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RESEARCH ARTICLE

Information design for weighted voting

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Abstract

We consider a sender who wishes to persuade multiple receivers to vote in favor of a proposal and sends them private correlated messages that are conditional on the true state of the world. The receivers share a common prior, wish to implement the outcome that matches the true state, and have homogeneous preferences. However, they are heterogeneous in their voting weights. We consider both behavioral and sophisticated voters. When voters are behavioral, public communication is optimal if and only if there is a veto player. For sophisticated voters, we establish lower bounds on the sender's gain from persuasion for general voting quotas and show that the sender can often improve upon public communication. Finally, in an extension, we show that even when behavioral voters have heterogeneous prior beliefs, public communication is optimal if and only if there is a veto player.

Keywords Information design · Bayesian persuasion · Strategic voting · Shareholder voting · Private communication

JEL Classification C72 · D72 · D82 · D83

1 Introduction

How should the CEO of a company communicate with its shareholders in order to convince them to accept an executive decision? How effectively can a lobbyist communicate with the members of the UN Security Council to sway them in favor of a proposal? What is an optimal way of communicating with voter blocs in electoral campaigns? The voting applications of Bayesian persuasion (e.g. Wang 2013; Schnakenberg 2015; Alonso and Câmara 2016b; Bardhi and Guo 2018; Chan et al. 2019; Heese and Lauermann 2021; Kerman and Tenev 2021; Guo 2021; Kerman et al. 2024) so far do not provide an answer to these questions, as the situations above involve

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collective decision making under *weighted voting* rather than voters with *equal voting power*.[1] In contrast, weighted voting has been a prominent topic both in cooperative game theory and in the social choice literature. It has been analyzed from a theoretical and computational point of view (e.g. Banzhaf III 1964; Taylor and Zwicker 1992, 1993; Snyder Jr et al. 2005; Elkind et al. 2008; Lindner 2008; Freixas and Molinero 2009; Chalkiadakis et al. 2012), but also in terms of its actual applications to political (e.g. Barbera and Jackson 2006; Kurz et al. 2017; Baharad et al. 2022) and financial decision-making (e.g. Strand and Rapkin 2005; Bar-Isaac and Shapiro 2020). For example, the United Nations Security Council, the Council of the European Union under the Nice Treaty, and the selection of the managing director of the IMF all involve non-uniform voting weights (Mayer and Napel 2020).[2] Similarly, some companies offer dual class shares, which give the owners different voting rights.[3] Alternatively, the shareholders could have voting power *proportional* to their total investment in the company. Decision-making in these cases is often shaped by targeted lobbying campaigns aimed at implementing a specific outcome (Wright 1990; Bergan 2009; Hall and Reynolds 2012; Schnakenberg 2017). Therefore, weighted voting setups present a natural application of information design that warrants a thorough investigation.

To the best of our knowledge, this is the first attempt at incorporating weighted voting into an information design framework. Optimal communication in our framework exhibits a similar property to ones in other voting applications of information design (Alonso and Câmara 2016b; Chan et al. 2019; Kerman et al. 2024), which is to target minimal winning coalitions with positive probability. As these studies consider agents who have the same voting weights, one can easily determine the defining features of minimal winning coalitions once the number of receivers is known. On the other hand, extending the standard case to heterogeneous voting weights poses a significant challenge since computing all minimal winning coalitions in a weighted voting game is an NP-hard problem (Aziz and Paterson 2008; Elkind et al. 2008, 2009). Therefore our problem necessitates using a different method.

We initially do not impose any structure on the voting problem and allow for any voting weights and *quota* (minimum number of approval votes required to pass a proposal). In this case, presenting the optimal communication problem as a tractable linear programming problem (a standard approach in the information design literature) does not produce a closed-form solution. Instead, as an alternative approach, while keeping the voting weights flexible, we consider particular voting quotas (e.g. minority, majority) that represent reasonable voting situations. Consequently, while analyzing the problem in its entirety is very challenging, parsing it significantly reduces its complexity and allows us to derive meaningful insights about the efficiency of private communication. In particular, by construction of communication strategies we establish lower bounds on the value which improve upon public communication and do not rely on a closed-form solution. Finally, we consider specific voting weights

[1] For consistency, throughout the paper we use the term *information design* instead of *Bayesian persuasion*.

[2] The use of weighted voting dates back at least to the Roman Republic, where the system of corporate voting allowed voters to have different influences (Mouritsen 2001).

[3] E.g. Alphabet (Google) and Meta (Facebook) offer dual-class shares. https://www.nytimes.com/2009/11/25/technology/internet/25facebook.html, https://www.wsj.com/articles/BL-CFOB-8866.

(in the case of apex games) that can be interpreted as different voting situations (e.g. shareholder voting, committee voting, voter blocs, parliamentary voting) and provide a full characterization of optimal communication.

The main results of this paper address both behavioral and sophisticated voters and provide bounds and characterizations for the sender's gain from persuasion under different voter behavior. This facilitates the comparison between private and public communication and establishes that the sender is very often (strictly) better off communicating privately. In fact, when voters are behavioral, public communication is optimal if and only if there exists a veto player, without whose support no positive decision can be made (Theorem 1). This can be especially important for particular voting situations such as in the UN Security Council: while in substantive votes permanent members have the right to veto, in procedural votes no country has veto power.[4] Our result implies that while in substantive votes a lobbyist cannot improve her chances of persuasion by communicating privately with the members, in procedural votes private communication can be effective.

In case a minority is sufficient to implement a proposal or in case of majority voting without veto players, the sender's gain from private persuasion can be significant, irrespective of the distribution of weights, often even when voters are sophisticated. We establish this by finding the lower bounds on the sender's expected utility in these cases. Finally, we consider apex games, a specific application of our model setup which imposes structure on the voting weights and the quota, and we fully characterize optimal communication for it (Theorem 2). The structure of an apex game is reminiscent of shareholder voting situations, where there can be a major shareholder and many others with a small number of shares. While intuitively a major shareholder should hinder the persuasion capabilities of a CEO since he has more influence on the outcome, interestingly, our result implies that this need not be the case (unless this is a controlling shareholder).

We analyze several possible extensions of the model. First, we consider heterogeneous payoff functions, i.e. receivers have different utilities from matching and mismatching the states. We first show that from the sender's perspective this is tantamount to receivers with heterogeneous prior beliefs. Afterwards, we show that Theorem 1 extends to this case as well (Theorem 3). This is an important consideration, since it is natural to consider that different groups of voters (e.g. voter blocs) might have different prior beliefs about the effect of a particular proposal (while these beliefs are close to homogeneous within every group). Our result shows that even when these priors diverge significantly, a campaign which utilizes private communication (e.g. sending different messages to distinct blocs) is more efficient than one that employs public communication, unless a group of voters has veto power. As a second extension we consider allowing the voters to abstain and show that our results do not change, i.e. excluding abstention is without loss of generality.

This paper frames its main focus as a voting problem, yet it can be interpreted in several different ways, for example in the broader context of *marketing*. In particular, the voting quota corresponds to achieving a *critical mass* of product sales and the voting weights represent different consumer demands. Many studies estab-

4 https://www.un.org/en/about-us/un-charter/chapter-5.

lish the importance of going over a particular threshold of adoptions to be able to take advantage of the network effects associated with maximum product diffusion in the market (e.g. see Terpstra 1983; Cabral et al. 1999; Fang et al. 2008; Peng 2010; Fernández-i Marín 2011). In this interpretation, the sender's objective is to find the *optimal targeted advertisement campaigns* to establish a product on the market. A similar persuasion problem arises in *crowdfunding* situations, where a certain level of investment is required for the product to be launched, which can come from any number of individual investors. Alternatively, as stated earlier, we can interpret the voting weights as a group of people who vote uniformly despite each member having a single vote, i.e. as *voter blocs*.[5]

Illustrative example

Consider a company that provides to its shareholders voting weights *proportional* to their total investments in the company.[6] The CEO of the company wishes to acquire a start-up in order to build her reputation in the industry (Masulis et al. 2009; Choi et al. 2020).[7] Suppose that a simple majority of votes is required to do so. The shareholders *initially* believe that the acquisition is profitable with probability $1/3$ and approve it when they have a belief of at least $1/2$ that this is so. Suppose that there are five shareholders with voting rights and let the voting weights be given by $(3, 1, 1, 1, 1)$, i.e. the total voting weight is 7. As the voting rule is simple majority, 4 votes are required to approve the acquisition.

The CEO commissions an economic evaluation of the proposal and *privately* communicates the results to the shareholders. The evaluation consists of information such as the potential profitability of the start-up, the expected costs of acquisition, the quality of the start-up's workers, etc. The process of communicating the economic evaluation can be designed so that it either always truthfully recommends to vote in favor of acquisition of the start-up, or it represents the facts skewed *in favor* of approving the acquisition. Mathematically, the communication process can be represented by an *experiment* π that sends correlated messages with a certain probability, conditional on the viability of the acquisition. The choice of π (i.e. the evaluation criteria) is observed by the shareholders. However, in each realization (i.e. vector of messages we call *signal*), shareholders only observe their own private *message*.

Denote the state in which the acquisition is profitable by P and the state in which it is not by N. The optimal experiment is given by π below.

[5] Many papers establish the existence and analyze the functioning of voter blocs, see Gormley and Murphy (2008), Evans and Tonge (2009), Jakulin et al. (2009), Spirling and Quinn (2010), Eguia (2011a, b), Grimmer et al. (2022).

[6] The problem of shareholder voting holds a prominent position in the literature on corporate governance. For example, it has been investigated in the context of trading shares/votes (Meirowitz and Pi 2022), as well as in the context of conflicts of interest (Dressler and Mugerman 2023).

[7] There are many other decisions that shareholders can vote on, such as merging with another firm or electing a new CFO to the company, all of which can present situations in which the *private* interest of the CEO does not align with the interest of the company.

(a) Weights (3, 1, 1, 1, 1).

π	P	N
$(p,\ p,\ p,\ p,\ p)$	1	0
$(p,\ p,\ \boldsymbol{n}, \boldsymbol{n}, \boldsymbol{n})$	0	$\frac{1}{8}$
$(p,\ \boldsymbol{n},\ p, \boldsymbol{n}, \boldsymbol{n})$	0	$\frac{1}{8}$
$(p,\ \boldsymbol{n}, \boldsymbol{n},\ p, \boldsymbol{n})$	0	$\frac{1}{8}$
$(p,\ \boldsymbol{n}, \boldsymbol{n}, \boldsymbol{n},\ p)$	0	$\frac{1}{8}$
$(\boldsymbol{n},\ p,\ p,\ p,\ p)$	0	$\frac{3}{8}$
$(\boldsymbol{n}, \boldsymbol{n}, \boldsymbol{n}, \boldsymbol{n}, \boldsymbol{n})$	0	$\frac{1}{8}$

(b) Weights (3, 2, 1, 1).

π'	P	N
$(p,\ p,\ p,\ p)$	1	0
$(p,\ p, \boldsymbol{n}, \boldsymbol{n})$	0	$\frac{1}{6}$
$(p,\ \boldsymbol{n},\ p, \boldsymbol{n})$	0	$\frac{1}{6}$
$(p,\ \boldsymbol{n}, \boldsymbol{n},\ p)$	0	$\frac{1}{6}$
$(\boldsymbol{n},\ p,\ p,\ p)$	0	$\frac{1}{3}$
$(\boldsymbol{n}, \boldsymbol{n}, \boldsymbol{n}, \boldsymbol{n})$	0	$\frac{1}{6}$

(c) Weights (4, 1, 1, 1).

π''	P	N
$(p,\ p,\ p,\ p)$	1	$\frac{1}{2}$
$(\boldsymbol{n}, \boldsymbol{n}, \boldsymbol{n}, \boldsymbol{n})$	0	$\frac{1}{2}$

Here, p represents the private message indicating that the acquisition is profitable and $\boldsymbol{n}$ represents the converse. Notice that under π, while the sender truthfully recommends p in state P, a *minimal winning coalition* of shareholders receives p in state N. Upon observing a private message p, a shareholder has a posterior belief $1/2$ that the acquisition is profitable. The acquisition is approved for *any* realization except $(\boldsymbol{n}, \boldsymbol{n}, \boldsymbol{n}, \boldsymbol{n}, \boldsymbol{n})$, so the ex-ante probability of accepting it under π is $1/3 \cdot 1 + 2/3 \cdot 7/8 = 11/12$.

