Acta Scientiarum Mathematicarum
https://doi.org/10.1007/s44146-024-00146-6

Acta Scientiarum Mathematicarum

Check for updates

# Lattice properties of strength functions

Andriamanankasina Ramanantoanina and Tamás Titkos

**Abstract.** This paper investigates an important functional representation of the cone of bounded positive semidefinite operators. It is known that the representation by strength functions turns the Löwner order into the pointwise order. However, very little is known about the structure of strength functions. Our main result says that the representation behaves naturally with the infimum and supremum operations. More precisely, we show that the pointwise minimum of two strength functions $f_A$ and $f_B$ is a strength function if and only if the infimum of $A$ and $B$ exists. This complements a recent result of L. Molnár stating that the pointwise maximum of $f_A$ and $f_B$ exists if and only if $A$ and $B$ are comparable, as this latter statement is equivalent to the existence of the supremum. The cornerstone of each argument in this paper is a fact that was discovered recently, namely that the strength function of the parallel sum $A : B$ (which is half of the harmonic mean) equals the parallel sum of the strength functions $f_A$ and $f_B$. We provide a new proof for this statement, and as a byproduct, in some special cases, we describe the strength function of the so-called (generalized) short.

**Mathematics Subject Classification.** Primary 47B65; Secondary 47A64.

**Keywords.** Parallel sum, Generalized short, Strength function, Löwner order.

**Contents**



A. Ramanantoanina is supported by the Hungarian National Research, Development and Innovation Office - NKFIH (grant no. K134944), T. Titkos is supported by the Hungarian National Research, Development and Innovation Office - NKFIH (grant no. K134944) and by the Momentum program of the Hungarian Academy of Sciences under grant agreement no. LP2021-15/2021.

## 1. Introduction and notations

Let $\mathcal{H}$ be a complex Hilbert space with the inner product $(\cdot,\cdot)$ and denote its unit sphere by $\mathbf{S}_{\mathcal{H}}$. A bounded operator $A : \mathcal{H} \to \mathcal{H}$ is called positive ($A \in \mathbf{B}_+(\mathcal{H})$, in symbols) if its quadratic form is non-negative, i.e. $(Ax, x) \geq 0$ for all $x \in \mathcal{H}$. In this paper $A \leq B$ always refers to the Löwner order, that is, $A \leq B$ if $B - A \in \mathbf{B}_+(\mathcal{H})$. For a given subset $\mathcal{T}$ of $\mathcal{H}$ the symbol $\mathbb{1}_{\mathcal{T}}$ denotes its indicator function, i.e. $\mathbb{1}_{\mathcal{T}}(x) = 1$ if $x \in \mathcal{T}$ and 0 otherwise. The strength function $f_A : \mathbf{S}_{\mathcal{H}} \to \mathbb{R}_+$ of an $A \in \mathbf{B}_+(\mathcal{H})$ is defined by

$$f_A(x) = \max\{t \geq 0 \,|\, tP_x \leq A\}, \quad x \in \mathbf{S}_{\mathcal{H}},$$

where $P_x$ is the rank one projection onto the one-dimensional subspace generated by $x$. This notion was originally introduced by Busch and Gudder in [4], and it turned out to be a great tool for investigating several preserver problems [6,12–15,17]. This is because the collection of strength functions (being nonnegative functions) comes equipped with a natural order which is the pointwise order, and the order structure among strength functions is isomorphic to that of the positive operators in the sense that $A \leq B$ if and only if $f_A(x) \leq f_B(x)$ for all $x \in \mathbf{S}_{\mathcal{H}}$. For the proof, we refer to [4, Theorem 3] or [16, Proposition 2]. (For an analogous notion in a more general context see [7].)

In [16], Molnár and the first author defined an analogue of the strength function, but for a different order structure. The partial order considered is the *spectral order* introduced by Olson in [19]. For positive operators, the spectral order can be defined as follow: $A, B \in \mathbf{B}_+(\mathcal{H})$, $A$ is less than $B$ in the spectral order ($A \preccurlyeq B$, in symbol) if and only if $A^n \leq B^n$, for positive integers $n$. The spectral order analogue of the strength function is defined in a very natural way then

$$\nu_A(x) = \max\{t \geq 0 \,|\, tP_x \preccurlyeq A\} \quad x \in \mathbf{S}_{\mathcal{H}}.$$

