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Guaranteed cost solution for discrete-time uncertain/nonlinear dynamic games

Éva Gyurkovics [a,1], Tibor Takács [b,*,1]

[a] Mathematical Institute, Budapest University of Technology and Economics, Budapest, Pf. 91, 1521, Hungary

[b] Corvinus University of Budapest, 1093 Budapest, 8 Fővám tér, Hungary

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ABSTRACT

Motivated by an example of fiscal and monetary policy interaction of a national economy, the problem of uncertain/nonlinear two players discrete-time noncooperative games is investigated. Since the models of the systems are uncertain, the notion of Nash equilibrium solution is not suitable, instead, new Nash guaranteeing strategies and Nash guaranteed costs are defined. The system's uncertainties and/or nonlinearities are assumed to be of quadratically bounded type. First, conditions of the Nash guaranteeing strategies are derived for general uncertain nonlinear systems. These results are specified for systems that have linear nominal part and quadratic cost functions. Approximate solutions are obtained by tractable quadratic matrix inequalities. To illustrate the application of the proposed method, two numerical examples are given.

1. Introduction

Dynamic games have been intensively researched for decades. There are several problems in the field of engineering and economics, where processes can be modelled as a result of interaction of different players (see e.g., [1–15], and the references therein). These processes can be considered as control problems with several agents having individual inputs and individual objective (or cost) functions. If these agents/players are not supposed to cooperate, very often a suitably defined Nash solution is determined, from which no player can deviate without increasing his/her cost (e.g., [2,4,5,9,10,12–14]). The modelled processes are typically uncertain. A usual way to model the uncertainties is the application of stochastic dynamic games (e.g., [6,11,15]). Other authors consider uncertain elements as external perturbations to avoid the necessity of stochastic assumptions [5,10,14,16]. Parametric uncertainty is investigated e.g. in [17], however a cooperative control protocol is investigated for multiagent systems with optimization of a global cost function in that paper. Both types of uncertain games are discussed in [2], where "optimality" is comprehended in different ways (as Nash, Stackelberg or cooperative games). Papers applying the robust techniques consider exogenous disturbances affecting the system dynamics, and players determine their strategies considering the worst case disturbances. However, it is a research gap that uncertainties due to the imperfect knowledge of system dynamics have not been considered yet. An exception is [16], where linear fractional uncertainties were considered for zero-sum difference games, but in the present paper a more general class of uncertainties is admitted in non-zero sum noncooperative games. The authors are aware of papers that applied the robust techniques to linear- (or linear-affine-) quadratic games. An open challenge is to extend these results, if the dynamic equation of the game is not necessary linear or linear-affine. The dynamic equation of the motivating example is such that the nonlinearities can be separated from the linear parts, and can be treated together with the deterministic unknown system uncertainties. Such technique

* Corresponding author.
E-mail addresses: gye@math.bme.hu (É. Gyurkovics), takacs.tibor@uni-corvinus.hu (T. Takács).

[1] Both authors have equal contribution to the submitted article.

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has not been applied yet to robust games, therefore a new Nash guaranteeing cost concept is introduced in this paper, and sufficient conditions for the Nash guaranteeing solution are derived in the form of highly nonlinear equations. Relaxing the requirement of getting an exact solution, an approximation is determined. The corresponding feedback strategies are determined by solving certain quadratic matrix inequalities, for which solvers are available. The authors are not aware of similar results with nonlinear terms in the game dynamics. Furthermore, the proposed approach can be applied to economic games as well, where the expectations of economic players affect the dynamics, however the standard assumptions of the stochastic approach usually do not hold true. This paper provides a new method to treat the expectations by robust technique getting rid of the stochastic assumptions.

The present paper contributes to the theory of uncertain discrete-time infinite horizon games. It has been motivated by an economic game problem modelling the conflicts of fiscal and monetary policy. The examined uncertain game is more general than others in the sense that it takes into consideration the model uncertainties. Furthermore, all these uncertainties and the nonlinear elements of the dynamics are modelled by a common unknown deterministic function, for which a quadratic boundedness condition is supposed. This uncertainty structure is the extension of several types of uncertainties. The contributions of the present paper are as follows.

- A new notion of Nash guaranteeing strategy is defined for uncertain dynamic games.
- All system nonlinearities and uncertainties are modelled by common deterministic but unknown functions assumed to be quadratically constrained.
- Nash guaranteeing strategies are determined for general nonlinear uncertain games.
- The results are specified for games with linear nominal part and quadratic cost functions. An approximate solution is given by quadratic matrix inequalities.

The paper is organized as follows. Section 2 presents the motivational example. In Section 3, the conditions of the Nash guaranteeing strategies are given for general nonlinear systems. The Nash solution is determined in Section 4, when the nominal system is linear, and the cost functions are quadratic. Both the cases of general quadratically bounded uncertainties and the uncertainties of linear fractional form are discussed. Two numerical examples illustrate the results in Section 5. Finally, Section 6 concludes the paper.

In the paper, standard notations are applied. The transpose of matrix A is denoted by A^T, and $P > 0$ (≥ 0) denotes the positive (semi-) definiteness of P. Notation v^H is used for the complex conjugate transpose of a complex vector v, while $\mathbf{u}$ is used for the vector series $u_0, u_1, \ldots$, and I denotes the identity matrix of appropriate dimension. The notation of time-dependence is omitted, if it does not cause any confusion. Other notations are defined at the first appearance.

2. A motivational example

The fiscal and monetary policy interaction of a national economy can be considered as a game theoretical problem. The main mission of monetary policy is the maintenance of price stability, keeping inflation at a low level, while the implementation of a countercyclical policy is expected from the fiscal policy. Monetary policy should enjoy a degree of independence from fiscal policy, otherwise an irresponsible procyclical policy of the government may lead to an unwanted increase of inflation. This means that the two policies can be considered as two agents with different controls and with different goals. One of the most important controls of the fiscal policy is the balance of the central budget, but there are also other interventional tools affecting the output and the redistribution of incomes. Monetary policy also has several tools to control the money supply and to support the countercyclical policy. One of these tools is the base rate, which is determined autonomously by the central bank. Both groups of control affect the dynamics of the national economy, therefore the dynamic game can be a good approach to model the impacts of interactions of the two policies. There are several models of this type applied to the fiscal-monetary game known from the literature. Continuous-time models are considered by e.g. [18–20], while discrete models are applied by [21–24].

An objective is defined for each agent/player on the basis of which a dynamic Nash equilibrium is sought. Although there would be more options, a relatively simple generic model is presented here with the most important controls as a motivational example.

Economic processes are inherently uncertain; therefore, it is suitable to take this into consideration in the formal models, too. The uncertainties in macroeconomic models are usually considered in the literature in a stochastic framework. A couple of examples regarding the fiscal-monetary games are [19,22], and [23]. In economic models, less attention has been paid so far to robust techniques, which are widespread in technical applications though. The point of the robust approach is that no stochastic assumptions are needed, instead, uncertainties are modelled by unknown deterministic functions, for which only certain boundedness conditions must hold true. This approach is also suitable to treat the expectations, which are crucial in economic models including inflation.

The dynamics of a generic fiscal-monetary game contains a dynamic equation describing the real sphere, and another equation modelling the price dynamics. Our motivational example is an uncertain model applying the deterministic approach of modelling the uncertain expectations. Assume that a reference trajectory of the nominal GDP is denoted by x_t^*, and let g_t^* denote a reference path of the central budget balance, which may be identically zero ($t = 0, 1, \ldots$). Consider the uncertain dynamics

$$z_{t+1} = -\alpha_1 \left(i_t - E\left[\pi_{t+1}\right] \right) + \alpha_2 g_t, \tag{1}$$

$$\pi_{t+1} = \beta E\left[z_{t+1}\right] + E\left[\pi_{t+1}\right], \tag{2}$$

where $z_t = \frac{x_t - x_t^*}{x_t^*}$, $g_t = \frac{\overline{g}_t - g_t^*}{x_t^*}$ with the nominal balance $\overline{g}_t$, i_t is the nominal interest rate, π_t is the inflation, α_1, α_2, and β are positive constant parameters. $E[.]$ denotes the expectation, which is considered to be uncertain:

$$E\left[z_{t+1}\right] = z_t + p_1(t, z_t, \pi_t), \tag{3}$$

$$E\left[\pi_{t+1}\right] = \pi_t + p_2(t, z_t, \pi_t) \tag{4}$$

with unknown but bounded p_i's possibly depending on time and on the two state variables. Eqs. (3)–(4) mean that the expectations are basically naive, i.e. the previous values affect the expectations, but also nonlinearities may occur as a result of non-foreseen developments. The p_i's therefore may be zero or nonzero for different t's with certain boundedness constraints. This seems to be a more realistic approach than assuming any specific probability distributions for the expectations, although the obviously existing uncertain fluctuations should be modelled. If the robust approach is applied, only certain bounds are given for these uncertain fluctuations, but their specific forms should not necessarily be known. The boundedness of uncertainties itself without specifying a probability distribution with known parameters (e.g. white noise with known variance) seems to be more flexible in economic applications.

Let π^* be a reference path for the inflation (inflation target) and assume that $i^* = \pi^*$ is a constant reference interest rate. The objective function of the fiscal policy is

$$J_F = \sum_{t=0}^{\infty} \left(\gamma_1 z_t^2 + \gamma_2 g_t^2\right), \tag{5}$$

while the objective function of he monetary policy is

$$J_M = \sum_{t=0}^{\infty} \left(\varrho_1 \left(\pi_t - \pi^*\right)^2 + \varrho_2 \left(i_t - i^*\right)^2\right), \tag{6}$$

with positive coefficients $\gamma_1, \gamma_2, \varrho_1$, and ϱ_2. The minimization of the objective function J_F supports the fiscal policy to keep the GDP as close as possible to a predetermined target trajectory x_t^*, and also to keep the central budget balance as a share of the GDP around the target path g_t^* to prevent the accumulation of a large state debt. The minimization of the objective function J_M supports the monetary policy to prevent the harmful deviation from its inflation target π^*, and to stabilize the base rate i^*, which is supposed to be consistent with this target. Since the uncertainties represented by functions p_1, p_2 are unknown, the minima of the objectives cannot be determined, only a guaranteed cost can be expected for both. The problem is to define an appropriate Nash guaranteeing cost solution, and to find it to compare different catching-up development paths for an economy.