Now, suppose that shareholder 5 sells his share and shareholder 2 acquires it, so the voting weights are given by $(3, 2, 1, 1)$. Since now there are *fewer* shareholders, the CEO's capability to communicate in private *decreases*, i.e. the number of minimal winning coalitions the CEO can target in state N is lower. In this case, the optimal experiment π' is given above, under which the probability of approving the proposal is $8/9$. Hence, the CEO can achieve her goal with a *lower* probability relative to the initial voting weights.

Alternatively, suppose that shareholder 5 sells his share to shareholder 1 instead of 2, so that shareholder 1 holds the majority of the shares. The voting weights are given by $(4, 1, 1, 1)$; shareholder 1 is a *dictator* in the company's decisions. In this case, the CEO *must* persuade shareholder 1. Since shareholder 1 is in all winning coalitions, this is equivalent to persuading all shareholders (i.e. targeting the grand coalition), which implies that optimal communication is equally efficient as *public* communication, i.e. when all shareholders observe the same message within a signal. The optimal experiment is given by π'' under which the probability of approval is $2/3$, cf. Kamenica and Gentzkow (2011).

The discussion above illustrates the non-trivial difference that the change of hands of a single vote can have on the persuasion capabilities of the sender. More generally, it highlights the importance of weighted voting within the information design framework as this setup complicates the sender's optimal persuasion problem when compared to considering only unitary weights. Another interesting observation from the example is that increased concentration of votes (e.g. moving from (3,2,1,1) to (4,1,1,1)), can reduce the sender's payoff and increase the voters' payoff (i.e. the implemented outcome matches the state with a higher probability). However, we show that this is not true in general.

Related literature

To the best of our knowledge, this is the first paper within the information design literature which incorporates voting weights in its setup.

The first benchmark in our analysis comes from the seminal paper by Kamenica and Gentzkow (2011) who consider a single sender and a single receiver, which corresponds to public communication in our case. The setup and analysis in the current paper is closest to and builds upon (Kerman et al. 2024), who provide our second benchmark for comparison as they consider a model with a single sender and multiple receivers with *unitary* voting weights. As our setup analyzes the case of different voting weights, our model is a generalization of theirs. Third, our model also relates to the one in Kerman and Tenev (2021), whose model features limited information spillovers in a social network. Since receivers in a complete component possess the same information, these components can be interpreted as a single agent with a non-unitary voting weight (or as voter blocs). Hence, our analysis nests their result on networks with complete components, which obtain that result as a special case.

We consider both naive and sophisticated voters, which have been extensively analyzed in the voting literature. A plethora of studies shows that agents are behavioral in a variety of situations, including large elections. Sincere voting has been observed under multiple experimental setups (e.g. Herzberg and Wilson 1988; Dasgupta et al. 2008; Hobolt and Spoon 2012; Esponda and Vespa 2014; Bhattacharya et al. 2014; Grosser and Seebauer 2016; Bhattacharya et al. 2017). Theoretical studies have also verified the existence of equilibria in which agents behave sincerely (e.g. Krishna and Morgan 2012; Acharya and Meirowitz 2017; Kleiner and Moldovanu 2017, 2019).[8] We address sincere voting both as a behavioral assumption and as equilibrium behavior and contrasts the sender's gain from persuasion in these cases.

Our paper also contributes to the discussion of public vs. private communication. Arieli and Babichenko (2019) provide a characterization of public communication under a certain assumption on the sender's utility function. Translating their assumption to a voting context means that each agent can singlehandedly implement the sender-preferred outcome. Clearly, this is too strong an assumption in our model. In fact, we show that in our model this assumption is not necessary to characterize public communication. In a different study, Taneva (2019) considers conditionally independent private signals and shows that they are never strictly optimal.[9] In contrast, we consider private correlated messages and show that public communication is optimal if and only if there is a veto player when the agents are behavioral. In case of sophisticated agents, however, public communication might be optimal even when there is no veto player. One important feature of optimal private communication in our model is that when the voters are behavioral, the sender targets minimal winning coalitions.

[8] For other papers that observe sincere voting in experimental setups, see Felsenthal and Brichta (1985), Degan and Merlo (2007), Van der Straeten et al. (2010), Bassi (2015), Hix et al. (2017), Rich (2017), Lebon et al. (2018), Puppe and Rollmann (2021); for others considering applications of sincere voting in different theoretical models, see Benoit et al. (2000), Davidovitch and Ben-Haim (2010), Carmona (2012), Ginzburg (2017), Bouton and Ogden (2017). For studies that empirically show sincere voting in different elections, see Burden and Jones (2006), Groseclose and Milyo (2010).

[9] Wang (2013) shows a similar result in a voting setup.

A similar result has been first introduced by Alonso and Câmara (2016b) in the context of public communication and receivers with heterogeneous payoffs. Several other papers which consider private communication also establish this property (Arieli and Babichenko 2019; Chan et al. 2019; Kerman et al. 2024). While this provides an important starting point for the analysis of sophisticated voters with heterogeneous voting weights, in this case it is optimal for the sender to target slightly larger coalitions in order to avoid having pivotal agents. However, these coalitions still need to be minimal in some sense: they contain at least one pair of agents such that when removed the coalition becomes losing.

The structure of the paper is as follows. Section 2 describes the model, Sect. 3 simplifies the problem. Starting with some preliminaries, it delineates the sender's problem, proves its tractability, sketches the bounds for its solution and establishes connections to relevant results in the literature. Section 4 carries the main results. It reinterprets the optimal persuasion problem with the help of notions pertinent to weighted majority games in general and thus substantiates tighter limits on the sender's value. Section 5 considers extending the model in two dimensions: allowing for heterogeneous payoffs/priors and for abstention. Section 6 concludes.

2 The model

2.1 Communication and beliefs

Let $N = \{1, \ldots, n\}$ be the set of receivers and $\Omega = \{X, Y\}$ the set of states of the world. For any set S denote by $\Delta(S)$ the set of probability distributions over S with finite support. Sender and receivers share a common prior belief $\lambda^0 \in \Delta(\Omega)$ about the true state of the world.

Let S_i be a finite set of *messages* the sender can send to receiver i, and let $S = \prod_{i \in N} S_i$, where the elements of S are called *signals*. An *experiment* is a function $\pi : \Omega \to \Delta(S)$ which maps each state of the world to a joint probability distribution over signals. Denote the set of all experiments by Π. For each $\pi \in \Pi$, $s_i \in S_i$, and $\omega \in \Omega$, let $\pi_i(s_i|\omega) = \sum_{t \in S: t_i = s_i} \pi(t|\omega)$ be the probability that receiver i observes s_i given ω.

Define $S^\pi = \{s \in S | \exists \omega \in \Omega : \pi(s|\omega) > 0\}$. That is, S^π consists of the signals in S which are sent with positive probability by π. Similarly, for each π and $i \in N$, define $S_i^\pi = \{s_i \in S_i | \exists \omega \in \Omega : \pi_i(s_i|\omega) > 0\}$, which is the set of *messages* receiver i observes with positive probability under π.

For any $\pi \in \Pi$ and $s \in S^\pi$, the posterior belief profile $\lambda^s \in \Delta(\Omega)^n$ is defined by

$$\lambda_i^s(\omega) = \frac{\pi_i(s_i|\omega)\lambda^0(\omega)}{\sum_{\omega' \in \Omega} \pi_i(s_i|\omega')\lambda^0(\omega')}, \quad i \in N, \omega \in \Omega.$$

That is, $\lambda_i^s(\omega)$ is receiver i's posterior belief that the state is ω upon observing s_i.

2.2 Weighted voting

Given a set of receivers N, let $(w_i)_{i\in N}$ denote a *weight profile*, where $w_i \in \mathbb{N}$ and let the *vote total* be $\tau = \sum_{i\in N} w_i$. For each $i \in N$, let $A_i = \{x, y\}$ be the set of *actions* of receiver i. Let $A = \prod_{i\in N} A_i$ be the space of action profiles and $Z = \{x, y\}$ the set of voting outcomes.

Let $z^q : A \to Z$ be a map, where $z^q(a)$ is the outcome of the vote when the action profile is a and the quota is $q \in \mathbb{N}$ such that

$$z^q(a) = \begin{cases} x & \text{if } \sum_{i\in N:a_i=x} w_i \geq q, \\ y & \text{otherwise.} \end{cases}$$

In words, the outcome x is implemented if the sum of the weights of the agents who are voting for x is at least q (out of the vote total τ).

We assume that the sender's utility function $U : Z \to \{0, 1\}$ has value 1 if x is implemented and 0 otherwise. For each $i \in N$, let $u_i : Z \times \Omega \to \{0, 1\}$ be the utility function of receiver i such that $u_i(x, X) = u_i(y, Y) = 1$ and $u_i(x, Y) = u_i(y, X) = 0$. That is, each receiver wants the implemented outcome to match the true state. A *voting problem* is a tuple $P = (N, \lambda^0, q, (w_i)_{i\in N}, (u_i)_{i\in N}, U)$.

Throughout the paper we assume that $\lambda^0(X) < \lambda^0(Y)$, since otherwise there is no need of persuasion. For any $\pi \in \Pi$, let $\alpha_i^\pi : S_i^\pi \to A_i$ be agent i's *strategy*. Define the set of signals which implement x under π and α^π as $Z_x^q(\pi) = \{s \in S^\pi | z^q(\alpha^\pi(s)) = x\}$. Let $a \in A$ be an action profile and $z = z^q(a)$ be a voting outcome. The *value* of an experiment $\pi \in \Pi$ for quota q is defined as the sender's expected utility under π:

$$V_q^\pi(\lambda^0) = \mathbb{E}_{\lambda^0}\left[\mathbb{E}_\pi\left[U(z^q\left(\alpha^\pi(s)\right))\right]\right] = \lambda^0(X) \sum_{s\in Z_x^q(\pi)} \pi(s|X) + \lambda^0(Y) \sum_{s\in Z_x^q(\pi)} \pi(s|Y).$$

Thus, the value of an experiment is equal to the probability of implementing x under π and quota q. An experiment π^* is *optimal* in Π if $V_q^{\pi^*}(\lambda^0) = \sup_{\pi\in\Pi} V_q^\pi(\lambda^0)$.

2.3 Voting behavior

Throughout the paper, we consider naive and sophisticated agents. *Behavioral*/naive agents choose action x if they believe the true state is X with at least probability $1/2$. For any $\pi \in \Pi$, $i \in N$, and $s \in S^\pi$, it holds that $\alpha_i^\pi(s_i) = x$ if $\lambda_i^s(X) \geq 1/2$ and $\alpha_i^\pi(s_i) = y$ otherwise, i.e. receivers vote sincerely. Note that when there is no communication between the sender and receivers, the receivers do not have an incentive to vote against their belief when there are only two possible outcomes of the vote, i.e. if their posterior belief that the state is X is greater than $1/2$, it is optimal for them to vote for x. However, as we will argue later, this might not be true when they are informed by the sender via an observable information structure.

While sincere voting is important as a standalone analysis, it also alleviates the study of sophisticated voters, who take into account the probability of being pivotal.

Receiver i is *pivotal* in $s \in S^\pi$ if for any $a_i \in A_i$, $z^q(a_i, \alpha^\pi_{-i}(s_{-i})) = a_i$. That is, i is pivotal following realization s if i's vote determines the voting outcome given that all $j \neq i$ follow α^π_j.

Two trivial equilibria of the voting game always exist: (i) all agents vote in favor of x and (ii) all agents vote in favor of y, regardless of their private information. Hence to exclude such equilibria, as in Kerman et al. (2024), we follow the voting literature (Osborne and Slivinski 1996; Banks and Duggan 2000; Levy 2004) and consider equilibria in which agents vote sincerely.[10] We consider the sender's gain from persuasion both when agents are behavioral and when the sender implements sincere voting as a BNE.

Denote by $G(P, \pi)$ a game of incomplete information defined by a voting problem P and an experiment π.