The spectral strength functions determine the spectral order in the sense that $A \preccurlyeq B$ if and only if $\nu_A(x) \leq \nu_B(x)$ for all $x \in \mathbf{S}_{\mathcal{H}}$. The most remarkable property of the spectral order is that it induces $\mathbf{B}_+(\mathcal{H})$ with a lattice structure. So on one side, there are the lattice operations (which can also be viewed as operator means on $\mathbf{B}_+(\mathcal{H})$, see [18]) with respect to the spectral order. On the other side, the set of nonnegative functions on $\mathbf{S}_{\mathcal{H}}$ also comes with a natural partial order which makes it a lattice as well. For two functions $f, g : \mathbf{S}_{\mathcal{H}} \to \mathbb{R}$ the symbols $f \wedge g$ and $f \vee g$ stands for the pointwise minimum and pointwise maximum, respectively. It is a natural question to ask whether the order isomorphism $A \longmapsto \nu_A$ is also a lattice isomorphism. It was proved in [16, Proposition 11] that this map, despite not being a lattice isomorphism (lattice embedding would be a more appropriate term since we know that this map is only injective and not surjective), preserves the infimum in the sense that

$$\nu_{A \curlywedge B}(x) = \big(\nu_A \wedge \nu_B\big)(x), \quad x \in \mathbf{S}_{\mathcal{H}},$$

where $A \curlywedge B$ is the infimum with respect to the spectral order. A natural question to ask is whether the same holds true for the Löwner order and the usual strength function.

The situation with the Löwner order is much more complicated. First, the greatest lower bound and the least upper bound (shortly the infimum and supremum), which is guaranteed to exist with respect to the spectral order, may not exist with respect to the Löwner order. According to Kadison's famous result [11], the set of self-adjoint operators is an anti-lattice, that is, the greatest lower bound and the least upper bound of $A, B \in \mathbf{B}_{\mathrm{sa}}(\mathcal{H})$ exists if and only if they are comparable (i.e. if $A \leq B$ or $B \leq A$). The infimum problem in $\mathbf{B}_+(\mathcal{H})$ is even more subtle: it was a long-standing open problem, which was solved first in the finite-dimensional case by Gudder and Moreland in [8]. (See also [9,10].) In the general case, the problem was solved by Ando in [3]: it turned out that in contrast with the spectral order, the Löwner order on $\mathbf{B}_+(\mathcal{H})$ is very far from being a lattice order. However, our main result says that the representation by strength functions respects the infimum and the supremum (if these operators exist).

In Ando's approach to the infimum problem, two binary operations play a crucial role: a commutative operation called parallel sum, and a non-commutative one called generalized short. Therefore we are going to investigate how strength functions behave under these operations. Below we are going to explain this in greater detail.

## 2. Main results

The cornerstone of each proof in this paper is the fact that the representation by strength functions respects the parallel sum. This was initially proved in [16, Theorem 9], by Molnár and the first author. In this paper, we provide a new proof that is more natural and intuitive. Let us recall that for positive real numbers $a, b$, the parallel sum $a : b$ is the half of their harmonic mean $\frac{2}{\frac{1}{a}+\frac{1}{b}}$. One can extend this notion for non-negative real numbers as follows. We define $a : b$ as $a : b = \left(\frac{1}{a} + \frac{1}{b}\right)^{-1} = \frac{ab}{a+b}$, when $ab > 0$ and it is defined to be 0 when $ab = 0$. The parallel sum for strength functions (which are nonnegative functions) is then defined pointwise. We recall that the parallel sum $A : B$ of $A, B \in \mathbf{B}_+(\mathcal{H})$ is defined by its quadratic form:

$$\big((A : B)x, x\big) = \inf_{y \in \mathcal{H}} \Big\{ \big(A(x-y), x-y\big) + \big(By, y\big) \Big\}, \quad x \in \mathcal{H}.$$

If $A$ and $B$ are invertible positive operators, then in analogy with the scalar case we have $A : B = A(A+B)^{-1}B$. (For more details on the parallel sum we refer the reader to [3,20]). The key lemma reads as follows.

