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Shapley–Scarf Housing Markets: Respecting Improvement, Integer Programming, and Kidney Exchange

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Abstract. In a housing market of Shapley and Scarf, each agent is endowed with one indivisible object and has preferences over all objects. An allocation of the objects is in the (strong) core if there exists no (weakly) blocking coalition. We show that, for strict preferences, the unique strong core allocation "respects improvement"—if an agent's object becomes more desirable for some other agents, then the agent's allotment in the unique strong core allocation weakly improves. We extend this result to weak preferences for both the strong core (conditional on nonemptiness) and the set of competitive allocations (using probabilistic allocations and stochastic dominance). There are no counterparts of the latter two results in the two-sided matching literature. We provide examples to show how our results break down when there is a bound on the length of exchange cycles. Respecting improvements is an important property for applications of the housing markets model, such as kidney exchange: it incentivizes each patient to bring the best possible set of donors to the market. We conduct computer simulations using markets that resemble the pools of kidney exchange programs. We compare the game-theoretical solutions with current techniques (maximum size and maximum weight allocations) in terms of violations of the respecting improvement property. We find that game-theoretical solutions fare much better at respecting improvements even when exchange cycles are bounded, and they do so at a low efficiency cost. As a stepping stone for our simulations, we provide novel integer programming formulations for computing core, competitive, and strong core allocations.

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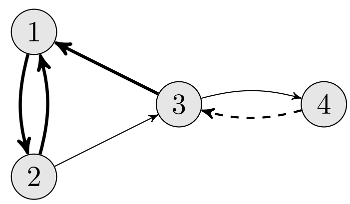
1. Introduction

Shapley and Scarf [47] introduce so-called "housing markets" to model trading in commodities that are inherently indivisible. Specifically, in a housing market, each agent is endowed with an object (e.g., a house or a kidney donor) and has ordinal preferences over all objects, including the agent's own. The aim is to find plausible or desirable allocations in which each agent is assigned one object. A standard approach in the literature is to discard allocations that can be blocked by a coalition of agents. Specifically, a coalition of agents blocks an allocation if they can trade their endowments so that each of the agents in the coalition obtains a strictly preferred allotment. Similarly, a coalition of agents weakly blocks an allocation if they can trade their endowments so that each of the agents in the coalition obtains a weakly preferred allotment and at least one of them obtains a strictly preferred allotment. Thus, an

Figure 1. The maximization of the number of transplants does not respect improvement.



allocation is in the (strong) core if it is not (weakly) blocked. A distinct but also well-studied solution concept is obtained from competitive equilibria, each of which consists of a vector of prices for the objects and a (competitive) allocation such that each agent's allotment is one of the agent's most preferred objects among those that the agent can afford. Interestingly, the three solution concepts are entwined: the strong core is contained in the set of competitive allocations, and each competitive allocation pertains to the core.

In a separate line of research, Balinski and Sönmez [11] study the classic two-sided college admissions model of Gale and Shapley [23] and prove that the student-optimal stable matching (SOSM) mechanism respects improvement of student's quality. This means that, under SOSM, an improvement of a student's rank at a college will, ceteris paribus, lead to a weakly preferred match for the student. The natural transposition of this property to (one-sided) housing markets requires that an agent obtains a weakly preferred allotment whenever the agent's object becomes more desirable for other agents. We study the following question: do the most prominent solution concepts for Shapley and Scarf's [47] housing market "respect improvement"? We obtain several positive answers to this question, which we describe in more detail in the next section.

The respecting improvement property is important in many applications in which centralized clearinghouses use mechanisms to implement barter exchanges. A leading example is kidney exchange programs (KEPs), in which end-stage renal patients exchange their willing but immunologically incompatible donors (Roth et al. [41]). In the context of KEPs, the respecting improvement property means that whenever a patient brings a "better" donor (e.g., younger or with universal blood type 0 instead of A, B, or AB) or registers an additional donor, the KEP should assign the patient the same or a better exchange donor.[1] In other words, the respecting improvement property incentivizes each patient to bring the best possible set of donors to the market. However, in current KEPs, the typical objective is to maximize the number of transplants and their overall qualities (see, e.g., Biró et al. [17]), which can lead to violations of the respecting improvement property. As an illustration, consider the maximization of the number of transplants in Figure 1, in which each node represents a patient–donor pair. A directed edge, from A to B say, indicates the compatibility of the donor in node B with the patient in node A. Patients may have different levels of preference over their set of compatible donors. Initially there are only continuous edges, on which a thick (thin) edge points to the most (least) preferred donor. For example, patient 3 has two compatible donors: donors 1 and 4, and donor 1 is preferred to donor 4. Obviously, the unique way to maximize the number of (compatible) transplants is obtained by picking the three-cycle (1, 2, 3). Suppose that patient 3 succeeds in bringing a second donor to the KEP, and this donor turns out to be compatible for patient 4. Then, the discontinuous edge is included as patient–donor pair 3 "improves." But, now, the unique way to maximize the number of (compatible) transplants is obtained by picking the two two-cycles (1, 2) and (3, 4), which means that patient 3 receives a kidney that is strictly worse than the kidney patient 3 would have received initially.

Similarly, the allocations induced by the standard objectives of KEPs need not pertain to the core. We refer to Example 1 for an illustration of this for the case of the maximization of the number of transplants. As a consequence, blocking coalitions may exist. This is an undesirable feature because patient groups could make a potentially justified claim that the matching procedure is not in their best interest. A particular instance could occur in the organization of international kidney exchanges if a group of patient–donor pairs, all citizens of the same country, learn that an internal (i.e., national) matching would yield a better match for all of them.

Next, we describe our contributions and review the related literature.

1.1. Contributions

Section 3 contains our theoretical results on the respecting improvement property. First, we show that, for strict preferences (Section 3.1), the unique strong core allocation (which coincides with the unique competitive allocation) respects improvement (Theorem 1).

In the case of preferences with ties (Section 3.2), we first analyze the set of competitive allocations. Because, typically, multiple competitive allocations exist, we have to make set-wise comparisons. Focusing on the agent's allotments obtained at competitive allocations, we establish a natural extension of our first result by using stochastic dominance and probabilistic allocations (Theorem 2). As a corollary, we obtain that the agent's most preferred allotment in the new market is weakly preferred to the agent's most preferred allotment in the initial market, and similarly, the agent's least preferred allotment in the new market is weakly preferred to the agent's least preferred allotment in the initial market (Corollary 1). Next, we focus on the (possibly empty) strong core. We prove that, when preferences have ties, the strong core respects improvement conditional on the strong core being nonempty. More precisely, under the assumption that strong core allocations exist in both the initial and new markets, we show that the agent under consideration weakly prefers each allotment in the new strong core to each allotment in the initial strong core (Theorem 3 and Corollary 2).

Finally, in Section 3.3, we relax an important assumption in the housing market of Shapley and Scarf [47], namely, that allocations can contain exchange cycles of any length; that is, cycles are unbounded. The definition of core and strong core can be naturally adjusted to the requirement that the length of exchange cycles does not exceed an exogenously given maximum. Wako [52] shows that the set of competitive allocations coincides with the core based on an antisymmetric weak domination concept. This equivalent definition, which we call the Wako-core, allows for a natural direct extension to the case of bounded exchange cycles. Unfortunately, when exchange cycles are bounded, the core (and, hence, also the set of competitive allocations and the strong core) can be empty.[2] Conditional on the existence of a core, competitive, or strong core allocation, we show that, even if preferences are strict, when the length of exchange cycles is limited (upper bound three or higher), the core, the set of competitive allocations, and the strong core do not respect improvement in terms of the most preferred allotment (Proposition 2).

In Section 4, we provide novel integer programming (IP) formulations for finding core, competitive, and strong core allocations, which serves as a stepping stone for our simulations in Section 5. For unbounded length exchanges, our novel edge formulation is much more efficient than the IP solution proposed by Quint and Wako [39]. Furthermore, our simple sets of constraints for the three solution concepts clearly show the hierarchy between them by pinpointing the additional requirements needed when moving from one solution concept to a stronger one. For bounded length exchanges, we obtain an improvement of the Quint–Wako formulations by focusing only on the feasible cycles. Our formulations are concise and useful for practical computations.

Section 5 complements our theoretical analysis and consists of computer simulations comparing core, competitive, and strong core allocations with maximum size and maximum total weight[3] allocations. To carry out our simulations, we draw markets from pools similar to those observed in KEPs and study both unbounded and bounded length exchange cycles.[4] The maximization of the size and the weight of the allocations correspond to the maximization of the number of transplants and overall quality of the transplants, respectively. In the simulations, we use our novel IP formulations for unbounded length exchanges and adjustments of the IP models developed in Klimentova et al. [28] for bounded length exchanges.

We first study the frequency of violations of the respecting improvement property (in terms of the most preferred allotment) for all models. We observe a large number of violations for maximum size and maximum weight allocations, whereas we only see a negligible amount of violations for core, competitive, and strong core allocations for bounded length cycles. In view of these findings, we analyze the potential trade-off between stability (no blocking) requirements and the maximum number of transplants. We find that, when the size of the instances increases, the trade-off decreases significantly: core allocations for instances with 150 patient–donor pairs yield a less than 1% reduction in the number of transplants. We complement this analysis by studying the number of weakly blocking cycles (or, equivalently, the number of violated stability constraints). Thus, we obtain an estimation of how much deficiency in terms of "robustness"/"fairness" we have to accept vis-à-vis the "ideal" (but potentially empty) strong core.

An important conclusion from our simulations is that, when kidney exchange programs are sufficiently large, one can take into account agents' preferences and largely ensure the respecting improvement property without a significant reduction in the number of transplants. Finally, our simulations also show that the novel IP formulations have a high potential of being used in practice as they prove to be efficient at finding optimal allocations for problems of practical size.

1.2. Literature Review

1.2.1. Housing Markets. The nonemptiness of the core is proved in Shapley and Scarf [47] by showing the balancedness of the corresponding nontransferable utility game and also in a constructive way by showing that David Gale's famous top trading cycles (TTC) algorithm always yields competitive allocations. Roth and Postlewaite [40] later show that, for strict preferences, the TTC results in the unique strong core allocation, which coincides with the unique competitive allocation in this case. However, if preferences are not strict (i.e., ties are present), the strong core can be empty or contain more than one allocation, but the TTC still produces all competitive allocations. Wako [50] shows that the strong core is always a subset of the set of competitive allocations. Quint and Wako [39] provide an efficient algorithm for finding a strong core allocation whenever there exists one. Their work is further generalized and simplified by Cechlárová and Fleiner [19], who use graph models. Wako [52] shows that the set of competitive allocations coincides with the core based on an antisymmetric weak domination concept, which we refer to as Wako-core in this paper. This equivalence is key for our extension of the definition of competitive allocations to the case of bounded exchange cycles.

1.2.2. Respecting Improvement. For Gale and Shapley's [23] college admissions model, Balinski and Sönmez [11] prove that SOSM respects improvement of student's quality. Kominers [29] generalizes this result to more general settings. Balinski and Sönmez [11] also show that SOSM is the unique stable mechanism that respects improvement of student quality. Abdulkadiroğlu and Sönmez [3] propose and discuss the use of TTC in a model of school choice, which is closely related to the college admissions model. Abdulkadiroğlu and Che [2] state and Hatfield et al. [25] formally prove that the TTC mechanism respects improvement of student quality.

Hatfield et al. [25] also focus on the other side of the market and study the existence of mechanisms that respect improvement of a college's quality. The fact that colleges can match with multiple students leads to a strong impossibility result: they prove that there is no stable or Pareto-efficient mechanism that respects improvement of a college's quality. In particular, the (Pareto-efficient) TTC mechanism does not respect improvement of a college's quality.

In the context of KEPs with pairwise exchanges, the incentives for bringing an additional donor to the exchange pool was first studied by Roth et al. [42]. In the model of housing markets their donor-monotonicity property boils down to the respecting improvement property. They show that so-called priority mechanisms are donor-monotonic if each agent's preferences are dichotomous, that is, the agent is indifferent between all acceptable donors. However, if agents have nondichotomous preferences, then any mechanism that maximizes the number of pairwise exchanges (so, in particular, any priority mechanism) does not respect improvement. This can be easily seen by means of Example 4 in Section 3.3.

1.2.3. IP Formulations for Matching. Quint and Wako [39] already give IP formulations for finding core and strong core allocations, but the number of constraints in their paper is highly exponential as their formulations contain a no-blocking condition for each set of agents and any possible exchanges among these agents. Other studies provide IP formulations for other matching problems. In particular, for Gale and Shapley's [23] college admissions model, Baïou and Balinski [9] already describe the stable admissions polytope, which can be used as a basic IP formulation. Further recent papers in this line of research focus on college admissions with special features (Ágoston et al. [5]), stable project allocation under distributional constraints (Ágoston et al. [6]), and the hospital–resident problem with couples (Biró et al. [15]) and ties (Delorme et al. [21], Kwanashie and Manlove [31]).

1.2.4. Kidney Exchange Programs. Starting with the seminal works Saidman et al. [44] and Roth et al. [43], initial research on KEPs focuses on IP models for selecting pairs for transplantation in such a way that maximum (social) welfare, generally measured by the number of patients transplanted, is achieved. Constantino et al. [20], Dickerson et al. [22], and Mak-Hau [32] propose new, compact formulations that, besides extending the models in Saidman et al. [44] and Roth et al. [43] to accommodate nondirected donors and patients with multiple donors, also aim to efficiently solve problems of larger size. The reader is referred to Ashlagi and Roth [8] for a recent operations perspective on KEPs.

In Europe, at least 10 countries have active national kidney exchange programs. Details of current practices and optimization aspects are summarized in Biró et al. [16, 17], respectively. Furthermore, there are already several international collaborations between European countries, which motivated a new line of research on group fairness (Carvalho et al. [18], Klimentova et al. [27], Mincu et al. [34]), in which agents (e.g., hospitals, regional and national programs) can collaborate. Allowing agents to control their internal exchanges, Carvalho et al. [18] study strategic interaction using noncooperative game theory. Specifically, for the two-agent case, they design a game such that some Nash equilibrium maximizes the overall social welfare. Considering multiple matching periods, Klimentova et al. [27] assume agents to be nonstrategic. Taking into account that, at each period, there can be multiple optimal allocations, each of which can benefit different agents, the authors propose an integer programming model to achieve an overall fair allocation. Finally, Mincu et al. [34] propose integer programming formulations for the case in which optimization goals and constraints can be distinct for different agents.

A recent line of research acknowledges the importance of considering patients' preferences (associated with, e.g., graft quality) over matches, raising the question of individual fairness. In the computer science and operations research literature, Biró and Cechlárová [13] consider a model for unbounded length kidney exchanges, in which patients most care about the quality of the graft they receive, but as a secondary factor, they prefer to be involved in an exchange cycle that is as short as possible. The authors show that, although core allocations can still be found by the TTC algorithm, finding a core allocation with the maximum number of transplants is a computationally hard problem (inapproximable unless $P = NP$). In two independent papers (Biró and McDermid [14], Huang [26]) stable exchanges are studied for bounded length cycles with NP-hardness results for the case of three-cycles. Recently, Klimentova et al. [28] provide integer programming formulations for the case in which each patient has preferences over the organs that the patient can receive. The authors focus on allocations that among all (strong) core allocations have maximum cardinality. Moving away from the (strong) core, they also analyze the trade-off between maximum cardinality and the number of blocking cycles. As we show through simulations in this paper, core allocations do not create a substantial number of violations of the respecting improvement property (for best allotments) and, thus, incentivize the participation in KEPs.

In the economics and game theory literature, the preferences of the recipients are dichotomous (i.e., either acceptable or unacceptable) in the classic papers, starting with Roth et al. [42]. Abbassi et al. [1] study a multiobject housing market under dichotomous preferences for both bounded and unbounded length exchange cycles with the aim of maximizing social welfare with truthful mechanisms in which it is a dominant strategy for each agent to report the true private information on the agent's own items offered for exchange and the agent's wish list. They show that, for the length-constrained variants, the problem is inapproximable. The first departure from this literature is by Nicolò and Rodríguez-Álvarez [36], who consider a setting in which the quality of potential transplants is given and each recipient can set an acceptability threshold. They prove an impossibility result for pairwise exchanges (that they later generalize in Nicolò and Rodríguez-Álvarez [37]), and they study conditions under which truthtelling is the unique protective strategy for the recipients. In a follow-up paper, Nicolò and Rodríguez-Álvarez [38] consider pairwise exchanges with age-based preferences and ties, and they propose a deterministic sequential priority rule that satisfies efficiency, strategy-proofness, and nonbossiness. Andersson and Kratz [7] consider a model motivated by the Swedish application, in which (1) ABO-incompatible transplants are allowed in the exchanges and (2) each ABO-incompatible recipient–donor pair only accepts a fully compatible donor. They study a priority matching rule on their trichotomous preference domain. Another proposal for giving incentives to compatible pairs by prioritizing patients in case of future graft failure was given by Sönmez et al. [48]. In a recent paper, Balbuzanov [10] considers bounded length exchange problems under strict preferences. He shows that there is no deterministic mechanism that satisfies individual rationality, ex post efficiency, and weak strategy-proofness. He also provides a random mechanism for pairwise exchanges that is individually rational, ordinally efficient, and anonymous.

2. Preliminaries

We consider housing markets as introduced by Shapley and Scarf [47]. Let $N = \{1, \ldots, n\}$, $n \geq 2$, be the set of agents. Each agent $i \in N$ is endowed with one object, which, with some abuse of notation, is denoted by i. Thus, N also denotes the set of objects. Each agent $i \in N$ has complete and transitive (weak) preferences R_i over objects.[5] We denote the strict part of R_i by P_i, that is, for all $j, k \in N$, jP_ik if and only if jR_ik and not kR_ij. Similarly, we denote the indifference part of R_i by I_i, that is, for all $j, k \in N$, jI_ik if and only if jR_ik and kR_ij. Let $R \equiv (R_i)_{i \in N}$. A (housing) market is a pair (N, R) or, if no confusion is possible, simply R.[6] Object $j \in N$ is acceptable to agent $i \in N$ if jR_ii. Agent i's preferences are called strict if they do not exhibit ties between acceptable objects; that is, for all acceptable $j, k \in N$ with $j \neq k$, we have jP_ik or kP_ij. A housing market has strict preferences if each agent has strict preferences. A housing market in which agents do not necessarily have strict preferences is often referred to as a housing market with weak preferences.

Given a housing market $M = (N, R)$ and a set $S \subseteq N$, the submarket M_S is the housing market in which S is the set of agents/objects and the preferences $(R_i)_{i \in S}$ are restricted to the objects in S.

The acceptability graph of a housing market $M = (N, R)$ is the directed graph $G_M = (N, E)$ or G for short, in which the set of nodes is N and (i, j) is a directed edge in E if j is an acceptable object for i, that is, jR_ii. In particular, all self-cycles (i, i) are in the graph (but, for convenience, they are omitted in all figures). Let $\tilde{N} \subseteq N$ and $\tilde{E} \subseteq E \cap (\tilde{N} \times \tilde{N})$. For each $i \in \tilde{N}$, the set of agent i's most preferred edges in graph $\tilde{G} \equiv (\tilde{N}, \tilde{E})$, or simply $\tilde{E}$, is the set $\tilde{E}^{T,i} \equiv \{(i, j) : (i, j) \in \tilde{E}$ and for each $(i, k) \in \tilde{E}, jR_ik\}$. The most preferred edges in graph $\tilde{G}$ is the set $\cup_{i \in \tilde{N}} \tilde{E}^{T,i}$.

Let $M = (N, R)$ be a housing market. An allocation is a redistribution of the objects such that each agent receives exactly one object; that is, an allocation is a vector $x = (x_i)_{i \in N} \in N^N$ such that

(1) For each $i \in N$, $x_i \in N$ denotes agent i's allotment, that is, the object that the agent receives.
(2) No object is assigned to more than one agent, that is, $\cup_{i \in N}\{x_i\} = N$.