3. Nonlinear uncertain difference games

Consider the uncertain dynamics

$$x_{t+1} = f\left(x_t, u_t^1, u_t^2\right), \tag{7}$$

where $f : \mathbf{R}^{n_x} \times \mathbf{R}^{n_{u^1}} \times \mathbf{R}^{n_{u^2}} \to \mathbf{R}^{n_x}$ is continuous, and u_t^i is the control of Player i ($i = 1, 2$). Function f is not known, but it is from a known set

$$\mathcal{F} \subset \left\{\varphi : \mathbf{R}^{n_x} \times \mathbf{R}^{n_{u^1}} \times \mathbf{R}^{n_{u^2}} \to \mathbf{R}^{n_x}, \varphi(0,0,0) = 0\right\}.$$

Introducing the notation $\mathbf{u}^i = \left(u_0^i, u_1^i, \ldots, u_t^i, \ldots\right)$, the objective function of Player i is

$$J_f^i\left(x_0, \mathbf{u}^1, \mathbf{u}^2\right) = \sum_{t=0}^{\infty} \mathcal{L}^i\left(x_t, u_t^1, u_t^2\right), \tag{8}$$

where $\mathcal{L}^i : \mathbf{R}^{n_x} \times \mathbf{R}^{n_{u^1}} \times \mathbf{R}^{n_{u^2}} \to \mathbf{R}$, and $\mathcal{L}^i\left(x, u^1, u^2\right) \geq 0$. Both players intend to minimize J_f^i applying feedback information pattern, i.e. $u_t^1 = \mu^1\left(x_t\right)$, $u_t^2 = \mu^2\left(x_t\right)$ with $\mu^i(0) = 0$, ($i = 1, 2$). The pair of strategies is called *admissible*, if the functions $x \to \mu^i(x)$ ($i = 1, 2$) are continuous, the closed loop system

$$x_{t+1} = f\left(x_t, \mu^1(x_t), \mu^2(x_t)\right) \tag{9}$$

is well-defined for any given x_0 and $f \in \mathcal{F}$, and

$$J_f^i\left(x_0, \mu^1, \mu^2\right) \in \mathbf{R} \cup \{\infty\}, \qquad \forall f \in \mathcal{F},\ i = 1, 2.$$

The set of admissible strategy pairs is denoted by $\mathcal{M}_{FB}$. Let $\overline{f} \in \mathcal{F}$ be given. A set $\mathcal{M}_{FB}^0(\overline{f}) \subset \mathcal{M}_{FB}$ is called the set of *admissible strategies with respect to* $\overline{f}$, if the origin is an asymptotically stable equilibrium of (9) considered with $f = \overline{f}$ and with the given strategy pair (μ^1, μ^2), moreover $J_{\overline{f}}^i\left(x_0, \mu^1, \mu^2\right) < \infty$ for any x_0. According to usual notations let $\hat{i}$ denote the 'other' player's index, i.e. $i = 1$, $\hat{i} = 2$, and $i = 2$, $\hat{i} = 1$.

Definition 1. (a) The strategy $\left(\mu^{1*}, \mu^{2*}\right) \in \mathcal{M}_{FB}$ is a guaranteed cost strategy with guaranteed cost $V^i(x_0)$ ($i = 1, 2$) if

$$J_f^i\left(x_0, \mu^{1*}, \mu^{2*}\right) \leq V^i\left(x_0\right), \qquad \forall f \in \mathcal{F},\ i = 1, 2.$$

(b) If, in addition, for (μ^{1*}, μ^{2*}) and for V^1, V^2 it is also true that there exist functions $\overline{f} \in \mathcal{F}$ and $\overline{\mathcal{L}}^i (x, u^1, u^2) \geq \mathcal{L}^i (x, u^1, u^2)$ such that $(\mu^{1*}, \mu^{2*}) \in \mathcal{M}^0_{FB}(\overline{f})$, and

$$V^i(x_0) = \overline{J}^i_{\overline{f}} (x_0, \mu^{1*}, \mu^{2*}) \leq \overline{J}^i_{\overline{f}} \left(x_0, u^i, \mu^{\hat{i}*}\right)$$

for all $\left(u^i, \mu^{\hat{i}*}\right) \in \mathcal{M}^0_{FB}(\overline{f})$, then (μ^{1*}, μ^{2*}) is a pair of Nash guaranteeing feedback strategies with guaranteed costs $V^1(x_0)$ and $V^2(x_0)$. (The objective function $\overline{J}^i_{\overline{f}}$ is determined by the running cost $\overline{\mathcal{L}}^i$.)

Remark 1. For notational convenience, two-players uncertain games are considered throughout the paper, but all results can be formulated for N-player games on account of slightly more complicated formulas.

Consider a fixed $f_0 \in \mathcal{F}$.

Theorem 1. *Consider system (7) with objective functions (8). Assume that there exist $(\mu^1, \mu^2) \in \mathcal{M}_{FB}$, $V^i : \mathbf{R}^{n_x} \to \mathbf{R}$, and $\mathcal{L}^i_+ : \mathbf{R}^{n_x} \times \mathbf{R}^{n_{u^1}} \times \mathbf{R}^{n_{u^2}} \to \mathbf{R}_+$, such that, for $i = 1, 2$,*

(A) $V^i(0) = 0$, $V^i(x) > 0$, if $x \neq 0$,

(B) $V^i \left(f \left(x, \mu^1(x), \mu^2(x)\right)\right) \leq V^i \left(f_0 \left(x, \mu^1(x), \mu^2(x)\right)\right) + \mathcal{L}^i_+ \left(x, \mu^1(x), \mu^2(x)\right), \quad \forall x \in \mathbf{R}^{n_x}, f \in \mathcal{F}$,

(C) $V^i(x) \geq V^i \left(f_0 \left(x, \mu^1(x), \mu^2(x)\right)\right) + \mathcal{L}^i \left(x, \mu^1(x), \mu^2(x)\right) + \mathcal{L}^i_+ \left(x, \mu^1(x), \mu^2(x)\right)$,

then (μ^1, μ^2) are guaranteed cost strategies with guaranteed costs $V^i(x_0)$.

(D) If, in addition, $(\mu^1, \mu^2) \in \mathcal{M}^0_{FB}(f_0)$ and

$$\begin{aligned} V^i(x) &= V^i \left(f_0(x, \mu^1(x), \mu^2(x))\right) + \widetilde{\mathcal{L}}^i \left(x, \mu^1(x), \mu^2(x)\right) \\ &\leq V^i \left(f_0(x, u^i(x), \mu^{\hat{i}}(x))\right) + \widetilde{\mathcal{L}}^i(x, u^i(x), \mu^{\hat{i}}(x)), \qquad \forall (u^i(x), \mu^{\hat{i}}(x)) \in \mathcal{M}^0_{FB}(f_0) \end{aligned}$$

holds true, where $\widetilde{\mathcal{L}}^i = \mathcal{L}^i + \mathcal{L}^i_+$, then (μ^1, μ^2) is a pair of Nash guaranteeing feedback strategies with Nash guaranteed costs $V^i(x_0)$.

Proof. Let f be an arbitrary element of $\mathcal{F}$, and $(\mu^1, \mu^2) \in \mathcal{M}_{FB}$ satisfy the conditions of the theorem. Let $x_{(\cdot)}$ be the trajectory of the closed system (9). Consider the forward difference for V^i. Applying conditions B and C subsequently, one obtains

$$\begin{aligned} V^i (x_{t+1}) - V^i (x_t) &= V^i \left(f \left(x_t, \mu^1(x_t), \mu^2(x_t)\right)\right) - V^i (x_t) \\ &\leq V^i(f_0(x_t, \mu^1(x_t), \mu^2(x_t))) + \mathcal{L}^i_+(x_t, \mu^1(x_t), \mu^2(x_t)) - V^i(x_t) \\ &\quad + \mathcal{L}^i(x_t, \mu^1(x_t), \mu^2(x_t)) - \mathcal{L}^i(x_t, \mu^1(x_t), \mu^2(x_t)) \\ &\leq V^i(x_t) - \mathcal{L}^i(x_t, \mu^1(x_t), \mu^2(x_t)) - V^i(x_t) = -\mathcal{L}^i(x_t, \mu^1(x_t), \mu^2(x_t)). \end{aligned} \tag{10}$$

Applying condition A, after rearranging and taking the sum from 0 to T, one obtains

$$\sum_{t=0}^{T} \mathcal{L}^i(x_t, \mu^1(x_t), \mu^2(x_t)) \leq V^i(x_0) - V^i(x_{T+1}) \leq V^i(x_0).$$

Since the sum of nonnegative terms on the left side has a bound independent of T, it is convergent as $T \to \infty$, and the limit is $J^i_f(x_0, \mu^1, \mu^2) \leq V^i(x_0)$, which verifies the first assertion of the theorem.

To show the assertion of part D, consider the trajectory $x^0_{(\cdot)}$ of (9) with $f = f_0$. Taking the sum of the equation part in D for $x = x^0_t$, $(t = 0, 1, \ldots, T)$ results in

$$V^i(x_0) - V^i(x^0_{T+1}) = \sum_{t=0}^{T} \widetilde{\mathcal{L}}^i(x^0_t, \mu^1(x^0_t), \mu^2(x^0_t)).$$

Since $(\mu^1, \mu^2) \in \mathcal{M}^0_{FB}(f_0)$, the second term of the left hand side converges to 0, as $T \to \infty$, while the right hand side converges to $\widetilde{J}^i_{f_0}(x_0, \mu^1, \mu^2)$, consequently, $V^i(x_0) = \widetilde{J}^i_{f_0}(x_0, \mu^1, \mu^2)$. On the other hand, applying the inequality part of D for the trajectory $x = x^{0,u}_t$ of (9) considered with $f = f_0$ and $(u^i, \mu^{\hat{i}}) \in \mathcal{M}^0_{FB}(f_0)$, it follows that

$$V^i(x_0) - V^i(x^{0,u}_{T+1}) \leq \sum_{t=0}^{T} \widetilde{\mathcal{L}}^i(x^{0,u}_t, u^i(x^{0,u}_t), \mu^{\hat{i}}(x^{0,u}_t)).$$

If $T \to \infty$, the second term of the left hand side converges to 0, while the right hand side converges to $\widetilde{J}^i_{f_0}(x_0, u^i, \mu^{\hat{i}})$. Therefore

$$\widetilde{J}^i_{f_0}(x_0, \mu^1, \mu^2) = V^i(x_0) \leq \widetilde{J}^i_{f_0}(x_0, u^i, \mu^{\hat{i}})$$

for all $(u^i, \mu^{\hat{i}}) \in \mathcal{M}^0_{FB}(f_0)$. Consequently, (μ^1, μ^2) is a pair of Nash guaranteeing feedback strategies with Nash guaranteed costs $V^1(x_0)$ and $V^2(x_0)$. □

In what follows we shall deal with the admissibility of a strategy pair (μ^1, μ^2) with respect to a function $f \in \mathcal{F}$.