**Key Lemma.** Let $A, B \in \mathbf{B}_+(\mathcal{H})$ be positive operators then $f_{A:B} = f_A : f_B$. That is,

$$f_{A:B}(x) = \begin{cases} \frac{f_A(x) f_B(x)}{f_A(x)+f_B(x)} & \text{if} \quad x \in \operatorname{ran} A^{1/2} \cap \operatorname{ran} B^{1/2} \cap \mathbf{S}_{\mathcal{H}}, \\ 0 & \text{otherwise.} \end{cases}$$

It is known that if $A \in \mathbf{B}_+(\mathcal{H})$ and $\mathcal{T} \subseteq \mathcal{H}$ is a closed linear subspace, then the set $\{C \in \mathbf{B}_+(\mathcal{H}) \,|\, 0 \leq C \leq A, \ \operatorname{ran} C \subseteq \mathcal{T}\}$ admits a largest element, denoted by $A_{\mathcal{T}}$ (see [1, Theorem 1]). It is called the short of $A$

to $\mathcal{T}$. Ando in [2] introduced the notion of generalized short as the strong limit $[B]A := \lim_{n\to\infty} A : nB$. The connection between these two notions is $A_{\mathcal{T}} = [P_{\mathcal{T}}]A$, where $P_{\mathcal{T}}$ is the orthogonal projection onto $\mathcal{T}$. But $[B]A$ makes sense even in the case when $\mathrm{ran}B$ is not closed. Recall that the sequence $(A : nB)_{n\in\mathbb{N}}$ is monotone increasing and bounded above by $A$, and thus $(f_{A:nB})_{n\in\mathbb{N}}$ is monotone increasing and bounded above as well. So the supremum in the ordered set of real-valued functions exists, the question is whether this coincides with the strength function of the supremum of the sequence $(A : nB)_n$, which is $[B]A$. The problem is that in general, the set of strength functions is not closed under taking monotone increasing limits. However, we can extract some non-trivial information about $f_{[B]A}$.

**Theorem 1.** *Let $A, B \in \mathbf{B}_+(\mathcal{H})$ positive operators. Then $f_{[B]A}(x) = f_A(x)$ for all $x \in \mathrm{ran}B^{1/2} \cap \mathbf{S}_{\mathcal{H}}$ and $f_{[B]A}(x) = 0$ for all $x \in (\mathcal{H} \setminus \overline{\mathrm{ran}B^{1/2}}) \cap \mathbf{S}_{\mathcal{H}}$. In particular, if $\mathrm{ran}B$ is closed, then $[B]A = A_{\mathrm{ran}B}$ and $f_{[B]A} = f_{A_{\mathrm{ran}B}} = f_A \cdot \mathbb{1}_{\mathrm{ran}B\cap\mathbf{S}_{\mathcal{H}}}$.*

Lastly, we discuss the lattice properties of the set of strength functions. As it was proved by Molnár, the pointwise maximum of two strength functions is a strength function (of some operator) if and only if the supremum of the two operators exists. We complement his result with the case of the pointwise minimum.

**Theorem 2.** *Let $A, B \in \mathbf{B}_+(\mathcal{H})$ be positive operators. The pointwise minimum of $f_A$ and $f_B$ is a strength function if and only if the greatest lower bound of $A$ and $B$ with respect to the Löwner order exists in $\mathbf{B}_+(\mathcal{H})$. Let us denote this operator by $A \wedge_L B$. Then we have*

$$f_{A\wedge_L B}(x) = \big(f_A \wedge f_B\big)(x), \qquad \text{for all } x \in \mathbf{S}_{\mathcal{H}}.$$

*In fact, $f_A$ and $f_B$ are comparable on $\mathcal{R}_{A,B} = \mathrm{ran}A^{1/2} \cap \mathrm{ran}B^{1/2} \cap \mathbf{S}_{\mathcal{H}}$, and thus*

$$f_{A\wedge_L B} = f_A \cdot \mathbb{1}_{\mathcal{R}_{A,B}} \quad \text{or} \quad f_{A\wedge_L B} = f_B \cdot \mathbb{1}_{\mathcal{R}_{A,B}}.$$

## 3. Strength functions and the parallel sum – Proof of the key Lemma

Now we turn to the proofs of our results. We start with the Key Lemma. Recall that this was first proved in [16, Theorem 9]. Our new proof is based on a version of Vigier's Theorem for strength functions. Vigier's Theorem on monotone convergence of self-adjoint operators is a fundamental result in operator theory. It guarantees that if $(A_n)_{n\in\mathbb{N}}$, is a monotone decreasing sequence of self-adjoint operators then the greatest lower bound of the family exists in the strong closure. Of course, the family $(f_{A_n})_{n\in\mathbb{N}}$ of the corresponding strength functions is monotone decreasing, and the greatest lower bound in the ordered set of real-valued functions exists. However, it is not obvious that this greatest lower bound is the strength function of the greatest lower bound of the operator sequence. Assume for a moment that this limiting property holds. Then the proof of the Key Lemma can be done as follows.