Definition 1 (*Sincere BNE*) Let $\pi \in \Pi$. The sincere strategy profile α^π constitutes a *Bayes-Nash equilibrium (BNE)* of $G(P, \pi)$ if for all $i \in N$, $s' \in S^\pi$, and $a_i \in A_i$ it holds that

$$\sum_{\omega \in \Omega} \lambda_i^{s'}(\omega) \sum_{s \in S^\pi : s_i = s'_i} \frac{\pi((s'_i, s_{-i})|\omega)}{\pi_i\left(s'_i|\omega\right)} u_i\left(z^q\left(\alpha^\pi_i(s'_i), \alpha^\pi_{-i}(s_{-i})\right), \omega\right)$$

$$\geq \sum_{\omega \in \Omega} \lambda_i^{s'}(\omega) \sum_{s \in S^\pi : s_i = s'_i} \frac{\pi((s'_i, s_{-i})|\omega)}{\pi_i\left(s'_i|\omega\right)} u_i\left(z^q\left(a_i, \alpha^\pi_{-i}(s_{-i})\right), \omega\right).$$

Let Π^{e} be the set of experiments such that the induced sincere strategy profile constitutes a BNE of the game of incomplete information.

3 Simplifying the problem

One can easily establish that it is without loss of generality for the sender to send recommendations to vote in favor of an alternative. This is known as straightforwardness.[11] Moreover, in our framework it is also still optimal for the sender to *truthfully recommend* x in state X, i.e. for all $i \in N$ it holds that $\pi_i(x|X) = 1$. Henceforth, Π denotes the set of all *straightforward* experiments to ease the notation.

It is possible to further simplify the search for an optimal experiment by restricting the set of signals that the sender can employ in state Y to *minimal winning coalitions*.[12] Let $W = \{T \subseteq N | \sum_{i \in T} w_i \geq q\}$ be the set of winning coalitions and define the set of minimal winning coalitions as $W^{\min} = \{T \in W | \forall R \subsetneq T : R \notin W\}$.

[10] This is, in spirit, similar to Brams and Fishburn (1978, 2002) in which the authors design a mechanism to implement sincere voting in equilibrium. There are studies that employ Bayes correlated equilibrium in similar setups, see for example Forges (1993), Bergemann and Morris (2016), Arieli and Babichenko (2019), Taneva (2019).

[11] An experiment $\pi \in \Pi$ is *straightforward* if for all $i \in N$: (i) $S_i^\pi \subseteq A_i$ and (ii) $\alpha_i^\pi(a_i) = a_i$ for all $a_i \in A_i$ (Kamenica and Gentzkow 2011).

[12] This is in line with the approach of Alonso and Câmara (2016b), Arieli and Babichenko (2019), Chan et al. (2019), Kerman et al. (2024).

For any $R \subseteq N$, let $\chi(R)$ denote the signal in which all agents in R observe x and agents not in R observe y. That is, for all $i \in R$ it holds that $\chi_i(R) = x$ and for all $j \notin R$ it holds that $\chi_j(R) = y$. In this case we say that the agents in R are *targeted* by the sender.[13] Define $S^{\min} = \{\chi(R) \in S | R \in W^{\min}\}$ as the set of signals that target minimal winning coalitions. Let $\bar{x} = \chi(N) = (x, \ldots, x)$ and $\bar{y} = \chi(\emptyset) = (y, \ldots, y)$. Let $S^{\min}_{+} = S^{\min} \cup \{\bar{x}, \bar{y}\}$. A table with the most relevant notation can be found in "Appendix A".

In state Y, an optimal experiment targets either minimal winning coalitions or no one (i.e. the signal $\bar{y}$ is realized with positive probability). We formalize this observation below. All proofs can be found in "Appendix C".

Lemma 1 *Let $\pi \in \Pi$. Then there exists $\pi^* \in \Pi$ with $\pi^*(\bar{x}|X) = 1$ and $S^{\pi^*} \subseteq S^{\min}_{+}$ such that $V_q^{\pi^*}(\lambda^0) \geq V_q^{\pi}(\lambda^0)$.*

Lemma 1 ensures the tractable representation of the the sender's optimal signaling problem.[14] Set $S^{\min} = \{s^1, \ldots, s^m\}$, where $m \in \mathbb{N}$. For any $i \in N$, define $S^{\min}_x(i) = \{s \in S^{\min} | s_i = x\}$, i.e. the set of signals in $S^{\min}$ which send x to agent i. Given $\pi \in \Pi$ and $j \in \{1, \ldots, m\}$, let $p_j = \pi(s^j|Y)$ and $p_0 = \pi(\bar{y}|Y)$.

Proposition 1 *Suppose voters are behavioral. An optimal experiment is a solution to*

$$\max_{p_1,\ldots,p_m} \quad 1 - \lambda^0(Y)p_0 \quad \textit{subject to}$$

$$p_j \geq 0, \quad \forall j \in \{0, \ldots, m\}, \tag{1}$$

$$\sum_{j=0}^{m} p_j = 1, \tag{2}$$

$$\sum_{j \in S^{\min}_x(i)} p_j \leq \lambda^0(X)/\lambda^0(Y), \quad \forall i \in N. \tag{3}$$

Inequality (1) represents the non-negativity constraints for the signals, equation (2) ensures that an optimal experiment is a probability distribution, and inequality (3) provides the obedience constraints for all agents (i.e. it ensures agents follow the recommended action). Inequality (3) implies that there are potentially n different obedience conditions, unlike the case of unitary weights.[15] In particular, unitary weights allow the sender to devise *anonymous* experiments, which target every minimal winning coalition with the same probability. Therefore, the obedience constraint in this case boils down to only one condition as in Kerman et al. (2024). The lack of anonymity

[13] For example, if $N = \{1, 2, 3\}$ and $R = \{1, 2\}$, then $\chi(R) = (x, x, y)$.

[14] Note that Lemma 1 has a direct implication about the optimal value for particular weight profiles. An agent $i \in N$ is a *dummy player* if for all $T \in W^{\min}$ it holds that $i \notin T$. That is, a dummy player is never in a minimal winning coalition. Examples of dummy players can be found in politics; in almost every country there are many small parties that only have a very small fraction of the total vote. Since it is optimal for the sender to only target minimal winning coalitions, by Lemma 1 a dummy player is never targeted by the sender. Therefore, it is as if the persuasion problem consists of $n - 1$ agents, which decreases the optimal value.

[15] Note that this is the case also when receivers have heterogeneous priors. See Sect. 5.1.

in our framework significantly complicates the sender's optimal signaling problem. Nevertheless, since the objective function is continuous and all inequalities are weak inequalities a solution *always exists* but it is *not unique* in general.

Given two voting problems, it is possible that the sender's optimization yields the same representation (i.e. the same winning coalitions) as provided by Proposition 1. We call such voting problems *equivalent*.

Corollary 1 *Let P and P' be equivalent voting problems under which optimal values are V and V', respectively. Then $V = V'$.*

While Corollary 1 is intuitive, it provides a powerful insight: a weight profile that represents a voting problem is not unique.

Example 1 (*Equivalence*) Let $\lambda^0(X) = 1/3$, and let $(w_i)_{i \in N} = (2, 1, 1, 1)$ with $q = 3$ and $(w_i')_{i \in N} = (3, 2, 1, 1)$ with $q = 4$ represent two voting problems (Taylor and Zwicker 1999). Since the minimal winning coalitions in both cases have an identical structure, the voting problems are equivalent, i.e. the constraints of the two optimization problems as defined above are identical. By Proposition 1 and Corollary 1 it holds that $V = V' = 8/9$. $\triangle$

An important aspect about Proposition 1 is that under an optimal experiment agents are *not pivotal* in state X, while they *are* pivotal in state Y upon observing x. This leads to the swing voter's curse (Feddersen and Pesendorfer 1996), i.e. voters are better off voting against their beliefs and therefore sincere voting is not a BNE under an optimal experiment obtained via Proposition 1. To circumvent the swing voter's curse, we follow the approach in Kerman et al. (2024) and consider agents who play the sincere BNE introduced in Definition 1. We incorporate this as an additional constraint into the sender's maximization problem. In this case, the sender's problem is to find $\pi \in \Pi^{\mathrm{e}}$ that maximizes her expected utility, i.e. her maximization problem is $\sup_{\pi \in \Pi^{\mathrm{e}}} V_q^{\pi}(\lambda^0)$. The sender can restrict her search to straightforward experiments and truthfully recommend x in state X also within the set of experiments such that the induced sincere strategy profile constitutes a BNE, i.e. Π^{e}.

The analysis of equilibrium in this case differs from the one about behavioral voters in only one significant dimension (and is therefore relegated "Appendix B"), which is the type of coalitions that are targeted in state Y. In particular, it is optimal for the sender to target coalitions in which *no agent* is pivotal.[16] Hence, the summation in inequality (3) is no longer over $S_x^{\min}(i)$, but over the set of signals that target coalitions that are slightly larger than minimal ones in which i observes x. Moreover, to ensure optimality, these signals need to be minimal in size, i.e. there exists at least one pair of agents such that removing them makes the coalition losing.[17] Note that these coalitions are not minimal winning coalitions, i.e. they are not in $W^{\min}$. For example, when the weights are unitary removing *any pair* from such a coalition makes it losing. For arbitrary weights, having at least one such pair ensures that the size of the coalition is minimal. In this way, under the optimal experiment, no agent is pivotal either in state

[16] Note that when $w_i = 1$ for all $i \in N$, this corresponds to targeting $q + 1$ agents.

[17] Notice that if for some $R \in W$ there is no $i, j \in R$ such that $R \backslash \{i, j\} \notin W$, then there exists $k \in R$ who is unnecessarily targeted and therefore, not targeting him weakly increases the value.

X or in state Y. Thus, no agent has an incentive to deviate from his recommendation and therefore the sincere strategy profile constitutes a BNE.

Notice that in Proposition 1, voting weights are *implicitly* included in $S_x^{\min}(i)$, the set of signals which target minimal winning coalitions. Given a weight profile, however, several combinations of agents with *different weights* may constitute a minimal winning coalition. Since the derivation of an optimal experiment crucially depends on the number and constitution of minimal winning coalitions, it is not possible to find a closed-form solution for an arbitrary weight profile. A similar observation holds if we consider the optimal experiment under sincere BNE. We illustrate this below and outline why some potential further simplifications do not hold in general.

Example 2 (*Behavioral vs. Sophisticated Voters*) Let $w = (6, 6, 6, 5, 5, 5)$ and $\lambda^0(X) = 1/3$. Let the quota be $q = 17$ and the vote total be $\tau = 33$. An optimal experiment π^* is given below.[18]

π^*	X	Y	π^e	X	Y
(x, x, x, x, x, x)	1	0	(x, x, x, x, x, x)	1	0
$(x, x, \boldsymbol{y}, \boldsymbol{y}, \boldsymbol{y}, x)$	0	$\frac{1}{10}$	$(x, x, x, x, \boldsymbol{y}, \boldsymbol{y})$	0	$\frac{1}{14}$
$(x, \boldsymbol{y}, x, \boldsymbol{y}, \boldsymbol{y}, x)$	0	$\frac{1}{10}$	$(x, x, x, \boldsymbol{y}, x, \boldsymbol{y})$	0	$\frac{1}{14}$
$(\boldsymbol{y}, x, x, x, \boldsymbol{y}, \boldsymbol{y})$	0	$\frac{2}{10}$	$(x, x, x, \boldsymbol{y}, \boldsymbol{y}, x)$	0	$\frac{1}{14}$
$(\boldsymbol{y}, x, x, \boldsymbol{y}, x, \boldsymbol{y})$	0	$\frac{2}{10}$	$(x, x, \boldsymbol{y}, x, x, x)$	0	$\frac{2}{14}$
$(x, \boldsymbol{y}, \boldsymbol{y}, x, x, x)$	0	$\frac{3}{10}$	$(x, \boldsymbol{y}, x, x, x, x)$	0	$\frac{2}{14}$
$(\boldsymbol{y}, \boldsymbol{y}, \boldsymbol{y}, \boldsymbol{y}, \boldsymbol{y}, \boldsymbol{y})$	0	$\frac{1}{10}$	$(\boldsymbol{y}, x, x, x, x, x)$	0	$\frac{2}{14}$
			$(\boldsymbol{y}, \boldsymbol{y}, \boldsymbol{y}, \boldsymbol{y}, \boldsymbol{y}, \boldsymbol{y})$	0	$\frac{5}{14}$

The optimal value is $V_q^{\pi^*}(\lambda^0) = 14/15$. However, if we consider strategic agents, to obtain an equilibrium the sender has to adjust the experiment as to make no agent pivotal in state Y. This gives rise to $\pi^{\mathrm{e}} \in \Pi^{\mathrm{e}}$, which leads to a sincere BNE and has value $V_q^{\pi^e}(\lambda^0) = 16/21$.[19] Thus, $V_q^{\pi^e}(\lambda^0) < V_q^{\pi^*}(\lambda^0)$.

