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# On monotone likelihood ratio of stationary probabilities in bonus-malus systems

KOLOS CSABA ÁGOSTON *
Institute of Mathematics and Statistical Modeling
Corvinus University of Budapest
H–1093, Fővám tér 13-15., Budapest, Hungary
and
MÁRTON GYETVAI†
Institute of Economics
Centre for Economic and Regional Studies
H–1097, Tóth Kálmán utca 4., Budapest, Hungary

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**Abstract**

Bonus-malus system is an often used risk management tool in the insurance industry, and it is usually modeled with Markov chains. Under mild conditions it can be stated that the bonus-malus system converges to a unique stationary distribution in the long run. The maximum likelihood ratio property is a well-known statistical concept and we define it for the stationary distribution of a bonus-malus system. For two special cases we could justify it algebraically. For other cases we describe a numerical method with which we can test this property in any case. With the help of the described method, we checked this property for cases that appear in actuarial practice.

**Mathematics Subject Classifications (2015).** 60J22

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## 1 Introduction

In the insurance industry it is a great challenge to insure everybody at their own risk. Insurance companies apply advanced statistical and/or data science models to describe more and more precisely the actual risk of a policyholder. This kind of risk classification has a great advantage for insurance companies. However, not all of the heterogeneity can be ceased in this way: two policyholders with the same demographic and behavioural feature can differ in their

*e-mail: kolos.agoston@uni-corvinus.hu
†e-mail: gyetvai.marton@krtk.hu

risk. For this reason, insurance companies are operating bonus-malus (BM) systems (most frequently —but not exclusively— in automobile coverages). If a policyholder has a claim, they get into a lower BM class, where the premium is higher; on the other hand, if they do not have any claim, they get into a better BM class, where the premium is less.

Bonus-malus system is usually modelled with the help of a Markov chain. Most frequently, it is assumed that the Markov chain is ergodic. Ergodic Markov chain is investigated widely in the literature, but Markov chain describing BM systems has special structure even in the class of ergodic Markov chains.

Monotone likelihood property is a well-known concept in statistics, and it appears in many decision problems as an assumption. The monotone likelihood ratio does not generally hold in Markov chains since there is no order of states. However, in the BM system, there is a strict order of BM classes. In BM systems we will define the monotone likelihood ratio. We cannot give a general mathematical proof (except for two special cases), but we present a method by which we check many of the relevant cases.

In Section 2 we give the most fundamental concepts in BM systems. In Section 3 we investigate the monotone likelihood ratio property in BM systems. We give analytical proof for two special cases and describe a numerical method for checking the property in other cases. With the help of the described method, we could check the monotone likelihood property of many cases. We could say that we covered the cases which is relevant in actuarial practice. In Section 4 we present a possible application of our results. In Section 5 we conclude our results.

## 2 Preliminaries

We investigate a BM system, where $K$ classes exist, indexed from 1 to $K$. In the BM system, in each year, the participants move $u$ class upward (or get into the top class if they cannot move $u$ class upward) if they do not cause claims and moves $d$ class downward (or get into the bottom class) if they have claims. The probability of claim is denoted by $q$ (in this paper we assume that more than one claim is not possible).

The element $t_{ij}$ of the transition matrix $T$ give the probability that a client who is in class $i$ in period $t$ will get to class $j$ in period $t+1$. Value $(c_i)_t$ gives the probability that a policyholder will be in BM class $i$ in period $t$. We arrange values $c_i^t$ into a vector $\mathbf{c_t}$. We know that $(\mathbf{c_t})^\top = (\mathbf{c_0})^\top \mathbf{T^t}$. As $t$ tends to infinity, $\mathbf{c_t}$ tends to a unique stationary distribution $\mathbf{c}$. We know $c$ is the solution of system of equations

$$(\mathbf{c})^\top \mathbf{T} = (\mathbf{c})^\top \ , \tag{1}$$

in other words $\mathbf{c}$ is the left eigenvector of eigenvalue 1 of matrix $T$, see Kaas et al. (2001), Kemeny and Snell (1976), Lemaire (1995). Stationary distribution is based on the probability of claim ($q$), so it will be denoted by $\mathbf{c}(q)$. The target of the investigation is how $c_i(q)$ depends on the probability of claim.