We focus on individually rational allocations, that is, allocations for which each agent receives an acceptable object. Then, an allocation x can equivalently be described by its corresponding cycle cover G^x of the acceptability graph G. Formally, $G^x = (N, E^x)$ is the subgraph of G in which $(i, j) \in E^x$ if and only if $x_i = j$. Thus, the graph G^x consists of disconnected trading or exchange cycles[7] that cover G. We often write an (individually rational) allocation in cycle notation, that is, as a set of exchange cycles (in which we sometimes omit self-cycles). We refer to Example 1 for an illustration.

An allocation x Pareto-dominates an allocation z if, for each $i \in N$, $x_i R_i z_i$, and for some $j \in N$, $x_j P_j z_j$. An allocation is Pareto efficient if it is not Pareto-dominated by any allocation. Two allocations x, z are welfare equivalent if, for each $i \in N$, $x_i I_i z_i$.

Next, we recall the definition of solution concepts that are studied in the literature. A nonempty coalition $S \subseteq N$ blocks an allocation x if there is an allocation z such that

(1) $\{z_i : i \in S\} = S$.
(2) For each $i \in S$, $z_i P_i x_i$.

An allocation x is in the core[8] of the market if there is no coalition that blocks x. For each market R, let $\mathcal{C}(R)$ denote its core.

A nonempty coalition $S \subseteq N$ weakly blocks an allocation x if there is an allocation z such that

(1) $\{z_i : i \in S\} = S$.
(2) For each $i \in S$, $z_i R_i x_i$.
(3) For some $j \in S$, $z_j P_j x_j$.

An allocation x is in the strong core[9] of the market if there is no coalition that weakly blocks x. For each market R, let $\mathcal{SC}(R)$ denote its (possibly empty) strong core.

A price vector is a vector $p = (p_i)_{i \in N} \in \mathbb{R}^N$, where p_i denotes the price of object i. A competitive equilibrium is a pair (x, p), where x is an allocation and p is a price vector such that

(1) For each agent $i \in N$, object x_i is affordable, that is, $p_{x_i} \leq p_i$.
(2) For each agent $i \in N$, each object the agent prefers to x_i is not affordable, that is, $j P_i x_i$ implies $p_j > p_i$.

An allocation is a competitive allocation if it is part of some competitive equilibrium. Because there are n objects, we can assume without loss of generality that prices are integers in the set $\{1, 2, \ldots, n\}$.

Remark 1. If (x, p) is such that

(1) For each $i \in N$, $p_{x_i} \leq p_i$ or
(2) For each $i \in N$, $p_i \leq p_{x_i}$,

then for each $i \in N$, $p_{x_i} = p_i$. This follows immediately by looking at each exchange cycle separately (see, e.g., the proof in Cechlárová and Fleiner [19, lemma 1]). Hence, at each competitive equilibrium (x, p), for each $i \in N$, $p_{x_i} = p_i$. □

Wako [52] proves that the set of competitive allocations can be defined equivalently as a different type of core. Formally, a nonempty coalition $S \subseteq N$ antisymmetrically weakly blocks an allocation x if there is an allocation z such that

(1) $\{z_i : i \in S\} = S$.
(2) For each $i \in S$, $z_i R_i x_i$.
(3) For some $j \in S$, $z_j P_j x_j$.
(4) For each $i \in S$, if $z_i I_i x_i$, then $z_i = x_i$.

Requirements 1–3 say that coalition S weakly blocks x. The additional requirement 4 is that, if an agent in S is indifferent between allotments at x and z, then the agent must get the very same object, that is, $z_i = x_i$. An allocation x is in the core defined by antisymmetric weak domination if there is no coalition that antisymmetrically weakly blocks x. Wako [52] proves that the set of competitive allocations coincides with the core defined by antisymmetric weak domination. Henceforth, we often refer to the set of competitive allocations as the Wako-core, and for each market R, we denote this set by $\mathcal{WC}(R)$. Note that, when preferences are strict, requirement 4 is redundant, and the equivalence of strong core and Wako-core follows immediately.

Note that the three blocking notions introduced are nested: blocking implies antisymmetrical weak blocking and antisymmetrical weak blocking implies weak blocking. Therefore, for each market R, $\mathcal{SC}(R) \subseteq \mathcal{WC}(R) \subseteq \mathcal{C}(R)$.[10]

The following lemma is helpful for computations and is also used in our IP formulations and simulations. It states that, for each of the three cores, to check whether it contains a given allocation, it is not necessary to check blocking by any possible coalition. It is sufficient to check potential blocking by coalitions that constitute cycles in the acceptability graph.[11]

Lemma 1. *The strong core, Wako-core, and core consist of individually rational allocations. Moreover, each of the cores is equivalently characterized by the absence of blocking by cycles in the acceptability graph $G = (N, E)$. Formally, let x be an individually rational allocation. Then, x is in the strong core/Wako-core/core if there is no coalition $\{i_1, \ldots, i_k\}$ with for each $l = 1, \ldots, k$ (mod k), $(i_l, i_{l+1}) \in E$, that weakly blocks x/antisymmetrically weakly blocks x/blocks x through some allocation z with for each $l = 1, \ldots, k$ (mod k), $z_{i_l} = i_{l+1}$.*

Proof. Individual rationality is immediate. To prove the statement for the strong core, let x be an individually rational allocation. Suppose there is a nonempty coalition T that weakly blocks x through some allocation w. Let $j \in T$ be such that $w_j P_j x_j$. Let $S \subseteq T$ be the agents that constitute the exchange cycle, say $(i_1, \ldots, i_k)$, in w that involves agent j, that is, $j \in S$. One immediately verifies that $S = \{i_1, \ldots, i_k\}$ weakly blocks x through the allocation z defined by

$$z_i \equiv \begin{cases} w_i & \text{if } i \in S; \\ x_i & \text{if } i \notin S. \end{cases}$$

This proves the statement for the strong core. The statements for the core and the Wako-core follow similarly. □

An individually rational allocation x is a maximum size allocation if, for each individually rational allocation z, $|\{i \in N : x_i \neq i\}| \geq |\{i \in N : z_i \neq i\}|$. We provide an example to illustrate the three cores and maximum size allocation.

Example 1. Let $N = \{1, \ldots, 6\}$ and let preferences be given by Table 1. Throughout the paper, we do not display agents' unacceptable objects. For instance, agent 1 is indifferent between objects 2 and 3 and strictly prefers both objects to object 5.

Figure 2 displays the induced acceptability graph.[12] Here, a thick edge denotes the most preferred object(s), and a thin edge denotes the second most preferred object (if any).

Consider the allocations defined in Table 2. For instance, x^d (in cycle notation but without self-cycles) is the allocation $x^d = (x_1^d, x_2^d, x_3^d, x_4^d, x_5^d, x_6^d) = (3, 1, 4, 2, 5, 6)$. Using Lemma 1, it can be easily verified that x^a is the unique strong core allocation, and x^a and x^b are the competitive allocations, whereas x^a, x^b, x^c, and x^d form the core. Hence, the strong core is a singleton and a proper subset of the set of competitive allocations, whereas the latter set is also a proper subset of the core. Finally, x^e is the unique maximum size allocation and does not pertain to the core. □

Shapley and Scarf [47] (see also page 135 Roth and Postlewaite [40]) show that the set of competitive allocations is nonempty and coincides with the set of allocations that are obtained through David Gale's top trading cycles algorithm,[13] which is discussed in Section 3.1. Roth and Postlewaite [40] show that, if preferences are strict, then there is a unique strong core allocation that coincides with the unique competitive allocation. In general, when preferences are not strict, the strong core can be empty (see, e.g., Endnote 14) or contain more than one allocation (see, e.g., Example 1).

If preferences are strict, the unique competitive allocation is Pareto efficient (because it is in the strong core) and Pareto dominates any other allocation (Roth and Postlewaite [40, lemma 1]); in particular, any other core allocation is Pareto inefficient. If preferences are not strict, it is possible that each competitive allocation is Pareto dominated by some allocation that is not competitive.[14]

Finally, competitive allocations need not be welfare equivalent; in fact, different agents can strictly prefer distinct competitive allocations (see, e.g., Endnote 14). However, Wako [51] shows that all strong core allocations are welfare equivalent. The latter result also immediately follows from Quint and Wako's [39] algorithm, which is discussed in Section 3.2.

Table 1. Preference in Example 1.

1	2	3	4	5	6
2, 3	1	2	3	2	1
5	3	4	2	6	6
1	2	3	4	5	

Figure 2. Acceptability graph in Example 1.

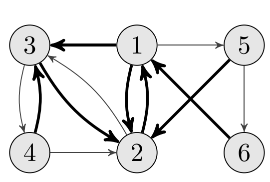


Table 2. Allocations in Example 1.

$x^a = \{(1,3,2)\}$
$x^b = \{(1,2),(3,4)\}$
$x^c = \{(1,5,2),(3,4)\}$
$x^d = \{(1,3,4,2)\}$
$x^e = \{(1,5,6),(2,3,4)\}$

3. Respecting Improvement

Let $R, \tilde{R}$ be two preference profiles over objects N. Let $i \in N$. We say that $\tilde{R}$ is an improvement for i with respect to R if the only difference between R and $\tilde{R}$ is that, at $\tilde{R}$, object i is ranked weakly higher by the other agents than at R.[15] In other words,

(1) Only agents different from i have possibly different preferences at $\tilde{R}$ and R.
(2) For each agent $j \neq i$, object i can become preferred to some additional objects.
(3) For each agent $j \neq i$ and each pair of objects different from i, preferences remain unchanged.

Formally,

(1) $\tilde{R}_i = R_i$.
(2) For all $j \neq i$ and all k with $k\,R_j\,j$, $i\,I_j\,k \Rightarrow i\,\tilde{R}_j\,k$ and $i\,P_j\,k \Rightarrow i\,\tilde{P}_j\,k$.
(3) For all $j \neq i$ and all $k, l \neq i$, $k\,R_j\,l \Longleftrightarrow k\,\tilde{R}_j\,l$.

As a simple example with $N = \{1,2,3,4,5\}$, let R be any preference profile such that $4\,P_5\,1\,I_5\,2\,I_5\,3\,P_5 5$. Let $\tilde{R}$ be the preference profile in which agents 1, 2, 3, and 4 have the same preferences as at R and let $\tilde{R}_5$ be defined by $1\,I_5 4\,P_5\,2\,I_5\,3\,P_5 5$. Then, $\tilde{R}$ is an improvement for agent 1 with respect to R.

Let $\mathscr{R}$ be a domain of markets. A single-valued allocation rule (on $\mathscr{R}$) is a map ϕ that associates with each market $R \in \mathscr{R}$ an allocation $\phi(R)$. For each $i \in N$, let $\phi_i(R)$ denote agent i's allotment at $\phi(R)$. We say that ϕ respects improvement (on $\mathscr{R}$) if, for each $i \in N$ and each pair of markets $R, \tilde{R} \in \mathscr{R}$ such that $\tilde{R}$ is an improvement for i with respect to R, we have that $\phi_i(\tilde{R}) R_i \phi_i(R)$. Respecting improvement is a natural and important property for applications of the housing markets model, such as kidney exchange programs. If the program employs an allocation rule that respects improvement, then it incentivizes all patients to bring the best possible set of donors to the market.

We also study a (generalized) respecting improvement property for multivalued allocation rules. A multivalued allocation rule (on $\mathscr{R}$) is a map Φ that associates with each market $R \in \mathscr{R}$ a (possibly empty) set of allocations $\Phi(R)$.

Because multivalued allocation rules yield sets of allocations, we have to compare sets of allotments for individual agents. Let X be a set of allocations and $i \in N$. The best allotments for agent i at X are the objects that the agent weakly most prefers among all objects in $\{x_i : x \in X\}$, denoted by X_i^+. Analogously, the worst allotments for i at X are the objects that the agent weakly least prefers among all objects in $\{x_i : x \in X\}$, denoted by X_i^-.

Let Φ be a nonempty[16] multivalued allocation rule (on $\mathscr{R}$). We say that Φ respects improvement for the best allotments or satisfies the RI-best property if, for each $i \in N$, each pair of markets $R, \tilde{R} \in \mathscr{R}$ such that $\tilde{R}$ is an improvement for i with respect to R and each pair of allotments $\tilde{x}_i \in \Phi_i^+(\tilde{R})$ and $x_i \in \Phi_i^+(R)$, we have $\tilde{x}_i R_i x_i$. Similarly, we say that Φ respects improvement for the worst allotments or satisfies the RI-worst property if, for each $i \in N$, each pair of markets $R, \tilde{R} \in \mathscr{R}$ such that $\tilde{R}$ is an improvement for i with respect to R, and each pair of allotments $\tilde{x}_i \in \Phi_i^-(\tilde{R})$ and $x_i \in \Phi_i^-(R)$, we have $\tilde{x}_i R_i x_i$.

Finally, in the case of a multivalued allocation rule Φ that, for some markets, can yield the empty set, we say that it conditionally respects improvement if the preceding requirements hold conditional on $\Phi(R), \Phi(\tilde{R}) \neq \emptyset$.

In the following three sections, we state and prove our main theoretical results. For a complete and transparent overview of all theoretical findings we refer to the summarizing Table 14 in Section 6.

3.1. Strict Preferences

We consider housing markets with strict and weak preferences separately. The reason is that, when preferences are strict, the strong core is always a singleton (which consists of the unique competitive allocation) so that the corresponding algorithm and notation are relatively simple. Tackling first the case of strict preferences also facilitates the discussion of the (general) case of weak preferences in the next section.

Before we present and prove our first main result, we describe the TTC algorithm for finding strong core allocations when preferences are strict. The graphs defined in the algorithm are crucial tools for the proof of Theorem 1.

Let $M = (N, R)$ be a housing market with strict preferences. We construct a subgraph G^{CP} of the acceptability graph G by using the TTC algorithm of David Gale (Shapley and Scarf [47]). The node set of G^{CP} is N, and its directed edges $E^{CP} = E^C \cup E^P$ are partitioned into two sets E^C and E^P, where E^C consists of the edges in the TTC cycles and E^P consists of all other edges that turn up during the execution of the algorithm and that point to more preferred objects.

3.1.1. TTC Algorithm: Construction of G^{CP}. Set $E^C \equiv \emptyset$, $E^P \equiv \emptyset$, and $M_1 \equiv M$. Let $G_1 = (N_1, E_1) \equiv (N, E)$ denote the acceptability graph of M_1. We iteratively construct "shrinking" submarkets M_t $(t = 2, 3, \ldots)$ whose acceptability graph is denoted by $G_t = (N_t, E_t)$. Set $t \equiv 1$.

Step 1. Let E_t^T be the set of most preferred edges in G_t.
Step 2. Let c_t be a (top trading) cycle in (N_t, E_t^T). Let C_t and E_t denote the node set and edge set of c_t, respectively.
Step 3. Add the edges of c_t to E^C, that is, $E^C \equiv E^C \cup E_t$.

Step 4. Let $E_t^T(\vec{C}_t)$ denote the subset of edges of E_t^T pointing to C_t from outside C_t. Formally, $E_t^T(\vec{C}_t) \equiv \{(i,j) \in E_t^T : i \in N_t \setminus C_t \text{ and } j \in C_t\}$. Add $E_t^T(\vec{C}_t)$ to E^P, that is, $E^P \equiv E^P \cup E_t^T(\vec{C}_t)$.

Step 5. If $N_t = C_t$, stop. Otherwise, let $N_{t+1} \equiv N_t \setminus C_t$, denote the submarket $M_{N_{t+1}}$ by M_{t+1}, and go to step 1.

When the algorithm terminates, the set of (top trading) cycles in E^C is the unique competitive allocation and, hence, the unique strong core allocation (see Roth and Postlewaite [40]). The following two facts about the graph G^{CP} are useful for later reference.

Fact 1. Cycles only contain edges in E^C. Each path that is not part of a cycle has an edge in E^P.

Fact 2. For any distinct $\ell, \ell' \in N$, if there is a path from agent ℓ to agent ℓ', then either the two agents are in the same cycle or agent ℓ' is removed from the market before agent ℓ.

For each profile of strict preferences R, let $\tau(R)$ denote the unique competitive allocation (or strong core allocation). Theorem 1 states that the allocation rule τ respects improvement. We provide a direct proof (based on the TTC algorithm) that is helpful to understand the similar but more complicated proof of Theorem 3 (with weak preferences). An anonymous reviewer suggested an alternative proof of Theorem 1 based on a two-sided matching model and by applying Hatfield et al. [25, theorem 9].[17] We include the details of the alternative proof in Appendix A as it discloses an interesting relationship between one- and two-sided matching problems.

Theorem 1. *When preferences are strict, the competitive allocation rule (or strong core allocation rule) τ respects improvement.*

Proof. Let $i \in N$ and $R, \tilde{R}$ profiles of strict preferences such that $\tilde{R}$ is an improvement for i with respect to R. Let $x = \tau(R)$ and $\tilde{x} = \tau(\tilde{R})$. We can assume that there is a unique agent $j \neq i$ with $\tilde{R}_j \neq R_j$ and prove that $\tilde{x}_i R_i x_i$. (If there is more than one such agent, we repeatedly apply the one-agent result to obtain the result.) We can also assume that $i \tilde{R}_j x_j$. (Otherwise, $x_j P_j i$, in which case all steps of the TTC algorithm are identical for R and $\tilde{R}$ so that $\tilde{x} = x$.)

Consider graph G^{CP} for market (N, R), that is, the graph that is obtained in the TTC algorithm for x. It follows from Facts 1 and 2 that agents i and j are related in (exactly) one of the following four ways:

(a) i and j are independent: there is no path from i to j nor from j to i.
(b) i and j are cycle members: i and j are in the same (top trading) cycle.
(c) i is a (noncycle) predecessor of j: there is a path from i to j with some edge in E^P.[18]
(d) j is a (noncycle) predecessor of i: there is a path from j to i with some edge in E^P.[19]

We distinguish among three cases, depending on the relation between agents i and j (see Figure 3). In each case, we describe if and how agent i's trading cycle changes to prove that $\tilde{x}_i R_i x_i$.

Case I: (a) i and j are independent or (d) j is a predecessor of i.

Let $F(i)$ be the set of followers of i in graph G^{CP}, that is, the nodes that can be reached from i through a path in G^{CP} (see Figure 3). We use the convention $i \in F(i)$. From (a) and (d), it follows that $j \notin F(i)$. Fact 2 implies that the TTC algorithm for R partitions the agents in $F(i)$ into trading cycles. Because, for each agent $\ell \in F(i)$, $\tilde{R}_\ell = R_\ell$, it follows that the TTC algorithm for $\tilde{R}$ partitions the agents in $F(i)$ into the same trading cycles. Hence, $\tilde{x}_i = x_i$.

Case II: (b) i and j are cycle members.

Let c be the cycle in graph G^{CP} that contains i and j. Let $p(i, j)$ be the unique path from i to j in the graph G^{CP} (see

Figure 3. (Color online) Graph G^{CP} (simplified) in the proof of Theorem 1. Each ellipse represents a top trading cycle.

