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COMPREHENSIVE ANALYSIS OF KERNEL-BASED INTERIOR-POINT METHODS FOR $P_*(\kappa)$ - LCP

ZSOLT DARVAY*, MARIANNA E.-NAGY†, GORAN LESAJA‡, PETRA RENÁTA RIGÓ§, AND ANITA VARGA¶

Abstract. We present an interior-point algorithmic framework for $P_*(\kappa)$-Linear Complementarity Problems that is based on a barrier function which is defined by a new class of univariate kernel functions called *Standard Kernel Functions* (SKFs). A unified, comprehensive complexity analysis of the generic interior-point method is provided and a general procedure to determine the iteration bounds for long-step and short-step versions of the method for the entire class of SKFs is developed. We illustrate the general procedure by determining the iteration bounds for several parametric SKFs which include all SKFs that appeared in the literature as special cases. In all cases, we matched the best iteration bounds obtained in the literature for these special cases of SKFs.

Key words. Linear complementarity problems, $P_*(\kappa)$-matrix, Interior-point methods, Kernel functions, Polynomial complexity

JEL codes. C61

1. Introduction. In this paper, we consider a class of Linear Complementarity Problems (LCPs) formulated in the standard form: given a matrix $M \in \mathbf{R}^{n\times n}$ and a vector $q \in \mathbf{R}^n$, find the vectors x and s such that

$$s = Mx + q, \quad x, s \geq 0, \quad xs = 0, \tag{1.1}$$

where xs denotes the componentwise (Hadamard) product of the vectors x and s.

Although this is a feasibility and not an optimization problem, it is closely related to optimization problems. It is well known that the Karush-Kuhn-Tucker (KKT) optimality conditions for Linear Optimization (LO) and Convex Quadratic Optimization can be written as LCPs. Moreover, many important practical problems in economics theory (equilibrium problems), game theory, transportation planning (assignment problems), optimal control, engineering, etc. can be directly formulated as LCPs. The comprehensive treatment of the theory and practice of LCPs can be found in the monographs of Cottle *et al.* [9], Fachinei and Pang [18], and Kojima *et al.* [20].

Due to the theoretical and practical importance of the LCPs, efficient methods for solving LCPs are of significant interest. The existing tradition of generalizing results for LO to LCP dates back to the early days of the development of simplex-type algorithms (pivoting algorithms), and it continues to this day. Interior-Point Methods (IPMs) that have been a great success for LO are no exception. Various IPMs for LO have been successfully generalized to LCPs. In addition, to the aforementioned

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monograph of Kojima *et al.* [20], and without any attempt to be complete, we mention a few other relevant references: [2, 3, 11, 17, 19, 21, 24, 26, 33].

Path-following IPMs can be classified as short-step (small-update) methods that take small steps near the central path and long-step (large-update) methods that take aggressive steps and deviate farther from the central path.

The theoretical iteration bound for most classical large-update methods based on the logarithmic barrier function is $O\left(n \ln \frac{n}{\varepsilon}\right)$, where n denotes the number of variables in the problem, and ε is the desired accuracy in terms of the objective value. The iteration bound for small-update IPMs is substantially better: $O\left(\sqrt{n} \ln \frac{n}{\varepsilon}\right)$. However, small-update IPMs are inefficient in practice, while large-update methods usually perform much better. This phenomenon is known as the *gap between theory and practice*, also called the *irony of IPMs* [28]. It was the primary motivation to try to design large-update IPMs with improved complexity, ideally with the same complexity as small-update methods.

In the effort to improve the theoretical complexity of long-step IPMs, several approaches can be observed. The first approach was considering higher-order methods based on the logarithmic barrier function. For example, the higher order methods proposed in [23, 27] have the complexity $O((1+\kappa)\sqrt{n} \ln \frac{n}{\varepsilon})$, do not depend on κ, and are superlinearly convergent even in the absence of strict complementarity. Parameter κ is called the handicap of the $P_*(\kappa)$-matrix which is defined in the next section.

Another direction was the IPM developed by Ai and Zhang [1]. Their approach is based on two key ideas: a new wide neighborhood definition and the decomposition of the Newton directions into two components. To determine the search directions, in each iteration, two linear systems need to be solved with the same coefficient matrix and different right-hand sides. It is also important that the steplengths assigned to the components are of different magnitudes. With an appropriate choice of parameters, they were able to close the gap and show that the iteration bound of their large-update method is the same as the iteration bound of short-step methods.

In order to define new search directions, Tuncel and Todd [30] introduced an early version of a reparametrization of the central path system. Later on, Darvay proposed a different method, called algebraically equivalent transformation (AET) technique [10], which consisted of dividing the perturbed complementarity equation by the barrier parameter μ and applying a continuously differentiable, invertible function to both sides of the equation. Subsequently, other mappings or function classes were introduced to design AET-based IPMs [11, 12, 17, 19]. In [14], the authors presented a new type of AET technique and discussed its relationship with the kernel function-based method. Darvay and Rigó [13] proposed an IPM based on the AET technique and defined the new concept of the positive-asymptotic kernel function.

One more direction to improve the theoretical complexity of long-step algorithms was to consider first-order IPMs based on barrier functions different from the logarithmic barrier function. The barrier functions were defined as separable functions of univariate functions called kernel functions. Peng *et al.* in [25] analyzed primal-dual IPMs for LO based on a class of barrier functions that are defined by so-called self-regular kernel functions. They managed to significantly improve the theoretical complexity of first-order large-update IPMs, obtaining the currently best-known iteration bound for these types of methods, namely $O(\sqrt{n} \ln n \ln \frac{n}{\varepsilon})$. Subsequently, Bai *et al.* [5, 6] presented primal-dual IPMs for LO based on another class of kernel functions called *eligible kernel functions* (EKFs), that are not necessarily self-regular. For some of them, they matched the best iteration bounds for kernel-based large-update IPMs.

These results were extended to $P_*(\kappa)$-LCPs for the entire class of EKFs in [21] and then further generalized to the Cartesian $P_*(\kappa)$-LCPs over symmetric cones in [22]. In [21], a unified and comprehensive convergence analysis was provided and a standardized Scheme was developed which streamlines the calculation of iteration bounds for specific EKFs. The Scheme was then used to find iterations bounds for small- and large-update IPMs for a list of specific EKFs. However, the derivation of the iteration bounds for some EKFs can be quite long and involved, which was the motivation to investigate whether this process can be further analyzed and developed, making the derivation of iteration bounds for specific EKFs simpler and more straightforward. This was one goal of the paper. The other goal was to define as large as possible subclass of EKFs for which a unified complexity analysis of long- and short-step IPMs for solving $P_*(\kappa)$-LCPs can be derived.

In addition to developing a unified analysis, we also aimed to understand the relationship between EKF properties better. In the literature, there is a sufficient condition for checking one of the EKF properties. We proved that this condition is sufficient (in combination with the other condition) but not necessary by giving an example. Furthermore, we proposed new sufficient conditions for checking this property. Analyzing the EKFs used in the literature, we observed that in most cases, they satisfy certain additional conditions. These conditions give lower and upper bounds on the kernel function and its first and second derivatives. They are not restrictive, however, they allow for the in-depth and unified analysis of the iteration complexity of the related IPMs. In this way, they are leading to a much simpler and straightforward procedure to derive iteration bounds for specific EKFs.

In this paper, we concentrate on EKFs with rational and exponential barrier terms. The EKFs with trigonometric barrier terms satisfy different conditions than the rational and exponential EKFs and they will be considered in a separate paper. Although all EKFs with rational or exponential barrier terms that appear in the literature satisfy the additional properties, we cannot claim that the whole class of EKFs satisfies these conditions. We call this subclass of EKFs the class of *Standard Kernel Functions* (SKFs) emphasizing that they frequently appear in the literature on this topic. We also provide examples of EKFs that are not SKFs. However, they have not appeared in the literature so far and the complexity results of IPMs based on these functions are not as good as of IPMs with some instances of SKFs. We provide a comprehensive and unified analysis of the iteration complexity for IPMs based on the SKFs and derive iteration bounds for several parametric SKFs, matching the iteration bounds obtained using the classical scheme in [21].

The outline of the paper is as follows. In Section 2, we review the $P_*(\kappa)$-LCP and generic kernel-based IPM for $P_*(\kappa)$-LCP. In Section 3, the class of SKFs is introduced and discussed. Section 4 is the main section of the paper where the iteration complexity of the generic IPM based on the SKF is analyzed and iteration bounds for long- and short-step versions of the method are derived. In Section 5, we derive iteration bounds for long- and short-step methods for several parametric SKFs with rational and exponential barrier terms. Section 6 contains concluding remarks and directions for future research. In the Appendix, we illustrate the procedure of verifying that EKF belongs to SKF for two specific EKFs, one with rational and one with exponential barrier term.

Some notations used throughout the paper are presented as follows. We use the standard notation, $\mathbf{R}^n$, $\mathbf{R}^n_+$ and $\mathbf{R}^n_{++}$ to denote the set of (real) vectors with n components, the set of nonnegative vectors, and the set of positive vectors, respectively. The 2-norm of the vector x is denoted by $\|x\|$. The bold symbol $\mathbf{e}$ always denotes the all-one vector with n components. Finally, for the real-valued functions $f(x) \geq 0$ and

$g(x)$ the notation $g(x) = O(f(x))$ means that there exists $\bar{x}$ such that $|g(x)| \leq \bar{c}f(x)$ for $x \geq \bar{x}$ and some positive constant $\bar{c}$. The notation $g(x) = \Theta(f(x))$ denotes that $c_1 f(x) \leq |g(x)| \leq c_2 f(x)$ for $x \geq \bar{x}$ and two positive constants c_1 and c_2.

2. The generic IPM for $P_*(\kappa)$-LCP. As indicated in the Introduction, in this paper, we consider LCPs in the standard form (1.1). It is known that for general matrices M the problem is NP-complete [8]. However, for the LCPs with $P_*(\kappa)$-matrix as a coefficient matrix, it has been shown that the LCP can be solved in polynomial time in the size of the problem and in the handicap of the matrix, which will be defined in the next paragraph.