Monotone likelihood ratio is well known in statistics and it appears in many decision problems as a necessary condition. Let $X$ be a random variable, its distribution depends on an outer parameter $q$. First let us suppose that $X$ has a probability density function: $f(x|q)$. The monotone likelihood property means that expression

$$\frac{f(x_0|q_1)}{f(x_0|q_0)} \geq \frac{f(x_1|q_1)}{f(x_1|q_0)} \tag{2}$$

holds for every $x_0 \leq x_1$ and $q_0 \leq q_1$. Rearranging (2):

$$f(x_0|q_1)f(x_1|q_0) \geq f(x_1|q_1)f(x_0|q_0) \tag{3}$$

Let $A$ be an interval: $[\underline{a}, \overline{a}]$, similarly $B$ is also an interval: $[\underline{b}, \overline{b}]$, furthermore $\overline{a} < \underline{b}$. Let us choose $\tilde{x}$ in a way that $\overline{a} < \tilde{x} < \underline{b}$.

Using (3) we can state that:

$$\int_A f(x|q_1)dx f(\tilde{x}|q_0) \geq f(\tilde{x}|q_1) \int_A f(x|q_0)dx$$

and

$$f(\tilde{x}|q_1) \int_B f(x|q_0)dx \geq \int_B f(x|q_1)dx f(\tilde{x}|q_0) \; . \tag{4}$$

Rearranging we can state that:

$$\frac{\int_A f(x|q_1)dx}{\int_A f(x|q_0)dx} \geq \frac{\int_B f(x|q_1)dx}{\int_B f(x|q_0)dx},$$

and

$$\frac{\int_A f(x|q_1)dx}{\int_B f(x|q_1)dx} \geq \frac{\int_A f(x|q_0)dx}{\int_B f(x|q_0)dx} \; . \tag{5}$$

Using (5) we can check the maximum likelihood ratio in case of a BM system. We have to check that the fraction

$$\frac{c_{K-i}(q)}{c_{K-(i+1)}(q)} \tag{6}$$

is monotone in $q$.

# 3 Monotone likelihood ratio in BM systems

## 3.1 One class upward, one class downward transition rule

If $u = 1$ and $d = 1$, the monotone likelihood ratio can be checked analytically. The first equation in the system of equations (1) is:

$$c_K = (1-q)c_K + (1-q)c_{K-1}$$

Rearranging we get:

$$c_{K-1} = \frac{q}{1-q} c_K \tag{7}$$

The second equation in the system of equations (1) is:

$$c_{K-1} = q c_K + (1-q) c_{K-2} \ .$$

Using (7) we get:

$$\frac{q}{1-q} c_K = q c_K + (1-q) c_{K-2} \ .$$

Rearranging:

$$c_{K-2} = \left(\frac{q}{1-q}\right)^2 c_K \ .$$

It can be seen easily, that

$$c_{K-i} = \left(\frac{q}{1-q}\right)^i c_K \ . \tag{8}$$

We would like to show that $\frac{c_{K-i}}{c_{K-(i+1)}}$ is decreasing in $(q)$. It is quite straightforward:

$$\frac{c_{K-i}}{c_{K-(i+1)}} = \frac{\left(\frac{q}{1-q}\right)^i c_K}{\left(\frac{q}{1-q}\right)^{i+1} c_K} = \frac{1-q}{q} \ ,$$

which is monotone decreasing in the interval $(0,1)$.

## 3.2 One class upward, $K$ class downward transition rule

Another case where we can determine a closed formula is $u = 1$, $d = K$ case, which practically means that if there is a claim, the policyholder gets into the lowest class. This rule is not really used in actuarial practice, but results in a simple closed formula that we think is worth mentioning.