There are a few notable aspects of π^* and π^{e}. First, it might not be optimal to employ all signals that target minimal winning coalitions, but a *subset* of them. This is also implied by the possibility of multiple solutions of the problem outlined in Proposition 1. Second, when the total weights of the agents who observe x are the same in two signals, they *need not* have equal probability in state Y. Finally, the number of x-observations for each voter in the employed signals is not the same. $\triangle$

A benchmark that we are going to use in the following sections is public communication. We call an experiment *public* if all agents observe the same message within a

[18] We provide a Matlab and Mathematica code on https://www.sites.google.com/view/aptenev/research that computes an optimal experiment for a specified weight profile.

[19] Recall that in constructing π^e, the sender targets coalitions in which there exists a pair such that when removed the coalition is losing. If this condition is imposed *for every* pair, then the coalitions such as $\{1, 2, 3, 4, 5\}$ are not targeted, leading to a lower value.

signal. That is, $\pi \in \Pi$ is public if for all $i, j \in N$ and $s \in S^\pi$ it holds that $s_i = s_j$. We denote the optimal public experiment by π^{pub}, where the value is

$$V^{\text{pub}}(\lambda^0) = \min\{2\lambda^0(X), 1\},$$

which follows from Kamenica and Gentzkow (2011) since voters are homogeneous. This provides a *lower bound* of what the sender can achieve, since $V^{\text{pub}}(\lambda^0)$ is *independent* of the underlying weight profile. Hence, we denote it by $V^{\text{pub}}(\lambda^0) = \underline{V}$.

In the next proposition, we also provide the *upper bound* of the sender's value, which is achieved when all voting weights are equal to 1 *and* voters vote sincerely. For $\lambda^0 \in \Delta(\Omega)$ and $\theta, \kappa \in \mathbb{Z}_+$ define the function

$$V(\theta, \kappa, \lambda^0) = \min\left\{\frac{\theta + \kappa}{\kappa}\lambda^0(X), 1\right\}.$$

Note that given a vote total τ, if $n = \tau$ (which implies that all weights are equal to one), it holds that the upper bound of the value is $V(\tau, q, \lambda^0)$, hence we denote it by $V(\tau, q, \lambda^0) = \overline{V}$. The observations about the bounds of the value are formalized below.

Proposition 2 (Bounds) *For any optimal* $\pi \in \Pi$ *it holds that*

$$\underline{V} = \min\{2\lambda^0(X), 1\} \leq V_q^\pi(\lambda^0) \leq \min\left\{\frac{\tau + q}{q}\lambda^0(X), 1\right\} = \overline{V}.$$

Note that the bounds hold also for experiments that lead to a BNE. An illustration of the bounds together with the optimal values when voters are behavioral and sophisticated in Example 2 can be seen in Fig. 1. In particular, if $n = 33$ and agents have unitary weights, then the upper bound of the optimal value is $\overline{V} = V(\tau, q, \lambda^0) = 50/51$. So, $2/3 = \underline{V} < V_q^{\pi^e}(\lambda^0) < V_q^{\pi^*}(\lambda^0) < \overline{V} = 50/51$. Note that the optimal equilibrium value is always (weakly) lower than the optimal value for behavioral voters.

The intuition behind establishing the upper and lower bounds follows immediately from Lemma 1. In particular, when all agents have the same unitary weight, the sender can fully utilize private communication by targeting the maximum number of minimal winning coalitions in state Y. When the voting weights of two agents are merged, *ceteris paribus* the number of minimal winning coalitions *weakly decreases*. This process exhibits *weak monotonicity*. That is, as the weights of agents are combined, the optimal value approaches $\underline{V}$. However, it should be emphasized that this property does not hold *strictly*; the upper bound $\overline{V}$ can still be achieved when not all voting weights are 1 as the illustrative example shows.

The discussion above prompts a surprising observation. When comparing the values of two weight distributions with the same vote total τ, it is possible that the more equal one makes the receivers *worse off*. Suppose that the measure of inequality that is applied to the weight distribution follows the Pigou-Dalton principle of transfers (Hindriks and Myles 2013) and one of the distributions can be obtained from the other

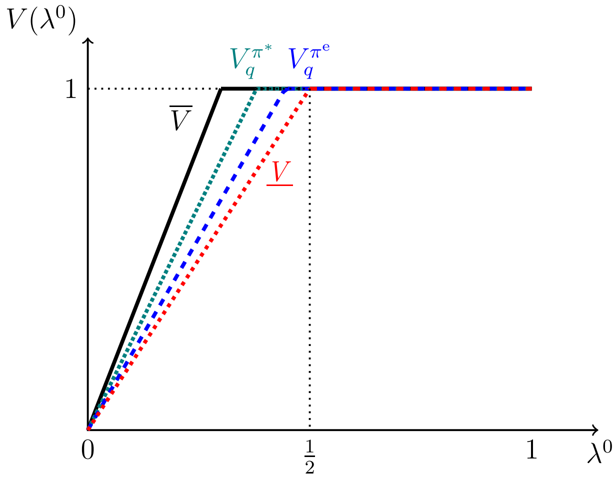

Fig. 1 Optimal Values in Example 2

one via a vote transfer.[20] In our motivating example, where $q = 4$, (4, 1, 1, 1) is the most *inequitable* distribution (as it contains a dictator). Weight profiles (3, 2, 1, 1) and (3, 1, 1, 1, 1) each are the result of a *progressive* transfer of one vote (in the latter case to someone who initially had no votes) rather than a merging the voting weights of two agents. As a result, the value of the sender *increases*, while the votes are distributed more *equally* among the receivers. This makes the receivers worse off due to the fact that the optimal value is *higher*: since it is optimal for the sender to truthfully recommend x in state X, a higher value corresponds to y being implemented in state Y with a *lower* probability (as the sender wishes to implement x), i.e. the outcome of the vote matches the true state *less* often.

However, it must be noted that a more inequitable distribution of votes is not always better for the sender, i.e. the relationship between equality and the sender's gain from persuasion is *non-monotonic*. As an example, recall that moving from weight profile (4, 1, 1, 1) to (3, 2, 1, 1) makes the sender better off (the optimal value changes from 2/3 to 8/9), while making the votes more equitable. Now suppose that in (3, 2, 1, 1), 1 vote is transferred from receiver 1 to receiver 3, so that the weight profile is (2, 2, 2, 1), which again makes the votes more equitable. In this case, receiver 4 becomes a dummy player (i.e. he is not in any minimal winning coalition) and the optimal value is lower than in the case of (3, 2, 1, 1) (it changes from 8/9 to 5/6). Hence, increasing equality in this case results in a lower gain from persuasion for the sender.[21]

[20] Informally, the Pigou-Dalton principle states that an inequality index must decrease if there is a transfer from the ones who have more to the ones who have less. It holds for inequality indices like the Gini coefficient or the Theil entropy measure.

[21] This non-monotonicity persists under different measures, e.g. the Herfindahl-Hirschman index, which estimates industry concentration and is used as a proxy for market power. An alternative way to obtain a ranking of weight distributions would be using mean-preserving spreads. However, this is a special case of ranking by second order stochastic dominance, which is equivalent to the Pigou-Dalton transfer principle for distributions with equal means (Aaberge et al. 2021).

Nevertheless, we can state simple sufficient conditions such that transferring votes to make a weight profile more/less equitable strictly increases/decreases the optimal value. Suppose that given a weight profile w, $\pi \in \Pi$ is optimal with $\underline{V} < V_q^{\pi}(\lambda^0) < \overline{V}$. Then, it is clear that if the votes are transferred to obtain the most equitable vote distribution (i.e. unitary weights), the sender benefits from it as now she can achieve the upper bound $\overline{V}$. On the other hand, if the votes are transferred to obtain a less equitable vote distribution, for example by creating a veto player, the sender is worse off since the optimal value is now $\underline{V}$, as we will show in Lemma 2 in the next section.

4 Results

The generic setup in the previous section considers a space of weight profiles which is very large and does not produce a closed-form solution. To be able to analyze real-life voting situations, our approach in this section is to split this space of voting weights using general characteristics of games such as *properness* or the existence of a *veto player* (Von Neumann and Morgenstern 1944; Shapley 1962). This allows a more comprehensive analysis, which characterizes optimal communication within concrete bounds but without imposing specific weights.

Interpreting the voting game agents play after observing their private message as a *weighted majority game* allows inferring some immediate results depending on the voting weights.[22] Recall that W is the set of winning coalitions.

Definition 2 (*Weighted Majority Game*) A *weighted majority game* (WMG) g is a tuple $(N, q, (w_i)_{i \in N})$ with quota $q > 0$ and weights $w_i \geq 0$ for all $i \in N$. We denote a WMG with weights $(w_i)_{i \in N}$ and quota q by $[q; w_1, \ldots, w_n]$.[23]

WMGs fit the general framework of voting applications of information design, especially when the sender has a state-independent utility function. In the following sections we denote the *value* of an experiment π on a *weighted majority game* g by $V_q^{\pi}(g)$. This section requires the intermittent introduction of notation which will be useful for specific statements, so we would like to refer the reader to "Appendix A", which presents an overview of most relevant notation.

4.1 Public communication

We start by considering WMGs that contain a player who has considerably more power than others (e.g. veto player, dictator). This hinders the persuasion capabilities of the sender very substantially since a *veto player* is in all winning coalitions.[24] For example, it is common in presidential systems for the president to be the veto player

22 Henceforth we borrow most definitions from Peleg and Sudhölter (2007).

23 Note that every voting problem P induces a WMG g.

24 Tsebelis (1995) studies different political systems in terms of their capacities to induce policy change, using a framework based on veto players.

or in parliamentary systems for some parties to be veto players.[25] We now introduce a lemma, which shows that the sender cannot improve upon public communication whenever there is an agent with sufficiently high voting power,[26]

Lemma 2 *If a WMG has a veto player, then (i) π^{pub} is optimal and (ii) $\pi^{\text{pub}} \in \Pi^{\text{e}}$.*

Intuitively, since a veto player is in all winning coalitions, the situation is equivalent to persuading all agents, making public communication optimal. It is interesting to note that this result is independent of the voter type (behavioral or sophisticated); the presence of a veto player nullifies the strategic considerations of other agents.

Note that veto players relate to our discussion of inequality. Voting situations with a veto player refer to some of the *most unequal* distributions of votes, yet the receivers *benefit the most* when the optimal value is public since the probability of implementing the correct outcome is highest among all optimal experiments under any WMG.[27] In fact, public communication is optimal when voters are behavioral *only if* there is a veto player.

Theorem 1 (Public Communication) *Suppose voters are behavioral. Then, π^{pub} is optimal if and only if there is a veto player.*

The 'if' part follows immediately from Lemma 2. Showing the 'only if' part employs insights from Proposition 4 and Proposition 6 that we introduce later, which broadly consider majority and minority voting. Thus, we postpone the discussion for it to Sect. 4.2. It is, however, easy to see why Theorem 1 holds when receivers have unitary weights. In this case, a receiver is a veto player if and only if $q = n$ (in fact, all receivers are veto players). Moreover, when voters are behavioral, unanimity is the only voting rule that does not allow the sender to improve upon public communication. Hence, public communication is optimal if and only if there is a veto player.

An important requirement of Theorem 1 is that voters are behavioral. However, note that if π^{pub} is the optimal strategy when voters are *sophisticated*, this does not necessarily mean that there is a veto player. Intuitively, sophisticated voters limit the persuasion capabilities of the sender and hence, if the sender wishes to implement a sincere BNE, she might need to employ public communication in a *broader* set of cases.