A matrix M satisfies the $P_*(\kappa)$-property if

$$\forall x \in \mathbf{R}^n \; : \; (1+4\kappa) \sum_{i \in \mathcal{I}^+(x)} x_i (Mx)_i + \sum_{i \in \mathcal{I}^-(x)} x_i (Mx)_i \geq 0 \; , \tag{2.1}$$

where

$$\mathcal{I}^+(x) = \{ i \; : \; x_i (Mx)_i > 0 \} \; , \; \; \mathcal{I}^-(x) = \{ i \; : \; x_i (Mx)_i < 0 \} \; .$$

The smallest parameter κ for which the $P_*(\kappa)$-property holds is called the *handicap* of the matrix M. We also define the class

$$P_* = \bigcup_{\kappa \geq 0} P_*(\kappa) \; . \tag{2.2}$$

The class of $P_*(\kappa)$ matrices was introduced by Kojima *et al.* in [20]. Especially interesting and nontrivial is the fact that P_* matrices are equivalent to the class of sufficient matrices defined by Cottle *et al.* [9]. This was proven by Väliaho in [31]. For more information on $P_*(\kappa)$ and sufficient matrices, we refer the reader to [16].

One of the reasons for considering the class of $P_*(\kappa)$-matrices is the fact that this is the largest class for which the convergence of the iterates of an IPM to the solution is guaranteed with the only condition that the strict interior of the feasible region is nonempty [20].

We make a standard assumption that the $P_*(\kappa)$-LCP satisfies the interior-point condition (IPC), i.e., there exists a point $x^0 > 0$ such that $s^0 = Mx^0 + q > 0$, which means that the strict interior of the feasible region is not empty. The IPC can be assumed without loss of generality. In [20] Kojima *et al.* presented a method of "reducing the LCP to an artificial LCP with an apparent interior feasible point".

The standard approach of IPMs for solving the $P_*(\kappa)$-LCP given in (1.1) is to consider the parameterized system

$$s = Mx + q, \quad x, s \geq 0, \quad xs = \mu \mathbf{e}, \tag{2.3}$$

where μ is a positive parameter.

Since we assume that IPC holds, and M is a $P_*(\kappa)$-matrix, the parameterized system (2.3) has a unique solution for each $\mu > 0$, see Lemma 4.3 in [20]. This solution is denoted as $(x(\mu), s(\mu))$ and it is called the μ-center of the LCP. The set of μ-centers (with μ running through all positive real numbers) gives a homotopy path, which is called *the central path* of the LCP. Under the above assumptions, if $\mu \to 0$, the limit of the central path exists and it is a solution of the LCP, see Theorem 4.4 in [20].

The IPMs trace the central path while reducing μ at each iteration. However, tracing the central path exactly would be too costly and inefficient. One of the main achievements of the theory of IPMs was the fact that it is sufficient to trace the central path approximately, within a certain neighborhood of the central path, while still achieving global convergence and favorable local convergence properties.

A wide variety of IPMs can be designed depending on the way the search direction is calculated, which is usually done using some version of Newton's type method; the way a neighborhood of the central path is defined, which influences the calculation of a step size; the choice of the parameters, etc.

We use the IPM based on the general barrier function which is a strictly convex function $\Psi(v),\ v \in \mathbf{R}^n_{++}$, where

$$v := \sqrt{\frac{xs}{\mu}}, \quad (x, s) > 0, \quad \mu > 0, \tag{2.4}$$

is a *variance vector* that plays an important role in the design and analysis of the algorithm. Note that the pair (x, s) coincides with the μ-center $(x(\mu), s(\mu))$ if and only if $v = \mathbf{e}$. Therefore, we require that $\Psi(v)$ is minimal at $v = \mathbf{e}$ and $\Psi(\mathbf{e}) = 0$, i.e.

$$\Psi(v) = 0 \quad \Leftrightarrow \quad \nabla\Psi(v) = 0 \quad \Leftrightarrow \quad v = \mathbf{e}. \tag{2.5}$$

Thus, the function $\Psi(v)$ serves as a proximity measure of closeness to the μ-center. Introducing a parameter $\tau > 0$ as a threshold value, the τ-neighborhood of the central path is defined as

$$\mathcal{N}_\Psi(\tau) = \{v \in \mathbf{R}^n_{++} : \Psi(v) \leq \tau\}. \tag{2.6}$$

In the analysis of the algorithm, we also use a norm-based proximity measure, which is defined as

$$\delta(v) := \frac{1}{2}\|\nabla\Psi(v)\|. \tag{2.7}$$

This is a proximity measure because $\delta(v) = 0 \Leftrightarrow v = \mathbf{e}$, that is, $\delta(v) = 0$ on the central path.

In this paper, we use the IPM based on the general barrier function described in Algorithm 2.1, which is the same method as in [21]. However, we assume that $\Psi(v)$ is expressed as a separable barrier function based on a kernel function, as is typical in existing methods from the literature. In the sequel, we call this algorithmic framework simply the *Algorithm*.

Note that the Algorithm can be started because due to the result given in [20], we can provide a strictly feasible starting point on the central path. In what follows, we give a brief overview of one iteration of the Algorithm. Suppose that the current iterate (x, s) is in the τ-neighborhood of the corresponding μ-center, i.e., $\Psi(v) \leq \tau$. The new iteration consists of two parts, the outer iteration followed by several inner iterations.

In the outer iteration, the value of the parameter μ is reduced by the factor $1 - \theta$, where $0 < \theta < 1$, which defines a new μ-center $(x(\mu), s(\mu))$ and changes the value of v according to (2.4). This will likely cause an increase in the value of the barrier function above the threshold value τ, i.e., $\Psi(v) \geq \tau$, with the new value of v.

In the second part of the main iteration, a sequence of inner iterations is performed to obtain the iterate that is again in the τ-neighborhood of the central path (2.6). Keeping the μ-parameter fixed, we obtain a new inner iterate (x_+, s_+) by firstly calculating the search directions Δx and Δs from the Newton system

$$\begin{aligned} -M\Delta x + \Delta s &= 0, \\ s\Delta x + x\Delta s &= -\mu v \nabla\Psi(v). \end{aligned} \tag{2.8}$$

Hence, the barrier function $\Psi(v)$ also plays a crucial role in calculating the search directions. Next, the update is calculated as

$$x_+ = x + \alpha\Delta x, \quad s_+ = s + \alpha\Delta s, \tag{2.9}$$

where $\alpha \in (0,1)$ denotes the step size, which has to be chosen appropriately to reduce the value of the barrier function. We use the default feasible steplength

$$\tilde{\alpha} = \frac{1}{(1+2\kappa)\psi''\left(\rho\left(2\delta(v)\right)\right)}, \tag{2.10}$$

where ψ is the kernel function defined in Definition 3.1, ρ is the inverse function of $-\frac{1}{2}\psi'$ (see Definition 4.3) and $\delta(v)$ is introduced in (2.7). This step length is derived in [21] and leads to a sufficiently good decrease of the barrier function guaranteeing global convergence and achieving best-known iteration bounds.

Algorithm 2.1 Algorithmic framework of IPM for LCP

Input:

A general kernel function ψ (see Definition 3.1) and its associated barrier function $\Psi(v) = \sum_{i=1}^{n} \psi(v_i)$ as introduced in (3.3);
a threshold parameter $\tau \geq 1$;
an accuracy parameter $\varepsilon > 0$;
a barrier update parameter $0 < \theta < 1$;
a starting point $(x^0, s^0), \mu^0 = \frac{(x^0)^T s^0}{n}, v^0 = \sqrt{\frac{x^0 s^0}{\mu^0}}$ s.t. $\Psi(v^0) \leq \tau$.

$x := x^0$; $s := s^0$; $\mu := \mu^0$;
while $n\mu \geq \varepsilon$ **do**
  $\mu := (1-\theta)\mu$;
  $v := \sqrt{\frac{xs}{\mu}}$;
  while $\Psi(v) > \tau$ **do**
    calculate the search direction $(\Delta x, \Delta s)$ using (2.8);
    calculate the default feasible steplength using (2.10);
    update $x := x + \tilde{\alpha}\Delta x$; $s := s + \tilde{\alpha}\Delta s$.
  end while
end while

This procedure is repeated until we find an iterate that again belongs to the τ-neighborhood (2.6) of the current μ-center. At this point, the current main iteration is completed and we start a new main iteration in the same way. This process is repeated until μ is small enough, $n\mu < \varepsilon$ for a certain (small) value of the accuracy parameter $\varepsilon > 0$.

As already mentioned, regarding the step length there are two types of methods: small-update (short-step) methods that take small steps close to the central path and large-update (long-step) methods, that take more aggressive steps farther away from the central path. The choice of the values for parameters τ and θ determines the type of the method. Large-update methods are characterized by the fact that θ is a fixed constant $\theta \in (0,1)$, independent of the dimension n of the problem, i.e. $\theta = \Theta(1)$ while $\tau = O(n)$, whereas small-update methods use a value of θ that depends on the dimension of the problem, with $\theta = \Theta\left(\frac{1}{\sqrt{n}}\right)$ while τ is a constant, i.e. $\tau = O(1)$.

The resulting iteration bound depends not only on a careful selection of the parameters in the Algorithm; it also heavily depends on the choice of the barrier function.

We will analyze the Algorithm for a wide class of barrier functions based on the class of kernel functions that we call SKFs which will be defined in the next section.

3. Standard Kernel Functions. As indicated in the previous section, the behavior of the Algorithm heavily depends on the choice of the parameters and the barrier function.