Assumption 1. Let $f \in \mathcal{F}$, and let $\overline{\overline{\mathcal{L}}}(x,u^1,u^2) \doteq \mathcal{L}^1(x,u^1,u^2) + \mathcal{L}^2(x,u^1,u^2)$.

(a) There exists a function $\Phi : [0,\infty) \to [0,\infty)$ such that $\overline{\overline{\mathcal{L}}}(x,u^1,u^2) \geq \Phi\left(\left\|u^1\right\| + \left\|u^2\right\|\right) > 0$, if $\left\|u^1\right\| + \left\|u^2\right\| \neq 0$, and $\Phi(0) = 0$.

(b) The function $x \to \overline{\overline{\mathcal{L}}}(x,0,0)$ can be written as $\overline{\overline{\mathcal{L}}}(x,0,0) = \overline{\overline{Q}}(h(x))$, where $\overline{\overline{Q}}(s) > 0$, if $s \neq 0$, $\overline{\overline{Q}}(0) = 0$, $h(0) = 0$, and system

$$x_{t+1} = f(x_t,0,0), \qquad y_t = h(x_t)$$

is zero-state detectable, i.e. if $h(x_t) = 0$ for all $t = t_0, t_0+1, \dots$ then $\lim_{t\to\infty} x_t \to 0$.

Corollary 1. *Suppose that the conditions A - C of Theorem 1 are satisfied and Assumption 1 is true for a function $f \in \mathcal{F}$. If the function $\overline{\overline{V}}(x) \doteq V^1(x) + V^2(x)$ is continuous and radially unbounded, then $(\mu^1,\mu^2) \in \mathcal{M}^0_{FB}(f)$. If in addition, Assumption 1 holds with $\overline{\overline{Q}}(h(x)) > 0$ for any $x \neq 0$, then $(\mu^1,\mu^2) \in \mathcal{M}^0_{FB}(f)$ for all $f \in \mathcal{F}$.*

Proof. The boundedness of $J^i_f(x,\mu^1,\mu^2)$, $(i = 1,2)$ follows from Theorem 1 for any $f \in \mathcal{F}$. By summing for $i = 1,2$ the left and right sides of inequality (10), one obtains that

$$\overline{\overline{V}}(x_{t+1}) - \overline{\overline{V}}(x_t) \leq -\overline{\overline{\mathcal{L}}}(x_t,\mu^1(x_t),\mu^2(x_t)).$$

Assumption 1 implies that $\overline{\overline{V}}(x_{t+1}) - \overline{\overline{V}}(x_t) \leq 0$, and $\overline{\overline{V}}(x_{t+1}) - \overline{\overline{V}}(x_t) = 0$ if and only if

$$x_t \in \mathcal{H} \doteq \left\{ x :\ \mu^1(x) = 0,\ \mu^2(x) = 0,\ \overline{\overline{Q}}(h(x)) = 0 \right\}.$$

Let $\mathcal{H}_1 \subset \mathcal{H}$ be the largest positively invariant set. Then Corollary 5.4.8 of [25] implies that any solution of (9) converges to the set $\mathcal{H}_1$ as $t \to \infty$. However, because of item (b) of Assumption 1, any trajectory starting in $\mathcal{H}_1$ converges to the origin. By continuity, one can easily prove that any trajectory converges to the origin, as well, i.e. $(\mu^1,\mu^2) \in \mathcal{M}^0_{FB}(f)$.

If Assumption 1 holds with $\overline{\overline{Q}}(h(x)) > 0$ for any $x \neq 0$, then $\mathcal{H} = \{0\}$, thus the origin is the only invariant subset of $\mathcal{H}$. Therefore, as a consequence of Corollary 5.4.9 of [25], the origin is an asymptotically stable equilibrium of (9) with any $f \in \mathcal{F}$. Thus $(\mu^1,\mu^2) \in \mathcal{M}^0_{FB}(f)$ for all $f \in \mathcal{F}$.

To find the Nash guaranteeing solution, one has

- to find functions f_0, V^i and $\mathcal{L}^i_+$ $(i = 1,2)$, for which conditions A), B) and C) are satisfied;
- to determine the Nash solution of the problem determined by f_0 and $\widetilde{\mathcal{L}}^i$ $(i = 1,2)$.

This is a difficult task in the general nonlinear case. In this paper we specifically present the solution for the linear–quadratic uncertain problem.

4. Linear–quadratic uncertain difference games

Consider a discrete time uncertain game of two players, where the game evolution is described by

$$x_{t+1} = Ax_t + B_1u^1_t + B_2u^2_t + Hp_t, \tag{11}$$

$$q_t = A_qx_t + Gp_t, \tag{12}$$

where $x \in \mathbf{R}^{n_x}$ is the state, $u^1 \in \mathbf{R}^{n_{u^1}}$ and $u^2 \in \mathbf{R}^{n_{u^2}}$ are the inputs of Player 1 and Player 2, A, B_1, B_2, H, A_q and G are given matrices of appropriate dimension. All system nonlinearities/uncertainties are represented by function p possibly depending on t and x. Function q is the uncertain output. The only available information about $p \in \mathbf{R}^{n_p}$ and $q \in \mathbf{R}^{n_q}$ is that their values are constrained by the set $\Omega = \Omega_1 \times \cdots \times \Omega_s$,

$$\Omega_i = \left\{ \begin{bmatrix} p_i \\ q_i \end{bmatrix} \in \mathbf{R}^{n_{p_i}+n_{q_i}} \ :\ \begin{bmatrix} p_i \\ q_i \end{bmatrix}^T \begin{bmatrix} Q_{0i} & S_{0i} \\ S^T_{0i} & R_{0i} \end{bmatrix} \begin{bmatrix} p_i \\ q_i \end{bmatrix} \geq 0 \right\}, \qquad i = 1,\dots,s, \tag{13}$$

where $Q_{0i} = Q^T_{0i}$, $R_{0i} = R^T_{0i}$ and S_{0i} are constant matrices, p, and q are partitioned appropriately. We shall use the notations $Q_0 = \mathrm{diag}\{Q_{01},\dots,Q_{0s}\}$, $R_0 = \mathrm{diag}\{R_{01},\dots,R_{0s}\}$, $S_0 = \mathrm{diag}\{S_{01},\dots,S_{0s}\}$.

Assumption 2. Inequalities

$$R_0 \geq 0, \tag{14}$$

$$\Xi_0 \doteq Q_0 + G^TS^T_0 + S_0G + G^TR_0G < 0 \tag{15}$$

hold true.

We note that condition (14), i.e. the positive semi-definiteness of R_0 assures that the system (11)–(12) is well posed, i.e. for any (x, u^1, u^2) there is a p so that $\left[p^T,\ q^T\right]^T \in \Omega$, while condition (15) guarantees that the origin is an equilibrium point of system (11)–(12). It is worth noting that the considered model of uncertainties involves several types of uncertainties frequently investigated in the literature. For example, if $Q_0 = 0$, $S_0 = I$ and $R_0 = 0$, then one speaks about positive real uncertainty, if $Q_0 = -I$, $S_0 = 0$ and $R_0 = I$, then one has norm-bounded or linear-fractional uncertainties depending on whether $G = 0$ or $G \neq 0$, and if $Q_0 = \frac{1}{2}(K_1^T K_2 + K_2^T K_1)$, $S_0 = \frac{1}{2}(K_1 + K_2)^T$ and $R_0 = I$, then one faces the case of sector-bounded uncertainties.

Consider furthermore the objective functionals

$$J_i(x_0, \mathbf{u}^1, \mathbf{u}^2) = \sum_{t=0}^{\infty} \left(x_t^T Q_i x_t + u_t^{1T} R_{i1} u_t^1 + u_t^{2T} R_{i2} u_t^2 \right), \qquad i = 1, 2, \tag{16}$$

with matrices $Q_i = Q_i^T \geq 0$ and $R_{ii} = R_{ii}^T > 0$, $R_{ij} = R_{ij}^T \geq 0$, $i, j = 1, 2$, $i \neq j$.

We wish to find Nash guaranteeing linear feedback strategies of Players 1 and 2 for the game (11)–(13) and (16), i.e. the problem is to find

$$\alpha_i(x_t) = K_i x_t, \qquad i = 1, 2,$$

which satisfy the conditions of Theorem 3.1. We note that $(\alpha_1, \alpha_2) \in \mathcal{M}_{FB}$.

4.1. The case of general quadratically constrained uncertainty

In what follows, we consider the game (11), (12), (16) with the general constraint set (13), and apply the following cast

$$\mathcal{L}^i(x, u^1, u^2) = x^T Q_i x + u^{1T} R_{i1} u^1 + u^{2T} R_{i2} u^2, \qquad i = 1, 2, \tag{17}$$

$$f_0(x, u^1, u^2) = Ax + B_1 u^1 + B_2 u^2, \tag{18}$$

$$f_p(x, u^1, u^2) = Ax + B_1 u^1 + B_2 u^2 + Hp, \tag{19}$$

and $f_p \in \mathcal{F}$, if and only if $\begin{bmatrix} p^T & q^T \end{bmatrix}^T \in \Omega$ with $q = A_q x + Gp$. The system with f_0 is called the nominal system.

In order to formulate the results, we need some notations. For any positive constants ω_j, $j = 1, \dots, s$ set

$$\underline{\omega} = \operatorname{diag}\left\{ \omega_1 I_{n_{p_1}}, \dots, \omega_s I_{n_{p_s}} \right\}, \qquad \underline{\underline{\omega}} = \operatorname{diag}\left\{ \omega_1 I_{n_{q_1}}, \dots, \omega_s I_{n_{q_s}} \right\}.$$

Furthermore, for any positive numbers τ_j^i and μ_j^i ($i = 1, 2$, $j = 1, \dots, s$), set $\mathcal{S}_0 = S_0^T + R_0 G$, and

$$\Xi(\underline{\tau}^i, \underline{\underline{\mu}}^i) = \underline{\tau}^i \left(\Xi_0 + \mathcal{S}_0^T \underline{\underline{\mu}}^i \mathcal{S}_0 \right) \tag{20}$$

Lemma 1. *Suppose that Assumption 1 holds true. Let $i = 1, 2$, and let $P_i = P_i^T > 0$, $\tau_j^i > 0$ and $\mu_j^i > 0$ ($j = 1, \dots, s$) be given so that inequalities*

$$\Xi_0 + \mathcal{S}_0^T \underline{\underline{\mu}}^i \mathcal{S}_0 < 0, \tag{21}$$

$$\widetilde{P}_i \doteq \left(P_i^{-1} + H \Xi(\underline{\tau}^i, \underline{\underline{\mu}}^i)^{-1} H^T \right)^{-1} > 0 \tag{22}$$

are satisfied. Then functions $V^i(x) = x^T P_i x$ and

$$\widetilde{\mathcal{L}}_+^i(x, u^1, u^2) \doteq \begin{bmatrix} x^T & u^{1T} & u^{2T} \end{bmatrix} \overline{Q}_i \begin{bmatrix} x \\ u^1 \\ u^2 \end{bmatrix} \tag{23}$$

with

$$\overline{Q}_i \doteq \begin{bmatrix} A^T & A_q^T \\ B_1^T & 0 \\ B_2^T & 0 \end{bmatrix} \begin{bmatrix} \overline{P_i} & 0 \\ 0 & \underline{\underline{\tau}}^i \left(R_0 + (\underline{\underline{\mu}}^i)^{-1} \right) \end{bmatrix} \begin{bmatrix} A & B_1 & B_2 \\ A_q & 0 & 0 \end{bmatrix}, \tag{24}$$

$$\overline{P_i} \doteq \widetilde{P}_i - P_i \tag{25}$$

satisfy conditions A and B of Theorem 1 for any feedback strategies $\alpha_1(x) = K_1 x$ and $\alpha_2(x) = K_2 x$. Moreover, $\widetilde{\mathcal{L}}_+^i(x, u^1, u^2) \geq 0$ for any x, u^1, u^2.

