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Procrastination and intertemporal consumption: A three-period extension of the CAPM with irrational agents

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ABSTRACT

In this paper, we investigate the capital asset pricing model (CAPM) derived from a three-period general equilibrium model incorporating time-inconsistent preferences. We define and consider two types of agents, i.e. they can be either sophisticated or naive. Sophisticated agents take into account their potentially changing future preferences when making a decision. Naive agents, on the other hand, do not anticipate this issue and their related self-control problems when they plan the consumption path.

We demonstrate that the derivation of the CAPM equation can be achieved even if the agents in the financial economy have time-inconsistent preferences.

1. Introduction

The capital asset pricing model, commonly referred to as CAPM in the literature, accurately estimates the relationship between the risk and the expected return of an asset. The CAPM model estimates the expected returns of risky assets in equilibrium.

Models based on neoclassical economic theory tend to assume that the agents in an economy are rational. However, the observed patterns often contradict these assumptions not only in our daily lives and the field of psychology but also in experimental and behavioral economics. Many of the decisions we make are deemed irrational by neoclassical economics, yet the agents in most models have remained rational, as they are said to describe the average decision-maker. However, it is now widely accepted that the so-called irrational behaviors are not just outlier instances but widely acknowledged social phenomena driven by cognitive biases, take, for example, Kahneman (2011). These empirical pieces of evidence suggest that the sole use of rational agents in economic models has become insufficient to describe real-world dynamics. These studies suggest that decision-makers do not necessarily follow constant discounting (Frederick et al., 2002).

To be able to capture some of these anomalies in intertemporal decision-making, one needs at least three time-periods. This is needed to provide agents with a point in time when they can change their minds. The three-period economic model allows us to model time-inconsistent behaviors.

As a foundation of our model, we use the well-known results described in the books by Magill and Quinzii (1996a) and LeRoy and Werner (2001), which we frequently use as building blocks in this study. Regarding the three-period model with rational agents,

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we also often reach for the results of Habis (2024). Here, we introduce two new variants of decision-makers: naive and sophisticated agents.[2] Both of these types can be modeled with hyperbolic discount factors, which allows us to describe real-world decisions more closely.

In Section 2 we give an outline of the model with rational agents. We derive the model with sophisticated and naive agents in Section 3, thereby asserting that it is possible and advantageous to insert irrational elements into financial-economic models.

2. The three-period CAPM with rational agents

In this section, we show how the β-pricing formula that relates the return of a risky asset to the return of the market portfolio can also be derived in a three-period finance general equilibrium model. We apply the Consumption Capital Asset Pricing Model (CCAPM), where the CAPM is centered on consumption. The CCAPM was first introduced by Rubinstein (1976), Lucas (1978), and Breeden (1979). The three-period structure described here is based on that of Habis (2024).

Let $t \in \{0,1,2\} = T$ denote the time periods. In periods $t = 1,2$ one event out of a finite set occurs. At every state $s \in \mathcal{S}$ we denote the date-event at period t by $s_t \in \mathcal{S}_t$, where the cardinality of $\mathcal{S}_t$ is S_t and $\mathcal{S} = \bigcup_t \mathcal{S}_t$ for all $t \in T$. For $t = 0$ we define $s_0 = 0$, which is the current state with no uncertainty. Let s_t^+ be the set of successors of s_t for all $t = 0,1$ and s_t^- the set of predecessors of s_t for all $t = 1,2$. Note, that s_1^- then becomes simply state 0. In each period there is a single, non-durable consumption good.

There is a finite number of agents $h \in H$ participating in the economy. Each agent h has initial endowments $(e^h_{s_t})_{s_t \in \{0\}\cup \mathcal{S}_1 \cup \mathcal{S}_2} \in \mathbb{R}^{(S_1+S_2+1)}$. Agents have preferences over consumption bundles $c^h_{s_t} \in \mathbb{R}^{(S_1+S_2+1)}$ where $s_t \in \mathcal{S}$.

We define the utility function of the rational agents (h) as follows:

$$u^h(c^h) = v_0^h(c_0^h) + \delta_1 \sum_{s_1 \in \mathcal{S}_1} \rho_{s_1} v^h_{s_1}(c^h_{s_1}) + \delta_1\delta_2 \sum_{s_1 \in \mathcal{S}_1} \rho_{s_1} \sum_{s_2 \in S_1^+} \rho_{s_2} v^h_{s_2}(c^h_{s_2}). \tag{1}$$

where ρ_{s_1} denotes the probability of occurrence of event s_1 and ρ_{s_2} denotes the probability of occurrence of event s_2 given event s_1 has occurred, δ_t is a one-period discount factor and $v^h_{s_t}$ is a Bernoulli function.

We apply the following assumptions throughout the paper.

Assumption 2.1. We assume that $\rho_{s_t} > 0$ for all $s_t \in \mathcal{S}_t$ and $\sum_{s_1 \in \mathcal{S}_1} \rho_{s_1} = 1$, $\sum_{s_2 \in \mathcal{S}_2} \rho_{s_2} = 1$, $\delta_1, \delta_2 > 0$, the probabilities and discount factors are identical across agents, and that the Bernoulli utility function is strictly increasing. Furthermore $c^h \in X^h$ where $X^h \subset \mathbb{R}^{1+S_1+S_2}$ and X^h is the vector of consumption bundles for agent h.

The constraint of $\rho_{s_t} > 0$ means that the agents only take into account the future outcomes for which the objective probability of occurrence is positive, i.e. unlikely events do not affect their utility. A further simplifying assumption is that all agents apply the same discount factors and have no satiation point.

There are J_{s_t} short-lived assets at each $s_t \in \{0\} \cup \mathcal{S}_1$. The set of assets at event s_t is $\mathcal{J}_{s_t}$. Each asset j pays (random) dividends $d_{s_{t+1},j}$ at date-events $s_{t+1} \in s_t^+$ and then it expires. We denote the vector of dividends by $d_{s_t} = (d_{s_t,1}, \dots, d_{s_t,J_{s_t^-}})$ where $s_t \in \mathcal{S}_1 \cup \mathcal{S}_2$, and the pay-off matrices by $A_{s_t} = (d_1, \dots, d_{J_{s_t}}) \in \mathbb{R}^{|s_t^+| \times J_{s_t}}$ where $s_t \in \{0\} \cup \mathcal{S}_1$.

The price of asset j at date-events $s_t \in \{0\} \cup \mathcal{S}_1$ is $q_{s_t,j} \in \mathbb{R}$. We denote the vector of asset prices by $q_{s_t} = (q_{s_t,1}, \dots, q_{s_t,J_{s_t}})$, and the collection of prices over date-events by $q = (q_{s_t})_{s_t \in \{0\}\cup\mathcal{S}_1}$. We assume that assets are in zero net supply. At date-event $s_t \in \{0\} \cup \mathcal{S}_1$ agent h chooses a portfolio-holding $\theta^h_{s_t} = (\theta^h_{s_t,1}, \theta^h_{s_t,2}, \dots, \theta^h_{s_t,J_{s_t}}) \in \mathbb{R}^{J_{s_t}}$.

Definition 2.2. The finance economy $\mathcal{E} = ((u^h, e^h)_{h=1,\dots,H}; (A_{s_t})_{s_t \in \{0\}\cup\mathcal{S}_1})$ is defined by the agents' utility functions and endowments, and the pay-off matrices.

Definition 2.3. A *competitive equilibrium* for an economy $\mathcal{E}$ is a collection of portfolio-holdings $\theta^* = (\theta^{1*}, \theta^{2*}, \dots, \theta^{H*}) \in \mathbb{R}^{H \times J \times (S_1+1)}$, consumption $c^* = (c^{1*}, c^{2*}, \dots, c^{H*}) \in \mathbb{R}^{H \times (S_1+S_2+1)}$ and asset prices $q^* \in \mathbb{R}^{J \times (S_1+1)}$ that satisfy the following conditions:

(a) For $h = 1, \dots, H$,

$$(c^{h*}, \theta^{h*}) \in \arg \max_{c^h \in X^h, \theta^h \in \mathbb{R}^{J\times(S_1+1)}} u^h(c^h) \tag{2}$$

$$\text{s. t. } c_0^h + q_0\theta_0^h = e_0^h,$$

$$c^h_{s_1} + q_{s_1}\theta^h_{s_1} = e^h_{s_1} + d_{s_1}\theta^h_0, \quad \text{for } s_1 \in \mathcal{S}_1,$$

$$c^h_{s_2} = e^h_{s_2} + d_{s_2}\theta^h_{s_2^-}, \quad \text{for } s_2 \in \mathcal{S}_2,$$

[2] Note, that in each case we consider homogeneous types of agents.