The classical *(scaled) logarithmic barrier function*

$$\Psi_c(v) := \sum_{i=1}^{n} \left(\frac{v_i^2 - 1}{2} - \ln v_i \right), \tag{3.1}$$

was used first and most frequently. Notice that (3.1) is a separable function with the univariate function

$$\psi_c(t) := \frac{t^2 - 1}{2} - \ln t \tag{3.2}$$

that is used for each component v_i of the variance vector v. The function $\psi_c(t)$ is called *logarithmic kernel function.*

Hence, we generalize the logarithmic barrier function and consider a class of barrier functions that are separable functions of the form

$$\Psi(v) := \sum_{i=1}^{n} \psi(v_i), \tag{3.3}$$

where the univariate function $\psi : (0, \infty) \to [0, \infty)$ is called a *kernel function* of $\Psi(v)$.

Definition 3.1. *The univariate twice continuously differentiable function ψ : $(0, \infty) \to [0, \infty)$ is called a* General Kernel Function *(GKF) if it satisfies the following conditions:*

$$\psi'(1) = \psi(1) = 0, \tag{GKF-a}$$

$$\psi''(t) > 0, \tag{GKF-b}$$

$$\lim_{t \downarrow 0} \psi(t) = \lim_{t \to \infty} \psi(t) = \infty. \tag{GKF-c}$$

Remark 3.2. *The barrier condition* $\lim_{t\downarrow 0} \psi(t) = \infty$ *can be relaxed by allowing $\psi(t)$ to have a finite value at $t = 0$, that is, $\psi(0) = \psi_0$, see [4]. The barrier condition* $\lim_{t\downarrow 0} \psi(t) = \infty$ *can be also replaced by* $\lim_{t\downarrow \xi} \psi(t) = \infty$*, $0 \le \xi < 1$. This leads to the notion of the positive-asymptotic kernel function, which was introduced in [13]. In this paper, we assume the classical barrier condition holds.*

Clearly, (GKF-a) and (GKF-b) indicate that $\psi(t)$ is a nonnegative strictly convex function such that $\psi(t)$ achieves its minimum at $t = 1$, i.e, $\psi(1) = 0$. This implies that since $\psi(t)$ is twice differentiable, it is completely determined by its second derivative:

$$\psi(t) = \int_1^t \int_1^\xi \psi''(\zeta)\, d\zeta d\xi. \tag{3.4}$$

Moreover, (GKF-c) indicates that $\psi(t)$ is coercive and has the barrier property. Furthermore, the conditions imply that $\psi(t)$ is decreasing for $0 < t < 1$ (barrier behaviour) and increasing for $t > 1$ (growth behavior).

The class of Eligible Kernel Functions (EKFs) was introduced in [5] for LO and then used in [21] to design large-update IPMs to solve $P_*(\kappa)$-LCP with improved iteration bounds of $O\left((1+2\kappa)\sqrt{n}\ln n \ln \frac{n}{\varepsilon}\right)$ as opposed to $O\left((1+2\kappa)n \ln \frac{n}{\varepsilon}\right)$ for logarithmic kernel function (3.2).

Definition 3.3. *A GKF that satisfies the following additional properties:*

$$\text{(EKF-a)} \qquad t\psi''(t) + \psi'(t) > 0, \quad t < 1,$$
$$\text{(EKF-b)} \qquad \psi'''(t) < 0, \quad t > 0,$$
$$\text{(EKF-c)} \qquad 2\psi''(t)^2 - \psi'(t)\psi'''(t) > 0, \quad t < 1,$$
$$\text{(EKF-d)} \qquad \psi''(t)\psi'(\beta t) - \beta\psi'(t)\psi''(\beta t) > 0, \quad t > 1,\ \beta > 1,$$

is called an Eligible Kernel Function (EKF).

Remark 3.4. *Several observations regarding the properties of EKFs are listed.*
(i) The following condition is also discussed in [5]:

$$t\psi''(t) - \psi'(t) > 0, \quad t > 1. \tag{3.5}$$

This condition is listed because properties (3.5) and (EKF-b) imply condition (EKF-d) (Lemma 4.4 in [5]). The reason for the introduction of condition (3.5) is that it is easier to check property (3.5) than (EKF-d) which is more technically involved.
(ii) Condition (EKF-a) is obviously satisfied if $t \geq 1$, because then $\psi'(t) \geq 0$. Similarly, condition (3.5) is satisfied if $t \leq 1$, since then $\psi'(t) \leq 0$. Another obvious but important consequence of condition (EKF-b) is that $\psi''(t)$ is decreasing for $t > 0$.
(iii) It is also shown in [5] that properties (EKF-a), (EKF-b), (EKF-c), and (3.5) are logically independent.

In the lemma below we provide an equivalent formulation of the property (EKF-d).

Lemma 3.5. *The property (EKF-d) holds if and only if for all $\beta > 1$ the function $\dfrac{\psi'(t)}{\psi'(\beta t)}$ is monotone increasing for $t > 1$.*

Proof. Since ψ is twice continuously differentiable and $\psi'(t) > 0$ for all $t > 1$, we can take the derivative of the above function:

$$\left[\frac{\psi'(t)}{\psi'(\beta t)}\right]' = \frac{\psi''(t)\psi'(\beta t) - \beta\psi'(t)\psi''(\beta t)}{(\psi'(\beta t))^2}.$$

The numerator of the right-hand side is exactly the expression in (EKF-d), while the denominator is positive, which proves the statement of the lemma. □

Note that the EKFs in the literature consist of two terms, the growth term $\psi_g(t)$ and the barrier term $\psi_b(t)$, i.e. $\psi(t) = \psi_g(t) + \psi_b(t)$.

To our best knowledge, three groups of barrier terms appeared in the literature: rational, exponential, and trigonometric barriers. In this paper, we deal with EKFs with rational and exponential barrier terms. Our aim was to consider parametric EKFs that cover the known EKFs with rational and exponential barrier terms from the literature, see Tables 1 and 3. EKFs with trigonometric barrier terms appeared, for example, in [32]. A comprehensive analysis of this class of EKFs will be considered in a separate paper.

The growth term governs the behavior of EKF for $t \geq 1$, while the barrier term governs the behavior of EKF for $t \leq 1$. To our best knowledge, all of the EKFs in the literature have the growth term $\psi_g(t) = \frac{t^{p+1}-1}{p+1}$, $p \in [0,1]$, in most of the cases with $p = 1$. Therefore, throughout the paper, we will focus on the case $\psi_g(t) = \frac{t^{p+1}-1}{p+1}$, $p \in [0,1]$. In this case, using (GKF-a), we have

$$\psi_b(1) = 0, \quad \psi_b'(1) = -1. \tag{3.6}$$

LEMMA 3.6. *Let $\psi(t) = \frac{t^{p+1}-1}{p+1} + \psi_b(t)$, where $p \in [0,1]$ and $\psi_b'(t) < 0$, $\psi_b''(t) > 0$, for $t > 1$. Then, (EKF-d) holds for ψ if the function $\frac{\psi_b'(\beta t)}{\psi_b'(t)}$ is monotone increasing for all $\beta > 1$, $t > 1$.*

Proof. Using Lemma 3.5, we need to prove that $\frac{\psi'(\beta t)}{\psi'(t)}$ is monotone decreasing.

$$\frac{\psi'(\beta t)}{\psi'(t)} = \frac{\beta^p t^p + \psi_b'(\beta t)}{t^p + \psi_b'(t)} = \beta^p + \frac{\psi_b'(\beta t) - \beta^p \psi_b'(t)}{t^p + \psi_b'(t)} = \beta^p + \frac{\beta^p - \frac{\psi_b'(\beta t)}{\psi_b'(t)}}{-\frac{t^p}{\psi_b'(t)} - 1}. \tag{3.7}$$

By assumption $\psi_b''(t) > 0$ for $t > 1$, implying that $\psi_b'(t)$ is increasing. From (3.6), it follows that $\psi_b'(t) > -1$ for $t > 1$. Therefore $0 < -\psi_b'(t) < 1 \leq t^p$. Hence, the denominator appearing in the last expression of (3.7) is positive for $t > 1$. Since $\beta t > t > 1$, and $\psi_b'(t)$ is increasing for $t > 1$ we conclude that $\psi_b'(\beta t) > \psi_b'(t)$. Furthermore, using assumption $\psi_b'(t) < 0$, we get $\frac{\psi_b'(\beta t)}{\psi_b'(t)} < 1 \leq \beta^p$. Hence, we obtain that the numerator of (3.7) is positive for $t > 1$. Since $\frac{\psi_b'(\beta t)}{\psi_b'(t)}$ is monotone increasing by assumption and the denominator appearing in (3.7) is positive, we conclude that the numerator in (3.7) is monotone decreasing. Furthermore, $\psi_b'(t)$ is monotone increasing, hence $-\frac{1}{\psi_b'(t)}$ is monotone increasing. Therefore, the denominator appearing in (3.7) is monotone increasing. The signs of the numerator and denominator stay the same for $t > 1$ implying that the fraction in (3.7) is a monotone decreasing function. Since β^p is a constant, it follows that $\frac{\psi'(\beta t)}{\psi'(t)}$ is monotone decreasing proving the statement. ∎

It is interesting to observe that kernel functions from Section 5 ψ_{p1}-ψ_{p3} and ψ_{e1}-ψ_{e4} with growth term $\frac{t^{p+1}-1}{p+1}$, $p \in [0,1]$ satisfy condition (3.5) only for $p = 1$, while for $p < 1$ the condition does not hold. Hence, these EKF functions with $p \in [0,1)$ are examples that (3.5) combined with (EKF-b) is only sufficient but not necessary condition for (EKF-d). This shows that Lemmas 3.5 and 3.6 are quite helpful in verifying the condition (EKF-d) for the kernel functions listed in Section 5 that all satisfy conditions $\psi_b'(t) < 0$, $\psi_b''(t) > 0$, for $t > 1$. However, for $p = 1$, the condition $\psi_b''(t) > 0$ automatically holds, since $\psi''(t) = \psi_b''(t)$ and $\psi''(t) > 0$ by (GKF-b). Furthermore, ψ_{e4} is an example that we cannot always use Lemma 3.6 to prove condition (EKF-d), because the fractional function appearing in the lemma is not monotone decreasing in this case, despite the fact that this is an EKF.