First we show that $f_{A:B} = f_A : f_B$ holds for positive definite operators. It is known that the strength function of a positive operator $A \in \mathbf{B}_+(\mathcal{H})$ is

$$f_A(x) = \frac{1}{\|A^{-1/2}x\|^2}, \qquad \text{if } x \in \operatorname{ran} A^{1/2} \cap \mathbf{S}_{\mathcal{H}}, \tag{3.1}$$

and is equal to 0 otherwise, where $A^{-1/2}$ is the (possibly unbounded) inverse of $A^{1/2}$ (densely) defined on the orthogonal of its kernel ([4, Theorem 4]). Let $A, B \in \mathbf{B}_+(\mathcal{H})$ be positive definite operators. Then by invertibility of $A$ and $B$, we see from the formula (3.1) above that

$$f_A(x) = \frac{1}{(A^{-1}x, x)}, \qquad \text{and,} \qquad f_B(x) = \frac{1}{(B^{-1}x, x)}, \qquad x \in \mathbf{S}_{\mathcal{H}}.$$

Computing the parallel sum of these two functions, we get

$$\begin{aligned} f_A(x) : f_B(x) &= \frac{1}{f_A(x)^{-1} + f_B(x)^{-1}} = \frac{1}{(A^{-1}x, x) + (B^{-1}x, x)} \\ &= \frac{1}{((A^{-1} + B^{-1})x, x)}. \end{aligned}$$

Since $A : B = (A^{-1} + B^{-1})^{-1}$ is also invertible, by formula (3.1) again, we have

$$f_{A:B}(x) = \frac{1}{((A : B)^{-1}x, x)} = \frac{1}{((A^{-1} + B^{-1})x, x)},$$

hence

$$f_A(x) : f_B(x) = f_{A:B}(x).$$

Now assume that $A, B \in \mathbf{B}_+(\mathcal{H})$ are positive operators, and set $A_n := A + \frac{1}{n}I$ and $B_n = B + \frac{1}{n}I$ for all $n \in \mathbb{N}$. Then $(A_n)_{n\in\mathbb{N}}$ and $(B_n)_{n\in\mathbb{N}}$ are monotone decreasing sequences of positive definite operators such that $A_n : B_n$ converges to $A : B$ in the strong operator topology (see e.g. [1, Theorem 8]). So by the above argument and the limiting property that we assumed to be true, we have that $f_{A:B} = f_A : f_B$.

What remains is to show that if $(A_n)_{n\in\mathbb{N}}$ is a decreasing family of positive operators such that $A_n$ converges to $A$ in the strong operator topology (in other words, $A$ is the greatest lower bound of the set $\{A_n \,|\, n \in \mathbb{N}\}$), then the corresponding family of strength functions $(f_{A_n})_{n\in\mathbb{N}}$ is monotone decreasing, and the greatest lower bound of $(f_{A_n})_{n\in\mathbb{N}}$ is the strength function of $A$, i.e.

$$f_A(x) = \inf_{n\in\mathbb{N}} f_{A_n}(x), \quad x \in \mathbf{S}_{\mathcal{H}}.$$

We first establish the following identity for the strength function of a positive operator $T \in \mathbf{B}_+(\mathcal{H})$: for $\|x\| = 1$,

$$f_T(x) = \inf_{(x,z)=1} (Tz, z). \tag{3.2}$$

To see this, denote by $m$ the infimum on the right hand side and let $y \in \mathcal{H}$, if $(x, y) = 0$ then $m|(x, y)|^2 = 0 \le (Ty, y)$. Otherwise, we denote $z = \frac{1}{(x,y)}y$ hence $(x, z) = 1$, and we have $m \le (Tz, z)$ which is equivalent to $m|(x, y)|^2 \le (Ty, y)$. We then showed that $m|(x, y)|^2 \le (Ty, y)$, for all $y \in \mathcal{H}$, i.e., the constant $m$ satisfies $mP_x \leq T$. Since $f_T(x)$ is the maximum of all such constants then $m \leq f_T(x)$.