Proof. Under the assumptions of the lemma, condition A of Theorem 1 is obvious.

Next, condition B will be investigated. In view of (18)–(19), one has to show that

$$V^i(\mathcal{A}x + Hp) \leq V^i(\mathcal{A}x) + \widetilde{\mathcal{L}}_+^i(x, K_1 x, K_2 x), \qquad \forall \begin{bmatrix} p \\ q \end{bmatrix} \in \Omega, \tag{26}$$

where $\mathcal{A} \doteq A + B_1 K_1 + B_2 K_2$. Let us write $\widetilde{\mathcal{L}}_+^i(x, K_1 x, K_2 x)$ as $\widetilde{\mathcal{L}}_+^i(x, K_1 x, K_2 x) = x^T \mathcal{Q}_i x$, where $\mathcal{Q}_i = \begin{bmatrix} I & K_1^T & K_2^T \end{bmatrix} \overline{Q}_i \begin{bmatrix} I & K_1^T & K_2^T \end{bmatrix}^T$.

Then, in view of (12) relation (26) is equivalent to

$$F_1(x,p) \doteq V^i(\mathcal{A}x + Hp) - x^T Q_i x - V^i(\mathcal{A}x) = \begin{bmatrix} x^T & p^T \end{bmatrix} \begin{bmatrix} -Q_i & \mathcal{A}^T P_i H \\ H^T P_i \mathcal{A} & H^T P_i H \end{bmatrix} \begin{bmatrix} x \\ p \end{bmatrix} \leq 0$$

for all $\begin{bmatrix} x^T & p^T \end{bmatrix}^T$ such that $\begin{bmatrix} p^T & (A_q x + Gp)^T \end{bmatrix}^T \in \Omega$.

On the other hand, $\begin{bmatrix} p \\ q \end{bmatrix} \in \Omega$ means that $\begin{bmatrix} p \\ q \end{bmatrix}^T \begin{bmatrix} Q_0 & S_0 \\ S_0^T & R_0 \end{bmatrix} \begin{bmatrix} p \\ q \end{bmatrix} \geq 0$, which implies that

$$\begin{bmatrix} p \\ q \end{bmatrix}^T \begin{bmatrix} \underline{\underline{\tau}}^i Q_0 & (\underline{\underline{\tau}}^i S_0) \\ (\underline{\underline{\tau}}^i S_0)^T & \underline{\underline{\tau}}^i R_0 \end{bmatrix} \begin{bmatrix} p \\ q \end{bmatrix} \geq 0, \qquad \forall \begin{bmatrix} p \\ q \end{bmatrix} \in \Omega, \quad \text{and} \quad \forall \tau_j^i > 0, \quad \begin{matrix} i = 1,2, \\ j = 1,\dots,s. \end{matrix} \tag{27}$$

Using (12), one can show that (27) is equivalent to

$$\begin{bmatrix} x \\ p \end{bmatrix}^T \begin{bmatrix} A_q^T(\underline{\underline{\tau}}^i R_0)A_q & A_q^T(\underline{\underline{\tau}}^i S_0) \\ (\underline{\underline{\tau}}^i S_0)^T A_q & \underline{\underline{\tau}}^i \Xi_0 \end{bmatrix} \begin{bmatrix} x \\ p \end{bmatrix} \geq 0, \qquad \forall \tau_j^i > 0, \quad \begin{matrix} i = 1,2, \\ j = 1,\dots,s, \end{matrix} \tag{28}$$

and for all $\begin{bmatrix} x^T & p^T \end{bmatrix}^T$ such that $\begin{bmatrix} p^T & (A_q x + Gp)^T \end{bmatrix}^T \in \Omega$.

It is well-known that for any matrices $X \in \mathbf{R}^{n\times m}$ and $Y \in \mathbf{R}^{m\times s}$ and symmetrical invertible matrix $\Lambda \in \mathbf{R}^{m\times m}$,

$$\begin{bmatrix} 0 & XY \\ Y^T X^T & 0 \end{bmatrix} \leq \begin{bmatrix} X\Lambda^{-2}X^T & 0 \\ 0 & Y^T \Lambda^2 Y \end{bmatrix}. \tag{29}$$

Applying (29) by choosing $\Lambda^2 = \underline{\underline{\mu}}^i(\underline{\underline{\tau}}^i)^{-1}$, with arbitrary $\mu_j^i > 0,\ i = 1,2,\ j = 1,\dots,s$, one can increase the left hand side of inequality (28) so that $\begin{bmatrix} p \\ q \end{bmatrix} \in \Omega$ implies

$$F_2(x,p,\underline{\underline{\tau}}^i,\underline{\underline{\mu}}^i) \doteq \begin{bmatrix} x \\ p \end{bmatrix}^T \begin{bmatrix} A_q^T \underline{\underline{\tau}}^i(R_0 + (\underline{\underline{\mu}}^i)^{-1})A_q & 0 \\ 0 & \underline{\underline{\tau}}^i(\Xi_0 + S_0^T \underline{\underline{\mu}}^i S_0) \end{bmatrix} \begin{bmatrix} x \\ p \end{bmatrix} \geq 0,$$

$$\text{and} \quad \forall \tau_j^i > 0, \quad \forall \mu_j^i > 0, \quad \begin{matrix} i = 1,2, \\ j = 1,\dots,s, \end{matrix}$$

and for all $\begin{bmatrix} x^T & p^T \end{bmatrix}^T$ such that $\begin{bmatrix} p^T & (A_q x + Gp)^T \end{bmatrix}^T \in \Omega$.

Thus, $F_1(x,p) \leq F_1(x,p) + F_2(x,p,\underline{\underline{\tau}}^i,\underline{\underline{\mu}}^i)$ for any $\begin{bmatrix} p^T & q^T \end{bmatrix}^T \in \Omega$, therefore, it is enough to show that under the choice of (23)–(25), $F_1(x,p) + F_2(x,p,\underline{\underline{\tau}}^i,\underline{\underline{\mu}}^i) \leq 0$. It can immediately be seen that

$$F_1(x,p) + F_2(x,p,\underline{\underline{\tau}}^i,\underline{\underline{\mu}}^i) = \begin{bmatrix} x \\ p \end{bmatrix}^T \Psi_i \begin{bmatrix} x \\ p \end{bmatrix},$$

where

$$\Psi_i = \begin{bmatrix} -Q_i + A_q^T \underline{\underline{\tau}}^i \left(R_0 + (\underline{\underline{\mu}}^i)^{-1} \right) A_q & \mathcal{A}^T P_i H \\ H^T P_i \mathcal{A} & H^T P_i H + \underline{\underline{\tau}}^i \left(\Xi_0 + S_0^T \underline{\underline{\mu}}^i S_0 \right). \end{bmatrix}$$

Let us investigate the condition $\Psi_i \leq 0$. Firstly, we shall show that the matrix in position (2,2) of Ψ_i is negative definite. Indeed, by (20) and (21) we have that $\Xi(\underline{\underline{\tau}}^i,\underline{\underline{\mu}}^i) < 0$. Using the Schur complement lemma twice, one obtains that

$$\begin{aligned} H^T P_i H + \underline{\underline{\tau}}^i(\Xi_0 + S_0^T \underline{\underline{\mu}}^i S_0) = H^T P_i H + \Xi(\underline{\underline{\tau}}^i,\underline{\underline{\mu}}^i) < 0 \\ \Leftrightarrow \qquad P_i^{-1} + H\Xi(\underline{\underline{\tau}}^i,\underline{\underline{\mu}}^i)^{-1}H^T > 0, \end{aligned}$$

which is true by (22). Thus the Schur-complement lemma is applicable to $\Psi_i \leq 0$, which results in the inequality

$$0 \geq -Q_i + A_q^T \underline{\underline{\tau}}^i \left(R_0 + (\underline{\underline{\mu}}^i)^{-1} \right) A_q - \mathcal{A}^T P_i \mathcal{A} + \mathcal{A}^T \left(P_i - P_i H \left(\Xi(\underline{\underline{\tau}}^i,\underline{\underline{\mu}}^i) + H^T P_i H \right)^{-1} H^T P_i \right) \mathcal{A}. \tag{30}$$

The application of the matrix inversion lemma to the last term of (30) gives that

$$\begin{aligned} 0 &\geq -Q_i + A_q^T \underline{\underline{\tau}}^i \left(R_0 + (\underline{\underline{\mu}}^i)^{-1} \right) A_q - \mathcal{A}^T P_i \mathcal{A} + \mathcal{A}^T \left(P_i^{-1} + H\Xi(\underline{\underline{\tau}}^i,\underline{\underline{\mu}}^i)^{-1}H^T \right)^{-1} \mathcal{A} \\ &= -Q_i + A_q^T \underline{\underline{\tau}}^i \left(R_0 + (\underline{\underline{\mu}}^i)^{-1} \right) A_q + \mathcal{A}^T \left(\widetilde{P}_i - P_i \right) \mathcal{A}. \end{aligned} \tag{31}$$

Remembering the definition of Q_i one can see that the right hand side of (31) is zero, thus $\Psi_i \leq 0$ is satisfied, which means that the first assertion of the lemma is true. To show the second assertion, one only has to observe that $P_i > 0$, (21) and the definition of $\widetilde{P}_i$ in (23) imply that $\widetilde{P}_i \geq P_i$, i.e. $\overline{P}_i \geq 0$. Thus, by taking into account (24), $\widetilde{\mathcal{L}}_+^i(x,u^1,u^2) \geq 0$ for any $x,\ u^1,\ u^2$ is immediate. □

In what follows, we shall assume the following:

Assumption 3. The matrix pair $\left(A,\left[B_1,B_2\right]\right)$ is stabilizable.