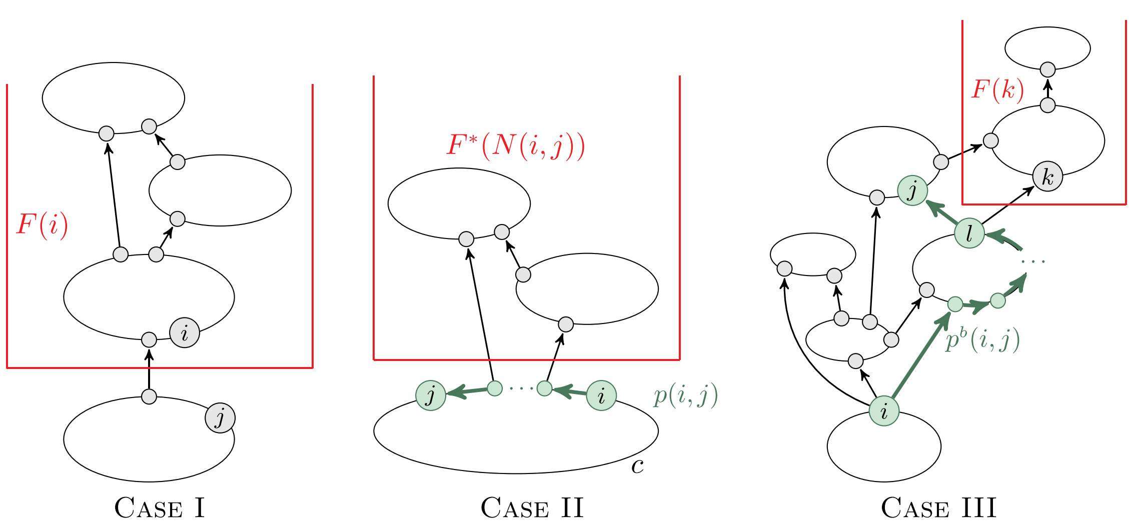


Figure 3). Obviously, $p(i, j)$ is part of c. Let $N(i, j)$ be the nodes on $p(i, j)$. (So $i, j \in N(i, j)$.) Let $F^*(N(i, j))$ be the followers outside of $N(i, j)$ that can be reached by some path in G^{CP} that (1) starts from some node in $N(i, j)$ and (2) does not contain edges in c. Fact 2 implies that the TTC algorithm for R partitions the agents in $F^*(N(i, j))$ into trading cycles. Because, for each agent $\ell \in F^*(N(i, j))$, $\tilde{R}_\ell = R_\ell$, it follows that the TTC algorithm for $\tilde{R}$ partitions the agents in $F^*(N(i, j))$ into the same trading cycles. Then, because $i\tilde{R}_j x_j$ and $\tilde{R}_j$ is obtained from R_j by shifting i up, the trading cycle of agent i at $\tilde{x}$ is the cycle $\tilde{c}$ that consists of the path $p(i, j)$ and the edge (j, i). Because $p(i, j)$ is part of the trading cycle c (in which i points to object x_i), it follows that agent i points to object x_i in trading cycle $\tilde{c}$, that is, $\tilde{x}_i = x_i$.

Case III: (c) i is a predecessor of j.

We define the best path from i to j to be the path from i to j in G^{CP}, where at each node $\ell \neq j$ on the path, the path follows agent ℓ's (unique) most preferred edge in

$$\{(\ell, \ell') \in E^{CP}: \text{ there is a path from } \ell' \text{ to } j \text{ using edges in } E^{CP}\}.$$

Let $p^b(i, j)$ denote the unique best path from i to j in G^{CP} (see Figure 3).

Because, for each $\ell \neq j$, $\tilde{R}_\ell = R_\ell$, $i\tilde{R}_j x_j$, and $\tilde{R}_j$ is obtained from R_j by shifting i up, it follows that, at some step in the TTC algorithm for $\tilde{R}$, agent j starts pointing to agent i and keeps doing so as long as agent i is present.

Next, consider the agent l with (l, j) on path $p^b(i, j)$. Let $k \in N$ with $kP_l j$ (see Figure 3). Because (l, j) is an edge in G^{CP} but agent l strictly prefers k to j, it follows that the TTC algorithm for R removes agent k before agent j. Fact 2 implies that the TTC algorithm for R partitions the agents in $F(k)$ (i.e., the followers of k, where $k \in F(k)$) into trading cycles. Because $j \notin F(k)$ and for each $\ell \in F(k)$, $\tilde{R}_\ell = R_\ell$, it follows that the TTC algorithm for $\tilde{R}$ partitions the agents in $F(k)$ into the same trading cycles. Recall that k is an arbitrary object with $kP_l j$. Thus, we can conclude that, at some step in the TTC algorithm for $\tilde{R}$, agent l starts pointing to agent j and keeps doing so as long as agent j is present.

We can repeat the same arguments until we conclude that each agent in the cycle $\tilde{c}$ formed by $p^b(i, j)$ and the edge (j, i), at some step in the TTC algorithm for $\tilde{R}$, starts pointing to its direct follower and keeps doing so as long as the follower is present. Thus, cycle $\tilde{c}$ is a trading cycle at $\tilde{x}$. Let i' be the direct follower of i in $\tilde{c}$. Note that, in graph G^{CP}, $(i, i') \in E^C$ or $(i, i') \in E^P$. If $(i, i') \in E^C$, then $i' = x_i$, in which case $\tilde{x}_i = i' = x_i$. If $(i, i') \in E^P$, then, by definition of E^P, $\tilde{x}_i = i' P_i x_i$. □

3.2. Weak Preferences

As discussed in Section 2, when preferences are weak, the strong core can be empty or contain more than one allocation, and it can be different from the set of competitive allocations (which is always nonempty). Therefore, the analysis of the case of weak preferences is divided into two parts accordingly.

3.2.1. Competitive Allocations.

For each market R, let $\mathcal{T}(R)$ denote the set of competitive allocations. We show that the multivalued allocation rule $\mathcal{T}$ satisfies the RI-best and RI-worst properties. However, we first prove a stronger result by focusing on probabilistic allocations. The RI-best and RI-worst properties then follow as a corollary.

Given a profile of weak preferences R, the TTC algorithm can be applied if we first break ties. Specifically, each indifference class in each agent's ranking of the objects is replaced by a strict ranking of the involved objects. Thus, different ways of breaking ties yield different profiles of strict preferences and, hence, also to potentially distinct outputs of the TTC algorithm. The set of allocations that can be generated by breaking ties and applying TTC equals the set of competitive allocations (see also Endnote 14).

One could conjecture that, if an agent's object improves in the preferences of the other agents so that at least one new competitive allocation is created, then at least one such allocation is weakly preferred to some old and removed competitive allocation. Surprisingly, as the following example shows, this need not be the case.

Example 2. Let $N = \{1, \ldots, 7\}$, let the initial preferences R be given by Table 3, and let the new preferences $\tilde{R}$ after the improvement of agent 7 be given by Table 4.

Table 3. Preferences R in Example 2.

1	2	3	4	5	6	7
2, 3	1	1	3	4	2	4
	6	4,7	5	7		5
	7					6

Table 4. Preferences $\tilde{R}$ in Example 2.

1	2	3	4	5	6	7
2, 3	1	1	3	4	2	4
	6,7	**7**	5	7		5
		4				6

Table 5. Allocations in Example 2.

$x^a = \{(1,3),(2,6),(4,5)\}$
$x^b = \{(1,3),(2,7,6),(4,5)\}$
$x^c = \{(1,2),(3,4),(5,7)\}$
$x^d = \{(1,2),(3,7,4)\}$

Consider the allocations defined in Table 5. It can be easily verified that the set of competitive allocations at R and $\tilde{R}$ is $\mathcal{T}(R) = \{x^a, x^c, x^d\}$ and $\mathcal{T}(\tilde{R}) = \{x^a, x^b, x^d\}$, respectively. Note that $\mathcal{T}(\tilde{R}) \setminus \mathcal{T}(R) = \{x^b\}$ and $\mathcal{T}(R) \setminus \mathcal{T}(\tilde{R}) = \{x^c\}$. Because $x^c_7 = 5\,P_7\,6 = x^b_7$, it follows that, for agent 7, each new competitive allocation after the improvement (i.e., x^b) is strictly worse than each old competitive allocation that was removed from the set of competitive allocations (i.e., x^c). □

Suppose that ties in weak preferences are broken uniformly at random. Then, the resulting probability distribution over profiles of strict preferences together with the TTC algorithm induce a probability distribution over (competitive) allocations and, hence, over allotments for each of the individual agents. Thus, we obtain a probabilistic allocation given by a doubly stochastic $n \times n$ matrix $\mathbf{a}^{TTC}(R)$ and in which, for each $(i,j) \in N \times N$, entry $\mathbf{a}^{TTC}_{ij}(R)$ denotes the probability that agent i receives object j.

Let $i \in N$. Let R_i be agent i's weak preferences. A probabilistic allocation $\mathbf{a}$ (first order) stochastically dominates another probabilistic allocation $\mathbf{a}'$ for agent i, denoted by $\mathbf{a} \succeq^{SD}_{R_i} \mathbf{a}'$, if for each object j, agent i obtains j or any other weakly preferred object with a higher probability under $\mathbf{a}$ than under $\mathbf{a}'$, that is,

$$\text{for each } j \in N, \quad \sum_{kR_ij} \mathbf{a}_{ik} \geq \sum_{kR_ij} \mathbf{a}'_{ik}.$$

A well-known important fact is that, if agent i has a cardinal utility function over objects that is consistent with the agent's weak preferences R_i, then $\mathbf{a} \succeq^{SD}_{R_i} \mathbf{a}'$ implies that the agent's expected utility at $\mathbf{a}$ is weakly higher than at $\mathbf{a}'$.

Theorem 2. *Let $i \in N$. Let $R, \tilde{R}$ be a pair of profiles of preferences such that $\tilde{R}$ is an improvement for i with respect to R. Then, $\mathbf{a}^{TTC}(\tilde{R}) \succeq^{SD}_{R_i} \mathbf{a}^{TTC}(R)$.*

Proof. It is sufficient to prove the result for the situation in which $\tilde{R}$ is a minimal improvement in the ranking of only one of the other agents', say j, that is, (a) object i is in an indifference class in R_j and moves right above it in $\tilde{R}_j$, or (b) object i is in itself an indifference class in R_j and moves into the indifference class right above it in $\tilde{R}_j$. In particular, for each $k \neq j$, $\tilde{R}_k = R_k$. Any other improvement can be obtained by a series of consecutive minimal improvements.

We first prove the statement for case (a). Let $T = \{i = i_1, i_2, \ldots, i_t\}$ with $t \geq 2$ be the objects in the indifference class in R_j of which i is a member. Let $\mathscr{R}^*$ be the set of strict profiles (i.e., profiles of strict preferences) generated by breaking all ties in R. Similarly, let $\tilde{\mathscr{R}}^*$ be the set of strict profiles generated by breaking all ties in $\tilde{R}$.

We define a function f from $\mathscr{R}^*$ to $\tilde{\mathscr{R}}^*$ as follows. Formally, let $R^* \in \mathscr{R}^*$ be a strict profile generated by breaking all ties in R. Then, define $f(R^*)$ as the strict profile obtained from R^* by moving object i right above all other objects in $T\backslash\{i\}$ (possibly it is already there) in R^*_j. One easily verifies that $f(R^*)$ is a strict profile that can be generated by breaking all ties in $\tilde{R}$, that is, $f(R^*) \in \tilde{\mathscr{R}}^*$. Hence, f is well-defined.

Let $R^* \in \mathscr{R}^*$. Note that there are exactly $t-1$ other strict profiles generated by R that f maps to $f(R^*)$ as well. (The only difference between these t strict profiles is the rank of object i in agent j's preferences. Because there are t objects in T, object i can take up any of the positions $1, \ldots, t$ in the ranking restricted to objects in T.) Conversely, one easily verifies that f is surjective: for each strict profile $\tilde{R}^* \in \tilde{\mathscr{R}}^*$, $f^{-1}(\tilde{R}^*)$ consists of exactly t strict profiles generated by breaking all ties in R.

Note that breaking all ties in R in all possible ways yields exactly t times more strict profiles than breaking all ties in $\tilde{R}$ in all possible ways, that is, $|\mathscr{R}^*| = t|\tilde{\mathscr{R}}^*|$. Because ties are broken uniformly at random, this implies that the probability that a given strict profile $\tilde{R}^*$ is generated by breaking all ties in $\tilde{R}$ equals the sum of probabilities of each of the t strict profiles in $f^{-1}(\tilde{R}^*)$ being generated from R.

Let $\tilde{R}^* \in \tilde{\mathscr{R}}^*$. We complete the proof of case (a) by comparing the allotment of agent i when TTC is applied to $\tilde{R}^*$ and when it is applied to any of the t profiles in $f^{-1}(\tilde{R}^*)$. Let $R^* \in f^{-1}(\tilde{R}^*)$. Because $\tilde{R}^*$ is an improvement for i with respect to R^* and both $\tilde{R}^*$ and R^* are strict profiles, Theorem 1 yields $\tau_i(\tilde{R}^*)\, R_i^*\, \tau_i(R^*)$. Because $R^* \in f^{-1}(\tilde{R}^*)$ is obtained from R by (only) breaking ties, we have that, for all objects $\ell, \ell' \in N$, $\ell\, R_i^*\, \ell' \Rightarrow \ell\, R_i\, \ell'$. Hence, $\tau_i(\tilde{R}^*)\, R_i\, \tau_i(R^*)$. In other words, agent i's TTC allotment at $\tilde{R}^*$ is weakly preferred to the TTC allotment at each of the t profiles in $f^{-1}(\tilde{R}^*)$. This, together with the previous observation on the corresponding probabilities, implies that $\mathbf{a}^{TTC}(\tilde{R}) \geq_{R_i}^{SD} \mathbf{a}^{TTC}(R)$.

Next, we prove the statement for case (b). Let $T = \{i = i_1, i_2, \ldots, i_t\}$ with $t \geq 2$ be the objects in the indifference class in $\tilde{R}_j$ of which i is a member. Let $\mathscr{R}^*$ be the set of strict profiles (i.e., profiles of strict preferences) generated by breaking all ties in R. Similarly, let $\tilde{\mathscr{R}}^*$ be the set of strict profiles generated by breaking all ties in $\tilde{R}$.

We define a function g from $\tilde{\mathscr{R}}^*$ to $\mathscr{R}^*$ as follows. Formally, let $\tilde{R}^* \in \tilde{\mathscr{R}}^*$ be a strict profile generated by breaking all ties in $\tilde{R}$. Then, define $g(\tilde{R}^*)$ as the strict profile obtained from $\tilde{R}^*$ by moving object i right below all other objects in $T\backslash\{i\}$ (possibly it is already there) in $\tilde{R}_j^*$. One easily verifies that $g(\tilde{R}^*)$ is a strict profile that can be generated by breaking all ties in R, that is, $g(\tilde{R}^*) \in \mathscr{R}^*$. Hence, g is well-defined.

Let $\tilde{R}^* \in \tilde{\mathscr{R}}^*$. Note that there are exactly $t-1$ other strict profiles generated by $\tilde{R}$ that g maps to $g(\tilde{R}^*)$ as well. (The only difference between these t strict profiles is the rank of object i in agent j's preferences. Because there are t objects in T, object i can take up any of the positions $1, \ldots, t$ in the ranking restricted to objects in T.) Conversely, one easily verifies that g is surjective: for each strict profile $R^* \in \mathscr{R}^*$, $g^{-1}(R^*)$ consists of exactly t strict profiles generated by breaking all ties in $\tilde{R}$.

Note that breaking all ties in $\tilde{R}$ in all possible ways yields exactly t times more strict profiles than breaking all ties in R in all possible ways, that is, $|\tilde{\mathscr{R}}^*| = t|\mathscr{R}^*|$. Because ties are broken uniformly at random, this implies that the probability that a given strict profile R^* is generated by breaking all ties in R equals the sum of probabilities of each of the t strict profiles in $g^{-1}(R^*)$ being generated from $\tilde{R}$.

Let $R^* \in \mathscr{R}^*$. We complete the proof of case (b) by comparing the allotment of agent i when TTC is applied to R^* and when it is applied to any of the t profiles in $g^{-1}(R^*)$. Let $\tilde{R}^* \in g^{-1}(R^*)$. Because $\tilde{R}^*$ is an improvement for i with respect to R^* and both $\tilde{R}^*$ and R^* are strict profiles, Theorem 1 yields $\tau_i(\tilde{R}^*)\, R_i^*\, \tau_i(R^*)$. Because R^* is obtained from R by (only) breaking ties, we have that, for all objects $\ell, \ell' \in N$, $\ell\, R_i^*\, \ell' \Rightarrow \ell\, R_i\, \ell'$. Hence, $\tau_i(\tilde{R}^*)\, R_i\, \tau_i(R^*)$. In other words, each of agent i's TTC allotments at the t profiles in $g^{-1}(R^*)$ is weakly preferred to the TTC allotment at R^*. This, together with the previous observation on the corresponding probabilities, implies that $\mathbf{a}^{TTC}(\tilde{R}) \geq_{R_i}^{SD} \mathbf{a}^{TTC}(R)$. □

Example 3. Consider again the markets and improvement discussed in Example 2. If, in line with Theorem 2, we also consider the probabilities of obtaining the competitive allocations in each of the two markets by breaking the ties in the TTC with uniform random probabilities, we get that for R, $Prob(x^a|R) = \frac{1}{2}$, $Prob(x^c|R) = \frac{1}{4}$, and $Prob(x^d|R) = \frac{1}{4}$, whereas for $\tilde{R}$, $Prob(x^a|\tilde{R}) = \frac{1}{4}$, $Prob(x^b|\tilde{R}) = \frac{1}{4}$, and $Prob(x^d|\tilde{R}) = \frac{1}{2}$. Because $x_7^a = 7$, $x_7^b = 6$, $x_7^c = 5$, $x_7^d = 4$, and $4\,P_7\,5\,P_7\,6\,P_7\,7$, we obtain $\mathbf{a}^{TTC}(\tilde{R}) >_{R_7}^{SD} \mathbf{a}^{TTC}(R)$; that is, for agent 7, the probabilistic allocation at $\tilde{R}$ strictly stochastically dominates the probabilistic allocation at R. Thus, at $\tilde{R}$, agent 7 has a higher expected utility than at R. □

As a corollary to Theorem 2, we obtain that the competitive allocation rule $\mathcal{T}$ satisfies the RI-best and the RI-worst properties.

Corollary 1. *Let $i \in N$. Let $R, \tilde{R}$ be a pair of profiles of preferences such that $\tilde{R}$ is an improvement for i with respect to R. Then,*

- *There is $\tilde{x} \in \mathcal{T}(\tilde{R})$ such that, for each $x \in \mathcal{T}(R)$, $\tilde{x}_i R_i x_i$.*
- *There is $x \in \mathcal{T}(R)$ such that, for each $\tilde{x} \in \mathcal{T}(\tilde{R})$, $\tilde{x}_i R_i x_i$.*

Proof. We first prove the first statement. Let $x_i \in \mathcal{T}_i^+(R)$. Then, there is some tie-breaking of R such that the associated TTC allocation gives allotment x_i to agent i. Hence, $\sum_{kR_ix_i} \mathbf{a}_{ik}^{TTC}(R) > 0$. From Theorem 2, $\sum_{kR_ix_i} \mathbf{a}_{ik}^{TTC}(\tilde{R}) \geq \sum_{kR_ix_i} \mathbf{a}_{ik}^{TTC}(R)$. So $\sum_{kR_ix_i} \mathbf{a}_{ik}^{TTC}(\tilde{R}) > 0$. Hence, there is some tie-breaking of $\tilde{R}$ such that the associated TTC allocation, say $\tilde{x} \in \mathcal{T}(\tilde{R})$, gives an allotment to agent i that the agent weakly prefers to x_i, that is, $\tilde{x}_i R_i x_i$. Because x_i is a best allotment for agent i among all allotments in $\{y_i : y \in \mathcal{T}(R)\}$, the first statement follows.

Next, we prove the second statement. Let $x_i \in \mathcal{T}_i^-(R)$. Then, x_i is a worst allotment for agent i among all allotments in $\{y_i : y \in \mathcal{T}(R)\}$. Hence, all TTC allocations obtained after tie-breaking of R give an allotment to agent i that the agent weakly prefers to x_i. Hence, $\sum_{kR_ix_i} \mathbf{a}_{ik}^{TTC}(R) = 1$. From Theorem 2, $\sum_{kR_ix_i} \mathbf{a}_{ik}^{TTC}(\tilde{R}) \geq \sum_{kR_ix_i} \mathbf{a}_{ik}^{TTC}(R)$. So $\sum_{kR_ix_i} \mathbf{a}_{ik}^{TTC}(\tilde{R}) = 1$. In other words, all TTC allocations obtained after tie-breaking of $\tilde{R}$ give an allotment to agent i that the agent weakly prefers to x_i. Hence, agent i's worst allotments in $\{y_i : y \in \mathcal{T}(\tilde{R})\}$ are weakly preferred to x_i, and the second statement follows. □

Note that, in general, there is no competitive allocation in which each agent receives the agent's most preferred allotment (among those that are obtained at competitive allocations); that is, agents do not unanimously agree on the best competitive allocation (see, e.g., agents 3 and 4 and competitive allocations x^a and x^b in Example 1). Nonetheless, Corollary 1 implies that any optimistic agent who believes that the agent always receives the best possible allotment subscribes to the thesis that the competitive correspondence respects any of the agent's potential improvements. A similar statement holds for any pessimistic agent who believes that the agent always receives the worst possible allotment.