On the other hand, since $f_T(x)P_x \leq T$, we have that $f_T(x)|(x,z)|^2 \leq (Tz,z)$, for all $z \in \mathcal{H}$, so we see that $f_T(x) \leq (Tz,z)$ whenever $(x,z) = 1$ and it is then clear that

$$f_T(x) \leq \inf_{(x,z)=1} (Tz,z).$$

Now if $(A_n)_{n\in\mathbb{N}}$ is a decreasing sequence converging to $A$ (which is actually the greatest lower bound of the operator set $\{A_n \,|\, n \in \mathbb{N}\}$), then the strength function of $A$ can be computed as follows. Since $A$ is the strong limit of $(A_n)_{n\in\mathbb{N}}$, we have

$$(Az,z) = \lim_{n\to\infty}(A_n z,z) = \inf_{n\in\mathbb{N}}(A_n z,z),$$

for all $z \in \mathcal{H}$, this holds in particular for all $z \in \mathcal{H}$ such that $(x,z) = 1$. From equation (3.2), we can compute the strength function of $A$ in the following way

$$\begin{aligned} f_A(x) &= \inf_{(x,z)=1} (Az,z) = \inf_{(x,z)=1} \inf_{n\in\mathbb{N}}(A_n z,z) \\ &= \inf_{n\in\mathbb{N}} \inf_{(x,z)=1} (A_n z,z) = \inf_{n\in\mathbb{N}} f_{A_n}(x). \end{aligned}$$

This completes the proof.

We remark that the key ingredient of the above proof was that the pointwise limit of a decreasing sequence of strength functions is a strength function. For the sake of completeness, let us remark that the analogue statement for increasing sequences is not true in general.

Consider a separable Hilbert space $\mathcal{H}$ and a basis $e_n, n \in \mathbb{N}$. If $P_i$ denotes the orthogonal projection onto the closed subspace $H_i$, generated by the first $i$ basis vectors, then $P_i$ is monotone increasing and converges to the identity operator $I$ in the strong operator topology. This implies that if the least upper bound of $(f_{P_i})_{i\in\mathbb{N}}$ is again a strength function, say it is the strength function of an operator $T$, then $I \leq T$ so that $f_T(x) \geq 1$ for all $x \in S_{\mathcal{H}}$. However, the unit vector

$$\frac{\sqrt{6}}{\pi}\sum_n \frac{1}{n} e_n$$

does not belong to any $H_i$ so that $f_{P_i}(x)$ is 0 for all $i \in \mathbb{N}$, but $f_T(x) \geq 1 > \sup_{i\in\mathbb{N}} f_{P_i}(x)$. Therefore, there is no such operator $T$. This shows that for upward-directed families of strength functions, the least upper bound is not necessarily a strength function.

## 4. Strength functions and (generalized) short – Proof of Theorem 1

Next we prove Theorem 1. Since we know how to compute the strength function of parallel sums, one could think that the strength function of $[B]A$ should follow easily. But as we just saw, the strength functions do not have the monotone convergence property for increasing sequences. In fact, it is not trivial to know at which point of the unit sphere does the strength of the generalized short vanishes. Despite these subtleties, we can extract some nontrivial information about $f_{[B]A}$, and if $B$ has closed range then we can calculate $f_{[B]A}$.

In order to calculate $f_{[B]A}$, we need some information about $\mathrm{ran}([B]A)^{1/2}$. We know from [5, Theorem 4.2] that

$$\mathrm{ran}(nB : A)^{1/2} = \mathrm{ran}A^{1/2} \cap \mathrm{ran}(nB)^{1/2} = \mathrm{ran}A^{1/2} \cap \mathrm{ran}B^{1/2} \subset \mathrm{ran}B^{1/2},$$

for all $n$, and thus

$$\mathrm{ran}([B]A)^{1/2} \subseteq \bigvee_n \mathrm{ran}(nB : A)^{1/2} \subseteq \overline{\mathrm{ran}B^{1/2}}.$$

This means that if $x \in \mathbf{S}_{\mathcal{H}}, x \notin \overline{\mathrm{ran}B^{1/2}}$, then $f_{[B]A}(x) = 0$. On the other hand, by the definition of $[B]A$ we always have $B : A \leq [B]A$, and thus $\mathrm{ran}(B : A)^{1/2} \subseteq \mathrm{ran}([B]A)^{1/2}$. But $\mathrm{ran}(B : A)^{1/2} = \mathrm{ran}A^{1/2} \cap \mathrm{ran}B^{1/2}$, so we get that

$$\mathrm{ran}A^{1/2} \cap \mathrm{ran}B^{1/2} \subseteq \mathrm{ran}([B]A)^{1/2}.$$

Let us calculate $f_{[B]A}(x)$ for an $x \in \mathrm{ran}A^{1/2} \cap \mathrm{ran}B^{1/2} \cap \mathbf{S}_{\mathcal{H}}$. We deduce from the previous paragraph that neither $f_B(x)$ nor $f_A(x)$ is zero. Since $(nB) : A \leq [B]A \leq A$ for all integer $n > 0$, the Key Lemma implies that

$$\frac{nf_B(x)f_A(x)}{nf_B(x) + f_A(x)} = \big(nf_B(x)\big) : f_A(x) = f_{nB:A}(x) \leq f_{[B]A}(x) \leq f_A(x).$$