Theorem 2. *Suppose that Assumption 3 and the assumptions of Lemma 1 are true, and function $\widetilde{\mathcal{L}}^i_+$ is chosen according to Lemma 1. If P_i, τ^i_j, μ^i_j, K_1 and K_2 satisfy additionally matrix inequality*

$$P_i \geq Q_i + \sum_{j=1}^{2} K_j^T R_{ij} K_j + A_q^T \underline{\underline{\tau}}^i \left(R_0 + (\underline{\underline{\mu}}^i)^{-1} \right) A_q + \mathcal{A}^T \left(P_i^{-1} + H \Xi(\underline{\tau}^i, \underline{\underline{\mu}}^i)^{-1} H^T \right)^{-1} \mathcal{A}, \qquad i = 1,2, \tag{32}$$

then $\alpha_1(x) = K_1 x$ and $\alpha_2(x) = K_2 x$ yield guaranteed cost strategies with guaranteed cost $V^i(x_0) = x_0^T P_i x_0$.

Proof. According to Theorem 1, we have to show that condition C is also valid. Substitution into the inequality of C gives that

$$\begin{aligned} V^i(f_0(x,K_1x,K_2x)) &+ \mathcal{L}_i(x,K_1x,K_2x) + \widetilde{\mathcal{L}}^i_+(x,K_1x,K_2x) \\ &= x^T \Bigg[\mathcal{A}^T P_i \mathcal{A} + Q_i + \sum_{j=1}^{2} K_j^T R_{ij} K_j + A_q^T \underline{\underline{\tau}}^i (R_0 + (\underline{\underline{\mu}}^i)^{-1}) A_q \\ &\quad + \mathcal{A}^T \left((P_i^{-1} + H \Xi(\underline{\tau}^i, \underline{\underline{\mu}}^i)^{-1} H^T)^{-1} - P_i \right) \mathcal{A} \Bigg] x. \end{aligned}$$

Thus, the assertion of the theorem is immediate. □

Assumption 4. The matrix pair $\left(A, Q_1 + Q_2\right)$ is detectable.

Corollary 2. *(i) Suppose that Assumption 4 and the conditions of Theorem 2 hold true. Then $(\alpha_1,\alpha_2) \in \mathcal{M}^0_{FB}(f_0)$, i.e.*

$$\left|\lambda(A + B_1K_1 + B_2K_2)\right| < 1.$$

(ii) If in addition $Q_1 + Q_2 > 0$, then $(\alpha_1,\alpha_2) \in \mathcal{M}^0_{FB}(f)$ for all $f \in \mathcal{F}$.

Proof. (i). Suppose, on the contrary, that matrix $\mathcal{A}$ has an eigenvalue λ with the eigenvector v such that $|\lambda| \geq 1$. Take the sum of both sides of (32) for $i = 1,2$, and multiply the obtained inequality by v^H and v from left and from right, respectively. After arranging and taking into consideration that $\widetilde{P}_i \geq P_i$, one obtains that

$$0 \geq v^H(Q_1+Q_2)v + \sum_{i=1}^{2} v^H (A_q^T \underline{\underline{\tau}}^i (R_0 + (\underline{\underline{\mu}}^i)^{-1}) A_q) v + \sum_{i,j=1}^{2} v^H K_i^T R_{ij} K_i v + (\bar{\lambda}\lambda - 1) v^H (P_1 + P_2) v. \tag{33}$$

Since all terms on the right hand side of (33) are nonnegative, it follows from (33) that $(Q_1+Q_2)v = 0$, $K_1 v = 0$, $K_2 v = 0$, and $A_q v = 0$, which contradicts to Assumption 4.

(ii) The statement is an immediate consequence of Corollary 1. □

Corollary 3. *Suppose that Assumptions 2 and 3 are true. Let $i = 1,2$ and let $\tau^i_j > 0$ and $\mu^i_j > 0$ $(j = 1,\dots,s)$ be given so that inequality (21) is satisfied.*

(i) If matrices $\widetilde{P}_i = \widetilde{P}_i^T > 0$, K_i are such that

$$0 \geq \begin{bmatrix} \Phi_i - \widetilde{P}_i & \mathcal{A}^T \widetilde{P}_i & \widetilde{P}_i H \\ \widetilde{P}_i \mathcal{A} & -\widetilde{P}_i & 0 \\ H^T \widetilde{P}_i & 0 & -H^T \widetilde{P}_i H + \Xi(\underline{\tau}^i, \underline{\underline{\mu}}^i) \end{bmatrix}, \qquad i = 1,2, \tag{34}$$

where $\Phi_i = Q_i + \sum_{j=1}^2 K_j^T R_{ij} K_j + A_q^T \underline{\underline{\tau}}^i (R_0 + (\underline{\underline{\mu}}^i)^{-1}) A_q$, then matrices P_1, P_2,

$$P_i = \left(\widetilde{P}_i^{-1} - H \Xi(\underline{\tau}^i, \underline{\underline{\mu}}^i)^{-1} H^T \right)^{-1}, \qquad i = 1,2 \tag{35}$$

are positive definite and satisfy inequality (32).

(ii) Conversely, if matrices $P_i = P_i^T > 0$, K_i are such that (22) and (32) hold true, then $\widetilde{P}_i = \widetilde{P}_i^T > 0$, K_i satisfy matrix inequality (34).

Proof. To show (i), first we observe that $\Xi(\underline{\tau}^i, \underline{\underline{\mu}}^i) < 0$ by assumptions, thus the positive definiteness of matrices P_i given by (35) is immediate. Next apply a congruence transformation to (34) with matrix $diag\{I, \widetilde{P}_i^{-1}, I\}$, then one obtains that

$$0 \geq \begin{bmatrix} \Phi_i - \widetilde{P}_i & \mathcal{A}^T & \widetilde{P}_i H \\ \mathcal{A} & -\widetilde{P}_i^{-1} & 0 \\ H^T \widetilde{P}_i & 0 & -H^T \widetilde{P}_i H + \Xi(\underline{\tau}^i, \underline{\underline{\mu}}^i) \end{bmatrix}, \qquad i = 1, 2.$$

Since $-H^T \widetilde{P}_i H + \Xi(\underline{\tau}^i, \underline{\underline{\mu}}^i) < 0$, the Schur-complement lemma is applicable, which yields with a short computation that

$$0 \geq \Phi_i + \mathcal{A}^T \widetilde{P}_i \mathcal{A} - \left[\widetilde{P}_i + \widetilde{P}_i H \left(\Xi(\underline{\tau}^i, \underline{\underline{\mu}}^i) - H^T \widetilde{P}_i H \right)^{-1} H^T \widetilde{P}_i \right]. \tag{36}$$

Now, the matrix inversion lemma shows that the expression in the square brackets in (36) is nothing else than P_i. Substituting the expression in the square brackets with P_i, and expressing $\widetilde{P}_i$ by P_i, one obtains from (36) the inequality (32).

To show (ii), one can apply analogous considerations in backward order, therefore the details are omitted. □

Theorem 3. *Suppose that Assumptions 2–4 and the assumptions of Lemma 1 are true, and function $\widetilde{\mathcal{L}}^i_+$ is chosen according to Lemma 1. If P_i, τ^i_j, μ^i_j, ($i = 1, 2,\ j = 1, \dots, s$), K_1 and K_2 satisfy matrix equations*

$$P_i = Q_i + \sum_{j=1}^{2} K_j^T R_{ij} K_j + A_q^T \underline{\tau}^i \left(R_0 + (\underline{\underline{\mu}}^i)^{-1} \right) A_q + \mathcal{A}^T \widetilde{P}_i \mathcal{A}, \qquad i = 1, 2, \tag{37}$$

$$\begin{bmatrix} R_{11} + B_1^T \widetilde{P}_1 B_1 & B_1^T \widetilde{P}_1 B_2 \\ B_2^T \widetilde{P}_2 B_1 & R_{22} + B_2^T \widetilde{P}_2 B_2 \end{bmatrix} \begin{bmatrix} K_1 \\ K_2 \end{bmatrix} = - \begin{bmatrix} B_1^T \widetilde{P}_1 \\ B_2^T \widetilde{P}_2 \end{bmatrix} A, \tag{38}$$

$$\widetilde{P}_i = \left(P_i^{-1} + H \Xi(\underline{\tau}^i, \underline{\underline{\mu}}^i)^{-1} H^T \right)^{-1}, \tag{39}$$

then condition D of Theorem 1 is satisfied, i.e. $\alpha_1(x) = K_1 x$ and $\alpha_2(x) = K_2 x$ are admissible with respect to f_0, and they yield Nash guaranteeing feedback strategies with Nash guaranteed cost $V^i(x_0) = x_0^T P_i x_0$ ($i = 1, 2$).

Proof. The admissibility with respect to f_0 of (α_1, α_2) follows from Corollary 2. We have seen that (32) implies the inequality in C of Theorem 1, thus (37) implies the equality part in condition D of Theorem 1. It remained to show that the inequality part in condition D of Theorem 1 holds true, as well. For simplicity, we consider the case $i = 1$, $\hat{\imath} = 2$. Suppose that $\alpha_2(x) = K_2 x$ is fixed with K_2 satisfying (38), and let u^1 be arbitrary. Applying the notation $A_{cl_2} = A + B_2 K_2$, a straightforward computation shows that

$$\begin{aligned} \Theta(x, u^1) &\doteq V^1\left(f_0(x, u^1, K_2 x)\right) + L^1(x, u^1, K_2 x) + \widetilde{L}^1_+(x, u^1, K_2 x) \\ &= \Theta_0(x) + \Theta_1(x, u^1) + \Theta_2(u^1), \end{aligned}$$

where the terms are collected so that Θ_i contains the i^{th} powers of u^1, namely

$$\begin{aligned} \Theta_0(x) &= x^T \left(Q_1 + K_2^T R_{12} K_2 + A_q^T \underline{\tau}^1 \left(R_0 + (\underline{\underline{\mu}}^1)^{-1} \right) A_q + A_{cl_2}^T \widetilde{P}_1 A_{cl_2} \right) x, \\ \Theta_1(x, u^1) &= 2 {u^1}^T B_1^T \widetilde{P}_1 A_{cl_2} x, \\ \Theta_2(u^1) &= {u^1}^T \left(R_{11} + B_1^T \widetilde{P}_1 B_1 \right) u^1. \end{aligned}$$

Using the necessary and sufficient condition of the minimum of multivariate functions gives that

$$\Theta(x, K_1^* x) = \min_{u^1} \Theta(x, u^1),$$

where

$$K_1^* = -\left(R_{11} + B_1^T \widetilde{P}_1 B_1 \right)^{-1} B_1^T \widetilde{P}_1 A_{cl_2}.$$

Expressing K_1 from the first equation of (38) with given K_2, one can see that $K_1 = K_1^*$, which verifies the inequality part in D of Theorem 1. Therefore, the statement follows from Theorem 1. □

Remark 2. We note that, in the uncertainty free case (i.e. for $H = 0,\ A_q = 0,\ G = 0$), Eq. (39) implies that $\widetilde{P}_i = P_i$, while Eqs. (37), (38) reduce to the equations (3.3a)–(3.4b) of Theorem 3.2 in [12]. In this way, Theorem 3 widens the range of solvable problems, while returning the results known from earlier literature for a narrower class of problems.