A rigorous and detailed convergence analysis of IPMs based on the EKFs is provided in [21]. The paper contains an eight-step Scheme, namely a guideline that describes the steps required to calculate the iteration bound for a given EKF, which in most cases can still be quite involved and complicated. This was a motivation to consider whether the complexity analysis of the IPMs for $P_*(\kappa)$-LCPs can be further unified and simplified. The goal was to define a large subclass of EKFs for which a unified and comprehensive complexity analysis of short- and long-step IPMs can be given, from where the iteration bounds can be directly derived. These directly derived iteration bounds were not accomplished in [21].

Analyzing the EKFs used in the literature, we observed that they all satisfy certain additional conditions. In this paper, we concentrate on the conditions for EKFs with rational and exponential barrier terms. We call EKFs that satisfy these conditions SKFs.

Definition 3.7. *An EKF is a Standard Kernel Function if there exist $p \in [0,1]$, $r \geq 1, \eta \geq 1, \sigma \geq 0, c > 0$ and a monotone increasing function $g : (0,1] \to (0,1]$ such that:*

$$\psi'(t) \leq \frac{1}{r}\Big(1 - g\left(t^{-\eta}\right)\Big), \quad t \leq 1, \tag{SKF-a}$$

$$\psi''(t) \leq c\big(1 - r\psi'(t)\big)t^{-\sigma}, \quad t \leq 1, \tag{SKF-b}$$

$$\psi(t) \leq \frac{t^{p+1} - 1}{p+1}, \quad t \geq 1, \tag{SKF-c}$$

$$\psi'(t) \geq \begin{cases} t^p - \frac{1}{t} & \text{if} \quad p \in [0,1), \\ t - 1 & \text{if} \quad p = 1, \end{cases} \qquad t \geq 1, \tag{SKF-d}$$

$$\psi(t) \geq \frac{1}{2t}\,(t-1)^2, \quad t \geq 1. \tag{SKF-e}$$

Remark 3.8. *We give brief comments on the properties (SKF-a)-(SKF-e).*
(i) We use $g(x) = x$ for SKFs with rational barrier term, and $g(x) = e^{x-1}$ for SKFs with exponential barrier term. Since the function g is monotone increasing, it has an inverse function $\hat{g}$. For $g(x) = x$, the inverse is obviously $\hat{g}(x) = x$, and for $g(x) = e^{x-1}$, the inverse function is $\hat{g}(x) = 1 + \ln x$.
(ii) Condition (SKF-c) gives a natural upper bound on the growth of $\psi(t)$ because usually in the literature we have $\psi_b(t) \leq 0$ for $t \geq 1$.
(iii) The lower bound on $\psi(t)$ in condition (SKF-e) is a generalization of the similar bounds discussed in [5, 21].
(iv) Conditions (SKF-a) and (SKF-b) represent an upper bound on the first and the second derivative of $\psi(t)$. They can be combined into one condition, see Lemma 4.5:

$$\psi''(t) \leq h\left(1 - r\psi'(t)\right), \quad \text{where} \quad h(x) = cx\hat{g}(x)^{\frac{\sigma}{\eta}}, \quad x \in [0,1], \tag{3.8}$$

however, it is easier to verify the separate conditions (SKF-a) and (SKF-b).
(v) Property (SKF-d) is a generalization of the similar conditions used in [5, 21].

In the rest of the paper we show that the known EKFs with rational and exponential barrier terms with published complexity results belong to the class of SKFs, see Tables 1 and 3.

As we have already mentioned, in this paper we consider EKFs with the growth term $\frac{t^{p+1}-1}{p+1}$ and rational and exponential barrier terms. They are listed in Section 5 in Tables 1 and 3. In this case, we can give sufficient conditions for (SKF-c)-(SKF-e).

Lemma 3.9. *Let $\psi(t) = \frac{t^{p+1}-1}{p+1} + \psi_b(t)$, where $p \in [0,1]$. Assume that for $t \geq 1$ the following inequalities are satisfied: $-\ln t \leq \psi_b(t) \leq 0$ and $\psi_b'(t) \geq -\frac{1}{t}$. Then, conditions (SKF-c)-(SKF-e) hold.*

Proof. (SKF-c) is equivalent to the assumption $\psi_b(t) \le 0$. (SKF-d) follows from the assumption $\psi_b'(t) \ge -\frac{1}{t}$ for $t \ge 1$. Indeed, $t-1 \le t - \frac{1}{t}$ for $t \ge 1$ gives the proof for the case $p = 1$.

To prove (SKF-e), we estimate the growth term using Taylor expansion of the function $f(x) = x^\alpha$ at $x = 1$

$$x^\alpha = 1 + \alpha \xi^{\alpha-1}(x-1), \tag{3.9}$$

where $\xi \in (1, x)$ for $x > 1$ or $\xi \in (x, 1)$ for $x < 1$. Let $x = t > 1$ and $\alpha = p+1 \ge 1$. Then, (3.9) implies $t^{p+1} \ge 1 + (p+1)(t-1)$ which can be written in the form

$$\frac{t^{p+1}-1}{p+1} \ge t - 1 \quad \text{for } t \ge 1. \tag{3.10}$$

Note that the above estimation for $p = 1$ can be derived directly.

On the other hand, by assumption $\psi_b(t) \ge -\ln t$. Furthermore, $\ln t \le \frac{1}{2}\left(t - \frac{1}{t}\right)$. Combining this inequality with (3.10) we obtain $\psi(t) \ge t - 1 - \frac{1}{2}\left(t - \frac{1}{t}\right) = \frac{1}{2t}(t-1)^2$, which is exactly (SKF-e). ∎

Remark 3.10. *Note that condition $\psi_b(t) \le 0$ for $t \ge 1$ follows from $\psi_b'(t) \le 0$ and (3.6).*

Suppose $p = 1$ and ψ is EKF. Then for (SKF-d), we need to show $\psi_b'(t) \ge -1$. Based on (3.6), it is enough to prove the monotone increasing property of $\psi_b'(t)$, namely $\psi_b''(t) \ge 0$. Hence, assuming the natural properties (monotone decreasing and convex) on the barrier function and $\psi_b(t) \ge -\ln t$, we need to check only the conditions (SKF-a) and (SKF-b) in order to decide whether the function is a SKF.

4. The complexity analysis of the *Algorithm*.

In this section, we discuss the derivation of iteration bounds for large- and small-update versions of the *Algorithm* based on the class of SKFs.

The following theorem is the main result of the paper.

Theorem 4.1. *The* Algorithm *with a SKF $\psi(t)$ which has a rational or an exponential barrier term finds an ε-solution of the $P_*(\kappa)$-LCP in*

- *(i)* $O\left((1+2\kappa)\frac{c}{\gamma} r \left[h(rn)\right]^{\frac{\sigma}{\eta}} n^\gamma \ln\frac{n}{\varepsilon}\right)$ *iterations with a long-step strategy, i.e.,* $\tau = O(n),\ \theta = \Theta(1)$,
- *(ii)* $O\left((1+2\kappa)\frac{c}{\gamma} r \left[h(r\psi''(1))\right]^{\frac{\sigma}{\eta}} \left[\psi''(1)\right]^\gamma \sqrt{n}\ln\frac{n}{\varepsilon}\right)$ *iterations with a short-step strategy, i.e.,* $\tau = O(1),\ \theta = \Theta(\frac{1}{\sqrt{n}})$,

where p, r, c, σ, η are parameters of the kernel function $\psi(t)$ from the conditions (SKF-a)-(SKF-e) and

$$\begin{aligned} \gamma &= \begin{cases} 1 - \frac{p}{p+1}\left(1 - \frac{\sigma}{\eta}\right) & \text{if } g(x) = x \\ \frac{1}{p+1} & \text{if } g(x) = e^{x-1}, \end{cases} \\ h(x) &= \begin{cases} 2r & \text{if } g(x) = x \\ 1 + \ln x & \text{if } g(x) = e^{x-1}. \end{cases} \end{aligned} \tag{4.1}$$

Remark 4.2. *Note that the complexity results obtained in Theorem 4.1 can be simplified if we know that r is constant and we have a constant upper bound on $\frac{\sigma}{\eta}$.*

In the sequel, we prove Theorem 4.1 through a series of lemmas that are mainly based on the results of the papers [5, 21].

Each iteration of the *Algorithm* consists of an outer iteration and a number of inner iterations per each outer iteration. Hence, the upper bound on the total number of iterations is a product of an upper bound on the number of outer iterations and an upper bound on the number of inner iterations per one outer iteration.

It is well known from [20, 29] that the number of outer iterations is bounded above by

$$\frac{1}{\theta}\ln\frac{n}{\varepsilon}, \tag{4.2}$$

where θ is the barrier parameter, an input parameter of the *Algorithm*.

After an outer iteration, due to the reduction of μ to $\mu_+ = (1-\theta)\mu$, the new value of the barrier function, $\Psi(v^+)$, where $v^+ = \frac{v}{\sqrt{1-\theta}}$ might become greater than the parameter τ. Hence, we need inner iterations in order to decrease the value of the barrier function until we get into the neighbourhood $\mathcal{N}_\psi(\tau)$ defined in (2.6).

In order to find the upper bound on the number of inner iterations per one outer iteration we need to find two bounds. One is related to the value $\Psi(v^+)$ of the barrier function after a μ-update which we denote by Ψ_0. The other bound is the decrease of the barrier function after a damped Newton-step in the inner iteration. This will be denoted by Δ, which is the same as $-f(\alpha)$ in [21]. Let us note that both Ψ_0 and Δ depend on v. Hence, in what follows, we need to find a suitable lower bound on Δ and a proper upper bound on Ψ_0 that are v-independent. We will follow the ideas presented in [21].

Before starting the calculation of the iteration bounds, let us define the following two functions that play a crucial role in the analysis of the *Algorithm*, see [21].