After simplifying by $nf_B(x)$ on the left-hand side and taking the limit we obtain that $f_{[B]A}(x) = f_A(x)$. If $x \in \mathbf{S}_{\mathcal{H}}$ and $x \in \mathrm{ran}B^{1/2} \setminus \mathrm{ran}A^{1/2}$, then $f_A(x) = 0$, and thus $0 \leq f_{[B]A}(x) \leq f_A(x) = 0$. Combining these observations together, we conclude that if $x \in \mathrm{ran}B^{1/2} \cap \mathbf{S}_{\mathcal{H}}$ then $f_{[B]A}(x) = f_A(x)$ and if $x \in (\mathcal{H} \setminus \overline{\mathrm{ran}B^{1/2}}) \cap \mathbf{S}_{\mathcal{H}}$ then $f_{[B]A}(x) = 0$. So the first part of the theorem is proved.

It is important to highlight that the above argument does not offer any information about $f_{[B]A}(x)$ if $x \in \overline{\mathrm{ran}B^{1/2}} \setminus \mathrm{ran}B^{1/2} \cap \mathbf{S}_{\mathcal{H}}$. This is the problematic subset of $\mathbf{S}_{\mathcal{H}}$ that shows the difference between the short to a closed subspace (for example a closed operator range) and the generalized short with respect to an operator with non-closed range. However, if $\mathrm{ran}B$ is closed, then $\mathrm{ran}B^{1/2} = \mathrm{ran}B$, and thus $\mathrm{ran}B^{1/2} = \overline{\mathrm{ran}B^{1/2}}$, so we have a complete description for $f_{[B]A}$. Namely $f_{[B]A}(x) = f_A(x)$ if $x \in \mathrm{ran}B \cap \mathbf{S}_{\mathcal{H}}$ and $f_{[B]A}(x) = 0$ otherwise. In other words, $f_{[B]A}(x) = f_A(x) \cdot \mathbb{1}_{\mathrm{ran}B \cap \mathbf{S}_{\mathcal{H}}}(x)$.

Finally, recall that the short of $A \in \mathbf{B}_+(\mathcal{H})$ to a closed subspace $\mathcal{T}$ (denoted by $A_{\mathcal{T}}$) is the supremum of the set $\{C \in \mathbf{B}_+(\mathcal{H}) \,|\, 0 \leq C \leq A,\ \mathrm{ran}C \subseteq \mathcal{T}\}$. Obviously, if $C$ belongs to this set, then $0 \leq f_C \leq f_A$, so the natural candidate for $A_{\mathcal{T}}$ is the operator whose strength function is $f := f_A \cdot \mathbb{1}_{\mathcal{T} \cap \mathbf{S}_{\mathcal{H}}}$. The question is whether such an operator exists. The above argument just tells that for any $T$ with $\mathrm{ran}T = \mathcal{T}$ the operator $[T]A$ can play this role, i.e. $f_{[T]A} = f_{A_{\mathrm{ran}T}} = f$. In particular, if $\mathrm{ran}B$ is closed, then $A_{\mathrm{ran}B} = [B]A$.

## 5. Strength functions and lattice operations – Proof of Theorem 2

In order to prove Theorem 2, first we have to recall Ando's theorem on the existence of the infimum in $\mathbf{B}_+(\mathcal{H})$. He showed in [3] that the infimum of $A$ and $B$ exists in $\mathbf{B}_+(\mathcal{H})$ if and only if the generalized shorts $[A]B$ and $[B]A$ are comparable, that is, $[A]B \leq [B]A$ of $[B]A \leq [A]B$ (and the infimum is the smaller one).

Recall that if $\mathcal{H}$ is finite-dimensional, then every linear subspace is closed, so we can use Theorem 1 to calculate strength functions. In this case, $f_{[B]A} = f_A \cdot \mathbb{1}_{ranB}$. But $f_A$ vanishes outside $\mathrm{ran}A^{1/2} = \mathrm{ran}A$, so $f_{[B]A} = f_A \cdot \mathbb{1}_{\mathrm{ran}A \cap \mathrm{ran}B}$. And similarly, $f_{[A]B} = f_B \cdot \mathbb{1}_{\mathrm{ran}A \cap \mathrm{ran}B}$. So Ando's theorem says that the infimum of $A$ and $B$ exists if and only if the strength functions $f_A$ and $f_B$ are comparable on $\mathrm{ran}A \cap \mathrm{ran}B$.

