposed with regard to the total number of local, European and other students selected.

We decided to investigate the integer programming techniques for solving these problems motivated by both applications. We had at least three reasons for choosing this technique. The first is that with IP formulations we can easily encode those distributional requirements that the organisers requested, so this solution method is robust to accommodate special features. The second reason is that the computational problem became NP-hard as the companies submitted lists with ties. Using ties in the ranking was by our recommendation to the companies, because ties give us more flexibility when finding a stable solution under the distributional constraints. We describe this issue more in detail shortly. Finally, our third reason for choosing IP techniques was that it facilitates multi-objective optimisation, e.g. finding a most-stable solution if a stable solution does not exist under the strict distributional constraints.

The usage of integer programming techniques for solving two-sided stable matching problems is very rare in the applications, and the theoretical studies on this topic have only started very recently. The reason is that the problems are relatively large in most applications, and the Gale-Shapley type heuristics are usually able to find stable solutions, even in potentially challenging cases. A classical example is the resident allocation problem with couples, which has been present in the US application for decades, and it is still solved by the Roth-Peranson heuristic [31]. The underlying matching problem is NP-hard [28], but heuristic solutions are quite successful in practice, see also [11] on the Scottish application. However, integer programming and constraints programming techniques have been developed very recently and they turned out to be powerful enough to solve large random instances [13], [14] and [15]. Similarly encouraging results have been obtained for some special college admission problems, which are present in the Hungarian higher education system. These special features also makes the problem NP-hard in general, but at least one of these challenging features, turned out to be solvable even in a real data involving more than 150,000 applicants [5]. Finally, the last paper that we highlight with regard to this topic deals with the problem of finding stable solutions in the presence of ties [25]. However, we are not aware of any papers that would study IP techniques for the problem of distributional constraints.

Distributional constraints are present in many two-sided matching markets. In the Japanese resident allocation the government wants to ensure that the doctors [22, 23, 20] are evenly distributed across the country, and to achieve this they imposed lower quotas on the number of doctors allocated in each region. Distributional objectives can also appear in school choice programs [2, 12, 17], where the decision makers want to control the socio-ethnical distribution of the students. Furthermore, the same kind of requirements are implemented in college admission schemes with affirmative action [1] such as the Brazilian college admission system [6] and the admission scheme to Indian engineering schools [7].

When stable solution does not exists for the strict distributional constraints then we either need to relax stability or to adjust the distributional constraints. In this study we will consider the trade-off between these two goals, and develop some reasonable solution concepts.

2 Definitions and preliminaries

Many-to-one stable matching markets have been defined in many context in the literature. In the classical college admissions problem by Gale and Shapley [19] the students are matched to colleges. In the computer science literature this problem setting is typically called Hospital / Residents problem (HR), due to the NRMP and other related applications. In our paper we will refer the two sets as *applicants*

$A = \{a_1, \dots, a_n\}$ and *companies* $\{C = c_1, \dots c_m\}$. Let u_j denote the upper quota of company c_j.

Regarding the preferences, we assume that the applicants provide strict rankings over the companies, but the companies may have ties in their rankings. This model is sometimes referred to as Hospital / Residents problem with Ties (HRT) in the computer science literature, see e.g. [26]. In our context, let r_{ij} denote the rank of company c_j in a_i's preference list, meaning that applicant a_i prefers c_j to c_k if and only if $r_{ij} < r_{ik}$. Let s_{ij} be an integer representing the score of a_i by company c_j, meaning that a_i is preferred over a_k by company c_j if $s_{ij} > s_{kj}$. Note that here two applicants may have the same score at a company, so $s_{ij} = s_{kj}$ is possible. Let $\bar{s}$ denote the maximum possible score at any company and let E be the set of applications. A *matching* is a subset of applications, where each applicant is assigned to at most one company and the number of assignees at each company is less than or equal to the upper quota. A matching is said to be *stable* if for any applicant-company pair not included in the matching either the applicant is matched to a more preferred company or the company filled its upper quota with applicants of the same or higher scores.

In the classical college admission problem, that we refer to as HR, a stable solution is guaranteed to exist, and the two-versions of the Gale-Shapley algorithm [19] find either a student-optimal or a college optimal solution, respectively. Furthermore, this algorithm can be implemented to run in linear time in the number of applications. Moreover, the student-proposing variant was also proved to be strategyproof for the students [29], which means that no student can ever get a better partner by submitting false preferences. Finally, the so-called Rural Hospitals Theorem [30] states that the same students are matched in every stable solution, the number of assignees does not vary across stable matchings for any college, and for the less popular colleges where the upper quota is not filled the set of assigned students is fixed.