(b)

$$\sum_{h=1}^{H} \theta^{h*} = 0, \tag{3}$$

(c)

$$\sum_{h=1}^{H} c^{h*} = \sum_{h=1}^{H} e^{h}. \tag{4}$$

Note that the market clearing (b) for the financial contracts imply that the consumption c^* is feasible; i.e. condition (c) is always satisfied when conditions (a) and (b) are. The feasibility of c^* means that in a finance economy (a single commodity model) when the financial markets clear, the demand for the commodity induced by the portfolio-holdings equals its supply in each state; thus the commodity market clears in each state (Magill and Quinzii, 1996b).

If Assumption 2.1 is met (i.e. agents have strictly increasing utility functions) equilibrium prices exclude arbitrage opportunities in the following sense.

Definition 2.4. Asset prices q of short-lived assets are *arbitrage-free* if there is no $\theta^h = (\theta^h_{s_t})_{s_t \in \{0\} \cup \mathcal{S}_1}$ such that

$$q_0 \theta^h_0 \leq 0, \tag{5}$$

$$\forall s_t \in \mathcal{S}_1 \cup \mathcal{S}_2 : q_{s_t} \theta^h_{s_t} \leq A_{s_t^-} \theta^h_{s_t^-}, \tag{6}$$

with at least one strict inequality.

Definition 2.5. Markets are *complete* if for every income stream $y \in \mathbb{R}^{S_1 + S_2}$ there exists a portfolio plan $(\theta^h_{s_t})_{s_t \in \{0\} \cup \mathcal{S}_1}$ such that

$$\forall s_1 \in \mathcal{S}_1 \ : \ d_{s_1} \theta^h_0 - q_{s_1} \theta^h_{s_1} = y_{s_1};$$

$$\forall s_2 \in \mathcal{S}_2 \ : \ d_{s_2} \theta^h_{s_2^-} = y_{s_2}.$$

That is, for each date-event $s_t \in \{0\} \cup \mathcal{S}_1$ and arbitrary payoffs in immediate successors of s_t, there exists a portfolio that generates those payoffs. Such a portfolio exists if and only if A_{s_t} has rank $|s_t^+|$, which is stated in the following proposition:

Proposition 2.6. *Markets are complete if and only if for every $s_t \in \{0\} \cup \mathcal{S}_1$ the following condition is met*

$$rank(A_{s_t}) = |s_t^+|. \tag{7}$$

Proof. The proof is given in Habis and Herings (2011). □

Proposition 2.7. *If there are no arbitrage opportunities in the financial markets and the markets are complete, then there exists a unique, strictly positive state price vector $(\pi_{s_t})_{s_t \in \{0\} \cup \mathcal{S}_1} \in \mathbb{R}^{S_1 + 1}$ such that*

$$q_{s_t} = \pi^\top_{s_t} \cdot A_{s_t}. \tag{8}$$

Proof. The proof is given in Magill and Quinzii (1996b). □

The following additional assumptions will be made throughout this section:

Assumption 2.8. We assume that

1. asset 1 is risk free, so $d_{s_t,1} = 1 \ \forall s_t \in \mathcal{S}_1 \cup \mathcal{S}_2$, and its return is $R^f = 1/q_{s_t,1}$,
2. and $\{c^h \in X^h | u^h(c^h) \geq u^h(e^h)\} \subset \text{int}(X^h)$, which prevents the solution of the agent's maximization problem from occurring at the boundary of the consumption set.

We use $E_{s_t}(c_{s_t^+})$ to denote the expectation of $c_{s_t^+}$ conditional on date-event s_t, so $E_{s_t}(c_{s_t^+}) = \sum_{s_{t+1} \in s_t^+} \rho_{s_t} c_{s_t}$.

Agent h maximizes this utility subject to her constraints on endowments, income, and costs. Since markets are complete, it is known that there exists a unique and strictly positive state price vector π_{s_t}. The asset price vector $q_{s_t} = \pi^T_{s_t} \cdot A_{s_t}$ then follows from the agents' optimization problem:

$$\mathcal{L}^h = u^h(c^h) - \lambda^h_0(c^h_0 - e^h_0 + q_0 \theta^h_0) - \lambda^h_{s_1}(c^h_{s_1} + q_{s_1}\theta^h_{s_1} - e^h_{s_1} - d_{s_1}\theta^h_0) - \lambda^h_{s_2}(c^h_{s_2} - e^h_{s_2} - d_{s_2}\theta^h_{s_2^-}), \tag{9}$$

where $\lambda^h_{s_t}$ denote the Lagrange multipliers. Solving this problem for q_{s_t} we get:

$$q_{s_t} = A_{s_t} \frac{\delta_{t+1} \sum_{s_{t+} \in S_t^+} \rho_{s_t^+} \partial v^h_{s_t^+}(c^h_{s_t^+}) / \partial c^h_{s_t^+}}{\partial v^h_{s_t}(c^h_{s_t}) / \partial c^h_{s_t}}. \tag{10}$$

It becomes apparent that what we get is the *marginal rate of substitution* (MRS) between the consumption levels of the different periods. Eq. (10) means that for each $s_t \in \{0\} \cup \mathcal{S}_1$ date-event, an agent h invests in j assets, such that the marginal cost of each additional $q_{s_t,j}$ unit equals its marginal utility, which is the present value of the future dividends of agent h.

Applying the definition of the expected value yields[3]

$$q_{s_t} = \frac{\delta_{t+1} E_{s_t}[\partial_{c_{s_t^+}} v^h_{s_t^+}(c^{h*}) A_{s_t}]}{\partial_{c_{s_t}} v^h_{s_t}(c^{h*})} = E(\mathrm{MRS}^h_{s_t} A_{s_t}), \quad \text{for all } s_t \in \{0\} \cup \mathcal{S}_1. \tag{11}$$

Eq. (11) asserts that each agent h invests in each asset j at each date-event $s_t \in \{0\} \cup \mathcal{S}_1$ in such a way that the marginal cost of an additional unit of the security $q_{s_t,j}$ is equal to its marginal benefit, the present value for agent h of its future stream of dividends. Although the $MRS^h_{s_t}$ of each agent can be different as a result of the shape of the utility function (e.g. based on their attitude towards risk), they cannot disagree on asset prices in equilibrium. If an individual $MRS^h_{s_t}$ is projected onto the marketed subspace $\langle A_{s_t} \rangle$ one obtains a unique pricing vector, given that $q_{s_t} = \pi^\top_{s_t} \cdot A_{s_t}$ which is the one defined in (11). Note, that if asset markets are complete, then c^* is Pareto-optimal in the competitive equilibrium (Habis, 2024).

For asset prices q_{s_t}, we define the one-period return $r_{s_t^+},\ _{\theta_{s_t}}$ for a portfolio θ_{s_t}, with $q_{s_t}\theta_{s_t} \neq 0$, by

$$r_{s_t^+,\theta_{s_t}} = \frac{A_{s_t}\theta^h_{s_t}}{q_{s_t}\theta^h_{s_t}}. \tag{12}$$

This reflects the general definition of returns: we divide the pay-offs of the securities in the portfolio by their price.

Following the steps of Habis (2024), we can arrive at the CAPM-pricing formula:

$$E_{s_t}[r_{s_t^+,\theta}] - R^f_{s_t} = \beta_{\theta_{s_t}}(E_{s_t}[r^M_{s_t^+}] - R^f_{s_t}); \tag{13}$$

which is, in fact, the formula of the *security market line*:

$$E_{s_t}[r_{s_t^+,\theta}] = R^f_{s_t} + \beta_{\theta_{s_t}}(E_{s_t}[r^M_{s_t^+}] - R^f_{s_t}). \tag{14}$$

As it is also stated in LeRoy and Werner (2001), the assumption that the equilibrium consumption choice is in the span of the market return and the risk-free return is trivial in a *representative-agent economy*. This is because the optimal consumption of each agent in the economy is equal to the per capita pay-off of the market portfolio.