Expression (7) is the same in this case as well; using the system of equations (1) for $c_{K-i}$ we have a simple formula:

$$c_{K-i} = \frac{q}{(1-q)^i} c_K \ ,$$

if $i < K-1$. It is straightforward that if (and only if) a claim occurs, the policyholder gets into the lowest class, so $c_1 = q$.

The fraction $\frac{c_{K-i}}{c_{K-(i+1)}}$ will result in a very easy expression: $1-q$, which is strictly monotone decreasing in the interval $[0,1]$. It is trivial for $i < K-2$, but we have to check these as well

$$\frac{c_2}{c_1} = \frac{c_{K-(K-2)}}{c_1} = \frac{\frac{q}{(1-q)^{(K-2)}} c_K}{q} = \frac{c_K}{(1-q)^{(K-2)}} \ .$$

To check the monotone likelihood ratio we have to calculate $c_K$. We know that: $\sum_{i=1}^{K} c_i = 1$ and also know that $c_1 = q$. So,

$$1-q = \sum_{i=2}^{K} c_i = c_K + \sum_{j=1}^{K-2} \frac{q}{(1-q)^j} c_K = c_K \left(1 + \left(\frac{q}{1-q}\right) \frac{1-\left(\frac{1}{1-q}\right)^{K-2}}{1-\frac{1}{1-q}}\right)$$

and after rearranging we get: $c_K = (1-q)^{K-1}$, so $\frac{c_2}{c_1} = (1-q)$ as well.

## 3.3 One class upward, two class downward transition rule

Let us see another transition rule: in case of a claim the policyholder moves two classes downward ($d = 2$), otherwise one class upward ($u = 1$). By the way, this is the actual transition rule in third party automobile liability insurance in Hungary and in many other countries.

In this case, the first equation in the system of equations (1) is the same as before, expression (7) holds in this case as well. The second one is different:

$$c_{K-1} = (1-q)c_{K-2} \ ,$$

so

$$c_{K-2} = \frac{q}{(1-q)^2} c_K \ . \tag{9}$$

The third equation in the system of equations (1) is: $c_{K-2} = qc_K + (1-q)c_{K-3}$, using (9) we get:

$$c_{K-3} = \frac{2q^2 - q^3}{(1-q)^3} c_K \tag{10}$$

The fourth equation in the system of equations (1) is: $c_{K-3} = qc_{K-1} + (1-q)c_{K-4}$, using equation (10) we get

$$c_{K-4} = \frac{q^2 + q^3 - q^4}{(1-q)^4} c_K \ . \tag{11}$$

Surely, we can continue the recursive expression. Up to $i-1$ we have an expression $c_{K-(i-1)} = \frac{P_{i-1}(q)}{(1-q)^{i-1}} c_K$, where $P_i(q)$ is a polynomial up to degree $i-1$. The $(i-1)$th equation in the system of equations (1) is: $c_{K-(i-1)} = qc_{K-(i-3)} + (1-q)c_{K-i}$. Rearranging, we get:

$$c_{K-i} = \frac{c_{K-(i-1)}}{(1-q)} - \frac{qc_{K-(i-3)}}{(1-q)} = \frac{P_{i-1}(q)}{(1-q)^i} c_K - \frac{qP_{i-3}(q)}{(1-q)^{i-2}} c_K = \frac{P_{i-1}(q) - q(1-q)^2 P_{i-3}(q)}{(1-q)^i} \ ,$$

so

$$P_i = P_{i-1}(q) - q(1-q)^2 P_{i-3}(q) \ . \tag{12}$$

To get the impression, we give the expression up to $c_{K-10}$:

$$c_{K-5} = \frac{3q^3 - 2q^4}{(1-q)^5} c_K$$

$$c_{K-6} = \frac{q^3 + 3q^4 - 4q^5 + q^6}{(1-q)^6} c_K$$

$$c_{K-7} = \frac{4q^4 - 2q^5 - 2q^6 + q^7}{(1-q)^7} c_K$$

$$c_{K-8} = \frac{q^4 + 6q^5 - 9q^6 + 3q^7}{(1-q)^8} c_K$$

$$c_{K-9} = \frac{5q^5 - 9q^7 + 6q^8 - q^9}{(1-q)^9} c_K$$

$$c_{K-10} = \frac{q^5 + 10q^6 - 15q^7 + 3q^8 + 3q^9 - q^{10}}{(1-q)^{10}} c_K$$

We could not discover any rules. Thus we were not able to give a closed formula for $c_{K-i}$ as in the two previously investigated cases.

Although a closed formula is not found, it can be handled easily numerically. We know that $P_i(q)$ is a polynomial up to degree $i$:

$$P_i(q) = \alpha_0 + \alpha_1 q + \alpha_2 q^2 + \ldots + \alpha_i q^i$$

To handle it in an easier way, we represent $P_i(q)$ as a vector $\mathbf{p_i}$, its $j$th component ($\mathbf{p}_{i[j]}, j = 1, \ldots, i+1$) is $\alpha_{j-1}$. We know: $P_0(q) = 1$ ($\mathbf{p}_0 = (1)$), $P_1(q) = q$ ($\mathbf{p}_1 = (0,1)^\top$) and $P_2(q) = q$ ($\mathbf{p}_2 = (0,1,0)^\top$). From $i = 3$ we use the recursive expression (12), so

$$\mathbf{p}_{i[j]} = \mathbf{p}_{i-1[j]} - \mathbf{p}_{i-3[j-1]} + 2\mathbf{p}_{i-3[j-2]} - \mathbf{p}_{i-3[j-3]} \ , \tag{13}$$

where $\mathbf{p}_{i[j]} = 0$, if $j \leq 0$.

Remark 1 Let $i = 2k + \ell$, where $k$ is nonnegative integer and $\ell$ is zero or one. In the polynomial $P_i(q)$ all coefficients $\alpha_m$ will be zero for $m < k + \ell$, furthermore $\alpha_k = 1$ if $\ell = 0$ ($i$ is even) and $\alpha_k = k + 1$ if $\ell = 1$ ($i$ is odd). We checked this property for all $i \leq 100$.

Now we can calculate:

$$\frac{c_{K-i}}{c_{K-(i+1)}} = \frac{\frac{P_i(q)}{(1-q)^i} c_K}{\frac{P_{i+1}(q)}{(1-q)^{i+1}} c_K} = \frac{(1-q)P_i(q)}{P_{i+1}(q)} \ . \tag{14}$$

We have to check whether $\frac{(1-q)P_i(q)}{P_{i+1}(q)}$ is monotone in $q$ or not. To do it, we calculate a derivative of the expression (14):

$$\left( \frac{(1-q)P_i(q)}{P_{i+1}(q)} \right)' = \frac{-P_i(q)P_{i+1}(q) + (1-q)P_i'(q)P_{i+1}(q) - (1-q)P_i(q)P_{i+1}'(q)}{(P_{i+1}(q))^2} \tag{15}$$

The denominator in expression (15) is always positive, so the sign of the fraction depends on the nominator. The expression in the nominator is a polynomial itself ($N_i(q)$) up to degree $2i+1$. We can give the coefficient (arranged into the vector $\mathbf{n}_i$) in matrix arithmetic form:

$$\mathbf{n}_i = -\mathbf{p}_i \otimes \mathbf{p}_{i+1} + \left( \begin{pmatrix} 1 \\ -1 \end{pmatrix} \otimes D_i \mathbf{p}_i \right) \otimes \mathbf{p}_{i+1} - \left( \begin{pmatrix} 1 \\ -1 \end{pmatrix} \otimes \mathbf{p}_i \right) \otimes (D_{i+1}\mathbf{p}_{i+1}) \,, \tag{16}$$