3.2.2. Strong Core. We now turn to the strong core, which, in the case of weak preferences, is a (possibly strict) subset of the set of competitive allocations.

Similarly to the case of strict preferences, before we state and prove our generalization of Theorem 1 to the domain of weak preferences, we first describe the efficient algorithm of Quint and Wako [39] for finding a strong core allocation whenever there exists one. The graphs defined in the algorithm are crucial tools for the proof of Theorem 3.

Let $M = (N, R)$ be a housing market with weak preferences. We use the simplified interpretation of Cechlárová and Fleiner [19] and construct a subgraph G^{SP} of the acceptability graph G with node set N and edge set $E^{SP} \equiv E^S \cup E^P$, which are useful for our later analysis.

A strongly connected component of a directed graph is a subgraph in which there is a directed path from each node to every other node. An absorbing set is a strongly connected component with no outgoing edge.[20] Note that each directed graph has at least one absorbing set.

3.2.3. Quint–Wako Algorithm: Construction of G^{SP}. Set $E^S \equiv \emptyset$, $E^P \equiv \emptyset$, and $M_1 = M$. Let $G_1 = (N_1, E_1) \equiv (N, E)$ denote the acceptability graph of M_1. We iteratively construct shrinking submarkets M_t ($t = 2, 3, \ldots$) whose acceptability graph is denoted by $G_t = (N_t, E_t)$. Set $t \equiv 1$.

Step 1. Let E_t^T be the set of most preferred edges in G_t.

Step 2. Let S_t be an absorbing set in (N_t, E_t^T). Let $N_t(S_t)$ and $E_t^T(S_t)$ denote the node set and edge set of S_t.

Step 3. Add the edges of S_t to E^S, that is, $E^S \equiv E^S \cup E_t^T(S_t)$.

Step 4. Let $E_t^T(\vec{S_t})$ denote the subset of edges of E_t^T pointing to $N_t(S_t)$ from outside $N_t(S_t)$. Formally, $E_t^T(\vec{S_t}) \equiv \{(i,j) \in E_t^T : i \in N_t \setminus N_t(S_t) \text{ and } j \in N_t(S_t)\}$. Add $E_t^T(\vec{S_t})$ to E^P, that is, $E^P \equiv E^P \cup E_t^T(\vec{S_t})$.

Step 5. If $N_t = N_t(S_t)$, stop. Otherwise, let $N_{t+1} \equiv N_t \setminus N_t(S_t)$ denote the submarket $M_{N_{t+1}}$ by M_{t+1}, and go to step 1.

The following two facts about the graph G^{SP} are useful for later reference.

Fact 1*. Absorbing sets only contain edges in E^S. Each path that is not part of an absorbing set has an edge in E^P.

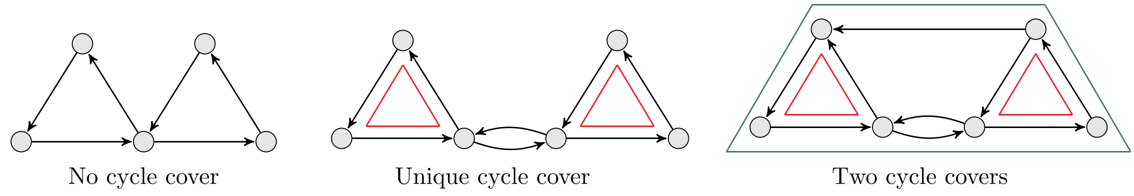
Fact 2*. For any distinct $\ell, \ell' \in N$, if there is a path from agent ℓ to agent ℓ', then either the two agents are in the same absorbing set or agent ℓ' is removed from the market before agent ℓ.

Quint and Wako [39] prove that there is a strong core allocation for M if and only if, for each absorbing set S_t defined in the algorithm there exists a cycle cover, that is, a set of cycles covering all the nodes of S_t. See Figure 4 for an illustration. Finding a cycle cover, if one exists, can be done with the classic Hungarian method (Kuhn [30]) for finding a perfect matching for the corresponding bipartite graph in which the objects are on one side, the agents are on the other side, and there is an undirected arc between an object–agent pair if the object is among the agent's most preferred objects (which might include the agent's own object). We refer to Abraham et al. [4], Quint and Wako [39], and Cechlárová and Fleiner [19] for further details on this reduction.

Remark 2. If, for each absorbing set S_t defined in the preceding algorithm, there exists a cycle cover, then the set of cycle covers (one cycle cover for each absorbing set) constitutes a strong core allocation. Conversely, as shown in the proof of Quint and Wako [39, theorem 5.5], each strong core allocation can be written as a set of cycle covers (one for each absorbing set S_t). Therefore, if the strong core is nonempty, all its allocations can be obtained by selecting all possible cycle covers in the algorithm. □

Remark 3. In the Quint–Wako algorithm, each agent obtains the same welfare at any two cycle covers in which the agent is involved (because the agent is indifferent between any two of the agent's outgoing edges in an

Figure 4. (Color online) Three absorbing sets.



absorbing set). Together with Remark 2, this immediately proves Wako [51, theorem 2(2)], which states that all strong core allocations are welfare equivalent. □

We can now show that the multivalued allocation rule $\mathcal{SC}$ conditionally respects improvement.

Theorem 3. *Let $i \in N$. Let $R, \tilde{R}$ be a pair of profiles of preferences such that $\tilde{R}$ is an improvement for i with respect to R. If $\mathcal{SC}(R), \mathcal{SC}(\tilde{R}) \neq \emptyset$, then, for each $\tilde{x} \in \mathcal{SC}(\tilde{R})$ and each $x \in \mathcal{SC}(R)$, $\tilde{x}_i R_i x_i$.*

Proof. Let $x \in \mathcal{SC}(R)$. It follows from Remark 3 that it is sufficient to show that there exists $\tilde{x} \in \mathcal{SC}(\tilde{R})$ with $\tilde{x}_i R_i x_i$. We can assume that there is a unique agent $j \neq i$ with $\tilde{R}_j \neq R_j$. (If there is more than one such agent, we repeatedly apply the one-agent result to obtain the result.) We can also assume that $i\tilde{R}_j x_j$. (Otherwise, $x_j \tilde{P}_j i$, in which case all steps of the Quint–Wako algorithm are identical for R and $\tilde{R}$ so that $x \in \mathcal{SC}(R) = \mathcal{SC}(\tilde{R})$.)

Consider graph G^{SP} for market (N, R), that is, the graph that is generated in the Quint–Wako algorithm to obtain x. It follows from Facts 1* and 2* that agents i and j are related in (exactly) one of the following four ways:

(a) i and j are independent: there is no path from i to j nor from j to i.
(b) i and j are absorbing set members: i and j are in the same absorbing set.
(c) i is a predecessor of j: there is a path from i to j in G^{SP} with some edge in E^P.[21]
(d) j is a predecessor of i: there is a path from j to i in G^{SP} with some edge in E^P.[22]

We distinguish among three cases, depending on the relation between agents i and j.

Case I: (a) i and j are independent or (d) j is a predecessor of i.

Let $F(i)$ be the followers of i in graph G^{SP}, that is, the nodes that can be reached through a path in G^{SP}. We use the convention $i \in F(i)$. From (a) and (d), it follows that $j \notin F(i)$. Fact 2* implies that the Quint–Wako algorithm for R partitions the agents in $F(i)$ into a collection of absorbing sets. Because, for each agent $k \in F(i)$, $\tilde{R}_k = R_k$, it follows that the Quint–Wako algorithm for $\tilde{R}$ partitions the agents in $F(i)$ into the same collection of absorbing sets. Because $\mathcal{SC}(\tilde{R}) \neq \emptyset$, it follows from Remark 2 that there exists $\tilde{x} \in \mathcal{SC}(\tilde{R})$ such that, for each agent $k \in F(i)$, $\tilde{x}_k = x_k$. In particular, $\tilde{x}_i = x_i$.

Case II: (b) i and j are absorbing set members.

Let S_t be the absorbing set that contains i and j in the Quint–Wako algorithm for R. Note that (i, x_i) is an edge in the edge set $E_t^T(S_t)$ of the absorbing set S^t (possibly $x_i = i$). Let $F^*(N_t(S_t))$ be the followers outside of $N_t(S_t)$ that can be reached by some path in G^{SP} that starts from a node in $N_t(S_t)$. Then, $x_i \notin F^*(N_t(S_t))$. Fact 2* implies that the Quint–Wako algorithm for R partitions the agents in $F^*(N_t(S_t))$ into a collection of absorbing sets. Because, for each agent $\ell \in F^*(N_t(S_t))$, $\tilde{R}_\ell = R_\ell$, it follows that the Quint–Wako algorithm for $\tilde{R}$ partitions the agents in $F^*(N_t(S_t))$ into the same absorbing sets. Then, because $i\tilde{R}_j x_j$ and $\tilde{R}_j$ is obtained from R_j by shifting i up, when the Quint–Wako algorithm is applied to $\tilde{R}$, the absorbing set that contains i again contains x_i and j and its edge set again contains (i, x_i). Thus, at each $\tilde{x} \in \mathcal{SC}(\tilde{R}) \neq \emptyset$, agent i receives an object $\tilde{x}_i$ such that $\tilde{x}_i I_i x_i$.

Case III: (c) i is a predecessor of j.

A path from i to j in G^{SP} is said to be a best path from i to j if, at each node $\ell \neq j$ on the path, the path follows one of agent ℓ's most preferred edges in

$$\{(\ell, \ell') \in E^{SP} : \text{ there is a path from } \ell' \text{ to } j \text{ using edges in } E^{SP}\}.$$

Let $P^b(i,j)$ denote the set of best paths from i to j in G^{SP}.

Because, for each $\ell \neq j$, $\tilde{R}_\ell = R_\ell$, $i\tilde{R}_j x_j$, and $\tilde{R}_j$ is obtained from R_j by shifting i up, it follows that at some step in the Quint–Wako algorithm for $\tilde{R}$, agent j starts pointing to agent i and keeps doing so as long as agent i is present.

Next, let $p^b(i,j) \in P^b(i,j)$ be any best path. Consider the agent l with (l, j) on path $p^b(i,j)$. Let $k \in N$ with $kP_l j$. Because (l, j) is an edge in G^{SP} but agent l strictly prefers k to j, it follows that the Quint–Wako algorithm for R removes agent k before agent j. Fact 2* implies that the Quint–Wako algorithm for R partitions the agents in $F(k)$ (i.e., the followers of k, where $k \in F(k)$) into absorbing sets. Because $j \notin F(k)$ and for each $\ell \in F(k)$, $\tilde{R}_\ell = R_\ell$, it follows that the Quint–Wako algorithm for $\tilde{R}$ partitions the agents in $F(k)$ into the same absorbing sets. Recall that k is an arbitrary object with $kP_l j$. Thus, we can conclude that, at some step in the Quint–Wako algorithm for $\tilde{R}$, agent l starts pointing to agent j and keeps doing so as long as agent j is present.

We can repeat the same arguments until we conclude that each agent in the cycle formed by $p^b(i,j)$ and the edge (j, i), at some step, starts pointing to its direct follower and keeps doing so as long as the follower is present. Hence, at some step of the algorithm, the cycle formed by $p^b(i,j)$ and the edge (j, i) is part of an absorbing set. Let i^b be the direct follower of agent i in path $p^b(i,j)$. Thus, at each $\tilde{x} \in \mathcal{SC}(\tilde{R}) \neq \emptyset$, agent i receives an object $\tilde{x}_i$ such that $\tilde{x}_i I_i i^b$. Note that, in graph G^{SP}, $(i, i^b) \in E^S$ or $(i, i^b) \in E^P$. If $(i, i^b) \in E^S$, then $i^b I_i x_i$, in which case $\tilde{x}_i I_i x_i$. If $(i, i^b) \in E^P$, then by definition of E^P, $i^b R_i x_i$, in which case $\tilde{x}_i R_i x_i$. □

An immediate corollary to Theorem 3 is that the strong core satisfies the conditional RI-best and RI-worst properties.

Corollary 2. *For each $i \in N$ and each pair of profiles of preferences $R, \tilde{R}$ such that $\mathcal{SC}(R), \mathcal{SC}(\tilde{R}) \neq \emptyset$ and $\tilde{R}$ is an improvement for i with respect to R,*

- *There is $\tilde{x} \in \mathcal{SC}(\tilde{R})$ such that, for each $x \in \mathcal{SC}(R)$, $\tilde{x}_i R_i x_i$.*
- *There is $x \in \mathcal{SC}(R)$ such that, for each $\tilde{x} \in \mathcal{SC}(\tilde{R})$, $\tilde{x}_i R_i x_i$.*

3.3. Bounded Length Exchange Cycles

Motivated by kidney exchange programs, we consider housing markets in which the length of allowed exchange cycles in allocations is limited. We provide several examples to demonstrate the possible violations of the respecting improvement property (or variants/extensions of the property) in the setting of bounded length exchanges.

3.3.1. Definitions. Let $M = (N, R)$ be a housing market. Let k be an integer that indicates the maximum allowed length of exchange cycles. An allocation is a k-allocation if each exchange cycle has length at most k, that is, there exists a partition of $N = S_1 \cup S_2 \cup \cdots \cup S_q$ such that, for each $p \in \{1, \ldots, q\}$, $|S_p| \leq k$ and $\{x_i : i \in S_p\} = S_p$. Assuming that blocking coalitions are subject to the same length of allowed exchange cycles, the definition of the three cores can be adjusted accordingly as well.[23] Specifically, the k-core consists of the k-allocations for which there is no blocking coalition of size at most k; the strong k-core consists of the k-allocations for which there is no weakly blocking coalition of size at most k; the Wako-k-core consists of the k-allocations that are not antisymmetrically weakly dominated through a coalition of size at most k.

Similarly to the unbounded case, the three blocking notions are nested. Hence, the strong k-core is a subset of the Wako-k-core, and the Wako-k-core is a subset of the k-core. Moreover, again similarly to the unbounded case, it follows easily that, for strict preferences, the strong k-core coincides with the Wako-k-core.

To keep notation as simple as possible, whenever the context is clear, we omit "k" from k-allocation, k-core, etc., and instead refer to k-housing markets to invoke the restriction on exchange cycles, blocking coalitions, allocations, and cores.

The practically important case of pairwise exchanges, that is, $k = 2$, is known as the stable roommates problem (introduced in Gale and Shapley [23]):

- Strict preferences: the strong core, Wako-core, and core coincide and correspond to the set of stable matchings.
- Weak preferences: the Wako-core and core coincide and correspond to the set of weakly stable matchings, whereas the strong core corresponds to the set of strongly stable matchings.

We refer to Manlove [33] for more details. It is important to note that any of the cores can be empty when exchange cycles are bounded even if preferences are strict. Therefore, the following analysis necessarily concerns conditional respecting improvement properties.

3.3.2. Pairwise Exchanges ($k = 2$). As mentioned in Section 1, the maximization of the number of pairwise exchanges does not respect improvement. Example 4 proves this formally. A consequence is that the priority mechanisms studied by Roth et al. [42] need not be donor-monotonic if agents' preferences can be nondichotomous.

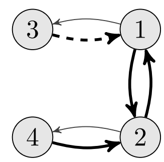
Table 6. Preferences R in Example 4.

1	2	3	4
2	1	3	2
3	4		4
1	2		

Table 7. Preferences $\tilde{R}$ in Example 4.

1	2	3	4
2	1	**1**	2
3	4	3	4
1	2		

Figure 5. Acceptablity graph in Example 4.



Example 4. Let $N = \{1,2,3,4\}$. Let the initial preferences R be given by Table 6 and the new preferences $\tilde{R}$, in which object 1 becomes acceptable for agent 3, by Table 7.

Initially, at R, there are two ways to maximize the number of pairwise exchanges, namely, by picking either of the two-cycles (1, 2) and (2, 4). Assume, without loss of generality, that (1, 2) is selected. (In case (2, 4) is selected, similar arguments can be employed.) Now, suppose the discontinuous edge (in Figure 5) is included so that agent 1 "improves" and we obtain $\tilde{R}$. Then, the unique way to maximize the number of pairwise exchanges is obtained by picking the two two-cycles (1, 3) and (2, 4), which means that agent 1 is strictly worse off than in the initial situation. □

The following example shows that the (strong, Wako-) core violates the conditional RI-worst property even if preferences are strict.

Example 5. Let $N = \{1,2,3,4\}$. Let the initial preferences R be given by Table 8 and the new preferences $\tilde{R}$, in which object 1 becomes acceptable for agent 3, by Table 9.

Initially, at R, the unique (strong, Wako-) core allocation is $x^a = \{(1,2),(3,4)\}$. Now, suppose the discontinuous edge (in Figure 6) is included so that agent 1 "improves" and we obtain $\tilde{R}$. Then, another (strong, Wako-) core allocation is created, $x^b = \{(1,3),(2,4)\}$, which is strictly worse for agent 1. Hence, the (strong, Wako-) core violates the conditional RI-worst property under strict preferences. □

The next example shows that when preferences are weak, the core/Wako-core also violates the conditional RI-best property.

Example 6. Let $N = \{1,2,3,4\}$. Let the initial preferences R be given by Table 10 and the new preferences $\tilde{R}$, in which object 4 becomes acceptable for agent 1, by Table 11.

Initially, at R, there exist two (Wako-) core allocations $x^a = \{(3,4)\}$ and $x^b = \{(1,3),(2,4)\}$. The best allotment for agent 4 is object 3, obtained at allocation x^a. Now, suppose the discontinuous edge in Figure 7 is included so that

Table 8. Preferences R in Example 5.

1	2	3	4
2	4	4	3
3	1	3	2
1	2		4

Table 9. Preferences $\tilde{R}$ in Example 5.

1	2	3	4
2	4	**1**	3
3	1	4	2
1	2	3	4

Figure 6. Acceptability graph in Example 5.

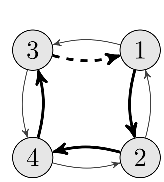


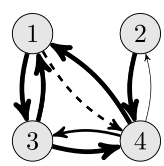
Table 10. Preferences R in Example 6.

1	2	3	4
3	4	1, 4	1
1	2	3	3
			2
			4

Table 11. Preferences $\tilde{R}$ in Example 6.

1	2	3	4
3	4	1,4	1
4	2	3	3
1			2
			4

Figure 7. Acceptability graph in Example 6.



agent 4 "improves" and we obtain $\tilde{R}$. Then, the new cycle (1, 4) blocks allocation x^a, whereas (1, 3) blocks the (unique) new feasible allocation $x^c = \{(1,4)\}$. Thus, the (Wako-) core consists of the unique allocation x^b. Hence, agent 4's allotment at the unique (Wako-) core allocation ($x_4^b = 2$) is strictly worse than the best allotment ($x_4^a = 3$) initially. Hence, the (Wako-) core violates the conditional RI-best property under weak preferences. □

We summarize these two findings in the following statement.

Proposition 1. *Under strict preferences and pairwise exchanges, the (strong, Wako-) core violates the conditional RI-worst property. Under weak preferences and pairwise exchanges, the (Wako-) core violates the conditional RI-best property.*

3.3.3. Three-Way and Longer Bounded Exchanges ($k \geq 3$). In the following example, we exhibit two housing markets, and we prove that, for each housing market, the three cores coincide (and are nonempty). Subsequently, we use the example to show that the three cores do not conditionally respect improvement for the best allotments when the maximum allowed length of exchange cycles is three.

Example 7. Throughout the example, we focus on the core. However, because all blocking arguments can be replaced by weak blocking arguments, all statements also hold for the strong core and, hence, also for the Wako-core. Let $N = \{1, \ldots, 10\}$ be the set of agents. We consider two housing markets that only differ in preferences. First, consider the housing market (N, R), or simply R for short, with "cyclic" strict preferences given in Table 12.