3. The three-period CAPM with irrational agents

In this section, we will extend the model of the previous section: we will derive the *security market line* formula from a three-period finance economy but replace the rational agent with a type of irrational agent. By irrational agent, we refer to agents with time-inconsistent preferences which we define the following way.

Definition 3.1. The preferences of an agent are time-inconsistent if there are two different t_A and t_B time periods and two different x_1 and x_2 baskets of goods for which both of the following hold:

$$x_1 \succsim_{t_A} x_2$$
$$x_1 \precsim_{t_B} x_2$$

i.e.: the agent changes her mind from one period to another. This is a non-standard preference type, and thus it is a form of irrationality.

We will introduce two kinds of irrational agents that are time-inconsistent: the naive and the sophisticated. In Pollak (1968), naive and sophisticated agents had already been characterized, but optimal consumption trajectories were not described. Rohde (2006) defined the equilibria of the two consumption trajectories following Pollak's results. Since, however, neither of them defined the two types of agents, we introduce the description of naive agents in Definition 3.2 and sophisticated agents in Definition 3.3 using the points discussed above and O'Donoghue and Rabin (1999).

Definition 3.2. The *naive agent* is a decision-maker who cannot foresee their potentially changing future preferences and their related self-control problems. As a result, in each period, they make decisions based solely on their current preferences. Then, they might change this decision in future periods due to changed period-dependent preferences.

Definition 3.3. The *sophisticated agent* is a decision-maker who can foresee their potentially changing future preferences and they make their decisions being conscious of this knowledge. This way, they make decisions that they stick to in future periods as well.

[3] For the sake of clearer notation, we will substitute the traditional notation ($\frac{\partial f(x)}{\partial x}$) of the partial derivative of any function $f(x)$ with respect to x variable by simply writing $\partial_x f(x)$.

In this Section, we will derive the CAPM with irrational agents, first in an economy with naive agents and then with sophisticated agents.

To introduce these irrational behaviors into the model, we need to modify the utility function by applying the hyperbolic discount factor instead of the exponential. The hyperbolic discount factor, based on Harvey (1986) and Loewenstein and Prelec (1992), defined for our use case, is

$$f(\tau) = \frac{1}{1+\gamma\tau} \tag{15}$$

where τ denotes the relative time and γ is a parameter controlling the intensity of the discounting.

As we will shortly show, the representation of the above-described irrational behavior requires at least three time periods, so we can model how the agents change their preferences over time. Appendix A.1 illustrates the decision-making processes of each type in a simple setup.

3.1. The naive agent

Similarly to the rational case, the utility functions of the naive agents $v^n_{s_t}(c^n_{s_t})$ need to be in the same quadratic form.

Assumption 3.4. Let each naive agent have the $v^n_{s_t}(c^n_{s_t}) = \xi_t c^n_{s_t} - \frac{1}{2}\alpha_t (c^n_{s_t})^2$ quadratic Bernoulli utility function $v^n_{s_t}(c^n_{s_t}) = \xi_t c^n_{s_t} - \frac{1}{2}\alpha_t (c^n_{s_t})^2$, $\forall s_t \in \{0\} \cup \mathcal{S}_1 \cup \mathcal{S}_2$.

Observe, that the naive agents behave similarly to the rational ones in period $t = 0$, they optimize their consumption by maximizing their utility function;

$$u^n(c^n) = v^n_0(c^n_0) + \frac{1}{1+\gamma} \sum_{s_1 \in \mathcal{S}_1} \rho_{s_1} v^n_{s_1}(c^n_{s_1}) + \frac{1}{1+2\gamma} \sum_{s_1 \in \mathcal{S}_1} \rho_{s_1} \sum_{s_2 \in \mathcal{S}_1^+} \rho_{s_2} v^n_{s_2}(c^n_{s_2}). \tag{16}$$

However, we need to address each period one by one, because as time passes, the naive agent can potentially change the original decision of period 0.

Similarly to the rational case, we apply the Lagrangian of the utility function

$$\mathcal{L}^n = u^n(c^n) - \lambda^n_0(c^n_0 - e^n_0 + q_0\theta^n_0) - \lambda^n_{s_1}(c^n_{s_1} + q_{s_1}\theta^n_{s_1} - e^n_{s_1} - d_{s_1}\theta^n_0) - \lambda^n_{s_2}(c^n_{s_2} - e^n_{s_2} - d_{s_2}\theta^n_{s_2^-}), \tag{17}$$

and we take its partial derivatives with respect to the consumption and the portfolio holdings at the different date-events. For details, see Appendix A.2.

We solve them again for q_{s_t}:

$$q_0 = d_{s_1}\frac{\lambda^n_{s_1}}{\lambda^n_0}, \text{ s.t.} \lambda^n_0 \neq 0. \tag{18}$$

From here, we use the respective version of Eq. (10) to obtain Eq. (11) for the naive agent, also in $t = 0$:

$$q_0 = \frac{\frac{1}{1+\gamma} E_0[\partial_{c_{s_1}} v^n_{s_1}(c^{n*}) d_0]}{\partial_{c_0} v^n_0(c^{n*})}. \tag{19}$$

With the help of the previously defined one-period return r_{s_1,θ_0}, and the usual formula of covariance, we arrive at the equation of the consumption capital asset pricing model. The one-period return of the risk-free security is

$$R^f_0 = \frac{\partial_{c_0} v^h_0(c^{n*})}{\frac{1}{1+\gamma} E_0[\partial_{c_{s_1}} v^h_{s_1}(c^{n*})]},$$

which is the same as the one seen in the case of the rational agent, replacing the discount factor with the hyperbolic for the respective period. Thus, the capital asset pricing equation at $t = 0$ is

$$E_0[r_{s_1,\theta_0}] = R^f_0 - \frac{1}{1+\gamma} R^f_0 \frac{cov_0(\partial_{c_{s_1}} v^n_{s_1}(c^{n*}), r_{s_1,\theta_0})}{\partial_{c_0} v^n_0(c^{n*})}. \tag{20}$$

From Assumption 3.4, we substitute the respective derivative of $v_0(c_0)^n = \xi_0 c^n_0 - \frac{1}{2}\alpha_0(c^n_0)^2$ in Eq. (20) and achieve

$$E_0[r_{s_1,\theta_0}] = R^f_0 - \frac{1}{1+\gamma} R^f_0 \frac{cov_0(\xi_1 - \alpha_1 c^n_{s_1}, r_{s_1,\theta_0})}{\xi_0 - \alpha_0 c^n_0}, \tag{21}$$

which means that the expected return of an arbitrary asset j is

$$E_0[r_{s_1,j}] = R^f_0 + \frac{\frac{1}{1+\gamma}\alpha_1 R^f_0}{\xi_0 - \alpha_0 c^n_0} cov_0(c^n_{s_1}, r_{s_1,j}). \tag{22}$$

As we have witnessed before, this holds for the market return $r^M_{s_t^+}$ as well, and thus

$$E_0[r^M_{s_1}] = R^f_0 + \frac{\frac{1}{1+\gamma}\alpha_1 R^f_0}{\xi_0 - \alpha_0 c^n_0} cov_0(c^n_{s_1}, r^M_{s_1}) \tag{23}$$

also holds. Rearranging the equations similarly as in Section 2 and with the definition of β_{θ_0}

$$\beta_{\theta_0} = \frac{cov_0(r^M_{s_1}, r_{s_1,\theta})}{var(r^M_{s_1})} \tag{24}$$

in period 0, the CAPM formula can also be expressed in the case of the naive agent as well for all $\theta^n_0 \in \mathbb{R}^{J_0}$

$$E_0[r_{s_1,\theta}] - R^f_0 = \beta_{\theta_0}(E_0[r^M_{s_1}] - R^f_0). \tag{25}$$