where $\mathbf{a} \otimes \mathbf{b}$ stands for convolution of vectors $\mathbf{a}$ and $\mathbf{b}$ and $D_i$ is the 'derivative matrix'. Let $\mathbf{a} \in \mathbb{R}^{\mathbf{n}}$ and $\mathbf{b} \in \mathbb{R}^{\mathbf{m}}$ then $\mathbf{a} \otimes \mathbf{b} \in \mathbb{R}^{n+m-1}$;

$$(\mathbf{a} \otimes \mathbf{b})_{[\ell]} = \sum_{k=\max(1,\ell+1-m)}^{\min(\ell,n)} \mathbf{a}_{[k]} \mathbf{b}_{[\ell-k+1]}$$

$D_i$ is a $(i-1) \times i$ matrix:

$$D_i = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 2 & \dots & 0 \\ \vdots & & & & \\ 0 & 0 & 0 & \dots & i \end{pmatrix}$$

Again, to get an impression we give $N_i(q)$ up to degree 10.

$$N_1(q) = -q^2$$

$$N_2(q) = -2q^2 + 2q^3 - q^4$$

$$N_3(q) = -5q^4 + 6q^5 - 2q^6$$

$$N_4(q) = -3q^4 + 4q^5 - 6q^6 + 6q^7 - 2q^8$$

$$N_5(q) = -14q^6 + 28q^7 - 23q^8 + 10q^9 - 2q^{10}$$

$$N_6(q) = -4q^6 + 4q^7 - 18q^8 + 44q^9 - 42q^{10} + 18q^{11} - 3q^{12}$$

$$N_7(q) = -30q^8 + 72q^9 - 81q^{10} + 68q^{11} - 45q^{12} + 18q^{13} - 3q^{14}$$

$$N_8(q) = -5q^8 - 48q^{10} + 186q^{11} - 265q^{12} + 200q^{13} - 90q^{14} + 24q^{15} - 3q^{16}$$

$$N_9(q) = -55q^{10} + 132q^{11} - 165q^{12} + 242q^{13} - 333q^{14} + 284q^{15} - 138q^{16} + 36q^{17} - 4q^{18}$$

Remark 2 Let $i = 2k + \ell$, where $k$ is a nonnegative integer and $\ell$ is zero or one. In the polynomial $N_i(q)$ all coefficients $\alpha_m$ will be zero for $m < 2(k + \ell)$. We checked this property numerically for $i \leq 100$.

We have to check whether $N_i(q)$ is positive or not. Since $0 < q < 1$ so both $q^k$ and $(1-q)^k$ is positive for all positive integer $k$. We will deconvulate $N_i(q)$ into a negative of sum of polynomials $q^k$ and $(1-q)^k$. Again let $i = 2k + \ell$, where $k$ is a nonnegative integer and $\ell$ is zero or one. If we find nonnegative weights $\alpha_{jm}$ and $\beta_j$ so that

$$N_i(q) = -q^{2(k+\ell)} \left( \sum_{j=1}^{2k} \sum_{m=1}^{2k+1-j} q^{m-1}\alpha_{jm}(1-q)^j + \sum_{j=0}^{2k} \beta_j q^j \right)$$

then we can be sure that $N_i(q)$ is negative (all of $\alpha$ and $\beta$ cannot be zero at the same time).

We can find nonnegative weights with the help of linear programming:

$$\sum_{\substack{j,m,r \\ j\geq 1, m\geq 1, r\geq 0, \\ r\leq j, m\leq 2k+1-j, \\ j<=2k, m+r-1=s}} \alpha_{jm}(-1)^r B(j,r) + \beta_s = -\mathbf{n}_i[2(k+\ell)+s] \ , \quad s = 0,\dots,2k \quad (17)$$