It is a well-known result that a game has a *nonempty core* if and only if there is a veto player. Therefore, we can relate Theorem 1 to Theorem 3 in Arieli and Babichenko (2019). However, while both results indicate public communication is optimal if and only if the core of the game in the respective setup is nonempty, their result holds

[25] If the set of veto players is not a singleton, then it is called an *oligarchy* A WMG is *dictatorial* if there exists $j \in N$ ("the dictator") such that $T \in W$ if and only if $j \in T$. Veto players, oligarchies, and dictators are *blocking coalitions*, since their complements are losing.

[26] If a WMG is *convex* then it is a *unanimity game*, in which case optimal communication is also public. Lemma 2 also relates to *big-boss games* (Muto et al. 1988) and *clan games* (Potters et al. 1989) since they contain veto players.

[27] An intuition for this observation, which is outside of the scope of this paper, is that an agent with a high voting weight might be more informed about the state of the world, which in turn would increase the probability of implementing the correct outcome. Following a similar logic, if the receivers could strategically and costlessly form coalitions, then they would form the grand coalition.

if the sender's utility function is *singleton-positive*. In our setup, this corresponds to any voter *singlehandedly* being able to implement the proposal. This is a very strong requirement in a voting setup since it implies that the proposal is *not* implemented if and only if everyone votes *against* it.

Finally, when there is a veto player the sincere BNE has an interesting property; while veto players who vote for x are pivotal in *both* states, the others are *never* pivotal. This provides an intermediate case between (Chan et al. 2019) where all agents are pivotal in equilibrium with positive probability and Kerman et al. (2024) where none of them are. We illustrate this in the next example.

Example 3 (*Veto players in BNE*) Suppose a lobbyist wishes to persuade the United Nations Security Council to pass a proposal. The council consists of 15 countries with a corresponding WMG [39; 7, 7, 7, 7, 7, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1].[28] The coalition comprising all agents with weight 7 consists of only *veto* players (i.e. it is an oligarchy). Therefore, it is optimal for the lobbyist to employ public communication, which constitutes a sincere BNE. Note that while *all veto players* are pivotal under the sincere BNE induced by π^{pub}, the agents with weight 1 are *never* pivotal. △

4.2 Private communication

We showed in the previous section that when the voters are behavioral, the sender can improve upon public communication if and only if there is no veto player. When voters are sophisticated, the sender can benefit from private communication *less* often. In this section, we determine the minimal improvement upon public communication for both voter types, i.e. we determine *lower bounds* for the sender's value. In most parts, we do not impose a particular structure on the voting weights, rather our focus lies on different voting quotas. More precisely, we consider optimal communication for minority, simple majority, and qualified majority voting rules.

Before imposing structure on the voting quota, we consider a WMG that is structurally closest to the unitary voting weights setup: A WMG is *symmetric* if for any $T \in W$ and $R \subseteq N$ it holds that $|T| = |R|$ implies $R \in W$. In such games the sender can restrict attention to *anonymous* experiments, which allows for a specific characterization of the value.[29] In other words, every minimal winning coalition consists of the same number of agents k (irrespective of their voting weights) and the situation is equivalent to persuading k out of n agents who have voting weights equal to one. The symmetry ensures that agents are pivotal with equal probability, i.e. they appear in minimal winning coalitions the same number of times. Recall that $V(\theta, \kappa, \lambda^0) = \min\left\{\frac{\theta+\kappa}{\kappa}\lambda^0(X), 1\right\}$.

[28] Note that while the voting system in the UN Security Council states that each member has one vote, since *permanent* members have the right to veto, the voting system is equivalent to a WMG with a voting weight of 7 for permanent and 1 for non-permanent members with a fixed quota of 39. https://www.un.org/securitycouncil/content/voting-system.

[29] This result is similar to Proposition 4.7 in Kerman and Tenev (2021). However, while there it is *assumed* that all minimal winning coalitions have the same cardinality, in this case this is *implied* by the symmetry of the WMG.

Proposition 3 (Equal Representation) *Let g be a symmetric WMG and $k = |T|$ for any $T \in W^{\min}$. Then there exist (i) $\pi \in \Pi$ with $V_q^\pi(g) = V(n, k, \lambda^0)$ and (ii) $\pi \in \Pi^{\mathrm{e}}$ with $V_q^\pi(g) = V(n, k+1, \lambda^0)$ is optimal.*

For a simple illustration of Proposition 3, consider the WMG [5; 4, 3, 2] and $\lambda^0(X) = 1/3$. In this case every $T \in W^{\min}$ has $|T| = 2$ and $V_q^\pi(g) = V(3, 2, 1/3) = 5/6$ for behavioral voters and $V_q^\pi(g) = V(3, 3, 1/3) = 2/3 = V^{\mathrm{pub}}$ for sophisticated voters.

Now we impose structure on the voting quota and start by considering majority voting.

Majority voting

We define majority voting as $q \geq (\tau + 1)/2$, i.e. it encompasses both simple and qualified majority. When the quota is at least simple majority, it follows that the complement of any winning coalition is losing. WMGs with this property are known as *proper*.

Definition 3 (*Proper Game*) A WMG is *proper* if $T \in W$ implies $N \setminus T \notin W$.

Note that assuming majority voting imposes a very mild restriction on the structure of the game, as it does not explicitly determine voting weights. Since finding the optimal experiment is contingent on specific voting weights, we provide lower bounds instead and demonstrate that the sender can improve upon public communication even without knowing the exact specification of the WMG.

If there is no veto player, then the voting rule is not unanimity, which implies that there are multiple minimal winning coalitions. Properness of the game ensures that the intersection of any two minimal winning coalitions is nonempty. To improve upon public communication, the sender must target at least three different winning coalitions (not necessarily minimal) in state Y such that no agent is in all of them (which is satisfied since there is no veto player). By Lemma 1, it follows that the fewer *non-minimal* winning coalitions the sender targets in state Y, the (weakly) higher the value. The experiment in the proof of Proposition 4 determines which *non-minimal* winning coalitions to target by a construction that uses the smallest intersection of two winning coalitions. In the case of behavioral voters this intersection is of two *minimal* winning coalitions and we denote its cardinality by φ; for sophisticated voters it is the intersection of two winning coalitions in $W^{\mathrm{np}} = \{T \in W | \forall i \in T : T \setminus \{i\} \in W\}$ (i.e. no agent is pivotal) and we denote its cardinality by β. Let

- $v^{\mathrm{P}}(\varphi) = \min\left\{\frac{2\varphi+3}{\varphi+1}\lambda^0(X), 1\right\}$, where $\varphi = \min\left\{|T \cap T'| : T, T' \in W^{\min}\right\}$,
- $\hat{v}^{\mathrm{P}}(\beta) = \min\left\{\frac{2\beta+3}{\beta+1}\lambda^0(X), 1\right\}$ if $|W^{\mathrm{np}}| \geq 2$ and $\hat{v}^{\mathrm{P}}(\beta) = \underline{V}$ otherwise, where $\beta = \min\left\{|T \cap T'| : T, T' \in W^{\mathrm{np}}\right\}$.

We use the superscript P to specify that the value is for proper games, where v and $\hat{v}$ are the values for behavioral and sophisticated voters, respectively.

Proposition 4 (Majority Voting) *If a WMG is proper and there is no veto player, the value is at least $v^{\mathrm{P}}(\varphi)$ for behavioral voters and at least $\hat{v}^{\mathrm{P}}(\beta)$ for sophisticated voters.*

Moreover, if the prior belief is sufficiently high, the sender can implement her preferred outcome with certainty.

Note that if $|W^{\text{np}}| \geq 2$, then the sender can improve upon public communication even with sophisticated voters. Secondly, if the prior belief is at least $(\beta+1)/(2\beta+3)$, then the sender can implement x with probability 1. For example, if $\beta = 3$, then the value is 1 for any $\lambda^0(X) \geq 4/9$.[30] This leads to another interpretation of the lower bounds; if agents have heterogeneous priors such that the lowest prior belief among the receivers is above this threshold, then the sender can still implement her preferred outcome with probability 1. In fact, even if the only information the sender has is that the lowest prior is above this threshold (i.e. if there is ambiguity about the priors), the sender can achieve the upper bound of the value.

Recall Example 2, where the game is proper and has no veto player. The parameters of the game induce $\varphi = 1$ and $\beta = 3$, so the lower bound on the value is $v^{\text{P}}(1) = 5/6$ for behavioral voters and $\hat{v}^{\text{P}}(3) = 3/4$ for sophisticated voters. Note that the lower bound for sophisticated voters, 3/4, is very close to the optimal value of $V_q^{\pi^e}(\lambda^0) = 16/21$.

The construction of the experiment which provides the lower bound in Proposition 4 relies on the game being proper, i.e. the complement of any winning coalition is losing. Alternatively, if the sender also knows that the complement of a losing coalition is winning, then she can achieve a higher lower bound than the one provided in Proposition 4. These particular types of games are known as *strong*.[31]

Definition 4 (*Strong WMG*) A WMG is *strong* if $T \notin W$ implies $N \setminus T \in W$.

Suppose that g is strong and fix a minimal winning coalition T and an agent $i \in T$. Since $T \in W^{\min}$ it follows that $T \setminus \{i\} \notin W$. But as g is strong, it holds that $(N \setminus T) \cup \{i\} \in W$. Note that since $i \in T$, there exist $|T|$ many such coalitions. Hence, in state Y the sender can target $|T| + 1$ many winning coalitions, just by knowing that g is strong. In constructing the experiment which provides the lower bound for strong games, we choose the T with the highest cardinality, which increases the number of winning coalitions that are targeted in state Y. Let $v^{\text{S}}(\delta) = \min\left\{\frac{3\delta-1}{\delta}\lambda^0(X), 1\right\}$, where $\delta = \max\left\{|T| : T \in W^{\min}\right\}$. We use the superscript S to specify that the value is for strong games.

Proposition 5 (Strong (Simple Majority) Voting) *If a WMG is strong, then the sender can achieve $v^{\text{S}}(\delta)$ for behavioral voters. Moreover, if the common prior is sufficiently high, then the sender can implement her preferred outcome with certainty.*

Note that Proposition 5 does not rely on the game being proper. However, if a game is both proper and strong observe that $v^{\text{S}}(\delta)$ improves upon $v^{\text{P}}(\varphi)$ since

$$3 - \frac{1}{\delta} = \frac{3\delta - 1}{\delta} \geq \frac{2\varphi + 3}{\varphi + 1} = 2 + \frac{1}{\varphi + 1} \iff 1 - \frac{1}{\delta} \geq \frac{1}{\varphi + 1},$$

30 To give a simple example, take [4; 1, 1, 1, 1, 1, 1, 1]. Here the smallest overlap between two winning coalitions such that no agent is pivotal is $\beta = 3$. The lower bound of the value is $\hat{v}^{\text{P}}(\beta) = \min\{9/4 \cdot \lambda^0(X), 1\}$, which is (weakly) lower than the optimal value $V(7, 4, \lambda^0) = \min\{11/4 \cdot \lambda^0(X), 1\}$.

31 The game we considered previously after Proposition 3, [5; 4, 3, 2], is proper and strong. As an example of a game which is proper but not strong, consider [6; 4, 3, 2]. In this case, the complement of the losing coalition {2, 3} is not winning.

which holds for any $\delta > 1$ and $\varphi \geq 1$.[32] We should note that if agents are *sophisticated* then the game being strong is too weak a condition to guarantee the existence of an equilibrium in which the sender improves upon public communication. This stems from the fact that in this case, a similar construction to the one in Proposition 5 would require to fix a coalition $T \in W^{\mathrm{np}}$ so that no agent is pivotal upon observing x in state Y. However, being strong does not rule out the possibility of W^{np} only consisting of the grand coalition, in which case the optimal value is $V^{\mathrm{pub}}(\lambda^0)$.

An additional insight of Proposition 5 is that as the number of receivers tends to infinity, the size of the minimal winning coalitions tend to infinity as well and thus $v^{\mathrm{S}}(\delta)$ approaches $\min\{3\lambda^0(X), 1\}$. Therefore, at the limit the sender can implement x with probability 1 for any $\lambda^0(X) \geq 1/3$.