Definition 4.3. *Given the kernel function ψ, we define the following functions:*
(*i*) $\varrho : [0,\infty) \to [1,\infty)$ *is the inverse function of $\psi(t)$ for $t \geq 1$;*
(*ii*) $\rho : [0,\infty) \to (0,1]$ *is the inverse function of $-\frac{1}{2}\psi'(t)$ for $t \leq 1$.*

In the following subsection, we analyze the decrease of the proximity measure after an inner iteration.

4.1. Lower bound on Δ. In the first part of the subsection, we give bounds on the functions ϱ and ρ introduced in Definition 4.3.

Lemma 4.4. *Consider a kernel function ψ satisfying (SKF-a). Then, $\rho(s) \geq \hat{g}(1+2rs)^{-\frac{1}{\eta}}$, where $\hat{g}$ is defined in Definition 3.7.*

Proof. Let $s = -\frac{1}{2}\psi'(t)$. By the definition of ρ, $t = \rho(s)$. Then, by using (SKF-a), we get

$$1 + 2rs = 1 - r\psi'(t) \geq g\left(t^{-\eta}\right).$$

Since g is invertible, we have

$$\hat{g}\left(1 - r\psi'(t)\right)^{-\frac{1}{\eta}} \leq t = \rho(s), \tag{4.3}$$

proving the lemma. ∎

Lemma 4.5. *Consider a kernel function ψ satisfying conditions (SKF-a) and (SKF-b). Then*

$$\psi''(t) \leq c(1 - r\psi'(t))\hat{g}\left(1 - r\psi'(t)\right)^{\frac{\sigma}{\eta}}.$$

Proof. Using (SKF-b) and (4.3), we get

$$\psi''(t) \leq c(1 - r\psi'(t))t^{-\sigma} \leq c(1 - r\psi'(t))\hat{g}\left(1 - r\psi'(t)\right)^{\frac{\sigma}{\eta}},$$

proving the lemma. ∎

Lemma 4.6. *Consider a kernel function ψ satisfying (SKF-c). Then, we have*

$$[1+(p+1)s]^{\frac{1}{1+p}} \leq \varrho(s).$$

Proof. Recall that $\psi : [1,\infty) \to [0,\infty)$ has an inverse function $\varrho : [0,\infty) \to [1,\infty)$, that is, $s = \psi(t)$ if and only if $t = \varrho(s)$. From condition (SKF-c), we obtain

$$(p+1)s = (p+1)\psi(t) \leq t^{p+1} - 1 = \varrho(s)^{p+1} - 1,$$

proving the lemma. □

Following the techniques of the kernel-based complexity analysis (see for example [5, 21]), we will use a lower bound on the norm-based proximity measure $\delta(v)$ in terms of the barrier function $\Psi(v)$.

Lemma 4.7 (Theorem 5.12 in [21] and Theorem 4.9 in [5]). *One has*

$$\delta(v) \geq \frac{1}{2}\psi'\left(\varrho\left(\Psi(v)\right)\right).$$

Now we are ready to derive the bound on $\delta(v)$ using the property (SKF-d) of the kernel function.

Lemma 4.8. *Consider a kernel function ψ satisfying (SKF-d). Then,*

$$\delta(v) \geq \frac{1}{6}\Psi(v)^{\frac{p}{p+1}}. \tag{4.4}$$

Proof. Let us denote $\Psi(v)$ as Ψ and $\delta(v)$ as δ. From Lemma 4.7, property (SKF-d), and considering the monotone increasing property of the function $t^p - \frac{1}{t}$ and $t-1$ with Lemma 4.6, we have the following two cases. In the case of $p \in [0,1)$, we have

$$\begin{aligned}\delta &\geq \tfrac{1}{2}\psi'(\varrho(\Psi)) \geq \tfrac{1}{2}\left(\varrho(\Psi)^p - \tfrac{1}{\varrho(\Psi)}\right) \geq \tfrac{1}{2}\left([1+(p+1)\Psi]^{\frac{p}{1+p}} - \tfrac{1}{[1+(p+1)\Psi]^{\frac{1}{1+p}}}\right)\\ &= \tfrac{1}{2}\tfrac{(1+p)\Psi}{[1+(p+1)\Psi]^{\frac{1}{1+p}}} \geq \tfrac{1}{2}\tfrac{\Psi}{[1+2\Psi]^{\frac{1}{1+p}}} \geq \tfrac{1}{6}\Psi^{\frac{p}{1+p}},\end{aligned}$$

since $1 \leq p+1 < 2$ and $\Psi > \tau \geq 1$. In the case of $p = 1$, we have

$$\delta \geq \tfrac{1}{2}\left(\varrho(\Psi) - 1\right) \geq \tfrac{1}{2}\left([1+2\Psi]^{\frac{1}{2}} - 1\right) = \tfrac{1}{2}\tfrac{2\Psi}{[1+2\Psi]^{\frac{1}{2}}+1} \geq \tfrac{\Psi}{(1+\sqrt{3})\sqrt{\Psi}} \geq \tfrac{1}{6}\Psi^{\frac{1}{2}},$$

where we used that $\Psi > \tau \geq 1$. □

Note that from any feasible steplength in the inner iteration, we can provide a lower bound on the maximal possible decrease of the barrier function. In [21] the following default feasible steplength is given:

$$\tilde{\alpha} = \frac{1}{(1+2\kappa)\psi''\left(\rho\left(2\delta\right)\right)}, \tag{4.5}$$

which leads to a sufficient decrease of the barrier function, as it is shown in the lemma below.

Lemma 4.9 (Theorem 5.11 in [21]). *If we consider the steplength $\tilde{\alpha}$ given in (4.5) in the inner iteration and $\delta = \delta(v)$, then we have*

$$\Delta \geq \frac{\delta^2}{(1+2\kappa)\psi''\left(\rho\left(2\delta\right)\right)}. \tag{4.6}$$

where the expression on the right side of the above inequality is monotone increasing in δ.

Using the previous two lemmas, we obtain a lower bound on Δ in terms of the barrier function $\Psi(v)$ that we denote as Ψ.

LEMMA 4.10. *Given a SKF ψ, the following inequality holds*

$$\Delta \geq \beta \Psi^{1-\gamma},$$

where Ψ is the associated barrier function and

(4.7)

$$\beta = \frac{\overline{\beta}}{36(1+2\kappa)c\left(1+\frac{2}{3}r\right)}, \quad \textit{with } \overline{\beta} = \begin{cases} \left(1+\frac{2}{3}r\right)^{-\frac{\sigma}{\eta}} & \textit{if} \quad g(x) = x \\ \left[1+\ln\left(1+\frac{2}{3}\Psi_0^{\frac{p}{p+1}}\right)\right]^{-\frac{\sigma}{\eta}} & \textit{if} \quad g(x) = e^{x-1} \end{cases}$$

and

$$\gamma = \begin{cases} 1-\frac{p}{p+1}\left(1-\frac{\sigma}{\eta}\right) & \textit{if} \quad g(x) = x \\ \frac{1}{p+1} & \textit{if} \quad g(x) = e^{x-1}. \end{cases} \tag{4.8}$$

Proof. Substituting (4.4) into (4.6), we obtain

$$\Delta \geq \frac{\frac{1}{36}\Psi^{\frac{2p}{p+1}}}{(1+2\kappa)\psi''\left(\rho\left(\frac{1}{3}\Psi^{\frac{p}{p+1}}\right)\right)}. \tag{4.9}$$

From Lemma 4.5, we have

$$\begin{aligned} \psi''(\rho(s)) &\leq c\left(1-r\psi'(\rho(s)\right)\hat{g}\left((1-r\psi'(\rho(s)\right)^{\frac{\sigma}{\eta}} \\ &= c\left(1+2r\left(-\frac{1}{2}\psi'(\rho(s)\right)\right)\hat{g}\left(1+2r\left(-\frac{1}{2}\psi'(\rho(s)\right)\right)^{\frac{\sigma}{\eta}} \\ &= c\left(1+2rs\right)\hat{g}\left((1+2rs\right)^{\frac{\sigma}{\eta}} \end{aligned} \tag{4.10}$$

since ρ is the inverse of the function $-\frac{1}{2}\psi'$.

If we substitute $s = \frac{1}{3}\Psi^{\frac{p}{p+1}}$ into (4.10) and use that $\Psi > 1$, we get

$$\begin{aligned} \psi''\left(\rho\left(\frac{1}{3}\Psi^{\frac{p}{p+1}}\right)\right) &\leq c\left(1+2r\frac{1}{3}\Psi^{\frac{p}{p+1}}\right)\hat{g}\left(1+2r\frac{1}{3}\Psi^{\frac{p}{p+1}}\right)^{\frac{\sigma}{\eta}} \\ &\leq c\left(1+\frac{2}{3}r\right)\Psi^{\frac{p}{p+1}}\hat{g}\left(1+\frac{2}{3}r\Psi^{\frac{p}{p+1}}\right)^{\frac{\sigma}{\eta}}. \end{aligned} \tag{4.11}$$

Now, we have two cases.

Case I: If $g(x) = x$, then the inverse function is the same, i.e. $\hat{g}(x) = x$. Hence,

$$\hat{g}\left(1+2r\frac{1}{3}\Psi^{\frac{p}{p+1}}\right) = 1+2r\frac{1}{3}\Psi^{\frac{p}{p+1}} \leq \left(1+\frac{2}{3}r\right)\Psi^{\frac{p}{p+1}}, \tag{4.12}$$

where the inequality above is because $\Psi > \tau \geq 1$. Substituting (4.11) and (4.12) into (4.9), we obtain

$$\Delta \geq \frac{1}{36(1+2\kappa)c}\frac{1}{\left(1+\frac{2}{3}r\right)^{1+\frac{\sigma}{\eta}}}\Psi^{\frac{p}{p+1}\left(1-\frac{\sigma}{\eta}\right)}. \tag{4.13}$$

Case II: If $g(x) = e^{x-1}$, then the inverse function is $\hat{g}(x) = 1 + \ln x$. Hence,

$$\hat{g}\left(1 + 2r\frac{1}{3}\Psi^{\frac{p}{p+1}}\right) = 1 + \ln\left(1 + \frac{2}{3}r\Psi^{\frac{p}{p+1}}\right) \le 1 + \ln\left(1 + \frac{2}{3}r\Psi_0^{\frac{p}{p+1}}\right), \tag{4.14}$$

where the inequality above is due to the fact that $\Psi_0 \ge \Psi > \tau \ge 1$. Substituting (4.11) and (4.14) into (4.9), we obtain

$$\Delta \ge \frac{1}{36(1+2\kappa)c\left(1+\frac{2}{3}r\right)} \frac{1}{\left[1 + \ln\left(1 + \frac{2}{3}r\Psi_0^{\frac{p}{p+1}}\right)\right]^{\frac{\sigma}{n}}} \Psi^{\frac{p}{p+1}}. \tag{4.15}$$

From (4.13) and (4.15), the values of β and γ stated in (4.7) and (4.8) directly follow and the lemma is proved. ∎

Using the bound obtained in Lemma 4.10, an upper bound is derived on the number of inner iterations per one outer iteration in terms of Ψ_0.