In period 0, we observed that all conditions are given to write up the *security market line* formula of the CAPM even if the agent is naive instead of rational

$$E_0[r_{s_1,\theta}] = R^f_0 + \beta_{\theta_0}(E_0[r^M_{s_1}] - R^f_0). \tag{26}$$

We arrived at a formula that on the surface appears to be the same as the SML of the rational agent. Nonetheless, there is a significant difference; in the expected return of the market portfolio $E_0[r^M_{s_1}]$ (see Eq. (23)) the discount factors are different; thus we will obtain somewhat different results. With this observation, we now continue to derive the SML with the assumption of a naive agent in period $t = 1$. We denote the date-event by $\bar{s}_1$ that occurred in period 1 of all the possible outcomes. The utility function now excludes the part that describes the utility of period 0

$$u^n(c^n) = v^n_{\bar{s}_1}(c^n_{\bar{s}_1}) + \frac{1}{1+1\cdot\gamma} \sum_{s_2 \in \bar{S}^+_1} \rho_{s_2} v^n_{s_2}(c^n_{s_2}). \tag{27}$$

As soon as the naive agents reach this date-event, they reconsider their period 0 choice, so we will go through the same equations, now with the new utility function and the most recent budget constraints. So the Lagrange function with one less constraint is

$$\mathcal{L}^n = u^n(c^n) - \lambda^n_{\bar{s}_1}(c^n_{\bar{s}_1} + q_{\bar{s}_1}\theta^n_{\bar{s}_1} - e^n_{\bar{s}_1} - d_{\bar{s}_1}\theta^n_0) - \lambda^n_{s_2}(c^n_{s_2} - e^n_{s_2} - d_{s_2}\theta^n_{s_2^-}), \tag{28}$$

and we solve its partial derivatives (detailed in Appendix A.2) for $q_{\bar{s}_1}$

$$q_{\bar{s}_1} = \frac{\frac{1}{1+\gamma} E_{\bar{s}_1}[\partial_{c_2} v^n_2(c^{n*}) d_{\bar{s}_1}]}{\partial_{c_{\bar{s}_1}} v^n_{\bar{s}_1}(c^{n*})}. \tag{29}$$

Following the previously practised steps, we use the expected return of the risk-free asset

$$R^f_{\bar{s}_1} = \frac{\partial_{c_{\bar{s}_1}} v^h_{\bar{s}_1}(c^{n*})}{\frac{1}{1+\gamma} E_{\bar{s}_1}[\partial_{s_2} v^h_{s_2}(c^{n*})]}$$

and the definition of covariance, we reach the consumption-based capital asset pricing equation

$$E_{\bar{s}_1}[r_{s_2,\theta_{\bar{s}_1}}] = R^f_{\bar{s}_1} - \frac{1}{1+\gamma} R^f_{\bar{s}_1} \frac{cov_{\bar{s}_1}(\partial_{c_{s_2}} v^n_{s_2}(c^{n*}), r_{s_2,\theta_{\bar{s}_1}})}{\partial_{c_{\bar{s}_1}} v^n_{\bar{s}_1}(c^{n*})}. \tag{30}$$

we substitute the Bernoulli function $v^n_{\bar{s}_1}(c^n_{\bar{s}_1}) = \xi_1 c^n_{\bar{s}_1} - \frac{1}{2}\alpha_1(c^n_{\bar{s}_1})^2$ from Assumption 3.4 into Eq. (30) and get

$$E_{\bar{s}_1}[r_{s_2,\theta_{\bar{s}_1}}] = R^f_{\bar{s}_1} - \frac{1}{1+\gamma} R^f_{\bar{s}_1} \frac{cov_{\bar{s}_1}(\xi_2 - \alpha_2 c^n_{s_2}, r_{s_2,\theta_{\bar{s}_1}})}{\xi_1 - \alpha_1 c^n_{\bar{s}_1}}. \tag{31}$$

For any asset j the expected value of the returns $r_{s_2,j}$ and $r^M_{s_2}$

$$E_{\bar{s}_1}[r_{s_2,j}] = R^f_{\bar{s}_1} + \frac{\frac{1}{1+\gamma}\alpha_2 R^f_{\bar{s}_1}}{\xi_1 - \alpha_1 c^n_{\bar{s}_1}} cov_{\bar{s}_1}(c^n_{s_2}, r_{s_2,j}), \tag{32}$$

$$E_{\bar{s}_1}[r^M_{s_2}] = R^f_{\bar{s}_1} + \frac{\frac{1}{1+\gamma}\alpha_2 R^f_{\bar{s}_1}}{\xi_1 - \alpha_1 c^n_{\bar{s}_1}} cov_{\bar{s}_1}(c^n_{s_2}, r^M_{s_2}). \tag{33}$$

With the demonstrated deduction, assuming $\beta_{\theta_{\bar{s}_1}} = \frac{cov_{\bar{s}_1}(r^M_{s_2}, r_{s_2,\theta})}{var(r^M_{s_2})}$, the SML of the CAPM in date-event $\bar{s}_1$ is again attained

$$E_{\bar{s}_1}[r_{s_2,\theta}] = R^f_{\bar{s}_1} + \beta_{\theta_{\bar{s}_1}}(E_{\bar{s}_1}[r^M_{s_2}] - R^f_{\bar{s}_1}). \tag{34}$$

This is a valuable development. We have proven that the capital asset pricing formula holds both in the case of a three-period time frame and with a time-inconsistent, in this case, naive agent type. We have not conducted empirical experiments, but this result inspires such testing and investigation.

In period $t = 2$, the date-event that occurred is denoted by $\bar{s}_2$ and the naive makes a new decision. The utility function has the simple formula of

$$u^n(c^n) = v^n_{\bar{s}_2}(c^n_{\bar{s}_2}) \tag{35}$$

since there is no more uncertainty in the model. Throughout the process of making a new consumption decision, the agent uses the rest of its endowment and dividend income fully for consumption; therefore, the optimal value of $c^n_{\bar{s}_2}$ is trivially

$$c^n_{\bar{s}_2} = e^n_{\bar{s}_2} + d_{\bar{s}_2}\theta^n_{\bar{s}_2^-}. \tag{36}$$

In this subsection on the naive agent, we derived all three periods with sufficient results. We can see that the methodologies employed in the case of the rational decision maker are working just as smoothly after replacing them with naive decision makers. Now, we will see how these methods work if the agent is sophisticated instead.

3.2. The sophisticated agent

We assume that the rational agent is now replaced solely by sophisticated agents in the model setting. Just as we did before, we continue with a version of Assumption 3.4 so that the utility function of the sophisticated agent is also that of the rational.

Assumption 3.5. Let the Bernoulli utility function of each sophisticated agent be defined by the $v^\zeta_{s_t}(c^\zeta_{s_t}) = \xi_t c^\zeta_{s_t} - \frac{1}{2}\alpha_t(c^\zeta_{s_t})^2$ quadratic utility function $\forall s_t \in \{0\} \cup \mathcal{S}_1 \cup \mathcal{S}_2$.