Note that if a game is strong, then the voting rule cannot be supermajority (i.e. with a quota greater than simple majority). Proposition 5 provides a bridge between majority and minority voting because it also applies to minority voting.[33] We show in the next section that if there is a minority voting rule then the sender can improve upon the lower bound $v^{\mathrm{S}}(\delta)$.

Minority voting

Now suppose that $q \leq (\tau + 1)/2$. While minority voting is less prominent than majority voting, it has multiple real-world applications. For example, the European Council has a blocking minority provision, whereby 4 out of 27 council members can block a particular proposal, even if it fulfils a qualified majority.[34] Hence, we can think of the sender as a lobbyist who wishes to block a proposal and therefore attempts to persuade a certain number of council members. An example of minority voting can also be found in the procedure of having a case heard by the U.S. Supreme Court, where it needs to be approved by at least four out of nine Justices. This approval is called a *grant of certiorari* (Taylor and Zwicker 1999).

Under minority voting, the complement of a winning coalition is not necessarily losing, i.e. the WMG with minority voting can be either proper or *improper*.[35] In case it is proper, then the lower bounds provided in Proposition 4 hold. If the game is improper, then the sender can target mutually exclusive winning coalitions and achieve another lower bound, which again *strictly* improves upon public communication. For example, if the WMG is [5; 3, 2, 2, 1, 1, 1], then the sender can target two disjoint winning coalitions, $\{1, 2\}$ and $\{3, 4, 5, 6\}$, to pass a certain proposal. For this purpose, let $\gamma = \max\{|R| : R \subseteq W \text{ s.t. } i \notin C_1 \cap C_2, \forall i \in N \text{ and } C_1, C_2 \in R\}$ be the maximum number of disjoint winning coalitions. Let $v^{\mathrm{I}}(\gamma) = \min\{(\gamma + 1)\lambda^0(X), 1\}$ and $\hat{v}^{\mathrm{I}}(\gamma) = \min\{V(\gamma, 2, \lambda^0), 1\}$. We use the superscript I to specify that the value is for improper games.

[32] Note that $\delta = 1$ is possible if either there is a dictator or if the game is improper. We exclude the former since otherwise there exists a veto player and we exclude the latter as we will consider improper games in the next subsection.

[33] The quota in the WMG [4; 4, 3, 2] represents minority voting, but [4; 4, 3, 2] is a strong game.

[34] https://www.consilium.europa.eu/en/council-eu/voting-system/qualified-majority/.

[35] An example of a proper WMG with minority voting is [4; 3, 3, 3].

Proposition 6 (Improper Minority Voting) *If a WMG is improper, then the value is at least* $v^{\mathrm{I}}(\gamma)$ *for behavioral voters and at least* $\hat{v}^{\mathrm{I}}(\gamma)$ *for sophisticated voters.*

Note that in the case of behavioral voters the value is at least $3\lambda^0(X)$ under private communication since $\gamma \geq 2$.[36] On the other hand, the optimal public communication value is $V^{\mathrm{pub}}(\lambda^0) = \min\{2\lambda^0(X), 1\}$. Hence, the sender can implement x at least 50% more efficiently compared to the case of public communication. However, when voters are sophisticated the sender can improve upon public communication only if $\gamma > 2$, since $\hat{v}^{\mathrm{I}}(2) = V(2, 2, \lambda^0) = \min\{2\lambda^0(X), 1\}$.

Interestingly, Proposition 6 implies that the sender can improve upon public communication even when she possesses much less information about the receivers. In particular, to achieve this, the sender would not need to know the exact voting weight profile, but only the total number of votes within certain groups of voters. More precisely, it is sufficient for the sender to know that there are two distinct groups of voters that can implement x on their own, to be able to benefit from private communication.

Now we can revisit Theorem 1 and provide intuition about the 'only if' part, i.e. optimal communication is public only if there exists a veto player. Assume to the contrary that there is no veto player. There are two options for the type of game, which are proper and improper. If the game is proper, then Proposition 4 implies that public communication is not optimal, a contradiction. In particular, assuming there is no veto player implies that there are sufficiently many (not necessarily minimal) coalitions that the sender can target in state Y to improve upon public communication. If the game is improper, then Proposition 6 implies that public communication is not optimal, once again a contradiction. Specifically, improperness of the game implies that there exist disjoint winning coalitions (which precludes the existence of a veto player). So, targeting these coalitions instead of the grand coalition improves upon public communication.

An application

Previously, we have first imposed no structure on voting weights or the quota and then only imposed some structure on the quota. This section considers a class of games which is known as *apex games*, that exhibits most of the characteristics that were analyzed so far and also imposes a particular weight structure. Apex games provide a natural concrete illustration of the general insights in the previous sections. In addition, because of their specific structure, apex games allow us to provide a complete characterization of optimal communication.

An apex game involves an *apex player* (e.g. a large political party or the chair of a committee) who generally has a higher voting weight than the *minor players* (e.g. smaller political parties or committee members).[37] In an apex game, winning coalitions consist of either: (i) the apex player and at least one minor player or (ii) at least all minor players. Apex games have been studied in the context of coalition formation (Von Neumann and Morgenstern 1944; Haller 1994; Bloch and Rottier 2002; Karos

[36] It is easy to see that $v^{\mathrm{I}}(\gamma)$ also improves upon $v^{\mathrm{S}}(\delta)$ since $3 > 3 - (1/\delta)$ for $\delta \geq 1$.

[37] The chair in committees often has a slightly larger voting power than the committee members (Chappell Jr et al. 1995; Chappell et al. 2004).

2014) and bargaining (Montero 2002), as well as within the experimental literature (Funk et al. 1980; Montero et al. 2016).

Definition 5 (*Apex Game*) We say that a WMG is an *apex game* with *apex player* $i \in N$ and *minor players* $N \setminus \{i\}$ if $T \in W$ when (i) $i \in T$ and $T \setminus \{i\} \neq \emptyset$, or (ii) $T = N \setminus \{i\}$, and $T \notin W$ otherwise.

Henceforth, we fix $i \in N$ to be the apex player.

First, note that apex games *do not* have a veto player and hence, with behavioral voters the sender can always improve upon public communication. Second, apex games are *proper* and therefore, the sender can achieve the lower bound $v^{\mathrm{P}}(\varphi)$ given by Proposition 4. However, as apex games are also *strong*, the lower bound $v^{\mathrm{S}}(\delta)$ provided in Proposition 5 also applies. It turns out that $v^{\mathrm{S}}(\delta)$ is the optimal value in an apex game as $\delta = n - 1$, so that $v^{\mathrm{S}}(n-1) = \min\{\frac{3(n-1)-1}{n-1}\lambda^0(X), 1\} = \min\{\frac{3n-4}{n-1}\lambda^0(X), 1\}$. We omit the proof of this result, since it will immediately follow from Theorem 2. It is not surprising that the optimal experiment and value are not explicitly dependent on the quota, since the quota is embedded in the definition of an apex game.

It is easy to see that the sender cannot improve upon public communication while ensuring a sincere BNE in apex games: the only coalition in which no agent is pivotal is the grand coalition. It is possible to generalize the idea of apex games to cover situations where a minimal winning coalition consists of either: (i) the apex player and multiple minor players or (ii) a strict subset of the minor players.[38] This intuitively connects to naturally occurring political situations. For example, a *subset* of the smaller parties could be a serious contender in elections or a party with a relatively larger voter base could need to form a coalition with *several* smaller parties to win.[39]

Definition 6 (*General Apex Game*) We say that a WMG is a *general apex game* with apex player $i \in N$ and minor players $N \setminus \{i\}$ with $0 < c_a < c_m \leq n - 1$ if $T \in W$ when (i) $i \in T$ and $|T \setminus \{i\}| = c_a$, or (ii) $i \notin T$ and $|T| = c_m$, and $T \notin W$ otherwise.

In words, when a coalition contains the apex player, it requires at least c_a minor players to be a winning coalition. On the other hand, if a coalition does not contain the apex player, then at least c_m minor players are required to form a winning coalition.[40] For example, [5; 3, 1, 1, 1, 1, 1, 1] defines a general apex game, where $c_a = 2$ and $c_m = 5$.

Unlike in a simple apex game, it is not immediate that assigning positive probability to all minimal winning coalitions is in fact optimal. However, a restricted version of anonymity holds in this case: under an optimal experiment, when fixing the message the apex player observes, it does not matter which minor players observe message x, but only the number of minor players who observe x. Therefore, it is optimal for the sender to employ *all* minimal winning coalitions under a general apex game.

Let $S^a = S_x^{\min}(i)$ be the set of all signals that target minimal coalitions which include the apex player and $S^m = S^{\min} \setminus S^a$ be the set of all signals that target minimal winning coalitions which consist of only minor players.

[38] Our definition is a special case of general apex games introduced by Karos (2014).

[39] The former was the case in recent elections held in Hungary, Israel, and Turkey.

[40] The definition of a general apex game implies that minor players are symmetric: given a WMG, j and k are *symmetric* if for all $T \subset N \setminus \{j, k\}$ it holds that $T \cup \{j\} \in W$ if and only if $T \cup \{k\} \in W$.

Theorem 2 (General Apex Game) *Let voters be behavioral and g be a general apex game. Then an optimal experiment is given by*

$$\pi^*(s|\omega) = \begin{cases} 1 & \textit{if } s = \bar{x} \textit{ and } \omega = X, \\ \frac{\lambda^0(X)}{\lambda^0(Y)} \binom{n-1}{c_a}^{-1} & \textit{if } s \in S^a \textit{ and } \omega = Y, \\ \frac{\lambda^0(X)}{\lambda^0(Y)} \frac{n-1-c_a}{n-1} \binom{n-2}{c_m-1}^{-1} & \textit{if } s \in S^m \textit{ and } \omega = Y, \\ 1 - \frac{\lambda^0(X)}{\lambda^0(Y)} \frac{n-1-c_a+c_m}{c_m} & \textit{if } s = \bar{y} \textit{ and } \omega = Y. \end{cases}$$

The optimal value is $V_q^{\pi^*}(g) = \min\left\{\frac{n-1-c_a+2c_m}{c_m}\lambda^0(X), 1\right\}$.

Note that if we set $c_a = 1$ and $c_m = n-1$, then it is reduced to an apex game and would yield an optimal experiment with value $\min\left\{\frac{3n-4}{n-1}\lambda^0(X), 1\right\}$. Additionally, Theorem 2 allows us to derive a known result.[41]

Corollary 2 (Majority of Unit Votes) *Suppose that the voting rule is at least simple majority, the receivers are behavioral, and there is a coalition of receivers who have unitary voting weights. If this coalition can implement x, then the sender can achieve* $\overline{V}$ *with private communication.*

Note that Corollary 2 is demonstrated by π in our illustrative example, where the upper bound on the value, $V(7, 4, 1/3)$, is achieved.[42] Thus, the higher voting weight of agent 1 does not impede the CEO's gain from persuasion, compared to a case where there are seven shareholders with one vote each.

As we mentioned before, the sender can improve upon public communication under general apex games while ensuring that sincere voting is a BNE. The structure of an optimal experiment in this case is similar to the structure of π^* in Theorem 2, but differs by targeting more minor players in state Y so as to make the apex player non-pivotal. For this purpose, let $W^a = \{T \in W | i \in T\}$ be the set of all winning coalitions that include the apex player and define $\mathcal{Q} = \{T \in W^a \cap W^{\text{np}} | T \backslash \{i, j\} \notin W, \forall j \neq i\}$. That is, the apex player is not pivotal in any coalition in $\mathcal{Q}$. We omit the proof as it immediately follows from Theorem 2.

Corollary 3 (BNE in General Apex Games) *Let g be a general apex game,* $T \in W^{\min} \cap W^a$, *and* $k = |\mathcal{Q}| - |T|$. *Then* $\pi^* \in \Pi^{\text{e}}$ *with* $V_q^{\pi^*}(g) = \min\left\{\frac{n+1-k-c_a+2c_m}{c_m+1}\lambda^0(X), 1\right\}$ *is optimal.*

[41] This result shows in a different context in Kerman and Tenev (2021) and hence Theorem 2 generalizes their result. Note also that all results in the current paper would be robust to limited information spillovers if agents are considered to be voter blocs *instead of individuals* From the same paper it follows that the sender can strictly gain from persuasion as long as information does not spill over to all members in a voter bloc. However, if we consider information spillovers between single receivers with different voting weight, the result would not apply.