Lemma 4.11 (Lemma 6.2 in [21]). *The number of inner iterations per one outer iteration of the* Algorithm *is bounded above by*

$$\left\lfloor \frac{\Psi_0^{\gamma}}{\beta\ \gamma} \right\rfloor, \tag{4.16}$$

where Ψ_0 is the value of the barrier function after the outer iteration, β and γ are given in Lemma 4.10.

In order to find an upper bound on the number of inner iterations which is independent of Ψ_0, we need to derive further estimates.

4.2. Upper bound on Ψ_0. The effect of the μ-update on the value of the barrier function Ψ was considered in [21].

Lemma 4.12 (Corollary 5.2 in [21]). *One has the following upper bounds on the barrier function value Ψ_0 after the μ-update in the outer iteration.*

$$\Psi_0 \le n\psi\left(\frac{\varrho\left(\frac{\tau}{n}\right)}{\sqrt{1-\theta}}\right) \le \frac{n}{2}\psi''(1)\left(\frac{\varrho(\frac{\tau}{n})}{\sqrt{1-\theta}} - 1\right)^2. \tag{4.17}$$

In what follows we further develop and simplify these bounds.

Lemma 4.13. *Consider a kernel function ψ satisfying (SKF-e). Then we have the following upper bounds on ϱ, the inverse function of ψ,*

$$\varrho(s) \le 1 + s + \sqrt{s(s+2)} \le 2(1+s). \tag{4.18}$$

Proof. From property (SKF-e), we have $(t-1)^2 \le 2t\psi(t)$. Let $t \ge 1$ and $s = \psi(t)$. Then, we get

$$t^2 - (2+2s)t + 1 \le 0,$$

which holds if $t \le 1 + s + \sqrt{(2+s)s}$. Since $t = \varrho(s)$, the first inequality in (4.18) is proved. The second inequality follows immediately from the arithmetic-geometric mean inequality. ∎

Combining Lemmas 4.12 and 4.13, we provide two different upper bounds on Ψ_0 that are independent of ϱ.

Lemma 4.14. *Consider a kernel function ψ satisfying (SKF-e). One has*

$$\Psi_0 \leq \frac{\psi''(1)}{2(1-\theta)n}\left[\theta n + \tau + \sqrt{2\tau n + \tau^2}\right]^2, \tag{4.19}$$

where $\theta \in (0,1)$ is the barrier update parameter.

Proof. Using the second inequality in Lemma 4.12 and the first inequality in Lemma 4.13, we have the following derivation

$$\Psi_0 \quad \leq \quad \tfrac{n}{2}\psi''(1)\left(\frac{1+\frac{\tau}{n}+\sqrt{\frac{\tau}{n}\left(2+\frac{\tau}{n}\right)}}{\sqrt{1-\theta}}-1\right)^2 \leq \tfrac{n}{2}\psi''(1)\left(\frac{\theta+\frac{\tau}{n}+\sqrt{\frac{\tau}{n}\left(2+\frac{\tau}{n}\right)}}{\sqrt{1-\theta}}\right)^2,$$

where in the second inequality we used $1-\sqrt{1-\theta} = \frac{\theta}{1+\sqrt{1-\theta}} \leq \theta$. ∎

Lemma 4.15. *Consider a kernel function ψ satisfying (SKF-c) and (SKF-e). One has*

$$\Psi_0 \leq \frac{4n}{1-\theta}\left(1+\frac{\tau}{n}\right)^2, \tag{4.20}$$

where $\theta \in (0,1)$ is the barrier update parameter.

Proof. From property (SKF-c), we obtain $\psi(t) \leq t^2$. Using this, the first inequality in Lemma 4.12, and the second inequality in Lemma 4.13, we have the following derivation

$$\Psi_0 \quad \leq \quad n\psi\left(\frac{\varrho\left(\frac{\tau}{n}\right)}{\sqrt{1-\theta}}\right) \leq n\left(\frac{\varrho\left(\frac{\tau}{n}\right)}{\sqrt{1-\theta}}\right)^2 \leq \tfrac{n}{1-\theta}\left(\varrho\left(\tfrac{\tau}{n}\right)\right)^2 \leq \tfrac{n}{1-\theta}\left[2\left(1+\tfrac{\tau}{n}\right)\right]^2.$$ ∎

We use Lemma 4.14 and Lemma 4.15 to estimate the upper bounds on Ψ_0 for the long- and short-step versions of the *Algorithm*.

Lemma 4.16. *Consider a SKF ψ. We have $\Psi_0 = O(\psi''(1))$ for the short-step version of the* Algorithm.

Proof. For the short-step version of the *Algorithm* we set $\theta = \Theta\left(\frac{1}{\sqrt{n}}\right)$ and $\tau = O(1)$. Using Lemma 4.14, we have

$$\Psi_0 \leq \frac{\psi''(1)}{2(1-\theta)}\left(\theta\sqrt{n} + \frac{\tau}{\sqrt{n}} + \sqrt{2\tau + \frac{\tau^2}{n}}\right)^2 = O(\psi''(1)),$$

since $\theta\sqrt{n} + \frac{\tau}{\sqrt{n}} + \sqrt{2\tau + \frac{\tau^2}{n}} = O(1)$ and $\frac{1}{1-\theta} = O(1)$. ∎

Let us note that using Lemma 4.15 instead, we have the following derivation:

$$\Psi_0 \leq \frac{4n}{1-\theta}\left(1+\frac{\tau}{n}\right)^2 = O(n),$$

since $\frac{\tau}{n} = O(1)$. This estimate is not good enough for short-step methods as it contains n. Hence, it is more advantageous to use the estimate that does not contain n, however, it does contain $\psi''(1)$, which may depend on parameters that could get large.

Lemma 4.17. *Consider a SKF ψ. One has $\Psi_0 = O(n)$ for the long-step version of the* Algorithm.

Proof. For the long-step version of the *Algorithm*, we have $\theta = \Theta(1)$ and $\tau = O(n)$. Using Lemma 4.15, we have

$$\Psi_0 \leq \tfrac{4n}{1-\theta}\left(1+\tfrac{\tau}{n}\right)^2 = O(n),$$

since in this case $\frac{1}{1-\theta} = O(1)$ and $\frac{\tau}{n} = O(1)$. ∎

Let us now use (4.19) in Lemma 4.15 instead, then we get the following:

$$\Psi_0 \quad \leq \quad \tfrac{\psi''(1)n}{2(1-\theta)}\left(\theta + \tfrac{\tau}{n} + \sqrt{\tfrac{2\tau}{n} + \tfrac{\tau^2}{n^2}}\right)^2 = O(\psi''(1)n).$$

The advantage of the first estimate over the second one is that it does not depend on $\psi''(1)$, which may depend on parameters that could get large.

4.3. Proof of the Main Theorem. We have all the ingredients to complete the proof of the Main Theorem. The number of outer iterations is bounded above by $\frac{1}{\theta}\ln\frac{n}{\varepsilon}$ and the number of inner iterations per one outer iteration is at most $\frac{\Psi_0^\gamma}{\beta\gamma}$. Hence, the total number of iterations is bounded above by

$$N \leq \frac{\Psi_0^\gamma}{\theta\beta\gamma}\ln\frac{n}{\varepsilon}. \tag{4.21}$$

For further estimates, we set a lower bound on β defined in Lemma 4.10.
For SKFs with rational barrier term:

$$\beta = \frac{1}{36(1+2\kappa)c}\frac{1}{\left(1+\frac{2}{3}r\right)^{1+\frac{\sigma}{\eta}}} \geq \frac{1}{36(1+2\kappa)c}\frac{1}{(2r)^{1+\frac{\sigma}{\eta}}}.$$

For SKFs with exponential barrier term:

$$\begin{aligned}\beta &= \frac{1}{36(1+2\kappa)c}\frac{1}{\left(1+\frac{2}{3}r\right)}\frac{1}{\left[1+\ln\left(1+\frac{2}{3}r\Psi_0^{\frac{p}{p+1}}\right)\right]^{\frac{\sigma}{\eta}}}\\ &\geq \frac{1}{72(1+2\kappa)c\,r}\frac{1}{\left[1+\ln(2r\Psi_0)\right]^{\frac{\sigma}{\eta}}}.\end{aligned}$$

Using the function h defined by (4.1), we can combine these two inequalities into the single bound

$$\beta \geq \frac{1}{72(1+2\kappa)c\,r}\frac{1}{h\left(2r\Psi_0\right)^{\frac{\sigma}{\eta}}}.$$

This means together with (4.21), that

$$N \leq 72(1+2\kappa)\frac{c}{\gamma}r\,h(2r\Psi_0)^{\frac{\sigma}{\eta}}\frac{1}{\theta}\Psi_0^\gamma\ln\frac{n}{\varepsilon}. \tag{4.22}$$

Finally, we give the magnitude of the right-hand side of (4.22) depending on the type of the algorithm.
For *short-step methods*, we have $\theta = \Theta\left(\frac{1}{\sqrt{n}}\right)$. Then, from Lemma 4.16, $\Psi_0 = O(\psi''(1))$ and we derive

$$h(2r\Psi_0)^{\frac{\sigma}{\eta}} = O\Big(h\big(r\psi''(1)\big)^{\frac{\sigma}{\eta}}\Big).$$

Substituting it into (4.22), we obtain

$$N = O\left((1+2\kappa)\frac{c}{\gamma} r \left[h(r\psi''(1)) \right]^{\frac{\sigma}{\eta}} \left[\psi''(1) \right]^{\gamma} \sqrt{n} \ln \frac{n}{\varepsilon} \right). \tag{4.23}$$

For *long-step methods*, we have $\theta = \Theta(1)$. Then, from Lemma 4.17, $\Psi_0 = O(n)$ and we derive

$$h(2r\Psi_0)^{\frac{\sigma}{\eta}} = O\Big(h\big(rn\big)^{\frac{\sigma}{\eta}} \Big).$$

Combining it with (4.22), we get

$$N = O\left((1+2\kappa)\frac{c}{\gamma} r \left[h(rn) \right]^{\frac{\sigma}{\eta}} n^{\gamma} \ln \frac{n}{\varepsilon} \right). \tag{4.24}$$

Then, (4.23) and (4.24) prove the Main Theorem 4.1.