When extending the classical college admission problem with the possibility of having ties in the colleges' rankings, that we referred to as an HRT instance, the existence of a stable solution is still guaranteed, since we can break the ties arbitrarily, and a stable solution for the strict preferences is also stable for the original ones. However, now the set of matched students and the size of the stable matchings can vary. Take just the following simple example: we have two applicants, a_1 and a_2 first applying to college c_1 with the same score and applicant a_2 also applies to college c_2 as her second choice. Here, if we break the tie at c_1 in favour of a_1 then we get the matching a_1c_1, a_2c_2, whilst if we break the tie in favour of a_2 then the resulting stable matching is a_2c_1 (thus a_1 is unmatched). The problem of finding a maximum size stable matching turned out to be NP-hard [27], and has been studied extensively in the computer science literature, see e.g. [26]. Note that when the objective of an application is to find a maximum size stable matchings, such as the Scottish resident allocation scheme [21], then the mechanism is not stategyproof. To see this, we just have to reconsider the above example, and assume that originally a_1 also found c_2 acceptable and would ranked it second, just like a_2. By removing c_2 from her list, a_1 is now guaranteed to get c_1 is the maximum size stable solution, however, for the original true preferences a_2 would have an equal chance to get her first choice c_1.

2.1 Introduction of lower quotas

In our first application the organisers of the project allocations wanted to ensure a minimum number of students for each company. Similar requirements have been imposed for the Japanese regions with regard to the number of residents allocated there. In our model, we introduce a lower quota l_j for each company c_j and we require that in a feasible matching the number of assignees at any company is between the lower and upper quotas. Stability is defined as before. We refer to the setting with strict preferences as Hospitals / Residents problem with Lower quotas (HRL) and the case with ties is referred to as Hospitals / Residents problem with Ties and Lower Quotas (HRTL).

Regarding HRL, the Rural Hospitals Theorem implies that the existence of a stable matching that obeys both the lower an upper quotas can be decided efficiently. This is because we just find one stable matching by considering the upper quotas only, and if the lower quotas are violated then there exists no stable solution under these distributional constraints. This problem can be still solved efficiently when the sets of companies have common lower and upper quotas in a laminar system, see [18].

However, the problem of deciding the existence of a stable matching for HRTL is NP-hard. To see this, we just have to remark that the problem of finding a complete stable matching for HRT with unit quotas is also NP-hard [27], so if we require both lower and upper quotas to be equal to one for all companies then the two problems are equivalent. Furthermore, no mechanism that finds a stable matching whenever there exists one can be strategyproof.

2.2 Adding types and distributional constraints

In our first application, the organisers want to distribute the foreign students across the projects almost equally. In our second application, there are target numbers for the total number of Hungarian, European and other participants and there are also specific lower quotas for Hungarian students by some companies. These applications motivate our problems with applicant types and distributional constraints.

Let $\mathcal{T} = \{T^1, \ldots, T^p\}$ be the set of types, where $t(a_i)$ denotes the type of applicant a_i. For a company c_j, let l_j^k and u_j^k denote the lower and upper quota for the number of assignees of type T^k. Furthermore, we may also set lower and upper quotas for any type of applicants for a set of companies. In particular, we denote the lower and upper quotas for the total number of applicants of type T^k assigned in the matching by L^k and U^k, respectively. The set of feasibility constraints for the matching is now extended with these lower and upper quotas. Yet, the original stability condition, which does not consider the types of the applicants, remains the same.

3 Solution concepts and integer programming formulations

In all of our formulations we use binary variables $x_{ij} \in \{0, 1\}$ for each application coming from applicant a_i to company c_j. This can be seen as a characteristic function of the matching, where $x_{ij} = 1$ corresponds to the case when a_i is assigned to c_j.

When describing the integer formulations, first we keep the stability condition fixed while we implement the set of distributional constraints. Then we investigate the ways one can relax stability or find most-stable solutions under the distributional constraints.

3.1 Finding stable solutions under distributional constraints

In this subsection we gradually add constraints to the model by keeping the classical stability condition.

Classical HR instance

First we describe the basic IP formulation for HR described in [8]. The feasibility of a matching can be ensured with the following two sets of constraints.

$$\sum_{j:(a_i,c_j)\in E} x_{ij} \leq 1 \text{ for each } a_i \in A \tag{1}$$

$$\sum_{i:(a_i,c_j)\in E} x_{ij} \leq u_j \text{ for each } c_j \in C \tag{2}$$

Note that (1) implies that no applicant can be assigned to more than one company, and (2) implies that the upper quotas of the companies are respected.