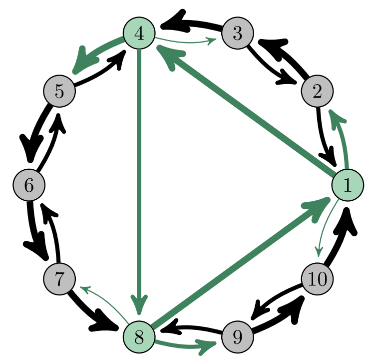
Table 12. Preferences R in Example 7.

1	2	3	4	5	6	7	8	9	10
2	3	4	5	6	7	8	9	10	1
10	1	2	3	4	5	6	7	8	9
1	2	3	4	5	6	7	8	9	10

Table 13. Preferences R^b in Example 7.

1	2	3	4	5	6	7	8	9	10
4	3	4	5	6	7	8	**1**	10	1
2	1	2	**8**	4	5	6	9	8	9
10	2	3	3	5	6	7	7	9	10
1			4				8		

Figure 8. (Color online) Acceptability graph for R^b in Example 7.



Because only directly neighboring objects (and one's own object) are acceptable, it follows that the only exchange cycles in which each agent is assigned an acceptable object are the 10 self-cycles and the 10 two-cycles $(i, i+1)$ (mod 10) in which agents i and $i+1$ swap their objects.[24] The core $\mathcal{C}(R) = \{x^a, x^b\}$ consists of the following two allocations:

$$x^a = \{(1,2),(3,4),(5,6),(7,8),(9,10)\} \text{ and}$$
$$x^b = \{(10,1),(2,3),(4,5),(6,7),(8,9)\}.$$

Next, we create an extended housing market R^b by inserting one three-cycle in R. Preferences R^b are provided in Table 13, in which the changes with respect to R are bold-faced and depicted in Figure 8.

Apart from the earlier mentioned self-cycles and two-cycles, the only additional exchange cycle with only acceptable objects in R^b is $c^b = (1,4,8)$. Allocation x^b is in the core of R^b because c^b does not block x^b: agent 4 obtains object 8 in c^b, which is strictly less preferred than the agent's assigned object 5 at x^b. In fact, x^b is the unique core allocation of R^b. To see this, note first that x^a is not in the core of R^b as c^b blocks it. And, second, the only new exchange cycle created in R^b, that is, c^b, cannot be part of a core allocation because, if it were, then to avoid blocking cycle (4, 5), the next two-cycle (5, 6) would have to be part of the allocation, in which case, 7 would remain unmatched (i.e., be a self-cycle) and cycle (6, 7) would block the allocation. Therefore, x^b is the unique core allocation of R^b, that is, $\mathcal{C}(R^b) = \{x^b\}$. □

Using the previous example, we can easily prove that, when $k = 3$, the strong core, Wako-core, and core violate the conditional RI-best property even if preferences are strict.

Proposition 2. *Suppose the maximum allowed length of exchange cycles is three. Then, there are three-housing markets with strict preferences (N, R) and $(N, \tilde{R})$ with*

- $X(R) \equiv \mathcal{SC}(R) = \mathcal{WC}(R) = \mathcal{C}(R) \neq \emptyset$ *and*
- $X(\tilde{R}) \equiv \mathcal{SC}(\tilde{R}) = \mathcal{WC}(\tilde{R}) = \mathcal{C}(\tilde{R}) \neq \emptyset$

such that, for some $i \in N$, $\tilde{R}$ is an improvement for i with respect to R but

- $X(\tilde{R}) \subseteq X(R)$,
- *For the unique $x \in X(R) \setminus X(\tilde{R})$ and for the unique $\tilde{x} \in X(\tilde{R}) \setminus X(R)$, $x_i P_i \tilde{x}_i$.*

Proof. Let $(N, \tilde{R})$ be the three-housing market with $N = \{1, \ldots, 10\}$ and $\tilde{R} = R^b$ from Example 7. Let (N, R) be the three-housing market that is obtained from $(N, \tilde{R})$ by making object 1 unacceptable for agent 8. Obviously, $\tilde{R}$ is an improvement for agent 1 with respect to R. As shown in Example 7, $\mathcal{SC}(\tilde{R}) = \mathcal{WC}(\tilde{R}) = \mathcal{C}(\tilde{R}) = \{x^b\} \neq \emptyset$. One also easily verifies that $\mathcal{SC}(R) = \mathcal{WC}(R) = \mathcal{C}(R) = \{x^a, x^b\} \neq \emptyset$. Finally, agent 1's most preferred allotment in $\mathcal{SC}(R) = \mathcal{WC}(R) = \mathcal{C}(R)$ is object 2, whereas agent 1's unique (hence, most preferred) allotment in $\mathcal{SC}(\tilde{R}) = \mathcal{WC}(\tilde{R}) = \mathcal{C}(\tilde{R})$ is object 10. Because agent 1 strictly prefers object 2 to object 10, the result follows. □

Remark 4. Example 7 and the proof of Proposition 2 can be adjusted to demonstrate the violation of the conditional RI-best property for any larger upper bound k on the length of the exchange cycles as follows. We keep the structure of the example with the double outer cycle and the embedded three-cycle but we extend the length of the outer cycle so that we do not create any new cycle of length at most k, also keeping the parity of the highest two nodes of the embedded three-cycle. That is, if 1 is the starting node and i, j are the other two nodes of the embedded three-cycle, then both i and j remain even. For example, for $k = 4$, we extend the double outer cycle to consist of 14 vertices, and we have an inner three-cycle that consists of agents 1, 6, and 10. □

Remark 5. When $k=3$, the strong core, Wako-core, and core also violate the conditional RI-worst property even if preferences are strict. This follows from Example 5 and the observation that there is no three-cycle with only acceptable objects. □

Remark 6. Motivated by kidney exchange programs, we consider a limitation on the length of allowed exchange cycles. All results (Examples 4–7 and Propositions 1 and 2) are negative; that is, we show violations of (adjusted versions of the) respecting improvement property. However, there can exist different types of restrictions (instead of bounded length), for instance, because of technological or logistical reasons that may exclude specific cyclical exchanges. We conjecture that such restrictions lead to additional negative results but leave an in-depth study for future research. □

4. Integer Programming Formulations

In this section, we propose new IP formulations for the core, the set of competitive allocations (i.e., the Wako-core), and the strong core. First, we propose novel edge formulations for the unbounded case for all three solution concepts. Second, we improve the formulations in Quint and Wako [39] by giving alternative cycle formulations for the core and strong core. Third, we provide a new formulation for the Wako-core for the case of bounded length exchange cycles. The novel IP formulations serve as a stepping stone for our computational experiments in Section 5.

4.1. Novel Edge Formulations

Let (N, R) be a housing market and $G \equiv G(N,R) = (N,E)$ its acceptability graph. Because all three cores only contain individually rational allocations, we can restrict attention to the edges of the acceptability graph. Specifically, with each edge $(i,j) \in E$, we associate a variable y_{ij} as follows:

$$y_{ij} = \begin{cases} 1 & \text{if agent } i \text{ receives object } j; \\ 0 & \text{otherwise.} \end{cases}$$

Then, the base model reads as follows:

$$\sum_{j:(i,j)\in E} y_{ij} = 1 \qquad \forall i \in N, \tag{1}$$

$$\sum_{j:(j,i)\in E} y_{ji} = 1 \qquad \forall i \in N, \tag{2}$$

$$y_{ij} \in \{0,1\} \qquad \forall (i,j) \in E. \tag{3}$$

Constraints (1) guarantee that agent i receives exactly one (acceptable) object (possibly the agent's own). Constraints (2) guarantee that object i is given to exactly one agent. Each vector $(y_{ij})_{(i,j)\in E}$ that satisfies (1)–(3) yields an allocation x defined by $x_i = j$ if and only if $y_{ij}=1$. Moreover, each allocation can be obtained in this way. So there is a one-to-one correspondence between allocations and vectors that satisfy (1)–(3).

We introduce for each $i \in N$ an additional integer variable p_i that represents the price of object i:

$$p_i \in \{1,\ldots,n\} \quad \forall i \in N. \tag{4}$$

In what follows, we give our IP formulations for the general case of weak preferences and explain how they can be simplified for strict preferences. We tackle the core, the set of competitive allocations (i.e., the Wako-core), and the strong core (in this order) by subsequently adding constraints. Given an allocation x, we say that x dominates an edge (i, j) in the acceptability graph G if agent i weakly prefers the agent's allotment x_i to object j, that is, $x_i \, R_i \, j$.

4.1.1. IP for the Core. It follows from Lemma 1 that an individually rational allocation x is in the core if and only if each cycle in G contains an edge that is dominated by x or, equivalently, there exists no cycle in G that consists of undominated edges. Note that the undominated edges in G form a cycle-free subgraph of G if and only if there is a topological order of the objects in the subgraph of G that consists of the undominated edges. The existence of this topological order is equivalent to the existence of prices of the objects such that, for each undominated edge (i, j), $p_i < p_j$. Therefore, an allocation x is in the core if and only if there exist prices $(p_i)_{i\in N}$ such that

$$(i,j) \in E \text{ is not dominated by } x \;\Rightarrow\; p_i < p_j. \tag{*}$$

Thus, core allocations are characterized by Constraints (1)–(4) together with (5):

$$p_i + 1 \leq p_j + n \cdot \sum_{k:kR_ij} y_{ik} \qquad \forall (i,j) \in E. \tag{5}$$

Proposition 3. *Let x be an allocation. Let y be the corresponding vector that satisfies (1)–(3). Allocation x is in the core if and only if there are prices $(p_i)_{i\in N}$ such that (4) and (5) hold.*

Proof. First, observe that, for each $(i,j) \in E$,

$$\begin{aligned}(i,j) \text{ is dominated by } x \quad &\Longleftrightarrow x_i R_i j \\ &\Longleftrightarrow \text{there is } k \in N \text{ with } k\, R_i\, j \text{ and } y_{ik} = 1 \\ &\Longleftrightarrow \sum_{k:kR_ij} y_{ik} = 1. \end{aligned} \tag{**}$$

Suppose x is in the core. Then, there exist prices $(p_i)_{i\in N}$ that satisfy (4) and (*). We verify that (5) holds. Let $(i,j) \in E$. If (i, j) is not dominated by x, then (5) follows immediately from (*). Suppose (i, j) is dominated by x. From (**), $\sum_{k:kR_ij} y_{ik} = 1$. Hence,

$$p_i + 1 \leq n + 1 \leq p_j + n = p_j + n \cdot \sum_{k:kR_ij} y_{ik}.$$

Suppose that there exist prices $(p_i)_{i\in N}$ such that (4) and (5) hold. We verify that (*) holds. Let $(i,j) \in E$ and suppose it is not dominated by x. From (**), $\sum_{k:kR_ij} y_{ik} = 0$. Hence, from (5), $p_i + 1 \leq p_j + n \cdot 0$, that is, $p_i < p_j$. □

4.1.2. IP for the Set of Competitive Allocations (Wako-Core). The set of competitive allocations is characterized by Constraints (1)–(5) together with (6):

$$p_i \leq p_j + n \cdot (1 - y_{ij}) \qquad \forall (i,j) \in E. \tag{6}$$

Proposition 4. *Let x be an allocation. Let y be the corresponding vector that satisfies (1)–(3). Allocation x is competitive if and only if there exist prices $(p_i)_{i\in N}$ such that (4)–(6) hold. Moreover, if such prices exist, then, together with x, they constitute a competitive equilibrium.*

Proof. Suppose x is competitive. Let $(p_i)_{i\in N}$ be prices such that (x, p) is a competitive equilibrium. Then, (4) and (*) hold. From the first part of the proof of Proposition 3, it follows that (5) holds. We now prove that (6) holds as well. Let $(i,j) \in E$. If $y_{ij} = 0$, then, immediately, $p_i \leq p_j + n = p_j + n \cdot (1 - y_{ij})$. If $y_{ij} = 1$, then $x_i = j$, and because (x, p) is a competitive equilibrium, it follows from Remark 1 that $p_i = p_{x_i} = p_j$.

Suppose that there exist prices $(p_i)_{i\in N}$ such that (4)–(6) hold. We verify that (x, p) is a competitive equilibrium. First, it follows from (6) that, for each $i \in N$, taking $j = x_i$ yields $p_i \leq p_{x_i} + n \cdot (1 - 1) = p_{x_i}$, that is, $p_i \leq p_{x_i}$. Hence, from Remark 1, for each $i \in N$, $p_i = p_{x_i}$. Second, let $j \in N$ be an object such that $j\, P_i\, x_i$. Then, $(i,j) \in E$ is not dominated by x. From the second part of the proof of Proposition 3, it follows that (*) holds. Hence, we obtain $p_i < p_j$. □

4.1.3. IP for the Strong Core. The strong core is characterized by Constraints (1)–(6) together with (7):

$$p_i \leq p_j + n \cdot \left(\sum_{k:kP_ij} y_{ik} \right) \qquad \forall (i,j) \in E. \tag{7}$$

Proposition 5. *Let x be an allocation. Let y be the corresponding vector that satisfies (1)–(3). Allocation x is in the strong core if and only if there exist prices $(p_i)_{i\in N}$ such that (4)–(7) hold. Moreover, if such prices exist, then, together with x, they constitute a competitive equilibrium.*

Proof. Suppose x is in the strong core. By Remark 2, x can be obtained in the Quint–Wako algorithm by choosing, for each absorbing set in the algorithm, a particular cycle cover. Hence, there exist price $(p_i)_{i\in N}$ such that (i) Constraints (4) are satisfied, (ii) all objects in the same absorbing set have the same price, and (iii) an absorbing set that is processed earlier by the algorithm has a strictly higher associated price (of its objects). It is easy to verify that (x, p) is a competitive allocation. Hence, from the first part of the proof of Proposition 4, it follows that (5) and (6) hold. Finally, to see that (7) holds, note that, from the definition of the prices, it follows that (i) if jR_ix_i, then $p_i \leq p_j$, and (ii) if x_iP_ij, then $p_i \leq n = n(\sum_{k:kP_ij} y_{ik})$.

Suppose that there exist prices $(p_i)_{i\in N}$ such that (4)–(7) hold. It follows from Proposition 4 that (x, p) is a competitive equilibrium. We prove that x is a strong core allocation. Suppose there is a coalition S that weakly blocks x through an allocation z. From Lemma 1, it follows that we can assume, without loss of generality, that $S = \{1, \dots, r\}$ and that, for each $i = 1, \dots, r-1$, $z_i = i+1$, $z_r = 1$, and $z_1 P_1 x_1$. Because x is individually rational, $r > 1$. Because (x, p) is a competitive equilibrium, $p_1 < p_2$. Because $3 = z_2 R_2 x_2$, we have $\sum_{k:kP_2 3} y_{2k} = 0$. Hence, from (7),

$$p_2 \le p_3 + n \cdot \left(\sum_{k:kP_2 3} y_{2k} \right) = p_3.$$

So $p_2 \le p_3$. By repeatedly applying the same arguments, we find $p_2 \le p_3 \le \dots \le p_r \le p_1$. Because $p_1 < p_2$, we obtain a contradiction. Therefore, there is no coalition that weakly blocks x. Hence, x is a strong core allocation. □

Remark 7. We note that, in the case of strict preferences, Constraints (7) are satisfied by any competitive equilibrium (x, p). To see this note that, if $y_{ij} = 1$, then (6) implies (7) because $1 - y_{ij} = 0$, and hence,

$$p_i \le p_j + n \cdot (1 - y_{ij}) = p_j \le p_j + n \cdot \left(\sum_{k:kP_i j} y_{ik} \right).$$

Otherwise, if $y_{ij} = 0$, then (5) implies (7) because, for strict preferences $\sum_{k:kP_i j} y_{ik} = \sum_{k:kP_i j} y_{ik} + y_{ij} = \sum_{k:kR_i j} y_{ik}$, and hence,

$$p_i < p_i + 1 \le p_j + n \cdot \left(\sum_{k:kR_i j} y_{ik} \right) = p_j + n \cdot \left(\sum_{k:kP_i j} y_{ik} \right).$$

Therefore, in either case, Constraints (7) are satisfied. This reflects the fact that, for strict preferences, the strong core is a singleton that consists of the unique competitive allocation. □

4.2. Quint and Wako's IP Formulations

To compare our IP formulations with the IP formulations for the core and strong core given by Quint and Wako [39], we describe the latter IP formulations using our notation.

First, for both the core and strong core, Quint and Wako [39] use the basic Constraints (1)–(3). We refer to their equations (9.2)–(9.4) as well as (8.2)–(8.4), together with an integrality condition.

Next, to obtain the core, Quint and Wako [39] impose the following additional no-blocking condition (see Quint and Wako [40, equation (9.1)]):

$$\sum_{i\in S} \left(\sum_{j:jR_i \pi_i} y_{ij} \right) \ge 1 \qquad \forall S \subseteq N, \pi \in \Pi_S. \tag{8}$$

Finally, to obtain the strong core, Quint and Wako [39] impose the following additional no-blocking condition (see Quint and Wako [39, equation (8.1)):

$$\sum_{i\in S} \left(\sum_{j:jP_i \pi_i} y_{ij} + \frac{1}{|S|} \sum_{j:jI_i \pi_i} y_{ij} \right) \ge 1 \qquad \forall S \subseteq N, \pi \in \Pi_S, \tag{9}$$

where Π_S is the set of allocations in the submarket M_S (so that π is an allocation in M_S).

Constraints (8) and (9) directly describe that no coalition S can block/weakly block through an allocation π, respectively. Both sets of constraints are highly exponential (in the number of agents) because they are required not only for all subsets S of N, but also for all possible redistributions within each S.

4.2.1. Alternative Cycle Formulations. In view of Lemma 1, it is sufficient to impose Constraints (8) and (9) for the cycles of the acceptability graph G. Based on this observation and results in Klimentova et al. [28], we describe alternative cycle formulations for the core and strong core. Furthermore, we provide a new proposition and IP formulation for the Wako-core.

Let $M = (N, R)$ be a housing market. Let $\mathcal{K}$ denote the set of exchange cycles in $G(N, R)$. For a cycle $c \in \mathcal{K}$, let $N(c)$ and $A(c)$ denote the set of nodes and edges in c, respectively, and let $|c|$ denote the size/length of c. We write $c_i = j$ if agent i receives object j in the exchange cycle c, that is, $(i, j) \in A(c)$.

Proposition 6 (Klimentova et al. [28]). *An allocation x is in the core if and only if, for each cycle $c \in \mathcal{K}$, for some agent $i \in N(c)$, $x_i R_i c_i$.*

The corresponding IP constraints, which reduce Constraints (8) to cycles, are as follows:

$$\sum_{(i,j)\in A(c)} \sum_{k:kR_ij} y_{ik} \geq 1 \qquad \forall c \in \mathcal{K}. \tag{10}$$

Next, we describe the alternative cycle formulation for the strong core. First, we focus on the special case of strict preferences.

Proposition 7 (Klimentova et al. [28]). *Suppose preferences are strict. Then, an allocation x is in the strong core if and only if, for each cycle $c \in \mathcal{K}$, c is an exchange cycle in x or for some agent $i \in N(c)$, $x_i P_i c_i$.*

Proposition 7 leads to the following constraints:

$$\sum_{(i,j)\in A(c)} y_{ij} + |c| \cdot \left[\sum_{(i,j)\in A(c)} \sum_{k:kP_ij} y_{ik}\right] \geq |c| \qquad \forall c \in \mathcal{K}. \tag{11}$$

The alternative cycle formulation for the strong core in the general case (in which preferences can have ties) is as follows.

Proposition 8 (Klimentova et al. [28]). *An allocation x is in the strong core if and only if, for each cycle $c \in \mathcal{K}$,*

(i) *c is an exchange cycle in x, or*
(ii) *For some agent $i \in N(c)$, $x_i P_i c_i$, or*
(iii) *For each agent $i \in N(c)$, $c_i I_i x_i$.*

The corresponding IP constraints, which reduce Constraints (9) to cycles, are as follows:

$$\sum_{(i,j)\in A(c)} \sum_{k:kI_ij} y_{ik} + |c| \cdot \left[\sum_{(i,j)\in A(c)} \sum_{k:kP_ij} y_{ik}\right] \geq |c| \qquad \forall c \in \mathcal{K}. \tag{12}$$

Finally, similarly to the core and strong core, we provide a new alternative characterization for the Wako-core.