The behavioral issue of the sophisticated agents stems from the same procrastination problem as the naive agents have. Importantly, however, the sophisticated agents are aware of their behavioral problems, and knowing them, they make their intertemporal decisions by thinking "backwards". They are not starting to make their decisions for period 0. Instead, they start with period $t = 2$ and then proceed by period 1, and finally with period 0. In the first optimization (in period 2), therefore, the sophisticated agent comes to the same decision as the naive. Next, they use the obtained results by substituting them into the utility function of the previous time period (which is period 1). It is worth noting that since a given date-event already occurred, we work with fixed values, but the equation still holds for any $s_2 \in \mathcal{S}_2$ which means that

$$c^\zeta_{s_2} = e^\zeta_{s_2} + d_{s_2}\theta^\zeta_{s_2^-} := \tilde{c}^\zeta_{s_2}.$$

Adopting this, the utility function in $t = 1$ is

$$u^\zeta(c^\zeta) = v^\zeta_{\bar{s}_1}(c^\zeta_{\bar{s}_1}) + \frac{1}{1+\gamma}\sum_{s_2\in\bar{S}_1^+}\rho_{s_2} v^\zeta_{s_2}(e^\zeta_{s_2} + d_{s_2}\theta^\zeta_{s_2^-}), \tag{37}$$

where we can see the fixed substituted expression in the second period. Consistently, the sophisticated also uses the Lagrangian utility maximization

$$\mathcal{L}^\zeta = u^\zeta(c^\zeta) - \lambda^\zeta_{\bar{s}_1}(c^\zeta_{\bar{s}_1} + q_{\bar{s}_1}\theta^\zeta_{\bar{s}_1} - e^\zeta_{\bar{s}_1} - d_{\bar{s}_1}\theta^\zeta_0), \tag{38}$$

which now has slightly differing partial derivatives detailed in Appendix A.3. However, this does not change the previous process, and so we solve the partial derivative with respect to $\theta^\zeta_{\bar{s}_1}$ for $q_{\bar{s}_1}$

$$q_{\bar{s}_1} = \frac{\frac{1}{1+\gamma}\sum_{s_2\in\bar{S}_1^+}\rho_{s_2} d_{s_2}\partial v^\zeta_{s_2}(e^\zeta_{s_2} + d_{s_2}\theta^\zeta_{s_2^-})/\partial\theta_{\bar{s}_1}}{\lambda^\zeta_{\bar{s}_1}}, \text{ s.t.} \lambda^\zeta_{\bar{s}_1} \neq 0.$$

Using the partial derivative with respect to consumption $c^\zeta_{\bar{s}_1}$ we can show that $\lambda^\zeta_{\bar{s}_1} = \frac{\partial v^\zeta_{\bar{s}_1}(c^\zeta_{\bar{s}_1})}{\partial c^\zeta_{\bar{s}_1}}$. With this expression

$$q_{\bar{s}_1} = \frac{\frac{1}{1+\gamma}\sum_{s_2\in\bar{S}_1^+}\rho_{s_2} d_{s_2}\partial v^\zeta_{s_2}(e^\zeta_{s_2} + d_{s_2}\theta^\zeta_{s_2^-})/\partial\theta_{\bar{s}_1}}{\partial v^\zeta_{\bar{s}_1}(c^\zeta_{\bar{s}_1})/\partial c^\zeta_{\bar{s}_1}},$$

the known formula of expected value, and with some changes in the notation of the derivatives we get a similar equation to Eq. (11) of the rational or Eq. (29) of the naive agent, as follows

$$q_{\bar{s}_1} = \frac{\frac{1}{1+\gamma}E_{\bar{s}_1}[\partial_{\theta_{\bar{s}_1}} v^\zeta_{s_2}(e^\zeta_{s_2} + d_{s_2}\theta^\zeta_{s_2^-})d_{s_2}]}{\partial_{c^\zeta_{\bar{s}_1}} v^\zeta_{\bar{s}_1}(c^\zeta_{\bar{s}_1})}. \tag{39}$$

We continue with the steps introduced in Section 2 and with the equations derived in Appendix A.4 we arrive at the expression

$$E_{\bar{s}_1}[r_{s_2,\theta_{\bar{s}_1}}] = \frac{\partial_{c^\zeta_{\bar{s}_1}} v^\zeta_{\bar{s}_1}(c^\zeta_{\bar{s}_1})}{\frac{1}{1+\gamma}E_{\bar{s}_1}[\partial_{\theta_{\bar{s}_1}} v^\zeta_{s_2}(\tilde{c}^\zeta_{s_2})]} - \frac{cov_{\bar{s}_1}[\partial_{\theta_{\bar{s}_1}} v^\zeta_{s_2}(\tilde{c}^\zeta_{s_2}), r_{s_2,\theta_{\bar{s}_1}}]}{E_{\bar{s}_1}[\partial_{\theta_{\bar{s}_1}} v^\zeta_{s_2}(\tilde{c}^\zeta_{s_2})]} \tag{40}$$

where, in service of future transparency, we apply the above-defined notion of $\tilde{c}^{\zeta}_{s_2}$ for the fixed consumption of period 2.

Notice that, due to the alterations in the partial derivatives, we now have a different expression for $q_{\bar{s}_1}$. Hence, this change will necessarily show up in the formula of the one-period risk-free asset's return

$$R^f_{\bar{s}_1} = \frac{\partial_{c^{\zeta}_{\bar{s}_1}} v^{\zeta}_{\bar{s}_1}(c^{\zeta}_{\bar{s}_1})}{\frac{1}{1+\gamma} E_{\bar{s}_1}[\partial_{\theta_{\bar{s}_1}} v^{\zeta}_{s_2}(\tilde{c}^{\zeta}_{s_2})]}.$$

We observe that, in the denominator, the partial derivative with respect to consumption is now replaced by the partial derivative with respect to portfolio holding $\theta_{\bar{s}_1}$. Nevertheless, this remains equivalent to the

$$\frac{1}{\sum_{s_t \in \{0\} \cup S_1 \cup S_2} q_{s_t}}$$

definition of the risk-free return, because it agrees with the reciprocal of Eq. (39) in the sense that the d_{s_2} payoff of the risk-free asset is 1. Accordingly, implementing a similar system as in the case of the naive agent, we can formulate the consumption-based capital asset pricing equation

$$E_{\bar{s}_1}[r_{s_2,\theta_{\bar{s}_1}}] = R^f_{\bar{s}_1} - \frac{1}{1+\gamma} R^f_{\bar{s}_1} \frac{cov_{\bar{s}_1}[\partial_{\theta_{\bar{s}_1}} v^{\zeta}_{s_2}(\tilde{c}^{\zeta}_{s_2}), r_{s_2,\theta_{\bar{s}_1}}]}{\partial_{c^{\zeta}_{\bar{s}_1}} v^{\zeta}_{\bar{s}_1}(c^{\zeta}_{\bar{s}_1})}. \tag{41}$$

We apply Assumption 3.5 and substitute the Bernoulli utility function in

$$E_{\bar{s}_1}[r_{s_2,\theta_{\bar{s}_1}}] = R^f_{\bar{s}_1} - \frac{1}{1+\gamma} R^f_{\bar{s}_1} \frac{cov_{\bar{s}_1}(\xi_2 - \alpha_2 \tilde{c}^{\zeta}_{s_2}, r_{s_2,\theta_{\bar{s}_1}})}{\xi_1 - \alpha_1 c^{\zeta}_{\bar{s}_1}}.$$

The expected return of any j asset is then

$$E_{\bar{s}_1}[r_{s_2,j}] = R^f_{\bar{s}_1} + \frac{\frac{1}{1+\gamma}\alpha_2 R^f_{\bar{s}_1}}{\xi_1 - \alpha_1 c^{\zeta}_{\bar{s}_1}} cov_{\bar{s}_1}(\tilde{c}^{\zeta}_{s_2}, r_{s_2,j}), \tag{42}$$

which we know to be holding for the expected return of the market portfolio as well

$$E_{\bar{s}_1}[r^M_{s_2}] = R^f_{\bar{s}_1} + \frac{\frac{1}{1+\gamma}\alpha_2 R^f_{\bar{s}_1}}{\xi_1 - \alpha_1 c^{\zeta}_{\bar{s}_1}} cov_{\bar{s}_1}(\tilde{c}^{\zeta}_{s_2}, r^M_{s_2}). \tag{43}$$

With the techniques explained previously, and taking β as defined in the case of the first-period problem of the naive agent, the *security market line* of the sophisticated agent in date-event $\bar{s}_1$ is

$$E_{\bar{s}_1}[r_{s_2,\theta}] = R^f_{\bar{s}_1} + \beta_{\theta_{\bar{s}_1}}(E_{\bar{s}_1}[r^M_{s_2}] - R^f_{\bar{s}_1}). \tag{44}$$

Just as in the case of the naive agent, we retrieved the SML formula of the CAPM from the period 1 utility maximization problem of the sophisticated agent, thereby arriving at a novel result again.

In the next step, the sophisticated agents use the results of the above maximization exercises to search for the optimal solution. This step is just as important, if not more than the previous one since this will help us prove that the pricing formula can be applied in a three-period setup with both naive and sophisticated agents.