5. Determining iteration bounds for specific SKFs. In this section, we determine the iteration bounds for different families of SKFs. Some of them are known EKFs from the literature, while some of them are generalizations of EKFs listed in [5, 6, 7, 15, 21]. We first verify whether these EKFs belong to the class of SKFs by determining suitable values for the parameters r, c, σ, and η such that the conditions (SKF-a)-(SKF-e) are satisfied. These values are summarized in Tables 1 and 3. Once the values of these parameters are determined, the iteration bounds are obtained by their substitution into (4.23) for short-step methods and (4.24) for long-step methods.

5.1. Iteration bounds for SKFs with rational barrier term. In Table 1 three parametric EKFs with rational barrier terms are listed. These EKFs were considered in [21]. EKF ψ_{p_3} is a generalization of the classical logarithmic kernel function $\psi(t) = \frac{t^2-1}{2} - \ln t$. Furthermore, ψ_{p_3} is a limiting case of ψ_{p_1} due to the fact that $\lim_{q\to 1} \frac{t^{1-q}-1}{1-q} = \ln t$. This is in accordance with the values of the parameters given in Table 1. The reason for considering these general parametric EKFs is that they contain all other EKFs with rational barrier terms that appear in the literature, including the EKFs in [21], as special cases.

We have verified that the above EKFs satisfy the rational form ($g(x) = x$) of conditions (SKF-a)-(SKF-e) and listed the values of the parameters in Table 1. In the penultimate of Table 1, we give the value of the parameter γ calculated from (4.1). Note that in the Appendix, we show the verification of the parameters for ψ_{p_1}.

p_i	Kernel functions $\psi_{p_i}(t)$	r	η	c	σ	γ	$\psi''_{p_i}(1)$
p_1	$\frac{t^{p+1}-1}{p+1} + \frac{t^{1-q}-1}{q-1}$	1	q	$p+q$	1	$\frac{q+p}{q(p+1)}$	$p+q$
p_2	$\frac{t^2-1}{2} + \frac{t^{1-q}-1}{q(q-1)} - \frac{q-1}{q}(t-1)$	q	q	$2q$	1	$\frac{q+1}{2q}$	2
p_3	$\frac{t^{p+1}-1}{p+1} - \ln t$	1	1	$p+1$	1	1	$p+1$

TABLE 1
EKFs with rational barrier terms with $p \in [0,1]$, $q > 1$

Substituting these parameters into the rational form ($h(x) = 2r$) of expressions (4.23) for short-step methods and (4.24) for long-step methods, we obtain the iteration bounds shown in Table 2.

p_i	Small-update methods	Large-update methods
p_1	$O\left((1+2\kappa)q^2\sqrt{n}\,\ln\frac{n}{\varepsilon}\right)$	$O\left((1+2\kappa)qn^{\frac{p+q}{q(1+p)}}\,\ln\frac{n}{\varepsilon}\right)$
p_2	$O\left((1+2\kappa)q\sqrt{n}\ln\frac{n}{\varepsilon}\right)$	$O\left((1+2\kappa)q^2n^{\frac{1+q}{2q}}\,\ln\frac{n}{\varepsilon}\right)$
p_3	$O\left((1+2\kappa)\sqrt{n}\ln\frac{n}{\varepsilon}\right)$	$O\left((1+2\kappa)n\ln\frac{n}{\varepsilon}\right)$

Table 2
Iteration bounds for SKFs with rational barrier terms

Let us note that the bounds for the EKFs listed in [5, 21], which are special cases of the parametric SKFs in Table 1 with appropriate values of parameters p and q, match exactly the iteration bounds in Table 2.

Remark 5.1. *Note that the function ψ_{p_2} with generalized growth term $\frac{t^{p+1}-1}{p+1}$, $p \in [0,1]$ is EKF, however, the condition (SKF-d) does not hold for $p < 1$. Hence, it is not a SKF. This example shows that the class of SKFs is a strict subclass of the class of EKFs. However, there is no complexity result for this function yet for $p < 1$.*

5.2. Iteration bounds for SKFs with exponential barrier term. In Table 3, three parametric EKFs with exponential barrier terms are listed. These EKFs are generalizations of the EKFs considered previously in [5, 7, 15, 21]. The special cases considered in the literature can be obtained by substituting appropriate values of the parameters p and q in the definitions of EKFs $\psi_{e_1} - \psi_{e_3}$.

We verified that the EKFs in Table 3 satisfy the exponential form ($g(x) = e^{x-1}$) of conditions (SKF-a)-(SKF-e) and we determined the values of the parameters. In the table, we also listed the value of the parameter γ determined from (4.1). Note that in the Appendix, we show the verification of the parameters in Table 3 for the SKF ψ_{e_1}.

e_i	Kernel functions $\psi_{e_i}(t)$	r	η	c	σ	γ	$\psi''_{e_i}(1)$
e_1	$\frac{t^{p+1}-1}{p+1} + \frac{e^{t^{-q}-1}-1}{q}$,	1	q	$2q+2$	$q+1$	$\frac{1}{p+1}$	$p+2q+1$
e_2	$\frac{t^2-1}{2} - \int_1^t e^{u^{-q}-1}du$,	1	q	$q+1$	$q+1$	$\frac{1}{2}$	$1+q$
e_3	$\frac{t^{p+1}-1}{p+1} + \frac{e^{t^{-q}-1}-1}{2q} - \frac{1}{2}\ln t$,	2	q	$2q+2$	$q+1$	$\frac{1}{p+1}$	$p+q+1$

Table 3
SKFs with exponential barrier terms with $p \in [0,1], q \geq 1$

Substituting the values of these parameters into the exponential form ($h(x) = 1+\ln x$) of expressions (4.23) for short-step methods and (4.24) for long-step methods, we obtain the following iteration bounds in Table 4.

e_i	Short-step methods	Long-step methods
e_1	$O\left((1+2\kappa)q^{1+\frac{1}{p+1}}(1+\ln q)^{\frac{q+1}{q}}\sqrt{n}\ln\frac{n}{\varepsilon}\right)$	$O\left((1+2\kappa)q\ n^{\frac{1}{p+1}}(\ln n)^{\frac{q+1}{q}}\ \ln\frac{n}{\varepsilon}\right)$
e_2	$O\left((1+2\kappa)q^{\frac{3}{2}}(1+\ln q)^{\frac{q+1}{q}}\sqrt{n}\ln\frac{n}{\varepsilon}\right)$	$O\left((1+2\kappa)q\ n^{\frac{1}{2}}(\ln n)^{\frac{q+1}{q}}\ \ln\frac{n}{\varepsilon}\right)$
e_3	$O\left((1+2\kappa)q^{1+\frac{1}{p+1}}(1+\ln q)^{\frac{q+1}{q}}\sqrt{n}\ln\frac{n}{\varepsilon}\right)$	$O\left((1+2\kappa)q\ n^{\frac{1}{p+1}}(\ln n)^{\frac{q+1}{q}}\ \ln\frac{n}{\varepsilon}\right)$

Table 4
Iteration bounds for SKFs with exponential barrier terms

The bounds for the special cases of EKFs in [5, 7, 15, 21] were obtained by substituting the appropriate values of the parameters p and q into the bounds for $\psi_{e_1}-\psi_{e_3}$. Note that in [7, 15] the iteration bounds that include $(\ln q)^{\frac{q+1}{q}}$ are problematic, because in the case of $q=1$ this term would be zero. Furthermore, the Remark 5.1 holds for the function ψ_{e_2} as well.

Remark 5.2. *We also consider the following EKF with exponential barrier term*

$$\psi_{e_4}(t)=\frac{t^{p+1}-1}{p+1}+\frac{(e-1)^2}{e}\frac{1}{e^t-1}-\frac{e-1}{e},\ p\in[0,1]. \tag{5.1}$$

Although ψ_{e_4} has an exponential barrier term, it turns out that it is more suitable to show that it belongs to the class of SKFs using the conditions for SKFs with rational barrier terms ($g(x)=x$). In this case, it still leads to obtaining the iteration bounds that match the best-known iteration bounds obtained in [21]. The values of the parameters are $r=1,\ \eta=2,\ c=5,\ \sigma=1$, which lead to the value of $\gamma=\frac{p+2}{2p+2}$. Furthermore, $\psi''_{e_4}(1)=\frac{2e}{e-1}$.

Substituting the values of these parameters into the rational form ($h(x)=2r$) of expressions (4.23) for short-step methods and (4.24) for long-step methods, we obtain $O\left((1+2\kappa)\sqrt{n}\log\frac{n}{\varepsilon}\right)$ for the short-step method and $O\left((1+2\kappa)n^{\frac{p+2}{2p+2}}\log\frac{n}{\varepsilon}\right)$ iteration bound for the long-step method.