Remark 3. Eqs. (37)–(39) are highly nonlinear in the decision variables, thus their direct solution with available standard tools is hard. If one solves the problem obtained by changing the equality in (37) to "≥", then one obtains an *approximation* to the Nash guaranteeing feedback problem. The advantage of this approach is that the new problem can be transformed to a more tractable problem.

Approximations to the Nash equilibrium problem has been studied e.g. in [3,13,26] (see also the references therein) for uncertainty free games. The concepts of ϵ_α-, ϵ_{x_0}-, $\epsilon_{\alpha,\beta}$-, etc Nash equilibrium solutions are introduced to characterize the deviation of the relaxed solution from the exact one by requesting some further constraints on the admissibility of feedbacks. Similar considerations can also be made in the case of the Nash guaranteeing problem. Since this is not one of the main goals of this paper, we only present the simplest concept of the ϵ_α-Nash guaranteed equilibrium solution (which is a slight modification of the $\epsilon_{\alpha,\beta}$-Nash equilibrium solution of [13]).

The pair $(\alpha_1, \alpha_2) \in \mathcal{M}^0_{FB}(f_0)$ $(\alpha_i(x) = K_i x,\ i = 1,2)$ is said to be an ϵ_ϱ-*Nash guaranteeing feedback strategy pair with approximate Nash guaranteed costs* $V^i(x_0)$ for the game (11)–(13) and (16), if there exists a constant $\epsilon_{x_0,\varrho} \geq 0$ parameterized in the initial condition $x(0) = x_0$ and constant $\varrho > 1$, such that

$$(i) \qquad J^i_{f_p}\left(x_0, \alpha_1, \alpha_2\right) \leq V^i(x_0), \qquad \forall f_p \in \mathcal{F},\ i = 1,2, \tag{40}$$

$$(ii) \qquad \widetilde{J}^i_{f_0}\left(x_0, \alpha_1, \alpha_2\right) \leq \widetilde{J}^i_{f_0}\left(x_0, \widehat{\alpha}_i, \alpha_{\hat{i}}\right) + \epsilon_{x_0,\varrho} \tag{41}$$

is satisfied for all such $\widehat{\alpha}_i, \alpha_{\hat{i}}$, $\widehat{\alpha}_i(x) = \widehat{K}_i x, \alpha_{\hat{i}}(x) = K_{\hat{i}} x$ for which $|\varrho\lambda_M(A_{cl})| < 1$, where $A_{cl} = A + B_i\widehat{K}_i + B_{\hat{i}}K_{\hat{i}}$ and $\lambda_M(C)$ denotes the eigenvalue of C with maximum absolute value, $\widetilde{J}^i_{f_0}$ are the cost functions defined by running costs $\widetilde{\mathcal{L}}^i = \mathcal{L}^i + \widetilde{\mathcal{L}}^i_+$, $\mathcal{L}^i$ and $\widetilde{\mathcal{L}}^i_+$ are defined by (17) and Lemma 1, respectively.

Claim. *Suppose that the conditions of Theorem 3 hold true.*

If P_i, $\tau^i_j > 0$, $\mu^i_j > 0$, $(i = 1,2,\ j = 1,\dots,s)$, K_1 and K_2 satisfy matrix inequality (32) and matrix Eqs. (38), (39), then $\alpha_1(x) = K_1 x$ and $\alpha_2(x) = K_2 x$ are ϵ_ϱ-Nash guaranteeing feedback strategies with approximate Nash guaranteed costs $V^i(x_0) = x_0^T P_i x_0,\ i = 1,2$.

Proof. It follows from (32) by Theorem 2 that $V^i(x_0) = x_0^T P_i x_0,\ i = 1,2$ satisfy (40). To show (41) one can follow the lines of the proofs of the corresponding theorems of [3,13], therefore the details are omitted.

The next corollary formulates the problem to be solved, which is quadratic in the decision variables.

Corollary 4. *Suppose that the conditions of Theorem 3 hold true. If $\widetilde{P}_i = \widetilde{P}_i^T > 0$, $\tau^i_j > 0$, $\nu^i_j > 0$, $(i = 1,2,\ j = 1,\dots,s)$, K_1 and K_2 satisfy for $i = 1,2$ matrix Eqs. (38) and matrix inequalities*

$$0 > \underline{\underline{\nu}}^i \Xi_0 + \underline{\underline{\tau}}^i S_0^T S_0 \tag{42}$$

$$0 \geq \begin{bmatrix} \widetilde{\Phi}_i - \widetilde{P}_i & \mathcal{A}^T\widetilde{P}_i & \widetilde{P}_i H & K_i^T & 0 \\ \widetilde{P}_i\mathcal{A} & -\widetilde{P}_i & 0 & 0 & 0 \\ H^T\widetilde{P}_i & 0 & -H^T\widetilde{P}_i H + \underline{\underline{\tau}}^i\Xi_0 & 0 & S_0^T\underline{\underline{\tau}}^i \\ K_i & 0 & 0 & -R_{ii}^{-1} & 0 \\ 0 & 0 & \underline{\underline{\tau}}^i S_0 & 0 & -\underline{\underline{\nu}}^i \end{bmatrix} \tag{43}$$

where $\widetilde{\Phi}_i = Q_i + K_{\hat{i}}^T R_{i\hat{i}} K_{\hat{i}} + A_q^T\left(\underline{\underline{\tau}}^i R_0 + \underline{\underline{\nu}}^i\right)A_q$, then P_i given by (35) satisfies matrix inequality (32), and $\alpha_1(x) = K_1 x$ and $\alpha_2(x) = K_2 x$ are ϵ_ϱ-Nash guaranteeing feedback strategies with approximate Nash guaranteed costs $V^i(x_0) = x_0^T P_i x_0,\ i = 1,2$.

Proof. We shall use here the results of Corollary 3. Accordingly, we have to investigate inequality (34). Using the definition of Φ_i and $\Xi(\underline{\underline{\tau}}^i, \underline{\underline{\mu}}^i)$, inequality (34) can equivalently be written as

$$0 \geq \begin{bmatrix} \widehat{\Phi}_i - \widetilde{P}_i & \mathcal{A}^T\widetilde{P}_i & \widetilde{P}_i H \\ \widetilde{P}_i\mathcal{A} & -\widetilde{P}_i & 0 \\ H^T\widetilde{P}_i & 0 & -H^T\widetilde{P}_i H + \underline{\underline{\tau}}^i\Xi_0 \end{bmatrix} + \begin{bmatrix} K_i^T & 0 \\ 0 & 0 \\ 0 & S_0^T \end{bmatrix} \begin{bmatrix} R_{ii} & 0 \\ 0 & \underline{\underline{\tau}}^i\underline{\underline{\mu}}^i \end{bmatrix} \begin{bmatrix} K_i & 0 & 0 \\ 0 & 0 & S_0 \end{bmatrix},$$

where $\widehat{\Phi}_i = Q_i + K_{\hat{i}}^T R_{i\hat{i}} K_{\hat{i}} + A_q^T\left(\underline{\underline{\tau}}^i R_0 + \underline{\underline{\tau}}^i(\underline{\underline{\mu}}^i)^{-1}\right)A_q$. The application of the Schur-complement lemma and a congruence transformation with $diag\{I, I, I, I, \underline{\underline{\tau}}^i\}$ thereafter yields

$$0 \geq \begin{bmatrix} \widehat{\Phi}_i - \widetilde{P}_i & \mathcal{A}^T\widetilde{P}_i & \widetilde{P}_i H & K_i^T & 0 \\ \widetilde{P}_i\mathcal{A} & -\widetilde{P}_i & 0 & 0 & 0 \\ H^T\widetilde{P}_i & 0 & -H^T\widetilde{P}_i H + \underline{\underline{\tau}}^i\Xi_0 & 0 & S_0^T\underline{\underline{\tau}}^i \\ K_i & 0 & 0 & -R_{ii}^{-1} & 0 \\ 0 & 0 & \underline{\underline{\tau}}^i S_0 & 0 & -\underline{\underline{\tau}}^i(\underline{\underline{\mu}}^i)^{-1} \end{bmatrix}$$

Introduce new variables by definition $\nu^i_j \doteq \tau^i_j/\mu^i_j$, $(i = 1,2\ j = 1,\dots,s)$, then one obtains (43). On the other hand, condition (21) is equivalent to $(\underline{\underline{\mu}}^i)^{-1}\underline{\underline{\tau}}^i\Xi_0 + \underline{\underline{\tau}}^i S_0^T S_0 < 0$, which is the same as (42) in view of the definition of ν^i_j.

4.2. The case of linear-fractional uncertainty

In this subsection we investigate the case of $Q_{0j} = -I$, $R_{0j} = I$ and $S_{0j} = 0$, $(j = 1, \ldots, s)$. Being a special case, the results of the previous section are naturally applicable for it. However, this special case is worth of additional investigation, because the nominal system can be given in this case in such a way that the results are "better" than in the general case. If Assumption 2 is valid, then $(I - GG^T) > 0$, (equivalently $(I - G^T G) > 0$,) which implies that $(I - \Delta G)$ is invertible for any Δ satisfying condition $\Delta^T \Delta \le I$, thus the game (11), (12), (16) with (13) can also be described as

$$\begin{aligned} x_{t+1} &= (A + \delta A_t)x_t + B_1 u_t^1 + B_2 u_t^2, \\ \delta A_t &= H(I - \Delta_t G)^{-1} \Delta_t A_q, \end{aligned}$$

where $\Delta_t^T \Delta_t \le I$. For simplicity, we shall assume that Δ_t consists of one single block, i.e. $s = 1$ (though all the considerations below may be performed with Δ having appropriate block diagonal structure with $s > 1$). According to Lemma 2.5 of [27], the set

$$\Upsilon = \left\{ \widetilde{\Delta} = (I - \Delta G)^{-1} \Delta : \ \Delta^T \Delta \le I \right\}$$

can also be written as

$$\Upsilon = \left\{ \widetilde{\Delta} = G^T (I - GG^T)^{-1} + \Pi (I - GG^T)^{-1/2} : \ \Pi^T \Pi \le (I - G^T G)^{-1} \right\}.$$

Let us introduce notations $A_0 \doteq A + HG^T(I - GG^T)^{-1} A_q$ and $\mathcal{A}_0 \doteq A_0 + B_1 K_1 + B_2 K_2$. Consider

$$\begin{aligned} f_0(x, u^1, u^2) &= A_0 x + B_1 u^1 + B_2 u^2, \\ f_\Delta(x, u^1, u^2) &= (A + \delta A_t)x + B_1 u^1 + B_2 u^2 + Hp, \end{aligned}$$

and $f_\Delta \in \mathcal{F}$, if and only if $\Delta_t^T \Delta_t \le I$. As before, f_0 is called the nominal system. Applying the same considerations as in the previous subsection, one can derive the following result.