Proposition 9. *An allocation x is in the Wako-core if and only if, for each cycle $c \in \mathcal{K}$,*

(i) *c is an exchange cycle in x, or*
(ii) *For some agent $i \in N(c)$, $x_i P_i c_i$, or*
(iii) *For some agent $i \in N(c)$, $c_i I_i x_i$ and $c_i \neq x_i$.*

The proof of Proposition 9 is omitted as it can be shown in a similar way as Proposition 8 (see Klimentova et al. [28]). Proposition 9 leads to the following constraints, which can be used to find competitive allocations (i.e., allocations in the Wako-core):

$$\sum_{(i,j)\in A(c)} y_{ij} + |c| \cdot \left[\sum_{(i,j)\in A(c)} \sum_{k:kR_ij,\,k\neq j} y_{ik}\right] \geq |c| \qquad \forall c \in \mathcal{K}. \tag{13}$$

To see the correctness of this new formulation, observe that the first term of (13) is equal to $|c|$ if condition (i) of Proposition 9 is satisfied and less than $|c|$ otherwise and the second term has value at least $|c|$ if condition (ii) or (iii) of Proposition 9 is satisfied and zero otherwise. Therefore, Constraint (13) is satisfied if and only if at least one of the three conditions of Proposition 9 holds.

4.3. Bounded Length Exchange Cycles

Note that the preceding cycle formulations are not very practical because of the exponentially large number of cycles. In fact, this justifies the novel IP formulations proposed in Section 4.1. However, the cycle formulations are practical for the case of bounded length exchange cycles.

One easily verifies that Lemma 1 can be extended to bounded length exchange cycles in a natural way: the strong core, Wako-core, and core of a k-housing market can be defined equivalently by the absence of corresponding blocking cycles of size at most k. In fact, Klimentova et al. [28] provide IP formulations for the core and strong core by adapting Constraints (10) and (12) to bounded exchange cycles. One can similarly adapt Constraints (13) to obtain an

IP formulation for the Wako-core of a k-housing market. In our simulations, we use the most efficient cycle-edge formulations by Klimentova et al. (see the detailed description in Klimentova et al. [28, section 3.3]).

5. Computational Experiments

This section is dedicated to computer simulations that use the IP formulations proposed in Section 4 and Klimentova et al. [28]. The main objective is the comparison of different solution concepts, in particular, with respect to the respecting improvement property. The simulations for strict and weak preferences were conducted separately, especially in view of our theoretical findings in Section 3.

Throughout, we consider two objective functions, namely, (1) the maximization of the size of the allocation (corresponding to the maximization of the number of transplants in the context of KEPs) and (2) the maximization of the total weight (weights of edges can be interpreted as the scores given to the corresponding transplants in a KEP, i.e., reflecting the quality of the transplants). Note that we use and distinguish between the two objectives for each of the cores as well as, in each of the cores, the allocations that yield a maximum number of transplants may be different from the allocations that yield maximum total weight.

The remainder of this section is organized as follows. In Section 5.1, we provide an overview of the test instances used for computational analysis and discuss the most relevant implementation details. In Section 5.2, we present our results on the frequency of violations of the (conditional) respecting improvement property for the best allotments with respect to all models. One important finding is that the strong core, Wako-core, and core perform much better than the size and weight maximization models. Then, to analyze the potential trade-off between stability requirements and size/weight, we study in Section 5.3 the reduction in size/weight of maximum size/weight allocations when ever more stringent stability (no blocking) requirements are imposed, that is, moving to core, then to competitive/Wako-core, and finally to strong core allocations. For the unbounded case, we furthermore analyze the price of fairness. Finally, as a counterpart to the analysis in Section 5.3, Section 5.4 computes for each model the average number of weakly blocking cycles. Thus, we obtain an estimation of how much robustness/fairness we have to give up vis-à-vis the strong core.

5.1. Test Instances and Implementation Details

Test instances were generated with the generator proposed in Santos et al. [45] and Klimentova et al. [27] to mimic the pools observed in KEPs and are available from https://doi.org/10.25747/xh4y-2r05. The generator creates compatibility (acceptability) graphs for KEPs with the set of agents N consisting of incompatible pairs and nondirected donors (NDDs), that is, donors with no associated recipient. Dummy edges were created from an NDD to each node to handle chains initiated by NDDs in the same way as cycles are operated. Thus, the preferences of the NDDs represent the interest of the patients on the waiting list. The size of an instance (i.e., number of agents/nodes $|N|$) ranged from 20 to 150; 50 instances of each size were generated. The weights associated with the edges of the graph were generated randomly within the interval (0, 1), and preferences were assigned in accordance with the weights: the higher the weight of an outgoing edge of a given node, the more preferred the corresponding (pointed) object for the (pointing) agent is. To generate instances with weak preferences, outgoing edges with weights within each interval of length $\frac{1}{|N|}$ were considered equally preferable.

For unbounded length exchange cycles, in order to speed up the running time of the IP formulations, we implemented the TTC algorithm and used its output as a starting allocation for all models. Even if this starting allocation was infeasible for the IP formulation (which can happen for the strong core) it was accepted by the solver.

All programs were implemented using Python programming language and tested using Gurobi as the optimization solver (Gurobi Optimization [24]). The code is available at https://gitlab.com/xenia.klimentova/housemarket_pub. Tests were executed on a MacMini 8 running macOS version 10.14.3 in an Intel Core i7 CPU with six cores at 3.2 GHz and 8 GB of RAM. Average CPU times required to solve an instance of a given size for each of the formulations in Section 4 are presented in Appendix C.

5.2. Violations of the Respecting Improvement Property

In this section, we conduct a computational analysis on how often the (conditional) RI property is violated for the best allotments under different models for both unbounded and bounded exchange cycles. For the unbounded case, we considered only the size and weight maximization models, that is, not the strong core, Wako-core, and core. The reason is that Theorem 1 and Corollary 2 show that the strong core satisfies the (conditional) RI-best property; Theorem 1 and Corollary 1 show that the Wako-core (competitive allocations) satisfies the RI-best property; and Schlotter et al. [46] prove that the core satisfies the RI-best property (for weak preferences).

For each model and instances with 20 and 30 nodes, we run Algorithm 1 to determine the number of violations of the (conditional) RI-best property. The algorithm proceeds as follows. For each pair of distinct agents i and j and starting from the original preferences, we let object i make consecutive improvements by moving it up in the preference list of agent j (until it is at the top). Specifically, let k be the lowest (least preferred) object that agent j strictly prefers to i. In the case of strict preferences, at each step of the **while** loop, object i is swapped with object k. In the case of ties (weak preferences), object i first becomes tied with (equally preferred to) object k. After each such improvement, the allocations (for the model under consideration) that provide the best allotments for i for the original (R) and improved ($\tilde{R}$) preferences are compared. If such an allocation does not exist for $\tilde{R}$, the algorithm continues with the next iteration of the **while** loop. If such an allocation does exist for $\tilde{R}$, then we check whether there is a violation of the RI-best property (i.e., whether agent i obtains a strictly worse allotment in the allocation for $\tilde{R}$).

In the formal description of the algorithm, we use the following definition and notation. For any agent i and any preferences R_i, we define for each object ℓ a rank $r^i_\ell \in \{1, \dots, |N|\}$ such that, for all objects ℓ, ℓ', we have $r^i_\ell \leq (<, =) r^i_{\ell'}$ if and only if $\ell R_i (P_i, I_i) \ell'$. In other words, objects with a smaller rank are more preferred.

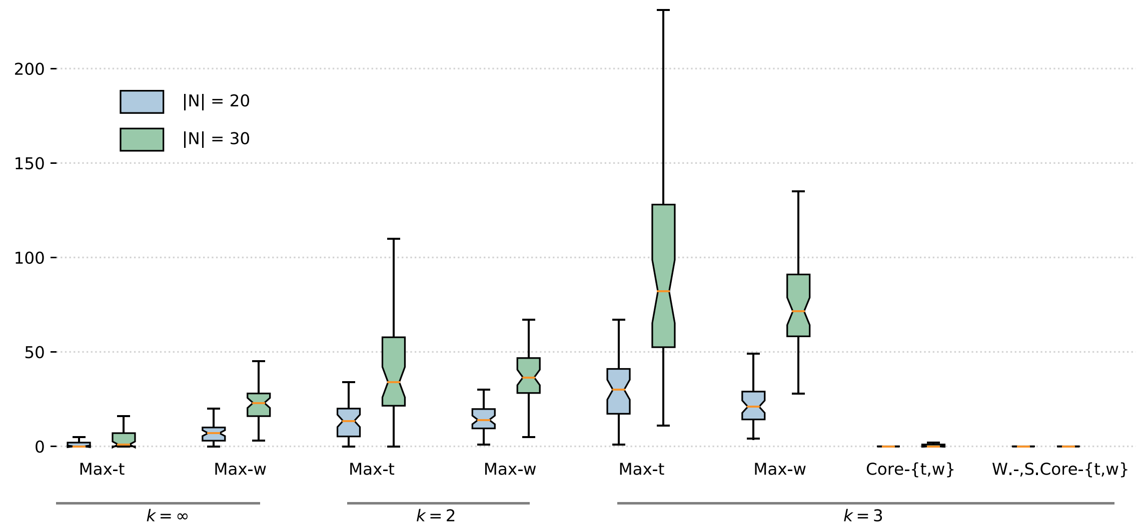
Algorithm 1 (Procedure for Checking the RI-Best Property)

Ensure: M number of violations of the RI-best property
$M \leftarrow 0$;
for $i \in N, j \in N, i \neq j$ **do**
  Let R be the current preferences of agents;
  Find an allocation with a best allotment for i with respect to R, denote the allocation by y;
  For $y_{i\ell} = 1$, denote $r = r^i_\ell$;
  while $\exists k P_j i$ **do**
    Let k be the first strictly preferred object for j that precedes i in R_j;
    if strict preferences **then**
      Swap i with k in the preferences of j;
    end if
    if weak preferences **then**
      Let i become equally preferred for j as k (i.e., $r^j_i \leftarrow r^j_k$);
    end if
    Denote the modified preferences by $\tilde{R}$;
    Find an allocation with a best allotment for i with respect to $\tilde{R}$, denote allocation by $\tilde{y}$;
    if core/Wako-core/strong core is empty **then**
      continue;
    end if
    For $\tilde{y}_{i\tilde{\ell}} = 1$, denote $\tilde{r} = r^i_{\tilde{\ell}}$;
    if $r < \tilde{r}$ **then**
      The RI-best property is violated: $M \leftarrow M + 1$;
    end if
    $r \leftarrow \tilde{r}$; $R \leftarrow \tilde{R}$;
  end while
end for

Figures 9 and 10 present box plots (in which outliers are omitted) for the number of violations of the RI-best property for strict and weak preferences, respectively, for models in which the RI-best property is violated at least once. Max-w refers to maximum weight allocations, and Max-t to allocations with maximum size, that is, maximum number of transplants. Similarly, Core-{t, w}, W.-Core-{t, w}, and S.Core-{t, w}, refer to the core, Wako-core and strong core, respectively.[25] Models that lead to the same result, independently of the considered objective, are plotted together. This is the case, for example, for Core-t and Core-w with $k=3$ and strict preferences (see Figure 9) and W.-Core and Core with $k=2, 3$ and weak preferences (see Figure 10).

It can be immediately observed that the (Wako-, strong) core models produced only a few cases of violations of the RI-best property. To give an indication, the total number of violations for all instances with weak preferences, $|N| = 30$, and $k=3$ was 4,549 for Max-t and 3,145 for Max-w but only 10 for Core-{t, w}, 20 for W.-Core-{t, w}, and 2 for S.Core-{t, w}. For maximum size and maximum weight allocations (Max-t and Max-w, respectively) for both the unbounded and bounded cases, one can observe a significant number of violations. These numbers increase with instance size. Interestingly, for the unbounded case, the number of violations for Max-t was lower than that for

Figure 9. (Color online) Total number of violations of the RI-best property of all instances of a given size with strict preferences.



Max-w. This can be explained by the fact that the former (size objective) problem usually induces multiple allocations that yield the same allotment for some agent, whereas the latter (weighted objective) problem usually induces a unique allocation. On the contrary, for the bounded case, maximum weight allocations tend to violate the RI-best property less often than maximum size allocations. A further inspection of the data shows that this difference between the unbounded and bounded case is mostly because of a higher number of violations for Max-t in the bounded case; the number of violations for Max-w is rather constant.[26]

5.3. Impact of Stability on the Number of Transplants

The most important finding in Section 5.2 is that, in terms of respecting improvement, the strong core, Wako-core, and core perform much better than the size and weight maximization models. Next, we analyze the potential trade-off between stability requirements and size/weight.

Focusing on the case of unbounded exchange cycles and weak preferences, Figure 11 depicts average maximum size and weight when increasingly stringent stability requirements are imposed. Starting off with no stability requirements (Max), we consecutively add the constraints required for core, competitive, and strong core allocations. We refrain ourselves from presenting the results for the case of strict preferences as all curves are similar (also recall that, for strict preferences, the competitive and strong core allocations coincide).

Figure 10. (Color online) Total number of violations of the RI-best property of all instances of a given size with weak preferences.

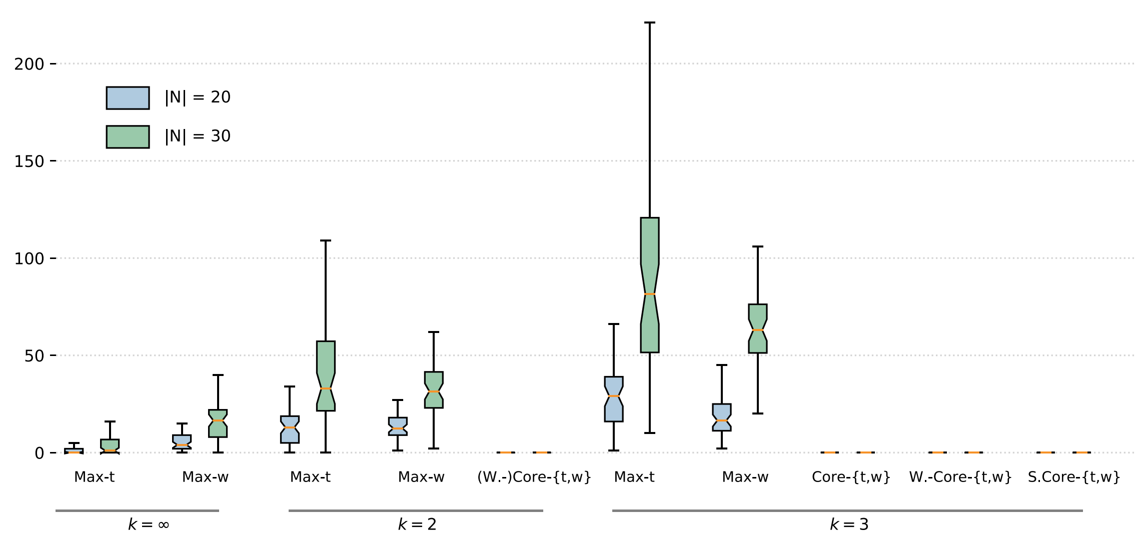
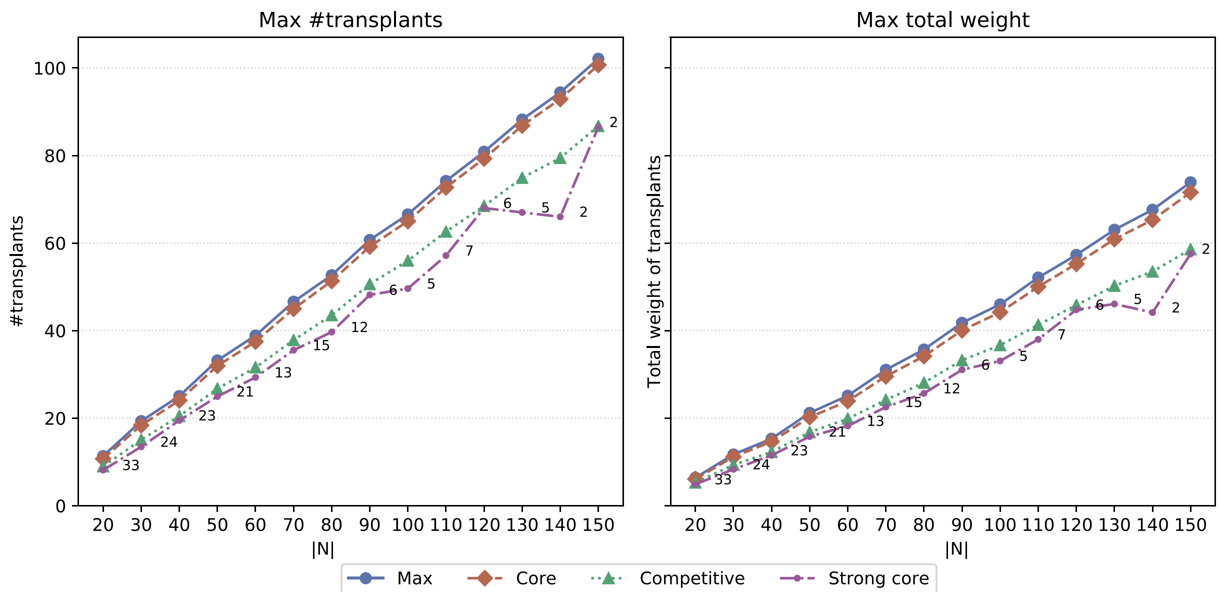


Figure 11. (Color online) Average number of transplants (left) and average total weight of transplants (right) for unbounded length and weak preferences. Each number indicates the number of instances (out of 50) for which a strong core allocation existed.



As expected, both the number of transplants and total weight decrease by increasing the number of constraints: when moving from Max to Core, then to Competitive, and finally to StrongCore, the corresponding curves shift downward. The StrongCore curve is nonmonotonic, which is explained by the nonexistence of strong core allocations for several instances. Next to the curve, we indicate the number of instances (out of the total 50) for which a strong allocation existed.

Figure 12 makes a similar analysis for the bounded case when $k = 2$ and $k = 3$, indicated as [k] next to the name of the curves. If the core, Wako-core, or strong core turned out to be empty, then we computed an allocation that minimizes the number of associated blocking cycles in the same way as described in Klimentova et al. [28].[27]

Figure 12. (Color online) Comparison of the average number of transplants (left) and the average total weight of transplants (right) for bounded length exchange cycles ($k = 2, 3$, indicated as [k] next to the name of the curves) and weak preferences. Solid lines are used for the unbounded case ([∞]), dotted lines for $k = 2$, and dashed lines for $k = 3$.