The situation in the infinite-dimensional case is similar, but we have to be slightly more cautious with the ranges. Recall that $\mathcal{R}_{A,B} = \mathrm{ran}A^{1/2} \cap \mathrm{ran}B^{1/2} \cap \mathbf{S}_{\mathcal{H}}$ and that the statement that we want to prove is the following: $f_A \wedge f_B$ is a strength function if and only if $A \wedge_L B$ exists. In that case, $f_{A \wedge_L B} = f_A \cdot \mathbb{1}_{\mathcal{R}_{A,B}}$ or $f_{A \wedge_L B} = f_B \cdot \mathbb{1}_{\mathcal{R}_{A,B}}$.

First, assume that the pointwise minimum $f_A \wedge f_B$ is a strength function of an operator $C \in \mathbf{B}_+(\mathcal{H})$. Then $f_C \leq f_A$ and $f_C \leq f_B$, so $C$ is a lower bound for $A$ and $B$. On the other hand if $D \in \mathbf{B}_+(\mathcal{H})$ satisfies $D \leq A$ and $D \leq B$, then $f_D \leq f_A$ and $f_D \leq f_B$, and thus $f_D \leq f_A \wedge f_B = f_C$, so $D \leq C$. This shows that $A \wedge_L B$ exists and equals to $C$.

Next, assume that $A \wedge_L B$ exists. As in the previous section, first we have to calculate the range of $(A \wedge_L B)^{1/2}$. Since $A \wedge_L B \leq A$, we have $\mathrm{ran}(A \wedge_L B)^{1/2} \subseteq \mathrm{ran}A^{1/2}$, and similarly, $\mathrm{ran}(A \wedge_L B)^{1/2} \subseteq \mathrm{ran}B^{1/2}$. On the other hand, $A : B \leq A$ and $A : B \leq B$, and thus $A : B \leq A \wedge_L B$. Again, using that $\mathrm{ran}(A : B)^{1/2} = \mathrm{ran}A^{1/2} \cap \mathrm{ran}B^{1/2}$, we conclude that $\mathrm{ran}(A \wedge_L B)^{1/2} = \mathrm{ran}A^{1/2} \cap \mathrm{ran}B^{1/2}$, which implies by (3.1) that $f_{A \wedge_L B}(x) = 0$ if $x \notin \mathcal{R}_{A,B}$. We also know by Ando's theorem that the existence of $A \wedge_L B$ implies that $[B]A \leq [A]B$ or $[A]B \leq [B]A$. By symmetry, we can assume that $[B]A \leq [A]B$. In this case $A \wedge_L B = [B]A$, which means two things: (1) $f_{[B]A}$ vanishes outside $\mathcal{R}_{A,B}$ (because $f_{A \wedge_L B}$ vanishes), and (2) $f_{A \wedge_L B} = f_{[B]A}$ on $\mathcal{R}_{A,B}$. But we know from Theorem 2 that if $x \in \mathcal{R}_{A,B}$ then $f_{[B]A}(x) = f_A(x)$. And similarly, $f_{[A]B}(x) = f_B(x)$. So we have that $f_A(x) = f_{[B]A}(x) \leq f_{[A]B}(x) = f_B(x)$ on $\mathcal{R}_{A,B}$. Furthermore, if $x \notin \mathcal{R}_{A,B}$ then at least one of $f_A$ and $f_B$ vanishes, so $\min\{f_A(x), f_B(x)\} = 0$. Putting these observations together we conclude that if $A \wedge_L B$ exists, then the pointwise minimum of $f_A$ and $f_B$ is a strength function, and $f_A \wedge f_B = f_{[B]A} \wedge f_{[A]B} = f_{A \wedge_L B}$. Furthermore, in this case $f_{A \wedge_L B} = f_A \cdot \mathbb{1}_{\mathcal{R}_{A,B}}$. This completes the proof.