Theorem 4. *Suppose that Assumption 2 holds true, $\left(A_0, [B_1, B_2]\right)$ is stabilizable, $\left(A_0, Q_1 + Q_2\right)$ is detectable. If $P_i = P_i^T > 0$ and $\mu^i > 0$ are such that, for $i = 1, 2,$*

$$\widetilde{P}_i \doteq \left(P_i^{-1} - \mu^i H(I - G^T G)^{-1} H^T\right)^{-1} > 0, \tag{44}$$

$$P_i = Q_i + \frac{1}{\mu^i} A_q^T \left(I - GG^T\right)^{-1} A_q + K_1^T R_{i1} K_1 + K_2^T R_{i2} K_2 + \mathcal{A}_0^T \widetilde{P}_i \mathcal{A}_0, \tag{45}$$

$$\begin{bmatrix} R_{11} + B_1^T \widetilde{P}_1 B_1 & B_1^T \widetilde{P}_1 B_2 \\ B_2^T \widetilde{P}_2 B_1 & R_{22} + B_2^T \widetilde{P}_1 B_2 \end{bmatrix} \begin{bmatrix} K_1 \\ K_2 \end{bmatrix} = \begin{bmatrix} B_1^T \widetilde{P}_1 \\ B_2^T \widetilde{P}_2 \end{bmatrix} A_0, \tag{46}$$

then $\alpha_1(x) = K_1 x$, $\alpha_2(x) = K_2 x$ are admissible with respect to the nominal system, and they are Nash guaranteeing strategies with Nash guaranteed costs $V^i(x_0) = x_0^T P_i x_0$.

Remark 4. A remark analogous to Remark 2 can be formulated concerning Theorem 4.

Remark 5. Clearly, (44)–(46) are highly nonlinear. Therefore, one has to be satisfied with an approximate solution obtained by replacing the equality in (45) with "≥". Keeping in mind the considerations in Remark 3 and applying the matrix inversion — and the Schur complement lemma, and introducing new variables with the definition $\nu^i \doteq 1/\mu^i$, the following corollary is obtained.

Corollary 5. *Suppose that the conditions of Theorem 4 hold true. If $\widetilde{P}_i = \widetilde{P}_i^T > 0$, $\nu^i > 0$, K_1 and K_2 satisfy matrix Eq. (46) and matrix inequality*

$$0 \ge \begin{bmatrix} \Phi_i - \widetilde{P}_i & \mathcal{A}_0^T \widetilde{P}_i & \widetilde{P}_i H & K_i^T \\ \widetilde{P}_i \mathcal{A}_0 & -\widetilde{P}_i & 0 & 0 \\ H^T \widetilde{P}_i & 0 & -H^T \widetilde{P}_i H - \nu^i (I - G^T G) & 0 \\ K_i & 0 & 0 & -R_{ii}^{-1} \end{bmatrix} \qquad i = 1, 2, \tag{47}$$

where $\Phi_i = Q_i + K_{\hat{i}}^T R_{i\hat{i}} K_{\hat{i}} + \nu^i A_q^T \left(I - GG^T\right)^{-1} A_q$, then

$$P_i = \left(\widetilde{P}_i^{-1} + H(I - G^T G)^{-1} H^T / \nu^i\right)^{-1} > 0, \qquad i = 1, 2, \tag{48}$$

$\alpha_1(x) = K_1 x$ and $\alpha_2(x) = K_2 x$ are admissible approximate ϵ_o-Nash guaranteeing feedback strategies with approximate Nash guaranteed cost $V^i(x_0) = x_0^T P_i x_0$, where P_i is given by (48).

Remark 6. Relations (46) and (47) have quadratic in the decision variables, thus they are more tractable with available software tools than (45)–(46) .

Table 1
Results for Example 1 with the first type of uncertainties obtained with Corollaries 4 and 5.

Data	Corollary 4		Corollary 5	
$H = 0.1$	$\widetilde{P}_1 = 1.3161$	$\widetilde{P}_2 = 0.3225$	$\widetilde{P}_1 = 1.3139$	$\widetilde{P}_2 = 0.2241$
$A_q = 0.1$	$P_1 = 1.3028$	$P_2 = 0.2222$	$P_1 = 1.3006$	$P_2 = 0.2218$
$G = 0.1$	$K_1 = -0.3225$	$K_2 = -0.1100$	$K_1 = -0.3227$	$K_2 = -0.1101$
$H = 0.5$	$\widetilde{P}_1 = 2.809$	$\widetilde{P}_2 = 0.4556$	$\widetilde{P}_1 = 2.4842$	$\widetilde{P}_2 = 0.3972$
$A_q = 0.5$	$P_1 = 2.0263$	$P_2 = 0.3302$	$P_1 = 1.8373$	$P_2 = 0.2937$
$G = 0.2$	$K_1 = -0.3731$	$K_2 = -0.1210$	$K_1 = -0.3865$	$K_2 = -0.1236$

Table 2
Results for Example 1 with the second type of uncertainties obtained with Corollary 4.

Data	Corollary 4		Data	Corollary 4	
$H = 0.1$	$\widetilde{P}_1 = 1.3130$	$\widetilde{P}_2 = 0.2240$	$H = 0.5$	$\widetilde{P}_1 = 2.3465$	$\widetilde{P}_2 = 0.3845$
$A_q = 0.1$	$P_1 = 1.2998$	$P_2 = 0.2218$	$A_q = 0.5$	$P_1 = 1.7599$	$P_2 = 0.2885$
$G = -1$	$K_1 = -0.3220$	$K_2 = -0.1100$	$G = -1$	$K_1 = -0.3631$	$K_2 = -0.1548$

5. Numerical examples

The effectiveness of the proposed method will be illustrated first by an example, which is the modification of an example considered in [12]. As a real world application, the monetary–fiscal game mentioned in Section 2 will be analysed in Example 2. We note that the methods published in the previous literature were not applicable in these cases. The computations have been performed by MATLAB and YALMIP.

Example 1 (*[12]*)**.** To illustrate the effectiveness of our approach we consider the example of [12] modified so as to allow uncertainties of types discussed above. Consider (11), (12), (16) and (13) with $A = 1$, $B_1 = 2$, $B_2 = 1$, $Q_1 = 1$, $Q_2 = 0.2$, $R_{11} = 2$, $R_{22} = 0.5$ and $R_{12} = R_{21} = 0$. First we consider the uncertainty free case both with Corollaries 4 and 5, i.e. $H = 0$, $A_q = 0$, $G = 0$, $Q_0 = -1$, $S_0 = 0$, and $R_0 = 1$. Both methods yielded the same results as [12] (see Remarks 2, 4): $P_1 = 1.2854$, $P_2 = 0.2197$, $K_1 = -0.3205$ and $K_2 = -0.1096$. Consider again $Q_0 = -1$, $S_0 = 0$, $R_0 = 1$ and nonzero uncertainties, the parameters of which and the results of computations are given in Table 1. One can see that both methods tolerate relatively large uncertainties, and making use of the special structure of the uncertainties at choosing the nominal system yields smaller guaranteed costs.

Secondly, consider the case of cone-bounded nonlinear uncertainty described by $Q_0 = 0$, $R_0 = 0$, $S_0 = 1$ and $G = -1$. It can be seen that this means the following bounds for p

$$-\left|\frac{A_q x}{2}\right| + \frac{A_q x}{2} \le p \le \left|\frac{A_q x}{2}\right| + \frac{A_q x}{2}.$$

The results obtained with Corollary 4 are given in Table 2.

Example 2. Consider the uncertain fiscal-monetary game, the dynamics of which is given by (1)–(4) with the objective functions (5) and (6). To determine the Nash guaranteeing feedback strategies and the corresponding guaranteed costs belonging to an initial point, parameters α_1, α_2 and β must be specified. We rely on some benchmark values found in the literature. Regarding parameter α_1, a cross-country analysis of [7] found that it is from the interval $[0.056; 0.180]$. Parameter α_2 expresses the impact of the balance of the central budget on the output gap, it is estimated around 0.3. [8] found that parameter β may take its value from interval $[0.18; 0.41]$. Introduce the new variables $\widetilde{\pi} = \pi - \pi^*$ and $\widetilde{i} = i - i^*$. By this transformation one obtains the dynamic system

$$\begin{bmatrix} z_{t+1} \\ \widetilde{\pi}_{t+1} \end{bmatrix} = \begin{bmatrix} 0 & \alpha_1 \\ \beta & 1 \end{bmatrix} \begin{bmatrix} z_t \\ \widetilde{\pi}_t \end{bmatrix} + \begin{bmatrix} \alpha_1 (\pi^* - i^*) \\ 0 \end{bmatrix} + \begin{bmatrix} -\alpha_1 & \alpha_2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \widetilde{i}_t \\ \widetilde{g}_t \end{bmatrix} + \begin{bmatrix} 0 & \alpha_1 \\ \beta & 1 \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \end{bmatrix}$$

If $\pi^* = i^*$, the identically zero solution, where $z_t \equiv 0$, $\widetilde{\pi}_t \equiv 0$, $\widetilde{i}_t \equiv 0$, $g_t \equiv 0$, is an equilibrium point of the system with $p_1 = 0$ and $p_2 = 0$.