where $B(j,r)$ is the binomial coefficient: $\frac{j!}{r!(j-r)!}$. The objective function is arbitrary since any feasible solution ensures the negativity of $N_i(q)$. The LP problem (17) is quite small considering the capacity of available solvers. Contrary to the fact that the size of LP is quite small, we can face numerical issues. The difficulty is based on the fact the binomial coefficient $B(j,r)$ can be quite large for large $j$. To avoid this problem we can set the constraint $\alpha_{jm} = 0$ when $j$ is large. This trick could extend the size of the solvable problem a bit, but itself in the polynomial $N_i(q)$ quite big coefficients appear, which cannot be avoided. For instance, in $N_{20}$ appears a coefficient greater than $10^6$; in $N_{29}(q)$ a value that is higher than $10^9$; moreover in $N_{39}(q)$ a coefficient greater than $10^{12}$.

For calculating the $N_i(q)$ polynomials, we used Python (version 3.7.6). We used an AMD Ryzen 5 2600 Six-Core CPU 3.40 GHz computer with 16 GB DDR4 RAM. For solving the LP and IP problems, we used Gurobi (8.1.0). Within one minute running time, we found a feasible solution for $i \leq 58$. This statement does not mean that there is no feasible solution for $i > 58$, but we face numerical problems due to the very large coefficients. It is worth mentioning that within 1 minute running time, we found an integer solution up to $i \leq 37$ (we consider the integer solution to be more stable). A feasible solution for the first 9 polynomials are:

$$N_2(q) = -q^2[2(1-q)+q^2] \ ,$$

$$N_3(q) = -q^4[2(1-q)+2(1-q)^2+1] \ ,$$

$$N_4(q) = -q^4[2(1-q)+2q^2(1-q)+(1-q)^2+2q^2(1-q)^2+q] \ ,$$

$$N_5(q) = -q^6[6q^2(1-q)+14(1-q)^2+2q^2(1-q)^2+q^2] \ ,$$

$$N_6(q) = -q^6[4(1-q)+5q^2(1-q)+3q^4(1-q)^2+13q^2(1-q)^3+q^5] \ ,$$

$$N_7(q) = -q^8[9(1-q)+2q^2(1-q)+3q^2(1-q)^2+3q^4(1-q)^2+21(1-q)^3+ \\ 13q^2(1-q)^3+q^5],$$

$$N_8(q) = -q^8[q^4 14(1-q) + 5(1-q)^2 + 3q(1-q)^2 + 28q^4(1-q)^2 + 2q^6(1-q)^2 + 7q(1-q)^3 + 70q^2(1-q)^3 + 20q^4(1-q)^3 + q^8] \ ,$$

$$N_9(q) = -q^{10}[2(1-q) + q(1-q) + 12q^4(1-q) + 47q^4(1-q)^2 + 4q^6(1-q)^2 + 53(1-q)^3 + 28q(1-q)^3 + 91q^2(1-q)^3 + 29q^4(1-q)^3 + q^7] \ .$$

### 3.4 One class upward, arbitrary class downward transition rule

After checking the $u = 1$, $d = 2$ transition rule, we can also check other transition rules similarly. When the policyholder gets $d$ class downward, expression (8) holds for $i < d$. For $i \geq d$ instead of (12) we can use the recursive expression:

$$P_i = P_{i-1}(q) - q(1-q)^d P_{i-1-d}(q) \ ,$$

and instead of (13) we have that

$$\mathbf{p}_{i[j]} = \mathbf{p}_{i-1[j]} - \sum_{\ell=0}^{d} (-1)^\ell B(d,\ell)\mathbf{p}_{i-1-d[j-1-\ell]} \ .$$

Expressions (15) and (16) remain true, so we can consider the LP problem (17). In Table 1 the reader can find information of the checked cases. Again, the values in Table 1 do not mean that above these values the monotone likelihood ratio property is not true, but above these values we face a serious numerical problem running the LP problem (17).

| | $d=3$ | $d=4$ | $d=5$ | $d=6$ | $d=7$ | $d=8$ | $d=9$ | $d=10$ |
|---|---|---|---|---|---|---|---|---|
| LP | 43 | 67 | 56 | 57 | 57 | 57 | 57 | 55 |
| ILP | 36 | 27 | 61 | 58 | 57 | 36 | 57 | 42 |

Table 1: Checked cases in various transition rules.