Let the period be at $t = 0$ now where the utility function looks similar to that of the naive agent in period 0 but we insert the fixed consumption $\tilde{c}^{\zeta}_{s_2}$-t from period 2, and the result of the period 1 maximization problem

$$\tilde{c}^{\zeta}_{\bar{s}_1} = e_{\bar{s}_1} + d_{\bar{s}_1}\theta^{\zeta}_{\bar{s}_1^-} - \frac{\frac{1}{1+\gamma} E_{\bar{s}_1}[\partial_{\theta_{\bar{s}_1}} v^{\zeta}_{s_2}(e^{\zeta}_{s_2} + d_{s_2}\theta^{\zeta}_{s_2^-}) d_{s_2}]}{\partial_{c^{\zeta}_{\bar{s}_1}} v^{\zeta}_{\bar{s}_1}(c^{\zeta}_{\bar{s}_1})} \theta^{\zeta}_{\bar{s}_1} \tag{45}$$

which is rewritten for the occurred $\bar{s}_1$ date-event, but is true for any date-event $s_1 \in \mathcal{S}_1$, thus we will discard the bar notation. The sophisticated agent now substitutes these consumption expressions from the first and second periods, together with c_0 defined by the budget constraints

$$\begin{aligned} u^{\zeta}(c^{\zeta}) &= v^{\zeta}_0(e^{\zeta}_0 - q_0\theta^{\zeta}_0) + \\ &\frac{1}{1+\gamma}\sum_{s_1 \in S_1} \rho_{s_1} v^{\zeta}_{s_1}\left(e_{s_1} + d_{\bar{s}_1}\theta^{\zeta}_{s_1^-} - \frac{\frac{1}{1+\gamma} E_{s_1}[\partial_{\theta_{s_1}} v^{\zeta}_{s_2}(e^{\zeta}_{s_2} + d_{s_2}\theta^{\zeta}_{s_2^-}) d_{s_2}]}{\partial_{c^{\zeta}_{s_1}} v^{\zeta}_{s_1}(c^{\zeta}_{s_1})} \theta^{\zeta}_{s_1}\right) + \\ &\frac{1}{1+2\gamma}\sum_{s_1 \in S_1} \rho_{s_1} \sum_{s_2 \in S_1^+} \rho_{s_2} v^{\zeta}_{s_2}(e^{\zeta}_{s_2} + d_{s_2}\theta^{\zeta}_{s_2^-}). \end{aligned}$$

Essentially, sophisticated agents can only make such a decision in $t = 0$ that they will adhere to in $t = 1$ which indicates that the price calculated in period 1 (i.e.: $\tilde{q}_{\bar{s}_1}$) has to be the equilibrium price as well. This observation leads us to the conclusion that we do not need to calculate the partial derivative with respect to θ_{s_1} because it will not affect the consumption choice.

We continue to search for maximum utility, without the discussed constraints, which means that we only have to take the partial derivative of the Lagrange function with respect to the portfolio holding θ_0 in Appendix A.3.

Following the procedure previously employed, we solve this derivative for q_0 and substitute $\tilde{c}^{\zeta}_{s_1}$

$$q_0 = \frac{\frac{1}{1+\gamma} \sum_{s_1 \in S_1} \rho_{s_1} d_{s_1} \partial v^{\zeta}_{s_1}(\tilde{c}^{\zeta}_{s_1}) / \partial \theta^{\zeta}_0}{\partial v^{\zeta}_0(c^{\zeta}_0) / \partial c^{\zeta}_0},$$

which is together with the expected value, and the known notion of partial derivatives can be expressed in the following form

$$q_0 = \frac{\frac{1}{1+\gamma} E_0[\partial_{\theta^{\zeta}_0} v^{\zeta}_{s_1}(\tilde{c}^{\zeta}_{s_1}) d_{s_1}]}{\partial_{c^{\zeta}_0} v^{\zeta}_0(c^{\zeta}_0)} \tag{46}$$

which is the respective variant of the period 1 Eq. (39). Considering that, except for the indexing, the two equations are the same, we skip some steps and use the covariance formula which provides us with the

$$R^f_0 = \frac{\partial_{c^{\zeta}_0} v^{\zeta}_0(c^{\zeta}_0)}{\frac{1}{1+\gamma} E_0[\partial_{\theta_0} v^{\zeta}_{s_1}(\tilde{c}^{\zeta}_{s_1})]}$$

formula. Referring to the same step in the first-period problem, we can write the pricing formula of the consumption-capital asset pricing model as

$$E_0[r_{s_1,\theta_0}] = R^f_0 - \frac{1}{1+\gamma} R^f_0 \frac{cov_0[\partial_{\theta_0} v^{\zeta}_{s_1}(\tilde{c}^{\zeta}_{s_1}), r_{s_1,\theta_0}]}{\partial_{c^{\zeta}_0} v^{\zeta}_0(c^{\zeta}_0)}. \tag{47}$$

We use Assumption 3.5 and substitute the derivatives of $v^{\zeta}_0(c^{\zeta}_0)$ and $v^{\zeta}_{s_1}(c^{\zeta}_{s_1})$ in Eq. (47)

$$E_0[r_{s_1,\theta_0}] = R^f_0 - \frac{1}{1+\gamma} R^f_0 \frac{cov_0(\xi_1 - \alpha_1 \tilde{c}^{\zeta}_{s_1}, r_{s_1,\theta_0})}{\xi_0 - \alpha_0 c^{\zeta}_0}$$

and this holds for any asset j and the market portfolio as well

$$E_0[r_{s_1,j}] = R^f_0 + \frac{\frac{1}{1+\gamma}\alpha_1 R^f_0}{\xi_0 - \alpha_0 c^{\zeta}_0} cov_0(\tilde{c}^{\zeta}_{s_1}, r_{s_1,j}),$$

$$E_0[r^M_{s_1}] = R^f_0 + \frac{\frac{1}{1+\gamma}\alpha_1 R^f_0}{\xi_0 - \alpha_0 c^{\zeta}_0} cov_0(\tilde{c}^{\zeta}_{s_1}, r^M_{s_1}).$$

Similarly to the previous period, with the known beta definition, once again all conditions are given to construct the *security market line*

$$E_0[r_{s_1,\theta}] = R^f_0 + \beta_{\theta_0}(E_0[r^M_{s_1}] - R^f_0). \tag{48}$$

With this expression, we have reached the end of our proof of deriving the β pricing formula in a three-period finance general equilibrium model with both rational and irrational agents. We also proved that we can arrive at an optimum in each utility maximization problem. It is a powerful result in itself that the capital asset pricing model can be extended to three periods but it is even more interesting that this is possible even if the agents of the economy have time-inconsistent preferences, i.e. they are either naive or sophisticated instead of the classically asserted rational ones. It is worthwhile to note, that the derivation of the CAPM model with time-inconsistent types does not require any more assumptions other than the ones needed for the classical, rational case.

Our findings could open the floor for many interesting lines of extensions of the CAPM and general equilibrium literature. The introduction of multiple periods makes it possible to incorporate long-lived assets into the model. It would be interesting to see if one still finds Pareto-efficient outcomes in that extension. Another possible field of future research could be the analysis of an economy with heterogeneous agents. There one would face the exciting question of how to define the market portfolio.

CRediT authorship contribution statement

Helga Habis: Writing – review & editing, Writing – original draft, Visualization, Validation, Supervision, Resources, Project administration, Methodology, Funding acquisition, Formal analysis, Conceptualization.

Data availability

No data was used for the research described in the article.