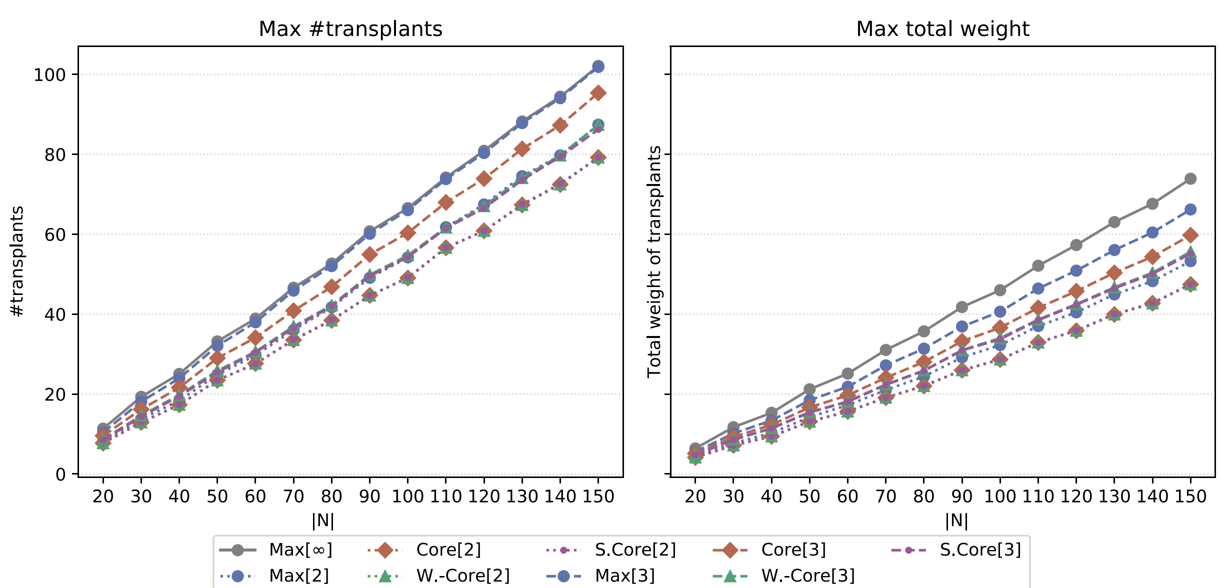
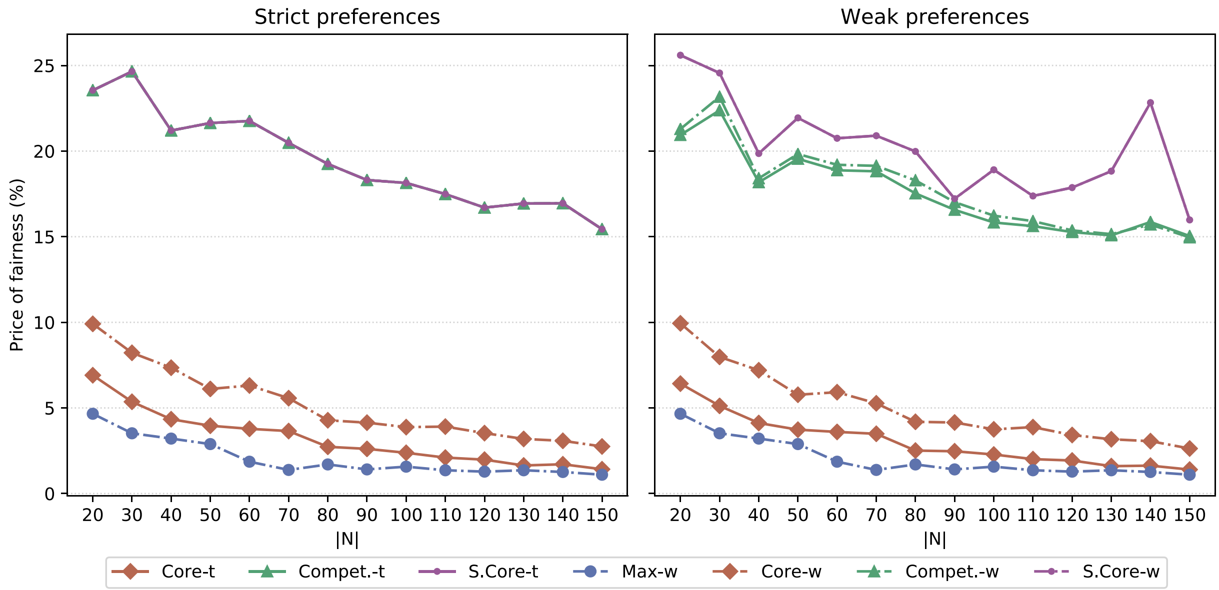


Figure 13. (Color online) Price of fairness with respect to the maximum number of transplants for maximum weight allocations as well as core, competitive, and strong core allocations with maximum number of transplants (-t) and maximum total weight (-w) objectives for strict (left) and weak (right) preferences and unbounded exchange cycles. Solid lines are used for models that maximize the number of transplants and dash-dotted lines for those that maximize the total weight.



To facilitate the comparison between the bounded and unbounded cases, Figure 12 also contains the two curves of the unbounded case from Figure 11 associated with maximum size/weight (Max), denoted by Max[∞], which provide upper bounds. Unsurprisingly, the curves associated with $k=2$ are located below those associated with $k=3$. We can observe that the maximum number of transplants for $k=3$ and unbounded k are very similar (see Figure 12 (left)). Notice also that, even though some curves overlap and seem identical, there are minor differences among them except for the case $k=2$ in which the core and Wako-core coincide. As before, we present results for weak preferences only as this is the more general case. In the case of strict preferences, for $k=3$, the curves are similar, whereas for $k=2$, the core, Wako-core, and strong core coincide.

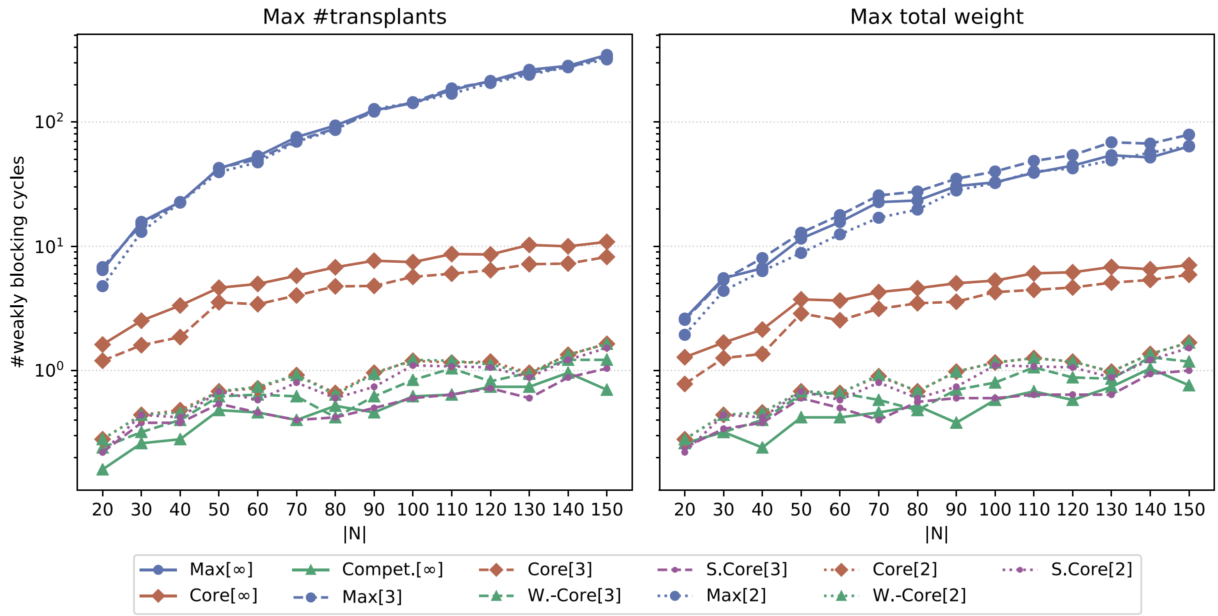
From a practical point of view, an interesting question to explore is the impact of (core) stability requirements on the achievable number of transplants. Although KEPs have many other key performance indicators, the achievable number of transplants is unarguably the most relevant one, as this criterion is optimized as a first objective in all European KEPs (Biró et al. [17]). Figure 13 depicts our findings on the price of fairness for unbounded exchange cycles. The price of fairness is calculated as the average percentage loss in the number of transplants for maximum weight allocations as well as for core, competitive, and strong core allocations under both objectives when compared with the maximum number of transplants achievable. Because the strong core can be empty for weak preferences, the corresponding curves in Figure 13 (right) are based on the instances (out of the 50 instances of each size) with a nonempty strong core. Note that for strict preferences (Figure 13, left) there exists a unique competitive equilibrium, which is also the unique strong core allocation. Therefore, the curves that correspond to the two objectives and both types of allocations (Compet.-t, Compet.-w, S.Core-t, and S.Core-w) coincide. For weak preferences (Figure 13, right), even though there may exist multiple strong core allocations, for all instances in our simulations, the number of transplants turns out to be the same for the two objectives. So the corresponding curves S.Core-t and S.Core-w coincide again.

As can be observed, the price of fairness for competitive and strong core allocations is significantly higher than for core allocations. It decreases with problem size for all allocation models and both objective functions. In particular, for the core with the maximum number of transplants objective (Core-t), when the size of instances is larger than 50, the loss in the number of transplants is less than 3% (decreasing to 1% for instances of size 150). This finding is of major practical relevance as it implies that when kidney exchange programs are sufficiently large, one can take into account preferences without a significant reduction in the number of transplants.

5.4. Analysis of the Number of Blocking Cycles

Finally, as a counterpart to the analysis in Section 5.3, we compute for each model the average number of weakly blocking cycles. Thus, we obtain an estimation of how much deficiency in terms of robustness/fairness we have to accept vis-à-vis the ideal (but potentially empty) strong core.

Figure 14. (Color online) Average number of weakly blocking cycles of length $l = 2$ for allocations with maximum number of transplants (left) and maximum total weight of transplants (right) for unbounded exchange cycles ([∞]) and exchange cycles of length up to $k = 2$ and $k = 3$ for weak preferences. Solid lines are used for the unbounded case, dotted lines for $k = 2$, and dashed lines for $k = 3$.



Specifically, we analyze the average number of weakly blocking cycles when their length can be up to $l = 2, 3, 4, 5$. When also the length of exchange cycles of allocations is bounded, say by k, the analysis is naturally restricted to the case $l \leq k$. Results on blocking cycles are very similar and, hence, omitted.

Figure 14 shows the average number of weakly blocking cycles of length $l = 2$ for Max, core, competitive/Wako-core, and strong core allocations. Figures for $l = 3$, 4, 5 are relegated to Appendix B as the conclusions drawn for these cases are similar to those for $l = 2$. If the core, Wako-core, or strong core turned out to be empty, then we computed an allocation that minimizes the number of associated blocking cycles in the same way as described in Klimentova et al. [28]. In particular, for the strong core in the case of bounded exchange cycles, following the same procedure as in Klimentova et al. [28], the corresponding two curves ($k = 2$, 3) are based on counting the minimum number of weakly blocking cycles[28] (hence, we register zero weakly blocking cycles if and only if an instance has a nonempty strong core). In the case of unbounded exchange cycles, the structure of the formulation is such that it prevents us from efficiently minimizing the number of weakly blocking cycles for the instances with an empty strong core. For that reason, the corresponding strong core curve is omitted altogether from our analysis.

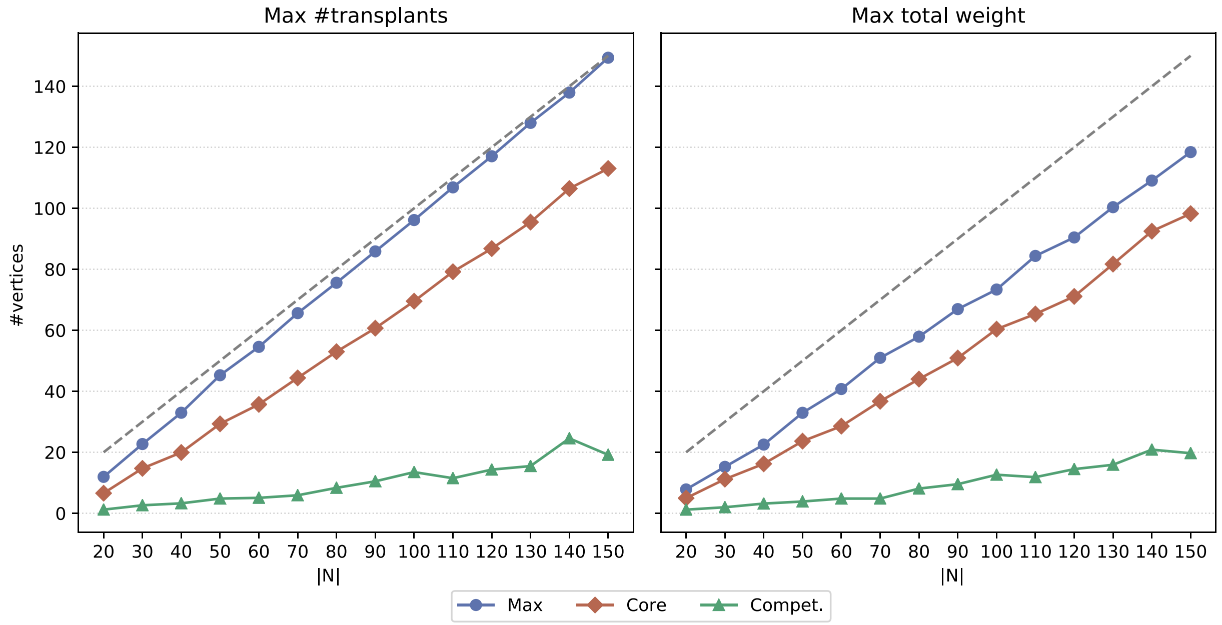
Interestingly, the "unstability" of the allocations that maximize the number of transplants (curves Max[∞], Max[2], Max[3] in Figure 14 (left)) barely depends on the maximum allowed length of exchange cycles. This is not true for the Core: the number of weakly blocking cycles is considerably smaller for $k = 2$. For this and all the remaining cases, the average number of weakly blocking cycles is very low—in most cases below one. It is worth noting that the average number of weakly blocking cycles tends to be smaller when the objective is to maximize the total weight (Figure 14 (right)). A possible explanation for this is that weights reflect patients' preferences, and therefore, an objective function that takes into account weights tends to create less weakly blocking cycles (which are determined by preferences).

Although these findings are already insightful, Figure 15 complements the analysis by focusing on the average number of agents that strictly prefer their allotments in at least one weakly blocking cycle (i.e., the number of patients that can receive a strictly better kidney). An important conclusion that can be drawn from the figure is that the maximization of total weight yields a lower number of agents that can obtain a better allotment in some weakly blocking cycle when compared with the maximum size allocations (compare curves Max in Figure 15 (left) and (right)). Comparing Figure 15 with Figure 13 gives insights into the reduction of the total number of transplants that would be necessary to meet a certain level of patients' preferences.

6. Conclusion

This paper advances the current state of the art in several lines of research on Shapley–Scarf housing markets. We prove that, in the case of strict preferences, the strong core (containing the unique competitive allocation) respects

Figure 15. (Color online) Weak preferences. Conditional on the existence of at least one weakly blocking cycle, average number of agents that receive a strictly better allotment in at least one weakly blocking cycle. The gray line is a reference line showing the number of nodes in an instance.



improvement. More importantly, we provide several extensions of the result to the case of weak preferences, for which there do not seem to exist parallel results in other matching models.

In a very recent paper, Schlotter et al. [46] tackle some of the questions that we leave open in our current paper. They prove that the core satisfies the RI-best property for unbounded exchanges and weak preferences (and also for a more general domain of partial orders) but that it violates the RI-worst property even for strict preferences. Similarly, they also show that the (strong, Wako-) core satisfies the conditional RI-best property for strict preferences.

We summarize our main theoretical findings and the additional results from Schlotter et al. [46] in Table 14.

We also contribute to the computation of the core, strong core, and set of competitive allocations by providing integer programming models that no longer involve an exponential number of constraints. These models assume that there is no limit on the size of an exchange cycle. However, because there are applications in which this assumption is unrealistic (for instance in kidney exchange programs), we also propose alternative IP models for bounded length cycles.

Finally, our new IP formulations constitute a practical stepping stone for our computational experiments, which provide several insights in the properties of allocation rules for kidney exchange programs. If a limit is set to the length of exchange cycles, then the proposed game-theoretical solutions need not satisfy the respecting improvement property. However, our computer simulation results show that violations of the property are remarkably less

Table 14. Summary of main theoretical results on the respecting improvement property.

	Housing market ($k = \infty$)	Roommates problem ($k = 2$)	$k = 3$
Strict preferences			
Strong core	RI (Theorem 1)	Cond. RI-best (Schlotter et al. [46]), no cond. RI-worst (Example 5)	No cond. RI (Proposition 2, Remark 5)
Wako-core	RI (Theorem 1)	Cond. RI-best (Schlotter et al. [46]), no cond. RI-worst (Example 5)	No cond. RI (Proposition 2, Remark 5)
Core	RI-best, no RI-worst (Schlotter et al. [46])	Cond. RI-best (Schlotter et al. [46]), no cond. RI-worst (Example 5)	No cond. RI (Proposition 2, Remark 5)
Weak preferences			
Strong core	Cond. RI (Theorem 3, Corollary 2)	No cond. RI-best (Schlotter et al. [46]), no cond. RI-worst (Example 5)	No cond. RI (Proposition 2, Remark 5)
Wako-core	RI-best/worst (Theorem 2, Corollary 1)	No cond. RI-best (Example 6), no cond. RI-worst (Example 5)	No cond. RI (Proposition 2, Remark 5)
Core	RI-best, no RI-worst (Schlotter et al. [46])	No cond. RI-best (Example 6), no cond. RI-worst (Example 5)	No cond. RI (Proposition 2, Remark 5)

frequent for the (Wako-, strong) core than for maximum size and weight allocations. In view of these findings, we analyze the potential trade-off between stability requirements and the maximum number of transplants. We find that, when the size of the instances increases, the trade-off decreases significantly: core allocations for instances with 150 patient–donor pairs entail a less than 1% reduction in the number of transplants. An important implication is that, when kidney exchange programs are sufficiently large, one can take into account agents' preferences and largely ensure the respecting improvement property without a significant reduction in the number of transplants.

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Appendix A. Alternative Proof of Theorem 1
We prove that, when preferences are strict, the competitive allocation rule (or strong core allocation rule) τ respects improvement by associating a two-sided school choice problem with each one-sided housing market and applying Hatfield et al. [25, online theorem 9].

We first provide some intuition/a sketch of the proof. There is a "standard" connection between the (classic) TTC for the housing market and the generalized TTC for the school choice model. Specifically, replace each agent i with a student–school pair (s_i, c_i), let each student inherit the preferences (of the corresponding agent) over the schools, and let each school have its student on the top of the priority list. It is well-known that the two top trading algorithms produce essentially the same outcome. Now consider a "reversed" construction, in which each student has the student's school as a top choice and each school inherits the preferences of the original corresponding agent as priorities. Again, the very same cycles are created in the TTC for this reversed school choice problem, only with the difference that now each student is assigned to the student's own school. The proof of Theorem 1 that is presented combines these two reductions by having the standard version for one agent only, say agent i, and the reversed version for all other agents. It is obvious that the very same cycles occur again as long as agent i is not involved. The key part of the proof is to show that, in the combined reduction, student s_i is assigned to the school that corresponds to the object agent i receives in the original housing market.

Formally, let $i \in N$. Let $R, \tilde{R}$ be two profiles of strict preferences over objects N such that $\tilde{R}$ is an improvement for i with respect to R.

Consider housing market (N, R). We construct an associated two-sided (school choice) problem $(S, C, R', \succ)$ with outside option $\emptyset$ as in Hatfield et al. [25] as follows. First, $S = \{s_k : k \in N\}$ is the set of students. Second, $C = \{c_k : k \in N\}$ is the set of schools, each of which has capacity one. Third, strict preferences $R' = (R'_{s_k})_{s_k \in S}$ and strict priority rankings $\succ = (\succ_{c_k})_{c_k \in C}$ satisfy the following conditions:

- Student s_i has strict preferences R'_{s_i} over schools C and the outside option $\emptyset$ such that, for all $k, l \in N$, $c_k R'_{s_i} c_l$ if and only if $k R_i l$ and for all $k \in N$, $c_k P'_{s_i} \emptyset$.
- For each $j \in N \backslash \{i\}$, student s_j has strict preferences R'_{s_j} over schools C and the outside option $\emptyset$ such that c_j is the most preferred school (and preferred to $\emptyset$).
- School c_i is endowed with a strict priority ranking $\succ_{c_i}$ over students S such that s_i is the agent with highest priority.
- For each $j \in N \backslash \{i\}$, school c_j is endowed with a strict priority ranking $\succ_{c_j}$ over students S such that, for all $k, l \in N$, $s_k \succ_{c_j} s_l$ if and only if $k R_j l$.

We similarly associate a problem $(S, C, R', \succ')$ with housing market $(N, \tilde{R})$ such that the only (possible) difference between problems $(S, C, R', \succ')$ and $(S, C, R', \succ)$ is that, for some $j \in N \backslash \{i\}$, $\succ_{c_j'} \neq \succ_{c_j}$. (This follows from the fact that that the only (possible) difference between the two housing markets $(N, \tilde{R})$ and (N, R) is that, for some $j \in N \backslash \{i\}$, $\tilde{R}_j \neq R_j$.)