Let

$$x_t = \begin{bmatrix} z_t \\ \widetilde{\pi}_t \end{bmatrix}, \qquad u_t^1 = g_t, \qquad u_t^2 = \widetilde{i}_t,$$

$$A = \begin{bmatrix} 0 & \alpha_1 \\ \beta & 1 \end{bmatrix}, \qquad B_1 = \begin{bmatrix} \alpha_2 \\ 0 \end{bmatrix}, \qquad B_2 = \begin{bmatrix} -\alpha_1 \\ 0 \end{bmatrix}, \qquad H = \begin{bmatrix} 0 & \alpha_1 \\ \beta & 1 \end{bmatrix},$$

$$A_{q_1} = \begin{bmatrix} \delta_1 & \delta_1 \end{bmatrix}, \qquad A_{q_2} = \begin{bmatrix} \delta_2 & \delta_2 \end{bmatrix},$$

$$Q_1 = \begin{bmatrix} \gamma_1 & 0 \\ 0 & 0 \end{bmatrix}, \qquad Q_2 = \begin{bmatrix} 0 & 0 \\ 0 & \gamma_2 \end{bmatrix}, \qquad R_1 = \varrho_1, \qquad R_2 = \varrho_2,$$

Table 3
Results for Example 2 with uncertainties (49) obtained with Corollary 4.

Data (1)	Corollary 4		Data (2)	Corollary 4	
$\alpha_1 = 0.118$	$\lambda_M(\widetilde{P}_1) = 1.690$	$\lambda_M(\widetilde{P}_2) = 1.289$	$\alpha_1 = 0.125$	$\lambda_M(\widetilde{P}_1) = 1.594$	$\lambda_M(\widetilde{P}_2) = 1.264$
$\alpha_2 = 0.3$	$\lambda_M(P_1) = 1.183$	$\lambda_M(P_2) = 1.246$	$\alpha_2 = 0.3$	$\lambda_M(P_1) = 1.068$	$\lambda_M(P_2) = 1.104$
$\beta = 0.295$	$K_1 = -[0.684$	$2.582]$	$\beta = 0.325$	$K_1 = -[0.692$	$2.406]$
	$K_2 = \quad [0.981$	$3.457]$		$K_2 = \quad [1.145$	$3.672]$

where the parameters of the running cost have been chosen as $\gamma_1 = 0.2$, $\gamma_2 = 0.2$, $\varrho_1 = 0.01$ and $\varrho_2 = 0.005$.

If no uncertainties are present, i.e. $\delta_1 = \delta_2 = 0$ and $H = 0$, then the following results are obtained with Corollary 4:
for $\alpha_1 = 0.118$, $\alpha_2 = 0.3$ and $\beta = 0.295$: $\lambda_M(P_1) = 0.8522$, $\lambda_M(P_2) = 0.9857$, $|\lambda(\mathcal{A})| \le 0.7877$,
for $\alpha_1 = 0.125$, $\alpha_2 = 0.3$ and $\beta = 0.325$: $\lambda_M(P_1) = 0.8288$, $\lambda_M(P_2) = 0.9436$, $|\lambda(\mathcal{A})| \le 0.7624$.
(Note the P_1 and P_2 coincide with $\widetilde{P}_1$ and $\widetilde{P}_2$ up to 15 digits.)

Suppose that nonzero uncertainties are present, and the scalar valued uncertainties p_1 and p_2 are bounded according to $Q_{01} = -1$, $R_{01} = 1$, $S_{01} = 0.5$, $G_1 = -1$, and $Q_{02} = 0$, $R_{02} = 0$, $S_{02} = 1$ and $G_2 = -1$. This means that

$$-\frac{\sqrt{5}}{2}\left|\frac{A_{q1}x}{2}\right| - \frac{A_{q1}x}{2} \le p_1 \le \frac{\sqrt{5}}{2}\left|\frac{A_{q1}x}{2}\right| - \frac{A_{q1}x}{2}, \qquad -\left|\frac{A_{q2}x}{2}\right| + \frac{A_{q2}x}{2} \le p_2 \le \left|\frac{A_{q2}x}{2}\right| + \frac{A_{q2}x}{2}. \tag{49}$$

One can see that $\Xi_{01} = -1$, $\Xi_{02} = -2$, the pair $\left(A, \left[B_1, B_2\right]\right)$ is controllable, $Q_1 + Q_2 > 0$ thus Assumptions 2–4 are satisfied. Under the choice of $\delta_1 = 0.08$ and $\delta_2 = 0.1$, the application of Corollary 4 yielded the results given in Table 3. For the initial values $x_0 = \begin{bmatrix}-0.07 & 0.15\end{bmatrix}$, $V^1(x_0) = 0.0253$ and $V^2(x_0) = 0.0182$ were obtained as guaranteed costs for the parameter set in column data (1) of Table 3, and $V^1(x_0) = 0.0227$ and $V^2(x_0) = 0.0172$ for the parameter set data (2). Simulations were performed with the uncertain functions

$$p_1 = \begin{cases} -\frac{y_1}{2} + \frac{\sqrt{5}|y_1|}{2}\sin(\frac{1}{|y_1|}), & \text{if } y_1 \neq 0, \\ 0, & \text{if } y_1 = 0, \end{cases} \qquad p_2 = \begin{cases} -\frac{y_2}{2} + \frac{|y_2|}{2}\sin(\frac{1}{|y_2|}), & \text{if } y_2 \neq 0, \\ 0, & \text{if } y_2 = 0, \end{cases}$$

where $y_i = A_{qi}x$, $(i = 1, 2)$ which satisfy the bounding conditions (49). The trajectories with the two parameter sets data (1) and data (2) of Table 3 are depicted in Fig. 1.

It can be seen that the proposed method is capable to compute the approximate Nash equilibrium solutions under relatively large uncertainties. Further, the monetary–fiscal game model is suitable to evaluate and compare different catch-up scenarios for an economy. If predetermined reference paths for the state and control variables, i.e. for x_t^*, π_t^*, g_t^* and i_t^* are given, one can compute the corresponding actual paths and other economically relevant quantities (as e.g. the government debt). A detailed analysis exceeds the frames of the present paper, it can be the topic of further research.

6. Conclusion

In this paper, the problem of uncertain/nonlinear two players discrete-time noncooperative games was investigated. The uncertainty of the model prevent to find the classical Nash equilibrium solution, instead, properly defined Nash guaranteeing strategies and Nash guaranteed costs were determined. The system's uncertainties/nonlinearities were assumed to be of quadratically bounded type. Conditions of the Nash guaranteeing strategies were derived for general uncertain nonlinear games with the aim of serving as a guideline to the solution process. These results are applied for games that have linear nominal part, quadratic cost functions and general quadratically bounded uncertainties/nonlinearities. The special case of linear-fractional uncertainties were also discussed. Approximate solutions were obtained by tractable quadratic matrix inequalities. To illustrate the application of the proposed method, first an academic numerical example was given. Secondly, the method was also applied to the monetary–fiscal game, which may support a sound economic policy for the catching-up economies to prevent countercyclical policies that may lead to unwanted increase of state debt and to high inflation weakening the domestic currency. A limitation of the proposed approach is that the nonlinearities do not depend on controls in the present model, and measurement errors are not taken into consideration either: these are open questions for further research. A further limitation concerns the economic model, in which the state debt is not endogenized in the monetary–fiscal game model, although it can be calculated for any simulated path. It is still an open question how it can be included in the dynamic equation, which still remains tractable applying the proposed approach.

CRediT authorship contribution statement

Éva Gyurkovics: Conceptualization, Formal analysis, Investigation, Methodology, Software. **Tibor Takács:** Conceptualization, Formal analysis, Investigation, Methodology.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

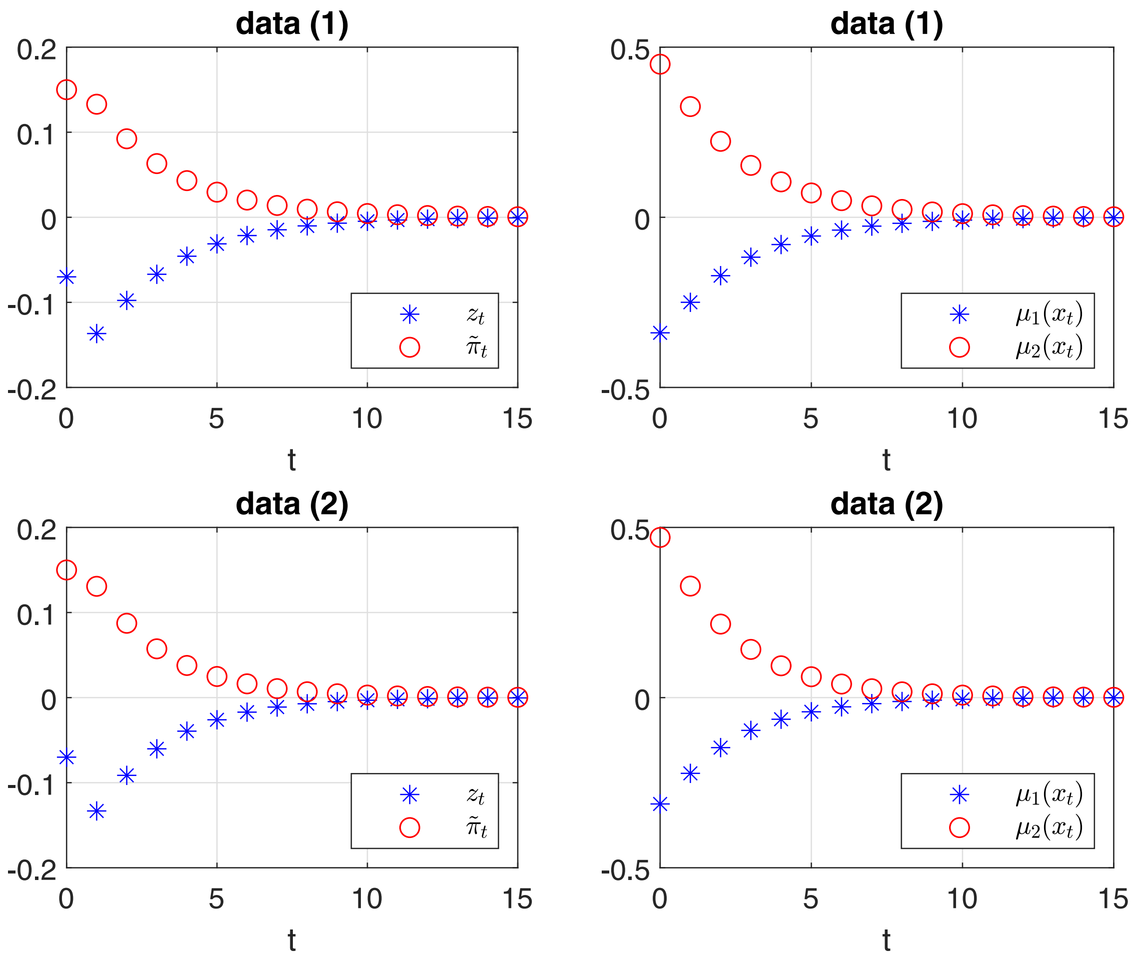

Fig. 1. Trajectories and feedback strategies with initial values $z_0 = -0.07$ and $\tilde{\pi}_0 = 0.15$.

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