To enforce the stability of a feasible matching we can use the following constraint.

$$\left(\sum_{k:r_{ik}\leq r_{ij}} x_{ik}\right) \cdot u_j + \sum_{h:(a_h,c_j)\in E, s_{hj}>s_{ij}} x_{hj} \geq u_j \text{ for each } (a_i,c_j) \in E \tag{3}$$

Note that for each $(a_i, c_j) \in E$, if a_i is matched to c_j or to a more preferred company then the first term provides the satisfaction of the inequality. Otherwise, when the first term is zero, then the second term is greater than or equal to the right hand side if and only if the places at c_j are filled with applicants with higher scores.

Among the stable solutions we can choose the applicant-optimal one by minimising the following objective function.

$$\sum_{(a_i,c_j)\in E} r_{ij} \cdot x_{ij}$$

Modification for HRT

When the companies can express ties the following modified stability constraints, together with the feasibility constraints (1) and (2), lead to stable matchings. Note that here the only difference between this and the previous constraint is that the strict inequality $s_{hj} > s_{ij}$ became weak.

$$\left(\sum_{k:r_{ik}\leq r_{ij}} x_{ik}\right) \cdot u_j + \sum_{h:(a_h,c_j)\in E, s_{hj}\geq s_{ij}} x_{hj} \geq u_j \text{ for each } (a_i,c_j)\in E \tag{4}$$

Extension with lower quotas

Here, we only add the lower quotas for every company.

$$\sum_{i:(a_i,c_j)\in E} x_{ij} \geq l_j \text{ for each } c_j \in C \tag{5}$$

Adding distributional constraints

As additional constraints we require the number of assignees of a particular type to be between the lower and upper quotas for that type at a company.

$$\sum_{i:t(a_i)=T^k,(a_i,c_j)\in E} x_{ij} \leq u_j^k \text{ for each } c_j \in C \text{ and } T^k \in \mathcal{T} \tag{6}$$

$$\sum_{i:t(a_i)=T^k,(a_i,c_j)\in E} x_{ij} \geq l_j^k \text{ for each } c_j \in C \text{ and } T^k \in \mathcal{T} \tag{7}$$

We can also add similar constraints for set of companies, or for the overall number of different assignees at all companies. We describe the latter, as we will use it when solving our second application.

$$\sum_{i,j:t(a_i)=T^k,(a_i,c_j)\in E} x_{ij} \leq U^k \text{ for each } T^k \in \mathcal{T} \tag{8}$$

$$\sum_{i,j:t(a_i)=T^k,(a_i,c_j)\in E} x_{ij} \geq L^k \text{ for each } T^k \in \mathcal{T} \tag{9}$$

3.2 Relaxing stability

Adding additional constraints to the problem can cause the lack of a stable matching, even if we added some flexibility with the ties.

One way to find a most-stable solution is to introduce nonnegative deficiency variables, d_{ij} for each application and add them to the left side of the stability constraint (4). By minimising the sum of these deficiencies as a first objective we can obtain a solution which is close to be stable.

$$\left(\sum_{k:r_{ik}\le r_{ij}} x_{ik}\right)\cdot u_j + \sum_{h:(a_h,c_j)\in E, s_{hj}\ge s_{ij}} x_{hj} + d_{ij} \ge u_j \text{ for each } (a_i,c_j)\in E \tag{10}$$

Note that here, if a pair (a_i, c_j) is blocking for the assignment then we need to add more compensation d_{ij} if the number of assignees at c_j that the company prefers to a_i is large. This approach can be reasonable if we want to avoid the refusal of a very good candidate at a company. We call this solution as *matching with minimum deficiency.*

Alternatively, if we just want to minimise the number of blocking pairs then we can set d_{ij} to be binary and minimise the sum of these variables under the following modified constraints.

$$\left(\sum_{k:r_{ik}\le r_{ij}} x_{ik}\right)\cdot u_j + \sum_{h:(a_h,c_j)\in E, s_{hj}\ge s_{ij}} x_{hj} + d_{ij}\cdot u_j \ge u_j \text{ for each } (a_i,c_j)\in E \tag{11}$$

Here, every blocking pair should be compensated by the same amount, so the number of blocking pairs in minimised. Note that this concept has already been studied in the literature for various models under the name of *almost stable matchings*, see e.g. [14].