[42] To obtain the upper bound with private communication from the value in Theorem 2, take $c_m = q$ agents with unit weights. This is equivalent to a WMG in which there are $n - 1$ agents with weight 1 and one agent with weight $\tau - n + 1$. Hence, $c_a = q - (\tau - n + 1)$. Substituting in the formula: $\min\{\frac{n-1-c_a+2c_m}{c_m}\lambda^0(X), 1\} = \min\{\frac{n-1-q+\tau-n+1+2q}{q}\lambda^0(X), 1\} = \min\{\frac{\tau+q}{q}\lambda^0(X), 1\}$, which is the upper bound on the sender's value specified in Proposition 2, $\overline{V}$.

Finally, we provide an example of a general apex game and illustrate that while the optimal value decreases under sincere BNE compared to behavioral voting, it still improves upon public communication.

Example 4 (*Improving Upon Public Communication in BNE*) Recall our illustrative example. Suppose the shareholder with the highest weight sells one share to a new investor, which induces a WMG [4; 2, 1, 1, 1, 1, 1]. This represents a general apex game, where shareholder 1 is the apex player, $c_a = 2$, and $c_m = 4$. By Theorem 2, the optimal value is $\frac{1}{3} \cdot \frac{1}{4}(6 - 1 - 2 + 2 \cdot 8) = 11/12 = V(7, 4, 1/3)$.

To achieve an equilibrium under sincere BNE, in every signal that implements x in state Y, the sender targets *four* minor players in addition to the apex player, i.e. $k = 2$. Thus, by Corollary 3 it follows that the optimal value is $11/15 > 2/3 = V^{\text{pub}}(1/3)$.
$\triangle$

5 Extensions and discussion

The literature on information design with multiple receivers considers heterogeneity in terms of either prior beliefs (Alonso and Câmara 2016a; Laclau and Renou 2017; Shimoji 2022; Senkov and Kerman 2024) or preferences (Alonso and Câmara 2016b; Bardhi and Guo 2018; Arieli and Babichenko 2019; Chan et al. 2019; Heese and Lauermann 2021). Allowing for such heterogeneity is also natural in our model, since receivers with different voting weights might have different priors or preferences; for example voters with a high voting weight might have a "bias" for a certain state. More precisely, each receiver might receive a different utility from matching state X and matching state Y, and the same might hold for mismatching them. We show that this is similar to a situation where agents have homogeneous preferences but hold different prior beliefs, and extend our main result to the heterogeneous priors case.[43]

Second, we consider allowing for abstention. We first show that in the current framework, not allowing for abstention is without loss of generality. Then, we provide alternative versions for the option to abstain and discuss how these affect the sender's gain from persuasion.

5.1 Heterogeneous preferences/priors

Suppose that for each $i \in N$ the payoff is of the following type: $\tilde{u}_i(x, X) = a_i, \tilde{u}_i(y, Y) = b_i$ and $\tilde{u}_i(x, Y) = c_i, \tilde{u}_i(y, X) = d_i$, where $a_i > d_i$ and $b_i > c_i$. That is, receivers prefer matching the state to mismatching it but have heterogeneous payoffs. This implies that receivers have different posterior belief thresholds for voting in favor of x, i.e. some agents are more difficult to persuade. This can be thought of as receivers having homogeneous preferences but different prior beliefs, since agents with lower priors are more difficult to persuade.

To show how the two problems relate to each other, let $(\bar{u}_i)_{i \in N}$ be the vector of utility functions such that $a_i = a, b_i = b, c_i = c, d_i = d$ for each $i \in N$, where

[43] A similar observation has also been made in a recent paper by Doval and Smolin (2024).

$a > d$ and $b > c$. That is, all receivers have the same utility function. Define two voting problems:

- $P = (N, \lambda^0, q, (w_i)_{i \in N}, (\tilde{u}_i)_{i \in N}, U)$ and
- $P' = (N, \lambda^0, \lambda_R^0, q, (w_i)_{i \in N}, (\bar{u}_i)_{i \in N}, U)$

With slight abuse of notation, let $\lambda_R^0 = (\lambda_i^0)_{i \in N}$ denote the vector of heterogeneous priors of receivers. Note that P' contains the sender's prior λ^0, as well as the additional parameter λ_R^0. Hence, $G(P, \pi)$ is a game of incomplete information in which receivers share a common prior but have heterogeneous preferences, whereas $G(P', \pi)$ is a game of incomplete information in which receivers have identical preferences but have heterogeneous priors. Denote the value of $\pi \in \Pi$ under voting problem P by $V[G(P, \pi)]$.

Lemma 3 *Let $\pi \in \Pi$ and suppose voters are behavioral. Then for any voting problem $P = (N, \lambda^0, q, (w_i)_{i \in N}, (\tilde{u}_i)_{i \in N}, U)$ there exists $P' = (N, \lambda^0, \lambda_R^0, q, (w_i)_{i \in N}, (\bar{u}_i)_{i \in N}, U)$ such that $V[G(P, \pi)] = V[G(P', \pi)]$.*

Therefore, from now on we assume that receivers have identical preferences but heterogeneous priors. In general, it is possible to formulate the problem of finding the optimal experiment with heterogeneous voting weights and heterogeneous priors. Once again, the only adjustment is the formulation of inequality (3) of Proposition 1. Namely, it should incorporate the heterogeneous prior beliefs of the voters:

$$\sum_{j \in S_x^{\min}(i)} p_j \leq \lambda_i^0(X)/\lambda_i^0(Y), \quad \forall i \in N. \tag{4}$$

This is still a tractable problem. However, as in the case of a common prior, it is very dependent on the specific beliefs to allow for a closed-form formulation of the value.

Similar to the bounds we provide in Sect. 4.2, in the case of heterogeneous priors we can establish lower bounds, which marginally improve upon public communication (see the proof of Theorem 3). However, if the common prior is $\lambda^0(X)$ and $\min \left\{\lambda_R^0(X)\right\} \geq \lambda^0(X)$, then the bounds we established in Sect. 4.2 still hold. This could provide an advantage to a sender who faces a great amount of uncertainty about the prior beliefs of the voters: only knowing the prior belief of the most pessimistic voter is sufficient to obtain the lower bounds and substantially improve upon public communication.

Finally, we state the generalization of Theorem 1 to receivers with heterogeneous priors.

Theorem 3 *Suppose voters are behavioral and have heterogeneous prior beliefs. Then π^{pub} is optimal if and only if there is a veto player.*

Note that since Lemma 2 does not depend on the receivers' priors, the 'if' part immediately follows. For the 'only if' part, similar to the proof of Theorem 1, we assume to the contrary that there exists a veto player despite that optimal communication is public and construct experiments that improve upon public communication to reach a contradiction. The constructions of experiments we provide differ from the ones in the proof of Theorem 1, since optimal public communication now depends on the prior of a particular receiver. More precisely, for the optimal public experiment the sender targets a winning coalition that consists of receivers with the highest possible priors, where we call the receiver with the lowest prior in the coalition the "cutoff" receiver. Therefore, under an optimal public experiment, the sender does not necessarily persuade all receivers, but only a subset of them. This way, the sender achieves a higher probability of persuasion relative to the case where she persuades all receivers, who potentially have lower priors. We show that in this case the sender can still gain from private persuasion by targeting multiple coalitions, which are slightly smaller than the grand coalition.

If we consider a model which has heterogeneity not only in weights but also in priors, it is natural to ask how the voting power and the preferences/prior beliefs of the receivers trade off. We explore this in the following example.

Example 5 Consider the apex game defined by [3; 2, 1, 1, 1]. First, note that if the sender and receivers share a common prior $\lambda^0(X) = 1/3$, then the value of the optimal signal prescribed by Theorem 2, call it π, is $8/9$. To see the difference between the effects of the apex player and one minor player having a higher/lower prior than others, we will consider four different prior belief distributions for the receivers (while fixing the sender's prior to $\lambda^0(X) = 1/3$): (i) $\lambda^1(X) = (1/3, 2/9, 1/3, 1/3)$, (ii) $\lambda^2(X) = (2/9, 1/3, 1/3, 1/3)$, (iii) $\lambda^3(X) = (1/3, 4/9, 1/3, 1/3)$, and (iv) $\lambda^4(X) = (4/9, 1/3, 1/3, 1/3)$. While in distributions λ^1 and λ^2 a minor player (receiver 2) and the apex player, respectively, has a slightly lower prior than others (i.e. 2/9), in distributions λ^3 and λ^4 a minor player (receiver 2) and the apex player, respectively, has a slightly higher prior than others (i.e. 4/9).

The table below outlines the optimal strategies for each case. Since the same set of minimal winning coalitions are used in all cases and in state X the experiment truthfully recommends x, the only difference is in state Y. Denote the optimal experiments in cases $(i) - (iv)$ by π'_m, π'_a, π''_m, and π''_a, respectively.

		π	π'_m	π'_a	π''_m	π''_a
	X	Y	Y	Y	Y	Y
(x, x, x, x)	1	0	0	0	0	0
$(x, x, \boldsymbol{y}, \boldsymbol{y})$	0	$\frac{1}{6}$	$\frac{1}{42}$	$\frac{2}{21}$	$\frac{11}{30}$	$\frac{3}{10}$
$(x, \boldsymbol{y}, x, \boldsymbol{y})$	0	$\frac{1}{6}$	$\frac{5}{21}$	$\frac{2}{21}$	$\frac{1}{15}$	$\frac{3}{10}$
$(x, \boldsymbol{y}, \boldsymbol{y}, x)$	0	$\frac{1}{6}$	$\frac{5}{21}$	$\frac{2}{21}$	$\frac{1}{15}$	$\frac{1}{5}$
$(\boldsymbol{y}, x, x, x)$	0	$\frac{1}{3}$	$\frac{11}{42}$	$\frac{17}{42}$	$\frac{13}{30}$	$\frac{1}{5}$
$(\boldsymbol{y}, \boldsymbol{y}, \boldsymbol{y}, \boldsymbol{y})$	0	$\frac{1}{6}$	$\frac{5}{21}$	$\frac{13}{42}$	$\frac{1}{15}$	0
		$\lambda^0(X)$	$\lambda^1(X)$	$\lambda^2(X)$	$\lambda^3(X)$	$\lambda^4(X)$

Overall, we observe that

$$V_3^{\pi''_a}(g) > V_3^{\pi''_m}(g) > V_3^{\pi}(g) > V_3^{\pi'_m}(g) > V_3^{\pi'_a}(g).$$

That is, more influential voters (e.g. the apex player) affect the value more positively if they have a higher belief, all else equal. In contrast, when such a player has a lower prior this decreases the value of the sender more than a minor player would. Here it is worth noting that the most influential player does not necessarily have a higher weight. A player can also have the lowest voting weight but equal voting power (e.g. in [3; 2, 2, 1]).

As the example illustrates, starting from a common prior and decreasing the prior belief of one voter to obtain heterogeneous priors, all else equal, the value should (weakly) decrease. The explanation is intuitive: the set of possible solutions shrinks as one of the conditions specified by inequality (4) becomes more restrictive, all else equal. Conversely, if one prior increases relative to the common prior case, the value (weakly) increases.

5.2 Abstention

Now suppose that voters have the option to abstain.[44] That is, the action set of agent $i \in N$ is given by $A_i = \{x, y, \emptyset\}$, where $\emptyset$ stands for abstention.

Let $s \in S$ be such that $s_i = x$. Note that while receiver i's expected utility of voting sincerely in favor of x is $\lambda_i^s(X) \cdot 1 + (1 - \lambda_i^s(X)) \cdot 0 = \lambda_i^s(X)$, his expected utilities of voting sincerely in favor of y and abstaining are equal and given by $\lambda_i^s(X) \cdot 0 + (1 - \lambda_i^s(X)) \cdot 1 = 1 - \lambda_i^s(X)$. Thus, i is indifferent between voting in favor of y and abstaining since they lead to the same expected utility.[45] Hence, excluding abstention is without loss of generality in our model, i.e. we can restrict attention to action sets only consisting of choosing x and choosing y.