Appendix. Appendices

A.1. Example

The following example is intended to show the behavior of the sophisticated, the naive, and the rational consumer within the framework of an intertemporal model. Consider 4 periods: $t \in T = \{0, 1, 2, 3\}$ where 0 denotes Wednesday, where only a decision process takes place, 1 denotes Thursday, and the numbering of the following days is understood accordingly. There are three actors in the economy, the sophisticated (s), the naive (n) and the rational (r), that is, $h \in H = \{s, n, r\}$, who must complete a task by Saturday. The costs can be linked to the completion of the task, so a given period cost arises only if work is carried out in that period. This is given by the following vector for each day: $c_t(c_1, c_2, c_3) = \{72.5, 100, 145\}$. Let τ denote the time elapsed from the time of the decision to t. The (hyperbolic) discount factor takes the following form for sophisticated and naive actors:

$$f(\tau) = \frac{1}{1+\alpha\tau}, \text{ where } \alpha = 0,5 \ \forall \tau. \tag{49}$$

Thus, the utility function becomes:

$$U^h(x_0^h, x_1^h, x_2^h, x_3^h) = x_0^h + (\frac{1}{1+1\alpha}) \cdot x_1^h + (\frac{1}{1+2\alpha}) \cdot x_2^h + (\frac{1}{1+3\alpha}) \cdot x_3^h \tag{50}$$

In the case of the rational consumer, we define a constant discount factor:

$$\delta^t = (\sqrt{0.5})^t \ \forall t \in T. \tag{51}$$

The utility function in the rational case:

$$U^h(x_0^h, x_1^h, x_2^h, x_3^h) = x_0^h + \sqrt{0.5} \cdot x_1^h + 0,5 \cdot x_2^h + \sqrt{0.5}^3 \cdot x_3^h \tag{52}$$

Next, we examine the optimization of each type at all periods.

Naive

The agent first finds the minimum cost using the hyperbolic discount factor and then makes a decision:

$t = 0$

$$min((\frac{1}{1+1\alpha}) \cdot c_1, (\frac{1}{1+2\alpha}) \cdot c_2, (\frac{1}{1+3\alpha}) \cdot c_3) = min(48\frac{1}{3}, 50, 58) = 48\frac{1}{3} \tag{53}$$

Based on cost minimization, she prepares the task in period 1.

$t = 1$

$$min(c_1, (\frac{1}{1+1\alpha}) \cdot c_2, (\frac{1}{1+2\alpha}) \cdot c_3) = min(72.5, 66\frac{2}{3}, 72.5) = 66\frac{2}{3} \tag{54}$$

After reaching the initial deadline, the agent decides to reconsider and delays the task for an additional period.

$t = 2$

$$min(c_2, (\frac{1}{1+1\alpha}) \cdot c_3) = min(100, \frac{145}{1+0.5}) = min(100, 96\frac{2}{3}) = 96\frac{2}{3} \tag{55}$$

We experience the same as in the previous period, the naive repeatedly postpones his task by one period, so he prepares it after one period, on Saturday he has no other choice, since in $t = 3$ there will be only one cost ($c_3 = 145$), and reaches the deadline. (For this reason, there is no need to perform a separate optimization for $t = 3$). It is important to note here that we can clearly see that the naive actor is not time-consistent. She has repeatedly reached the appointed time for completing the task, where she re-optimizes, succumbs to his behavioral problem, and postpones the work (in our case by a period), although she would not be satisfied with these decisions in the preceding periods.

Sophisticated

The case of the sophisticated agent is partly similar to that of the naive one since they both deal with the same behavioral problem and work with the same discount factor and utility function. However, the sophisticated actor is aware of his problem and decides accordingly for the future. That is why you need to think backwards, to avoid very high costs to the best of your ability.

Again, the actor starts with minimizing the cost, but now she makes a decision only after examining all periods and moves backwards in time.

$t = 3$

$$min(c_3) = c_3 = 145 \tag{56}$$

$t = 2$

$$min(c_2, (\frac{1}{1+1\alpha}) \cdot c_3) = min(100, \frac{145}{1+0.5}) = min(100, 96\frac{2}{3}) = 96\frac{2}{3} \tag{57}$$

$t = 1$

$$min(c_1, (\frac{1}{1+1\alpha}) \cdot c_2, (\frac{1}{1+2\alpha}) \cdot c_3) = min(72, 66\frac{2}{3}, 72.5) = 66\frac{2}{3} \tag{58}$$

$t = 0$

$$min((\frac{1}{1+1\alpha}) \cdot c_1, (\frac{1}{1+2\alpha}) \cdot c_2, (\frac{1}{1+3\alpha}) \cdot c_3) = min(48\frac{1}{3}, 50, 58) = 48\frac{1}{3} \tag{59}$$

Let us look at each case. There is no choice in $t = 3$, you have to do the task with the cost of 145. In $t = 2$, you would decide to postpone it by one period to the "cheaper" period 3. (which costs $96\frac{3}{2}$ here). We experience the same thing in $t = 1$, a shift by one period would be expected to the 2. period (here, however, the minimum cost is still 663). Finally, in $t = 0$, you would decide that it would be optimal to complete the task after a period, in $t = 1$ (then the cost would be $48\frac{1}{3}$).

The sophisticated agent foresees these and – based on this – performs the task at the first possible time immediately in period 1, because she sees that if he did it for the period 2, she would already slide into 3., which is very costly. So overall, the decision is to do it at $t = 1$, to prevent future procrastination and increasing costs.

Rational

Here again, the agent moves forward from $t = 0$, minimizes costs, and makes a decision.

$t = 0$

$$min(\delta \cdot c_1, \delta^2 \cdot c_2, \delta^3 \cdot c_3) = min(51.27, 50, 51.27) = 50 \tag{60}$$

Since the cost in period 2 is the smallest, the rational agent will complete the task at that time.

$t = 1$

$$\min(c_1, \delta \cdot c_2, \delta^2 \cdot c_3) = \min(72.5, \sqrt{0.5} \cdot 100, 0.5 \cdot 145) = \min(72.5, 70.71, 72.5) = 70.71 \tag{61}$$

The rational agent still considers period 2 to be optimal for the job.

$t = 2$

$$min(c_2, \delta \cdot c_3) = min(100, \sqrt{0.5} \cdot 145) = min(100, 102.53) = 100 \tag{62}$$

Period 2 is still optimal.

So, the rational consumer with a constant discount factor, as this example shows, is time-consistent. So whatever time-period you make your decision, it does not change (unless you have no choice but in $t = 3$), it is always considered optimal.

A.2. Partial derivatives: The naive agent

Period zero

The period 0 partial derivatives of the naive agent's Lagrange function:

$$\frac{\partial \mathcal{L}^n}{\partial c_0^n} = \frac{\partial v_0^n(c_0^n)}{\partial c_0^n} - \lambda_0^n = 0,$$

$$\frac{\partial \mathcal{L}^n}{\partial c_{s_1}^n} = \frac{\frac{1}{1+\gamma}\sum_{s_1 \in S_1} \rho_{s_1} \partial v_{s_1}^n(c_{s_1}^n)}{\partial c_{s_1}^n} - \lambda_{s_1}^n = 0,$$

$$\frac{\partial \mathcal{L}^n}{\partial c_{s_2}^n} = \frac{\frac{1}{1+2\gamma}\sum_{s_1 \in S_1} \rho_{s_1} \sum_{s_2 \in S_1^+} \rho_{s_2} \partial v_{s_2}^n(c_{s_2}^n)}{\partial c_{s_2}^n} - \lambda_{s_2}^n = 0,$$

$$\frac{\partial \mathcal{L}^n}{\partial \theta_0^n} = -\lambda_0^n q_0 + d_{s_1} \lambda_{s_1}^n = 0,$$

$$\frac{\partial \mathcal{L}^n}{\partial \theta_{s_1}^n} = -\lambda_{s_1}^n q_{s_1} + d_{s_2} \lambda_{s_2}^n = 0.$$

Period one

The period 1 partial derivatives of the Lagrange function of the naive agent:

$$\frac{\partial \mathcal{L}^n}{\partial c_{\bar{s}_1}^n} = \frac{\partial v_{\bar{s}_1}^n(c_{\bar{s}_1}^n)}{\partial c_{\bar{s}_1}^n} - \lambda_{\bar{s}_1}^n = 0,$$

$$\frac{\partial \mathcal{L}^n}{\partial c_{s_2}^n} = \frac{\frac{1}{1+1\cdot\gamma}\sum_{s_2 \in \bar{S}_1^+} \rho_{s_2} \partial v_{s_2}^n(c_{s_2}^n)}{\partial c_{s_2}^n} - \lambda_{s_2}^n = 0,$$

$$\frac{\partial \mathcal{L}^n}{\partial \theta_{\bar{s}_1}^n} = -\lambda_{\bar{s}_1}^n q_{\bar{s}_1} + d_{s_2} \lambda_{s_2}^n = 0.$$