3.3 Adjusting upper capacities, envy-free matchings

A different way of enforcing the lowers quota is to relax stability by artificially decreasing the capacities of the companies. This was also the solution in the resident allocation scheme in Japan [22], where the government introduced artificial upper quotas for each of the hospitals, so that in each region the sum of these artificial upper bounds summed up to the target capacity for that region. In the case of our motivating example of project allocation, one simple way of achieving the lower quotas was by reducing the upper quotas at every company.

In this solution what we essentially get is a so-called *envy-free matching*, studied in [33]. The matching is stable with respect to the artificial upper quotas, which means that the only blocking pairs that may occur with regard to the original upper quotas are due to the empty slots created by the difference between the original and the artificial quotas.

However, one may not want to reduce the upper quotas of the companies in the same way, perhaps some more popular companies should be allowed to have more students than the less popular ones. Furthermore, maybe the decision on which upper quotas should be reduced should be make depending on their effect of satisfying the lower quotas (or other requirements). Thus, we may not want to set the artificial upper quotas in advance, but keep them as variables, by ensuring envy-freeness in a different way. One alternative way of enforcing envy-freeness is by the following set of constraints.

$$\sum_{k:r_{ik}\le r_{ij}} x_{ik} \ge x_{hj} \quad \forall (a_i,c_j),(a_h,c_j)\in E, s_{ij} > s_{hj} \tag{12}$$

Constraints (12) will ensure envy-freeness, by making sure that if applicant a_h is assigned to company c_j and applicant a_i has higher score than a_h at c_j then a_i must be assigned to c_j or to a more preferred company.

3.4 Type-specific priorities

So far we have only considered different approaches of relaxing stability or enlarging the set of feasible solutions in order to satisfy the distributional constraints. In this subsection we study alternative solution concepts and methods for the case when the distributional constraints are type-dependent. This is the case also in our motivating application, where special requirements are set for the foreign students assigned to the companies.

When the number of students of a type does not achieve the minimum required at a place then there are two well-known approaches. For instance in a school choice scenario, where the ratio of an socio-ethnic group should be improved (see e.g. [2]) then one possible affirmative action is to increase the scores of that group of students as much as needed. The other usual solution is to set some reserved seats to those students (see e.g. [6]).

In our project allocation application our requirement is to have at least one foreign student assigned to every company. If in a stable solution this condition would be violated for a company then we can try to enforce the admission of a foreign student by increasing the scores of the foreign students at this company. We call such a solution as *stable matching with type-specific scores*, where the classical stability condition is required for the adjusted scores. The second approach is to devote one place at each company to foreign students. For this one seat the foreign students will have higher priority than the locals irrespective of their scores, but for the rest of the spaces the usual score-based rankings apply. We call this concept as *stable matching with reserved seats for types*. Note that neither of these two concepts can always ensure that we get at least one foreign student at each company, since they may all have high scores and they may all dislike a particular company. However, this situation changes if we also allow to decrease the scores of a group of students. We will describe this case after discussing the third approach.

Finally, as a third approach, we can also extend the concept of envy-free matchings for types. We do not require any stability with regard to students of different types, but we do require envy-freeness for students of the same type. Thus the so-called *type-specific envy-free matchings* will be those who satisfy the following set of constraints.

$$\sum_{k:r_{ik}\leq r_{ij}} x_{ik} \geq x_{hj} \quad \forall (a_i, c_j), (a_h, c_j) \in E, s_{ij} > s_{hj}, t(a_i) = t(a_h) = T^k \text{ for each } T^k \in \mathcal{T} \tag{13}$$

That is, if a_i and a_h have the same type and a_h is assigned to c_j then the higher ranked a_i must also be assigned to c_j or to a more preferred company. Note that with this modification we extend the set of feasible solutions compared to the set of envy-free matchings. Another important observation that is motivated by our project allocation problem is that under some realistic assumptions a type-specific envy-free matching always exists, that we will show in the following theorem.