For the sake of completeness, we end this paper by recalling Molnár's result on the pointwise maximum of strength functions (see [15, Proposition 3]. Kadison proved that the supremum in the set of self-adjoint operators exists if and only if the operators in question are comparable. Notice that two positive operators have a supremum in $\mathbf{B}_+(\mathcal{H})$ if and only if they have a supremum in the set of self-adjoint operators. This implies that $f_A \vee f_B$ is a strength function if and only if $f_A$ and $f_B$ are comparable. Indeed, if $f_A$ and $f_B$ are comparable then $f_A \vee f_B$ is either $f_A$ or $f_B$. On the other hand, if $f_A \vee f_B = f_C$ for some $C$, then $f_A \leq f = f_C$ and $f_B \leq f = f_C$, so $C$ is a common upper bound. Now take any $D$ such that $A \leq D$ and $B \leq D$. Then $f_A \leq f_D$ and $f_B \leq f_D$, and thus $f_C = f_A \vee f_B \leq f_D$. This implies that the supremum exists and thus $A \leq B$ or $B \leq A$, or equivalently, $f_A$ and $f_B$ are comparable.

**Funding** Open access funding provided by HUN-REN Alfréd Rényi Institute of Mathematics.

### Declarations

**Conflict of interest** On behalf of all authors, the corresponding author states that there is no Conflict of interest.



## References

[1] Anderson, W.N., Jr., Trapp, G.E.: Shorted Operators II. SIAM J. Appl. Math. **28**, 60–71 (1975)

[2] Ando, T.: Lebesgue-type decomposition of positive operators. Acta Sci. Math. (Szeged) **38**, 253–260 (1976)

[3] Ando, T.: Problem of infimum in the positive cone, Analytic and geometric inequalities and applications, pp. 1–12. Kluwer Academic Publishers, Dordrecht (1999)

[4] Busch, P., Gudder, S.P.: Effects as functions on projective Hilbert space. Lett. Math. Phys. **47**, 329–337 (1999)

[5] Fillmore, P.A., Williams, J.P.: On operator ranges. Adv. Math. **7**(3), 254–281 (1971)

[6] Gehér, G.P., Šemrl, P.: Coexistency on Hilbert space effect algebras and a characterisation of its symmetry transformations. Comm. Math. Phys. **379**, 1077–1112 (2020)

[7] Göde, Á., Tarcsay, Zs.: Operators on anti-dual pairs: supremum and infimum of positive operators. J. Math. Anal. Appl. **531**(2), 2 (2024)

[8] Gudder, S., Moreland, T.: Infima of Hilbert space effects. Linear Algebra Appl. **286**, 1–17 (1999)

[9] Gudder, S.: Lattice properties of quantum effects. J. Math. Phys. **37**, 2637–2642 (1996)

[10] Gudder, S., Greechie, R.: Effect algebra counterexamples. Math. Slovaca **46**(4), 317–325 (1996)

[11] Kadison, R.: Order properties of bounded self-adjoint operators. Proc. Amer. Math. Soc. **34**, 505–510 (1951)

[12] Molnár, L.: Maps preserving the geometric mean of positive operators. Proc. Am. Math. Soc. **137**, 1763–1770 (2009)

[13] Molnár, L.: Maps preserving the harmonic mean or the parallel sum of positive operators. Linear Algebra Appl. **430**, 3058–3065 (2009)

[14] Molnár, L.: Maps preserving general means of positive operators. Electron. J. Linear Algebra **22**, 864–874 (2011)

[15] Molnár, L.: Busch-Gudder metric on the cone of positive semidefinite operators and its isometries. Integral Equ. Opera. Theory **90**, 20 (2018)

[16] Molnár, L., Ramanantoanina, A.: On functional representations of positive Hilbert space operators. Integral Equ. Oper. Theory **93**, 2 (2021)

[17] Molnár, L., Szokol, P.: Transformations preserving norms of means of positive operators and nonnegative functions. Integral Equ. Oper. Theory **83**, 271–290 (2015)

[18] Ramanantoanina, A.: Characterisation of the lattice operations on positive operators with respect to the spectral order. Linear Algebra Appl. **656**, 446–462 (2023)

[19] Olson, M.P.: The selfadjoint operators of a von Neumann algebra form a conditionally complete lattice. Proc. Amer. Math. Soc. **28**, 537–544 (1971)

[20] Tarcsay, Zs.: On the parallel sum of positive operators, forms, and functionals. Acta Math Hungar. **147**, 408–426 (2015)

Andriamanankasina Ramanantoanina (✉)
Bolyai Institute
University of Szeged
Aradi Vértanúk Tere 1.
6720 Szeged
Hungary
e-mail: manana@aims.ac.za

Tamás Titkos
HUN-REN Alfréd Rényi Institute of Mathematics
Reáltanoda u. 13-15
Budapest 1053
Hungary
e-mail: titkos@renyi.hu

and

Corvinus University of Budapest
Department of Mathematics
Fővám tér 13-15
Budapest 1093
Hungary

Received: February 28, 2024.
Accepted: June 3, 2024.