A.3. Partial derivatives: The sophisticated agent

Period one

The period 1 partial derivatives of the sophisticated agent's Lagrange function, now with some alterations:

$$\frac{\partial \mathcal{L}^{\zeta}}{\partial c^{\zeta}_{\bar{s}_1}} = \frac{\partial v^{\zeta}_{\bar{s}_1}(c^{\zeta}_{\bar{s}_1})}{\partial c^{\zeta}_{\bar{s}_1}} - \lambda^{\zeta}_{\bar{s}_1} = 0,$$

$$\frac{\partial \mathcal{L}^{\zeta}}{\partial \theta^{\zeta}_{\bar{s}_1}} = \frac{\frac{1}{1+\gamma}\sum_{s_2 \in \bar{s}_1^{+}} \rho_{s_2} \partial v^{\zeta}_{s_2}(e^{\zeta}_{s_2} + d_{s_2}\theta^{\zeta}_{s_2^-})d_{s_2}}{\partial \theta_{\bar{s}_1}} - \lambda^{\zeta}_{\bar{s}_1} q_{\bar{s}_1} = 0.$$

Period zero

The period 0 partial derivatives of the sophisticated agent's Lagrange function:

$$\frac{\partial u^{\zeta}(c^{\zeta})}{\partial \theta^{\zeta}_0} = -\frac{\partial v_0(e^{\zeta}_0 - q_0\theta^{\zeta}_0)}{\partial \theta^{\zeta}_0} q_0 +$$

$$\frac{\frac{1}{1+\gamma}\sum_{s_1 \in S_1} \rho_{s_1} \partial v^{\zeta}_{s_1}\left(e_{s_1} + d_{\bar{s}_1}\theta^{\zeta}_{s_1^-} - \frac{\frac{1}{1+\gamma}E_{s_1}[\partial_{\theta_{s_1}} v^{\zeta}_{s_2}(e^{\zeta}_{s_2}+d_{s_2}\theta^{\zeta}_{s_2^-})d_{s_2}]}{\partial_{c^{\zeta}_{\bar{s}_1}} v^{\zeta}_{s_1}(c^{\zeta}_{s_1})}\theta^{\zeta}_{s_1}\right)}{\partial \theta^{\zeta}_0} d_{s_1} = 0.$$

A.4. In-depth: Deriving the equations of the sophisticated agent

These equations show that not only do the indices and notation change but there is a change in the partial derivatives. Formally, however, these equations are similar as we have seen in the case of rational and naive agents, thus the inclusion as an appendix.

We divide Eq. (39) by $q_{\bar{s}_1}$, moreover we use the formula of the one-period return

$$1 = \frac{\frac{1}{1+\gamma}E_{\bar{s}_1}[\partial_{\theta_{\bar{s}_1}} v^{\zeta}_{s_2}(e^{\zeta}_{s_2} + d_{s_2}\theta^{\zeta}_{s_2^-})r_{s_2,\theta_{\bar{s}_1}}]}{\partial_{c^{\zeta}_{\bar{s}_1}} v^{\zeta}_{\bar{s}_1}(c^{\zeta}_{\bar{s}_1})},$$

then following the introduced definitions and the covariance formula, we can write the equation

$$1 = \frac{\frac{1}{1+\gamma}E_{\bar{s}_1}[\partial_{\theta_{\bar{s}_1}} v^{\zeta}_{s_2}(e^{\zeta}_{s_2} + d_{s_2}\theta^{\zeta}_{s_2^-})]E_{\bar{s}_1}[r_{s_2,\theta_{\bar{s}_1}}]}{\partial_{c^{\zeta}_{\bar{s}_1}} v^{\zeta}_{\bar{s}_1}(c^{\zeta}_{\bar{s}_1})} + \frac{\frac{1}{1+\gamma}cov_{\bar{s}_1}[\partial_{\theta_{\bar{s}_1}} v^{\zeta}_{s_2}(e^{\zeta}_{s_2} + d_{s_2}\theta^{\zeta}_{s_2^-}), r_{s_2,\theta_{\bar{s}_1}}]}{\partial_{c^{\zeta}_{\bar{s}_1}} v^{\zeta}_{\bar{s}_1}(c^{\zeta}_{\bar{s}_1})}$$

which can be rearranged as

$$E_{\bar{s}_1}[r_{s_2,\theta_{\bar{s}_1}}] = \frac{\partial_{c^{\zeta}_{\bar{s}_1}} v^{\zeta}_{\bar{s}_1}(c^{\zeta}_{\bar{s}_1})}{\frac{1}{1+\gamma}E_{\bar{s}_1}[\partial_{\theta_{\bar{s}_1}} v^{\zeta}_{s_2}(e^{\zeta}_{s_2} + d_{s_2}\theta^{\zeta}_{s_2^-})]} - \frac{cov_{\bar{s}_1}[\partial_{\theta_{\bar{s}_1}} v^{\zeta}_{s_2}(e^{\zeta}_{s_2} + d_{s_2}\theta^{\zeta}_{s_2^-}), r_{s_2,\theta_{\bar{s}_1}}]}{E_{\bar{s}_1}[\partial_{\theta_{\bar{s}_1}} v^{\zeta}_{s_2}(e^{\zeta}_{s_2} + d_{s_2}\theta^{\zeta}_{s_2^-})]}.$$

References

Breeden, D., 1979. An intertemporal asset pricing model with stochastic consumption and investment opportunities. J. Financ. Econ. 7, 265–296. http://dx.doi.org/10.1016/0304-405X(79)90016-3.

Frederick, S., Loewenstein, G., O'Donoghue, T., 2002. Time discounting and time preference: A critical review. J. Econ. Lit. 40 (2), 351–401, URL http://www.jstor.org/stable/2698382.

Habis, H., 2024. A three-period extension of the CAPM. J. Econ. Stud. 51 (9), 200–211. http://dx.doi.org/10.1108/JES-11-2023-0640.

Habis, H., Herings, J.-J., 2011. Core concepts for incomplete market economies. J. Math. Econom. 47 (5), 595–609. http://dx.doi.org/10.1016/j.jmateco.2011.07.005.

Harvey, C.M., 1986. Value functions for infinite-period planning. Manage. Sci. 32 (9), 1123–1139. http://dx.doi.org/10.1287/mnsc.32.9.1123.

Kahneman, D., 2011. Thinking, Fast and Slow. Farrar, Straus and Giroux, New York.

LeRoy, S.F., Werner, J., 2001. Principles of Financial Economics. Cambridge University Press, http://dx.doi.org/10.1017/cbo9780511753787.

Loewenstein, G., Prelec, D., 1992. Anomalies in intertemporal choice: Evidence and an interpretation. Q. J. Econ. 107 (2), 573–597. http://dx.doi.org/10.2307/2118482.

Lucas, R., 1978. Asset prices in an exchange economy. Econometrica 46, 1429–1445. http://dx.doi.org/10.2307/1913837.

Magill, M., Quinzii, M., 1996a. Theory of Incomplete Markets, vol. 1, Massachusetts Institute of Technology, Cambridge and London.

Magill, M.J.M., Quinzii, M., 1996b. Theory of Incomplete Markets. MIT Press, Cambridge, Massachusetts.

O'Donoghue, T., Rabin, M., 1999. Doing it now or later. Am. Econ. Rev. 89 (1), 103–124. http://dx.doi.org/10.1257/aer.89.1.103.

Pollak, R.A., 1968. Consistent planning. Rev. Econom. Stud. 35 (2), 201–208. http://dx.doi.org/10.2307/2296548.

Rohde, K.I.M., 2006. Intertemporal choice and competitive equilibrium. Neth. Datawyse Maastricht.

Rubinstein, M., 1976. The valuation of uncertain income streams and the pricing of options. Bell J. Econ. Manage. Sci. 7, 407–425. http://dx.doi.org/10.2307/3003264.