## 4 Application

In this section, we give a possible application of the monotone likelihood property. In actuarial sciences, a well-studied problem to set optimal premium scale in a BM system, see Lemaire (1995), Heras et al. (2004), Ágoston and Gyetvai (2020)

For a small example, we assume that there are two equal-sized risk groups ($A$ and $B$) in a risk community. To the risk community, the premium is set based on a common BM system. The claim probability for group $A$ is less than for group $B$ ($q^A < q^B$). The objective of the problem is to find a premium value ($\pi$) for each BM class that minimizes the quadratic loss function:

$$c_i^A(q^A - \pi_i)^2 + c_i^B(q^B - \pi_i)^2 \to \min$$

Therefore the optimal premium scale is $\pi_i = \frac{c_i^A q^A + c_i^B q^B}{c_i^A + c_i^B}$. If for stationary probabilities the maximum likelihood property is true then $\pi_i \leq \pi_{i-1}$ (we do not need to explicitly prescribe this constraint as in Heras et al. (2004) for instance). To prove it let $\frac{c_i^A}{c_{i-1}^A} = r_i^A$ and $\frac{c_i^B}{c_{i-1}^B} = r_i^B$. Since the maximum likelihood ratio property holds we know that $r_i^A > r_i^B$. Now:

$$\pi_{i-1} = \frac{c_{i-1}^A q^A + c_{i-1}^B q^B}{c_{i-1}^A + c_{i-1}^B} = \frac{c_{i-1}^A q^A + c_{i-1}^B q^A + c_{i-1}^B (q^B - q^A)}{c_{i-1}^A + c_{i-1}^B} =$$
$$q^A + \frac{c_{i-1}^B (q^B - q^A)}{c_{i-1}^A + c_{i-1}^B} = q^A + \frac{c_{i-1}^B r_i^B (q^B - q^A)}{c_{i-1}^A r_i^B + c_{i-1}^B r_i^B} >$$
$$q^A + \frac{c_{i-1}^B r_i^B (q^B - q^A)}{c_{i-1}^A r_i^A + c_{i-1}^B r_i^B} = q^A + \frac{c_i^B (q^B - q^A)}{c_i^A + c_i^B} = \frac{c_i^A q^A + c_i^B q^B}{c_i^A + c_i^B} = \pi_i \ .$$

We can get a quite similar result if we use the absolute loss function. In this case, our problem is:

$$c_i^A |q^A - \pi_i| + c_i^B |q^B - \pi_i| \to \min \ .$$

Again, it is not hard to see that the optimal premium scale is

$$\pi_i = \begin{cases} q^A, & \text{if } c_i^A \geq c_i^B \\ q^B, & \text{otherwise} \ . \end{cases}$$

If the stationary probabilities fulfill the maximum likelihood ratio assumption, the premium scale will be monotonic. The proof is almost trivial: let suppose that $\pi_{i-1} = q^A$. Then:

$$\begin{aligned} & c_{i-1}^A \geq c_{i-1}^B \\ & \Downarrow \\ & c_{i-1}^A r_i^B \geq c_{i-1}^B r_i^B \\ & \Downarrow \ (\text{since } r_i^A > r_i^B) \\ & c_{i-1}^A r_i^A \geq c_{i-1}^B r_i^B \\ & \Downarrow \\ & c_i^A \geq c_i^B \ , \end{aligned}$$

so $\pi_i = q^A$ as well.

# 5 Conclusion

In this paper, we investigated the monotone likelihood ratio property in the case of BM systems. We could prove analytically that the stationary probabilities have the monotone likelihood ratio property for the two most extreme cases. We described a numerical method for checking this property for other cases and checked the monotone likelihood property for many cases. Although undoubtedly, an analytical proof would be more satisfactory, we covered those cases which appear in actuarial practice.

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