Next, we relate the top trading cycles algorithm τ for housing markets (N, R) and $(N, \tilde{R})$ with the top trading cycles algorithm φ^{TTC} for the associated two-sided problems $(S, C, R', \succ)$ and $(S, C, R', \succ')$ (for the definition of φ^{TTC} we refer to Hatfield et al. [25, section 2.1.3]).

Claim A.1. Let $k \in N$. Then, $\varphi^{\text{TTC}}_{s_i}(S, C, R', \succ) = c_k$ if and only if $\tau_i(N, R) = k$. Similarly, $\varphi^{\text{TTC}}_{s_i}(S, C, R', \succ') = c_k$ if and only if $\tau_i(N, \tilde{R}) = k$.

The difference between $(S, C, R', \succ)$ and $(S, C, R', \succ')$ is that student s_i is ranked higher (i.e., has higher priority) by some schools at $(S, C, R', \succ')$ relative to $(S, C, R', \succ)$. Hatfield et al. [25, online theorem 9] states that the top trading cycles algorithm for two-sided problems respects improvements of student quality. Hence,

$$\varphi^{\text{TTC}}_{s_i}(S, C, R', \succ') \, R'_{s_i} \, \varphi^{\text{TTC}}_{s_i}(S, C, R', \succ). \tag{A.1}$$

Moreover, note that R'_{s_i} finds all schools acceptable, and at both $(S, C, R', \succ)$ and $(S, C, R', \succ')$, the number of school seats equals the number of students. Hence,

$$\varphi^{\text{TTC}}_{s_i}(S, C, R', \succ') \neq \emptyset \neq \varphi^{\text{TTC}}_{s_i}(S, C, R', \succ). \quad \text{(A.2)}$$

Hence, (A.1), (A.2), Claim A.1, and the definition of R'_{s_i} yield $\tau_i(N, \tilde{R})\, R_i\, \tau_i(N, R)$. So τ respects improvement.

Proof of Claim A.1. It is sufficient to show that

$$\text{for all } k \in N, \quad \varphi^{\text{TTC}}_{s_i}(S, C, R', \succ) = c_k \text{ if and only if } \tau_i(N, R) = k. \quad \text{(A.3)}$$

(The statement that $\varphi^{\text{TTC}}_{s_i}(S, C, R', \succ') = c_k$ if and only if $\tau_i(N, \tilde{R}) = k$ follows similarly.)

We apply TTC to two-sided problem $(S, C, R', \succ)$ as well as to housing market (N, R). We show that, as long as agent i (in the housing market) or, equivalently, student s_i and college c_i (in the two-sided problem) are present, at each step of the algorithm, there is a one-to-one correspondence between cycles of the two-sided problem and of the housing market.

Consider the initial situation. We distinguish among three types of cycles.

First, if (s_i, c_i) is a cycle at $(S, C, R', \succ)$, then c_i is student s_i's most preferred school, and hence, i is a self-cycle at (N, R). Similarly, if i is a self-cycle at (N, R), then (s_i, c_i) is a cycle at $(S, C, R', \succ)$. In particular, (A.3) holds.

Second, let $j \in N \backslash \{i\}$. If (s_j, c_j) is a cycle at $(S, C, R', \succ)$, then student s_j has highest priority at school c_j, and hence, j is a self-cycle at (N, R). Similarly, if j is a self-cycle at (N, R), then (s_j, c_j) is a cycle at $(S, C, R', \succ)$. Obviously, removing these cycles is equivalent to removing student s_j, school c_j, and agent j.

Third, let $c = (s_{i_1}, c_{i_2}, s_{i_3}, \ldots, c_{i_\ell})$ with $\ell > 2$ be a cycle at $(S, C, R', \succ)$. Note that ℓ is even and $c_i \notin \{c_{i_2}, c_{i_4}, \ldots, c_{i_\ell}\}$ (otherwise, we are in the case of cycle (s_i, c_i) because, at the initial step, student s_i is present, the only student that can point to c_i is student s_i, and school c_i points to s_i).

Case I: $s_i \in \{s_{i_1}, s_{i_3}, s_{i_5}, \ldots, s_{i_{\ell-1}}\}$. Without loss of generality, we can assume that $i_1 = i$. Then, at cycle c,

- Student $s_{i_1} = s_i$ points to the student's most preferred school c_{i_2}.
- School c_{i_2} points to the school's highest priority student s_{i_3}.
- $i_3 = i_4$ because student s_{i_3} points to school c_{i_4}, but c_{i_3} is the student's most preferred school (which is present at the initial step), that is, $c_{i_4} = c_{i_3}$, which implies that $i_3 = i_4$.
- School c_{i_4} points to the school's highest priority student s_{i_5}.
- $i_5 = i_6$ (because of a similar argument).
- …
- School $c_{i_{\ell-2}}$ points to the school's highest priority student $s_{i_{\ell-1}}$.
- $i_{\ell-1} = i_\ell$.
- School c_{i_ℓ} points to the school's highest priority student $s_{i_1} = s_i$.

Thus, $(i_1, i_2, i_4, i_6, \ldots, i_\ell)$ is a cycle at (N, R).

Case II: $s_i \notin \{s_{i_1}, s_{i_3}, s_{i_5}, \ldots, s_{i_{\ell-1}}\}$. Then, at cycle c,

- $i_1 = i_2$.
- School c_{i_2} points to the school's highest priority student s_{i_3}.
- $i_3 = i_4$;
- School c_{i_4} points to the school's highest priority student s_{i_5}.
- $i_5 = i_6$.
- …
- School $c_{i_{\ell-2}}$ points to the school's highest priority student $s_{i_{\ell-1}}$.
- $i_{\ell-1} = i_\ell$.
- School c_{i_ℓ} points to the school's highest priority student s_{i_1}.

Thus, $(i_2, i_4, i_6, \ldots, i_\ell)$ is a cycle at (N, R).

Reversely, if $(i_1 = i, i_2, i_4, i_6, \ldots, i_\ell)$ with $\ell > 2$ is a cycle at (N, R), then $c = (s_{i_1}, c_{i_2}, s_{i_4}, c_{i_4}, s_{i_6}, c_{i_6}, \ldots, s_{i_\ell}, c_{i_\ell})$ is a cycle at $(S, C, R', \succ)$ (case I). Similarly, if $(i_2, i_4, i_6, \ldots, i_\ell)$ with $\ell > 2$ is a cycle at (N, R) and $i \notin \{i_2, \ldots, i_\ell\}$, then $c = (s_{i_2}, c_{i_2}, s_{i_4}, c_{i_4}, \ldots, s_{i_\ell}, c_{i_\ell})$ is a cycle at $(S, C, R', \succ)$ (case II).

In case I, we obtain (A.3). In case II, removing the cycle at (N, R) and the associated cycle at $(S, C, R', \succ)$ is equivalent to removing students $s_{i_1}, s_{i_2}, \ldots, s_{i_\ell}$, schools $c_{i_1}, c_{i_2}, \ldots, c_{i_\ell}$, and agents $i_1, i_2, \ldots, i_\ell$.

We can repeatedly apply similar arguments (as in the three types of cycles) to the reduced two-sided problem and the reduced housing market, remove cycles, etc., until we obtain (A.3). □

Appendix B. Analysis of the Number of Weakly Blocking Cycles of Length 3, 4, 5

Figure B.1 extends the results presented in Figure 14 by considering weakly blocking cycles of length up to $l = 3$. The conclusions drawn for $l = 2$ remain valid for this case.

For the unbounded case, the number of weakly blocking cycles is larger because one must consider also the cases when $l > 3$. Figure B.2 provides information on the number of weakly blocking cycles of length up to 4 and 5 (indicated by suffixes _4 and _5). We do not present results for $l > 5$ as searching for these larger weakly blocking cycles would lead to excessively long CPU time.

Figure B.1. (Color online) Average number of weakly blocking cycles of length up to $l = 3$ for allocations with maximum number of transplants (left) and maximum total weight of transplants (right) for unbounded exchange cycles ([∞]) and exchange cycles of length up to $k = 3$ for weak preferences.

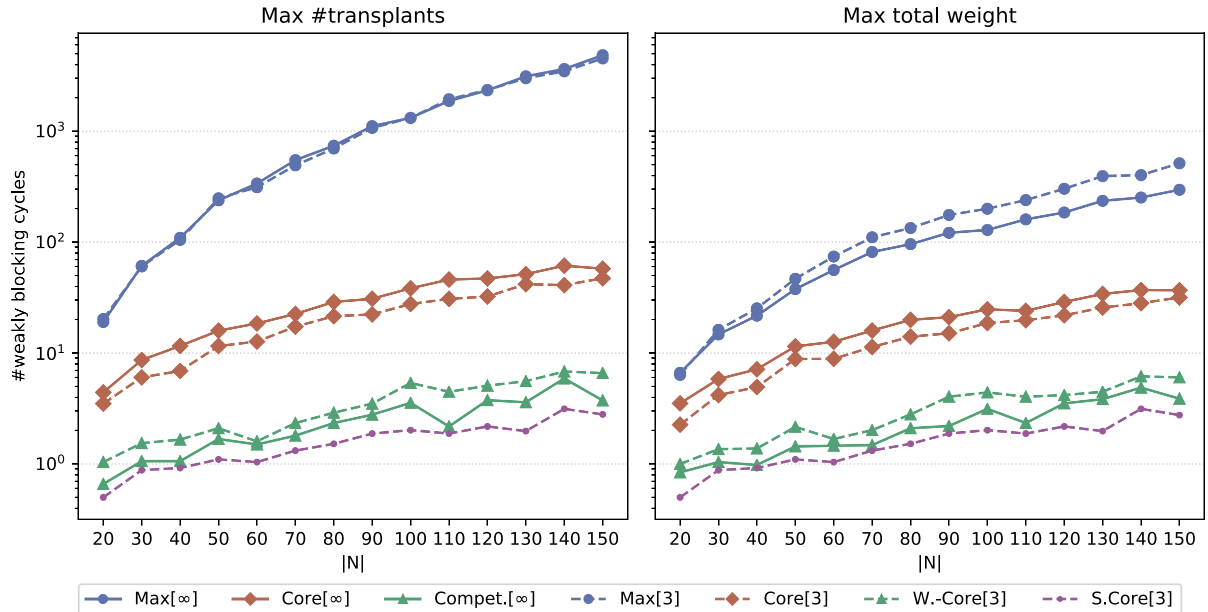
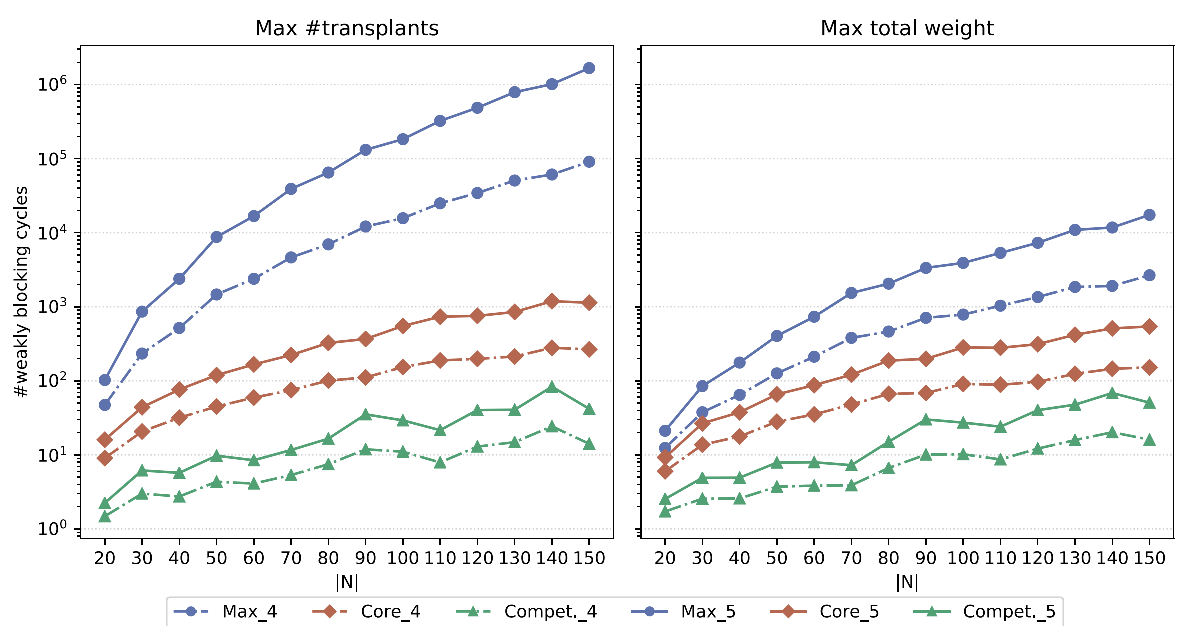


Figure B.2. (Color online) Average number of weakly blocking cycles of length up to $l = 4$ and $l = 5$, indicated as _l next to the name of a curve, for allocations with maximum number of transplants (left) and maximum total weight of transplants (right), for unbounded exchange cycles and weak preferences.



Appendix C. CPU Time for Unbounded Models

In Table C.1, we present the average CPU time for solving an instance of a given size with one of the three newly proposed IP models for the unbounded case. Recall that the allocation obtained by the TTC algorithm was used as a starting allocation for all formulations.

Table C.1. Average CPU time (in seconds) for solving an instance of a given size with the proposed formulation.

	Maximum number of transplants			Maximum total weight			Maximum number of transplants			Maximum total weight		
$\|N\|$	Core	Compet.	S. core	Core	Compet.	S. core	Core	Compet.	S. core	Core	Compet.	S. core
	Strict preferences						Weak preferences					
20	0.00	0.03	0.01	0.00	0.02	0.01	0.00	0.04	0.01	0.00	0.03	0.01
30	0.03	0.13	0.04	0.02	0.11	0.03	0.02	0.28	0.04	0.02	0.17	0.03
40	0.08	0.48	0.12	0.06	0.25	0.11	0.09	0.63	0.10	0.06	0.44	0.08
50	0.24	1.74	0.38	0.16	0.58	0.34	0.20	2.15	0.25	0.17	1.06	0.21
60	0.47	2.39	0.87	0.28	0.91	0.79	0.52	6.03	0.44	0.26	2.87	0.39
70	1.06	3.91	1.94	0.66	2.29	1.50	0.84	16.99	1.09	0.53	7.35	0.77
80	1.62	6.54	3.26	0.82	3.39	2.32	1.41	32.21	1.63	0.76	17.47	1.01
90	3.14	36.34	5.31	3.27	5.38	3.59	3.29	167.15	2.36	1.82	80.88	1.49
100	3.53	16.19	19.26	2.43	6.15	9.81	4.51	188.35	8.87	3.08	95.39	4.62
110	8.73	21.42	28.26	4.97	9.01	13.79	6.68	331.64	16.40	5.92	159.12	7.24
120	17.84	72.87	57.36	6.81	15.36	24.32	20.14	392.88	19.60	6.79	218.58	10.87
130	14.34	46.92	84.49	14.24	22.68	34.11	14.78	586.27	21.75	12.32	438.23	10.42
140	29.50	61.99	110.82	21.51	34.33	46.67	41.59	708.92	40.97	16.43	539.56	14.89
150	41.99	161.10	214.32	30.66	52.61	70.77	57.13	786.43	61.79	27.82	682.99	23.91

In the case of weak preferences, CPU times are much longer for the core and, especially, the competitive allocations. However, finding strong core allocations for weak preferences is faster than doing so for strict preferences. Moreover, surprisingly, finding the strong core is the most time-consuming task for strict preferences, whereas it is the least time-consuming task for weak preferences. Finally, we notice that the models for finding core and strong core allocations perform (with respect to CPU time) within the same ranges of magnitude compared with the corresponding models for the bounded case analyzed in Klimentova et al. [28].

Endnotes

1 Allowing for additional donors does not require an extension to a model in which agents can be endowed with multiple objects: an agent's set of donors can only be assigned to one other agent and this agent can only "consume" its most preferred element from the set.

2 The corresponding decision problem is NP-hard (Biró and McDermid [14], Huang [26]) even for tripartite graphs (also known as the cyclic 3-D stable matching problem (Ng and Hirschberg [35])).

3 In the literature on KEPs, it is often assumed that each edge has a weight representing the fit/quality of the donor's kidney for the receiving patient. The total weight of an exchange cycle is the sum of the weights associated with the edges involved in the exchange. The total weight of an allocation is the sum of the weights of its exchange cycles. Details are in Section 5.

4 In KEPs, all transplants in the same exchange cycle are usually carried out simultaneously. Obviously, if the number of surgical teams and operation rooms is small, some of the transplants have to be conducted in a nonsimultaneous way. In many countries, this "risky" solution is not allowed because of possible reneging (Biró et al. [16]). Thus, in practice, exchange cycles are usually bounded.

5 In other words, an agent can be indifferent between objects, including the agent's own endowment.

6 Therefore, when keeping the set of agents fixed, we interchangeably refer to R as the profile of preferences and the market.

7 Note that a (trading/exchange) cycle is a nonempty directed path in which only the first and last nodes are equal. A single node is a self-cycle, that is, a degenerate cycle.

8 In the literature, the core is sometimes called the weak core or regular core.

9 In the literature, the strong core is sometimes called the strict core.

10 Wako [51] shows that the strong core coincides with the set of competitive allocations if and only if any two competitive allocations are welfare equivalent. Hence, whenever the set of competitive allocations is a singleton, it coincides with the strong core.

11 This result and generalizations of it have appeared in the literature; see, for example, Biró [12, proposition 1.1.3]. We include a short, self-contained proof.

12 Throughout the paper, self-cycles are omitted from the acceptability graphs in the examples.

13 If preferences are not strict, then the top trading cycles algorithm is applied to the preference profiles that can be obtained by breaking ties in all possible ways.

14 Sotomayor [49, example 1], which is attributed to Jun Wako, is illustrative: $N = \{1,2,3\}$ with $2P_1 3P_1 1$, $1I_2 3P_2 2$, $2P_3 1P_3 3$. The set of competitive allocations consists of $x = \{(1,2),(3)\}$ and $x' = \{(1),(2,3)\}$, which are Pareto dominated by core allocations $\{(1,2,3)\}$ and $\{(1,3,2)\}$, respectively. Moreover, $x_1 P_1 x_{1'}$ and $x_{3'} P_3 x_3$. The strong core is empty.

15 We can relax the definition by allowing for differences in the relative ranking of unacceptable objects, but because we only study individually rational allocations, such a relaxation has no impact.

16 In other words, for each $R \in \mathscr{R}$, $\Phi(R) \neq \emptyset$.

17 We are very grateful to the reviewer for suggesting the alternative proof.

[18] It follows from Fact 2 that, in this case, j is removed from the market before i.

[19] It follows from Fact 2 that, in this case, i is removed from the market before j.

[20] In other words, there is no edge from a node in the absorbing set to a node outside the absorbing set.

[21] It follows from Fact 2^* that, in this case, j was removed from the market before i.

[22] It follows from Fact 2^* that, in this case, i was removed from the market before j.

[23] For the core and strong core, see also Biró and McDermid [14]. In view of Wako's [52] result, we similarly adjust the set of competitive allocations by using the (equivalent) Wako-core.

[24] So the core coincides with the set of stable matchings of the corresponding roommate problem (Gale and Shapley [23]).

[25] Recall that we use and distinguish between the objectives/suffixes t and w for each of the cores as well as, in each of the cores, the allocations that yield a maximum number of transplants may be different from the allocations that yield maximum total weight.

[26] We conjecture that this is related to the impact of objects that move up from being unacceptable to being acceptable. In the bounded case, the potential new trading cycles induced by a new acceptable object can easily increase the maximum size allocation but less easily the maximum weight allocation. Because a change of allocation brings along possibilities of violations of the RI-best property, Max-t is more likely to experience an increase.

[27] Note that, in the IP formulations, the stability requirements are written for each cycle; hence, the minimization of the number of blocking cycles is equivalent to the minimization of the number of violated constraints.

[28] This is among the allocations with the maximum number of transplants and maximum total weight of transplants in Figure 14 (left) and (right), respectively.

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