Theorem 1 *Suppose that all the companies are acceptable to every student and that the sum of the lower quotas with regard to each type is less than equal to the number of students of that type, and the sum of the lower quotas across types for a company is less than or equal to the upper quota of that company, then a complete within-type envy-free matching always exists and can be found efficiently.*

PROOF: We construct a within-type envy-free matching separately for each type and then we merge them. When considering a particular type T^k, we set artificial upper quotas at the companies to be equal to the type-specific lower quotas (i.e. l_j^k for company c_j) and we find a stable matching M_k for this type. This stable matching must exist, since we assumed that all the companies are acceptable to every student and the number of students in every type is at least as much as the sum of the lower quotas for that type. We create matching M by merging the stable matchings for the types, i.e. $M = M_1 \cup M_2 \cup \cdots \cup M_p$. Note that no upper quota is violated in M, since we assumed that the sum of the lower quotas across types for any company c_j is less than equal to the upper quota of c_j. By the stability of M_k for every type T^k it follows that matching M is within-type envy-free. If there is still a company c_j, where the overall lower quota (l_j) is not yet met, then we increase an artificial upper quota for some at c_j so that there is still some unmatched applicants of this type. Since the total number of applicants is greater or equal to the sum of the lower quotas, we have to achieve the lower quotas at all companies in this way. Finally, it there are still some unmatched applicants then we increase some artificial upper quotas for their types one-by-one, by making sure that we never exceed any overall upper quota. At the end of this iterative process we must reach a complete within-type envy free matching. □

Let us abbreviate a *complete within-type envy-free matching* as CWTEFM. Now, we will compare this concept of CWTEFM with stable matchings with type-specific scores and observe that they are essentially the same.

Theorem 2 *Under the assumptions of Theorem 1 a complete matching is within-type envy-free if and only if it is stable with type-specific scores.*

Proof: Suppose first that M is a complete stable matching with type-specific scores, we will see that M is also within-type envy-free by definition. Suppose for a contradiction that there is a student a_i who has justified envy against student a_h of the same type at company c_j, i.e. a_h is assigned to c_j whilst a_i has higher score at c_j than a_h and a_i is assigned to a less preferred company. This would mean that the pair $\{a_i, c_j\}$ is blocking for the adjusted scores, since both students get the same adjustment at c_j, contradicting with the stability of M.

Suppose now that M is a CWTEFM. Let us adjust the scores of the students according to their types at each company such that the weakest students admitted have the same scores across types. Matching M is stable with regard to the adjusted scores, because if a student a_i is not admitted to a company c_j and any better place of her preference that it must be the case that her score at c_j was less then or equal to the score of the weakest assigned student of the same type at c_j, which means that the adjusted score of a_i at c_j is less than or equal to the adjusted score of every assigned student at c_j. $\square$

Instead of using the above described processes of setting type-specific artificial upper quotas or making adjustments for the scores of different types, we can also get a CWTEFM directly by an IP formulation. We shall simply use the feasibility and distributional constraints together with (13) and with an objective function maximising the number of students assigned. This approach is not just more robust than the above described two heuristics, but it has also the advantage that we can enforce additional optimality or fairness criteria. Regarding optimality, we may want to minimise the total rank of the students, leading to a Pareto-optimal assignment under the constraints. As an additional fairness criterion we may aim to minimise the envy across types. We can achieve this by adding deficiency variables to the left hand side of constraints (12) for students of different types, as described in (14) below, and then minimising the sum of the deficiencies. We refer to this solution as MinDefCWTEFM, that is *complete within-type envy-free matching with minimum deficiency across types.*

$$\sum_{k:r_{ik}\leq r_{ij}} x_{ik} + d_{ih}^{j} \geq x_{hj} \quad \forall (a_i, c_j), (a_h, c_j) \in E, s_{ij} > s_{hj}, t(a_i) \neq t(a_h) \tag{14}$$

We remark that in our project allocation application the conditions of Theorem 1 are satisfied, since all the students have to rank (and accept) all the companies and we require to have at least one foreign student at each company, where the number of foreign students is more than the number of companies. Therefore a complete within-type envy-free matching always exists. Within this set of solutions we decided to minimise the envy across types, as suggested above as first objective. As a secondary objective we can choose to minimise the total rank of the students or as an alternative we can also minimise the open-slot blockings (i.e. the blockings due to unfilled positions with regard to the original upper quotas). The latter objective is useful to make sure that the popular companies always fill their upper quotas, and so the less popular companies will admit fewer students.

4 Further notes

We applied the above described solution concepts for our two motivating applications, in a project allocation problem at Corvinus University of Budapest and for a conference organisation case, that we will describe in details in an extension of this paper. We will also test these solution concepts on randomly generated instances to find out how large problems can be solved with the IP technique.

One could also try to come up with alternative solution concepts and different IP formulations for the same concepts that we proposed. Finally, it would be interesting to test the applicability of our approach in other applications, such as the resident allocation problem with regional quotas, controlled school choice, and college admissions with affirmative action or minority reserves.

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