Remark 5.3. *As observed in the previous remark, SKFs with exponential barrier terms can be considered using the conditions for SKFs with rational barrier terms. However, in most cases, the values of the parameters r,c,σ,η will lead to a weaker complexity result in comparison with the values obtained using conditions for exponential barrier terms and, in addition, more complicated to derive. To make the comparison more evident, we consider a special case of ψ_{e_1} with $p=1,\ q=1$. In that case, the values of the parameters obtained from the conditions for rational barrier terms are $r=1,\ \eta=3,\ c=4,\ \sigma=2$ and $\gamma=\frac{5}{6}$. Substituting the values of the parameters into the rational version of (4.23) and (4.24), we obtain $O\left((1+2\kappa)\sqrt{n}\ln\frac{n}{\varepsilon}\right)$ for the short-step method and $O\left((1+2\kappa)n^{\frac{5}{6}}\ln\frac{n}{\varepsilon}\right)$ for the long-step method. From Table 4, we get the same iteration bound for the short-step method while for the long-step-method, we get a much better iteration bound, $O\left((1+2\kappa)\sqrt{n}(\ln n)^2\ln\frac{n}{\varepsilon}\right)$.*

6. Concluding remarks. In this paper, we considered a kernel-based IPM framework for $P_*(\kappa)$-LCPs presented in Algorithm 2.1 that we called simply the *Al-*

gorithm. A unified and comprehensive iteration complexity analysis of the *Algorithm* for the new class of kernel functions called SKFs defined in Definition 3.7 was provided and summarized in the Main Theorem 4.1.

The SKFs are a proper subclass of EKFs. Once it is verified that an EKF satisfies additional conditions listed in the definition of SKF and the related parameters are calculated, the derivations of the iteration bounds for short- and long-step versions of the *Algorithm* are straightforward and follow immediately from the Main Theorem. Hence, the derivation of the iteration bounds for the whole class of SKFs, is much simpler than in [21] and other papers on this topic. The iteration bounds for the parametric SKFs analyzed in this paper are listed in Tables 2 and 4. It is important to mention that we matched the best iteration bounds obtained in the literature for the SKFs that are special cases of the SKFs listed in this paper.

Although the class of SKFs is a proper subset of the class of EKFs, the EKFs that appear in the literature are all SKFs. The ones that are not in the class of SKFs but are in the class of EKFs have not yet appeared in the literature. The reason might be that the complexity results that could be obtained for these functions using the current techniques in [21] would be weaker than for the SKFs.

Furthermore, we observed that the best iteration bounds are obtained considering kernel functions with growth term $\frac{t^2-1}{2}$, i.e. $p = 1$. In this case, by assuming natural properties on the barrier function, the definition of SKF becomes simpler, see Remark 3.10.

It is also worth mentioning that as a part of the analysis of the SKFs, we further analyzed the definition of EKFs and gave a new sufficient condition to check one of the EKF properties.

As already mentioned, future research includes developing similar iteration bound analysis for the EKFs with trigonometric barrier terms. Although some of the observed conditions are different than for the SKFs, preliminary results show that a similar comprehensive complexity analysis can be developed. Another interesting direction for future research is the generalization of these techniques to symmetric optimization and LCPs over symmetric cones.

Appendix A. Verification of SKF conditions for two families of EKFs. We illustrate the verification process for EKF to be an SKF for two families of EKFs, one with a rational barrier term, ψ_{p_1}, and one with an exponential barrier term, ψ_{e_1}. We verify all five conditions for EKF to be SKF with the given parameters from Tables 1 and 3.

Verification that ψ_{p_1} is SKF. From Table 1, we have the following parameters in Definition 3.7 for ψ_{p_1}: $r = 1$, $\eta = q$, $c = p + q$, $\sigma = 1$ and $g(x) = x$.
(SKF-a): Since $t \le 1$, we have

$$\psi'_{p_1}(t) = t^p - t^{-q} \le 1 - \frac{1}{t^q} = \frac{1}{r}\left(1 - \frac{1}{t^\eta}\right).$$

(SKF-b): Since $t \le 1$, $p \ge 0$ and $q > 1$, it follows that $t^p \le 1 \le t^{-q}$, and we have

$$\begin{aligned}\psi''_{p_1}(t) &= \left(pt^p + qt^{-q}\right)\frac{1}{t} \le \left[p + qt^{-q} + p\left(t^{-q} - t^p\right) + q\left(1 - t^p\right)\right]\frac{1}{t} \\ &= \left[(p+q) - qt^p - pt^p + (p+q)t^{-q}\right]\frac{1}{t} = c[1 - \psi'(t)]t^{-\sigma}.\end{aligned}$$

(SKF-c)-(SKF-e): We show that the assumptions of Lemma 3.9 hold.

In this case $\psi_b(t) = \frac{t^{1-q}-1}{q-1}$, which is nonpositive for $t \geq 1$, since $q > 1$.
Furthermore, $\psi_b'(t) = -t^{-q} \geq -\frac{1}{t}$ for $t \geq 1$.
To give a lower bound on the barrier term, we use Taylor expansion of the function $f(x) = t^x,\ t \geq 1$ at $x = 0$.

$$t^x = 1 + t^{\xi}(\ln t)x, \tag{A.1}$$

where $\xi \in (0, x)$ for $x > 0$ or $\xi \in (x, 0)$ for $x < 0$. Let $x = 1 - q < 0$. Then, (A.1) implies

$$t^{1-q} = 1 + t^{\xi}(\ln t)(1 - q) \geq 1 - (\ln t)(q - 1) \tag{A.2}$$

The inequality is due to the fact that $t^{\xi} \leq 1$ because $t \geq 1$ and $\xi \leq 0$. Hence, from (A.2), it follows that $\psi_b(t) \geq -\ln t$.

Iteration bounds: Substituting the verified parameters into (4.1), we get $\gamma = \frac{q+p}{q(p+1)}$ and $h(x) = 2$. Therefore, $\frac{\sigma}{\eta} < 1$, $\frac{c}{\gamma} \leq 2q$ and $\left(\psi''_{p_1}(1)\right)^{\gamma} < 2q$. Using the Main Theorem 4.1, we derive the iteration bounds $O\left((1 + 2\kappa)q^2\sqrt{n}\ \ln \frac{n}{\varepsilon}\right)$ for the short-step version and $O\left((1 + 2\kappa)qn^{\frac{p+q}{q(1+p)}} \ln \frac{n}{\varepsilon}\right)$ for the long-step version of the *Algorithm*, as listed in Table 2.

Verification that ψ_{e_1} is SKF. From Table 3, we have the following parameters in Definition 3.7: $r = 1$, $\eta = q$, $c = 2q + 2$, $\sigma = q + 1$ and $g(x) = e^x$.
(SKF-a): Since $t \leq 1$, $p \geq 0$, and $q > 1$, we have $t^p \leq 1$ and $-\frac{1}{t^{q+1}} \leq -1$, implying

$$\psi'_{e_1}(t) = t^p - \frac{e^{t^{-q}-1}}{t^{q+1}} \leq 1 - e^{t^{-q}-1} = \frac{1}{r}\left(1 - e^{t^{-\eta}-1}\right).$$

(SKF-b): Since $t \leq 1$, $0 \leq p \leq 1$, and $q > 1$ and using $1-t^p \geq 0$, and $\frac{1}{t^{q+1}}e^{t^{-q}-1} - t^p \geq 0$, we have

$$\begin{aligned}
\psi''_{e_1}(t) &= pt^{p-1} + \frac{(q+1)t^q + q}{t^{2q+2}}e^{t^{-q}-1} = \left(pt^{p+q} + \frac{(q+1)t^q + q}{t^{q+1}}e^{t^{-q}-1}\right)\frac{1}{t^{q+1}} \\
&\leq \left(p + \frac{(2q+1)}{t^{q+1}}e^{t^{-q}-1} + (2q+1)(1 - t^p) + \frac{1}{t^{q+1}}e^{t^{-q}-1} - t^p\right)\frac{1}{t^{q+1}} \\
&\leq (2q+2)\left(1 - t^p + \frac{e^{t^{-q}-1}}{t^{q+1}}\right)\frac{1}{t^{q+1}} = c(1 - \psi'(t))t^{-\sigma}. \qquad \text{(A.3)}
\end{aligned}$$

(SKF-c)-(SKF-e): We show that the assumptions of Lemma 3.9 hold.
In this case $\psi_b(t) = \frac{e^{t^{-q}-1}-1}{q}$, which is nonpositive for $t \geq 1$, since $q > 1$.
Furthermore, $\psi_b'(t) = -\frac{e^{t^{-q}-1}}{t^{q+1}} \geq -\frac{1}{t}$ for $t \geq 1$.
For the barrier term, we use the following inequalities

$$e^{x-1} \geq x \geq 1 + \ln x,\ \text{for } x > 0. \tag{A.4}$$

Substituting $x = t^{-q}$ into the (A.4), we obtain

$$e^{t^{-q}-1} \geq 1 + \ln t^{-q} = 1 - q\ln t,\ \text{for } t > 1.$$

Hence, $\psi_b(t) \geq -\ln t$.

Iteration bounds: Substituting the verified parameters into (4.1), we get $\gamma = \frac{1}{p+1}$ and $h(x) = 1+\ln x$. Therefore, $\frac{c}{\gamma} < 8q$ and $\psi''_{e_1}(1) < 4q$. Using the Main Theorem 4.1, we derive the iteration bounds $O\left((1+2\kappa)q^{1+\frac{1}{p+1}}(1+\ln q)^{\frac{q+1}{q}}\sqrt{n}\ln\frac{n}{\varepsilon}\right)$ for the short-step version and $O\left((1+2\kappa)q\; n^{\frac{1}{p+1}}(\ln n)^{\frac{q+1}{q}}\ln\frac{n}{\varepsilon}\right)$ for the long-step version of the *Algorithm*, as listed in Table 4.

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