Alternatively, suppose that voting is costly and voters have heterogeneous *voting costs*, where the cost of voting for agent i is $c_i > 0$. This leads to agents having different thresholds for voting in favor of x. In particular, agent i votes in favor of x if $\lambda_i^s(X) \geq (1 + c_i)/2$. Therefore, the situation is similar to voters having different preferences or different prior beliefs, which is covered by our discussion in Sect. 5.1. One aspect to note is that in this case, an agent would never vote for y when they think state Y is more likely, since abstaining leads to the same outcome and is costless.

Finally, we consider a broader interpretation of abstention. Suppose that prior to communication, voters can commit to abstain, which is observed by the sender. For example, The U.S. recently abstained from the vote calling for more humanitarian aid in Gaza (see https://www.pbs.org/newshour). It is likely that a lobbyist who wishes to influence the outcome of the vote would have insider information and know beforehand that the U.S. is going to abstain. In this case, the lobbyist would direct her resources to the persuasion of the remaining countries.

As in the UN Security Council, assume that the quota is fixed, i.e. it is not proportional to the number of *active* voters (the ones who have not abstained).[46] In this

[44] While the analysis would hold for heterogeneous beliefs, we assume a common prior for simplicity.

[45] This is intuitive given that x is implemented when the number of x votes reaches the quota; without any additional assumptions abstaining is equivalent to voting for y.

[46] Note that assuming a proportional quota does not have an effect on the equilibrium outcome. Since receivers are obedient in equilibrium, voting for x upon observing x yields a (weakly) higher payoff than

case, this type of abstention will be detrimental to the sender, as she now needs to persuade a sufficient number of countries (who satisfy the quota) from a smaller pool. To illustrate, consider the same game in Example 5 and suppose agents share a common prior of $\lambda^0(X) = 1/3$. Suppose that the apex player commits to abstaining before the vote. Then, the optimal value decreases from $8/9$ to $2/3$, as optimal communication becomes public.

Recall that the upper bound of the value is $\overline{V} = \min\left\{\frac{\tau+q}{q}\lambda^0(X), 1\right\}$. If voter i abstains, then the upper bound becomes $\min\left\{\frac{\tau+q-w_i}{q}\lambda^0(X), 1\right\}$. In fact, the decrease in the upper bound is monotonic in the number of voters who commit to abstaining.[47]

6 Conclusion

This paper considers a voting application of information design in which voters have heterogeneous voting weights. The model is applicable to different setups such as achieving a *critical mass* of consumers in a marketing campaign, securing a certain level of *crowdfunding* investment, and *persuading voter blocs*. Determining how a sender optimally communicates with receivers in this framework does not readily generalize on the basis of previous research. While the sender's optimization task remains tractable, a closed-form solution for an optimal communication protocol given arbitrary voting weights is not obtained with the standard tools. Instead, to deal with this issue we first impose some structure on the quota and then also on the voting weights.

Throughout the paper we consider both *behavioral* and *sophisticated* voters. We first show that when voters are behavioral *public communication is optimal* if and only if there is a *veto player*. This is not necessarily true when they are sophisticated, as sophisticated voters also condition their actions on being pivotal, which requires the sender to divulge more information. Next, while keeping the voting weights generic, we consider different quotas and provide *lower bounds* for the sender's gain from persuasion. We show that the sender can often *improve upon public communication* even when voters are sophisticated. Finally, we consider an application (*apex games*) which illustrates the previous results and makes them more concrete. In particular, we show that the lower bounds we provided before improve upon public communication significantly and can even coincide with the upper bound of the sender's expected utility.

Finally, we extend the model in two different directions. First, we consider receiver who have heterogeneous payoff functions and show that from the sender's perspective, this is as if the receiver's have heterogeneous prior beliefs. We show that public communication is optimal if and only if there is a veto player, even when receivers

Footnote 46 continued
any other action. Suppose x is being implemented. If an agent abstains instead of voting for x, then either the outcome of the vote does not change (which gives the same expected utility) or it changes to y (which makes the agent worse off).

47 Notice that this type of pre-commitment is an effective way to restrict the sender which is reminiscent of Tsakas et al. (2021), but different in its implementation as our framework has multiple receivers.

have heterogeneous priors. Second, we consider abstention and show that excluding it is without loss of generality.

This model opens several avenues for research. Given that in the current setup the most equitable representation is the most manipulable, future research can also analyze a policymaker who designs a voting system and wishes to balance equitable representation and robustness to propaganda. Furthermore, the current framework could be extended to incorporate frequently observed phenomena such as a finite and arbitrarily large state space and state-dependent utility functions.

Appendix A: Notation guide

Notation	Definition	Explanation
β	$\min\left\{\lvert T \cap T'\rvert : T, T' \in W^{\mathrm{np}}\right\}$	Minimal overlap of two winning coalitions without pivotal agents
γ	$\max\{\lvert R\rvert : R \subseteq W$ s.t. $i \notin C_1 \cap C_2$, $\forall i \in N$ and $C_1, C_2 \in R\}$	The maximum number of disjoint winning coalitions
δ	$\max\left\{\lvert T\rvert : T \in W^{\min}\right\}$	The maximum cardinality of a minimal winning coalition
φ	$\min\left\{\lvert T \cap T'\rvert : T, T' \in W^{\min}\right\}$	Minimal overlap of two minimal winning coalitions
$\lambda^0(X)$		Sender's prior/ common prior
$\lambda^0_a(X)$		Apex player's prior
$\lambda^0_m(X)$		Minor player's prior
$\lambda^s_i(\omega)$	$\frac{\pi(s_i \mid \omega)\lambda^0(\omega)}{\sum_{\omega' \in \Omega} \pi(s_i \mid \omega')\lambda^0(\omega')}$	Posterior of voter i
π^{pub}		Optimal public experiment
Π		Set of all experiments
Π^{e}		Experiments whose induced sincere strategy profile constitutes a BNE
S^π	$\{s \in S \mid \exists \omega \in \Omega : \pi(s \mid \omega) > 0\}$	Signals in S which are sent with positive probability by π
S^π_i	$\{s_i \in S_i \mid \exists \omega \in \Omega : \pi_i(s_i \mid \omega) > 0\}$	Messages receiver i observes with positive probability under π
$S^{\min}$	$\left\{\chi(R) \in S \mid R \in W^{\min}\right\}$	Signals that target minimal winning coalitions
$S^{\min}_+$	$S^{\min} \cup \{\bar{x}, \bar{y}\}$	
$S^{\min}_x(i)$	$\{s \in S^{\min} \mid s_i = x\}$	Signals in $S^{\min}$ which send x to agent i
S^{np}	$\{\chi(R) \in S \mid R \in W^{\mathrm{np}}$ and $\exists i, j \in R$ s.t. $R \setminus \{i, j\} \notin W\}$	Signals targeting the *smallest* winning coalitions without pivotal agents

Notation	Definition	Explanation
S_+^{np}	$S^{\mathrm{np}} \cup \{\bar{x}, \bar{y}\}$	
$S_x^{\mathrm{np}}(i)$	$\{s \in S^{\mathrm{np}} \| s_i = x\}$	Signals in S^{np} which send x to agent i
S^a	$S_x^{\min}(i)$	Signals targeting minimal coalitions which include the apex player
S^m	$S^{\min} \setminus S^a$	Signals targeting winning coalitions consisting of only minor players
$V(\theta, \kappa, \lambda^0)$	$\min\left\{\frac{\theta+\kappa}{\kappa}\lambda^0(X), 1\right\}$	Value of private experiment (common prior $\lambda^0(X)$, unitary weights)
$\overline{V}$	$V(\tau, q, \lambda^0)$	Upper bound of the value
$V^{\mathrm{pub}}(\lambda^0)$	$\min\{2\lambda^0(X), 1\}$	Value of the public experiment with common prior $\lambda^0(X)$
$\underline{V}$	$V^{\mathrm{pub}}(\lambda^0)$	Lower bound of the value
$V_q^{\pi}(\lambda^0)$	$\lambda^0(X)\sum_{s \in Z_x^q(\pi)} \pi(s\|X) +$ $\lambda^0(Y)\sum_{s \in Z_x^q(\pi)} \pi(s\|Y)$	Value (general definition)
$V_q^{\pi}(g)$		Value of an experiment π on a weighted majority game g
$v^{\mathrm{P}}(\varphi)$	$\min\left\{\frac{2\varphi+3}{\varphi+1}\lambda^0(X), 1\right\}$	lower bound of the value for *proper* games with *behavioral* voters
$\hat{v}^{\mathrm{P}}(\beta)$	$\min\left\{\frac{2\beta+3}{\beta+1}\lambda^0(X), 1\right\}$	Lower bound of the value for *proper* games with *sophisticated* voters
$v^{\mathrm{I}}(\gamma)$	$\min\{(\gamma+1)\lambda^0(X), 1\}$	Lower bound of the value for *improper* games with *behavioral* voters
$\hat{v}^{\mathrm{I}}(\gamma)$	$\min\{V(\gamma, 2, \lambda^0), 1\}$	Lower bound of the value for *improper* games with *sophisticated* voters
$v^{\mathrm{S}}(\delta)$	$\min\left\{\frac{3\delta-1}{\delta}\lambda^0(X), 1\right\}$	Lower bound of the value for *strong* games with *behavioral* voters
W	$\{T \subseteq N \| \sum_{i \in T} w_i \geq q\}$	The set of winning coalitions
$W^{\min}$	$\{T \in W \| \forall R \subsetneq T : R \notin W\}$	The set of minimal winning coalitions
W^{np}	$\{T \in W \| \forall i \in T : T \setminus \{i\} \in W\}$	Set of coalitions in which no agent is pivotal
W^a	$\{T \in W \| i \in T\}$	Set of winning coalitions that include the apex player
$\chi(R)$	$\chi_i(R) = x,\ \forall i \in R$ and $\chi_j(R) = y,\ \forall j \notin R$	The signal which targets coalition R with message x

Appendix B: Sincere BNE

One way to ensure that agents vote sincerely in equilibrium is to target (in state Y) winning coalitions of non-pivotal agents, that are minimal in size. Lemma 4 shows that this is without loss of generality. Formally, recall that $W^{\mathrm{np}} = \{T \in W | \forall i \in T :$

$T \setminus \{i\} \in W\}$ is the set of coalitions in which no agent is pivotal. Further, let $S^{\mathrm{np}} = \{\chi(R) \in S | R \in W^{\mathrm{np}}$ and $\exists i, j \in R$ s.t. $R \setminus \{i, j\} \notin W\}$ and $S_+^{\mathrm{np}} = S^{\mathrm{np}} \cup \{\bar{x}, \bar{y}\}$.[48] That is, S^{np} includes signals in which no agent is pivotal, but there exists *at least one* pair of agents such that removing them makes the coalition losing.[49] Note that when the weights are unitary, then removing *any* pair from such a coalition makes it losing. However, for arbitrary weights this condition is too restrictive, as under some weight profiles this leads to $S^{\mathrm{np}} = \emptyset$, whereas requiring the existence does not.[50]

Lemma 4 *Let* $\lambda^0 \in \Delta(\Omega)$ *and* $\hat{\pi} \in \Pi^{\mathrm{e}}$. *There exists* $\pi \in \Pi^{\mathrm{e}}$ *with* $S^{\pi} \subseteq S_+^{\mathrm{np}}$ *such that* $V_q^{\pi}(\lambda^0) \geq V_q^{\hat{\pi}}(\lambda^0)$.

In short, the lemma captures the following logic: if $\hat{\pi} \in \Pi^{\mathrm{e}}$, then either all agents are pivotal or non-pivotal in both states. Since setting $\hat{\pi}(\bar{x}|X) = 1$ is without loss, in the former case we can transform $\hat{\pi}$ into an experiment in which no agent is pivotal without decreasing the value. In the latter case, the construction follows from Lemma 1.

Let $S^{\mathrm{np}} = \{s^1, \ldots, s^m\}$ where $m \in \mathbb{N}$ and for any $i \in N$, let $S_x^{\mathrm{np}}(i) = \{s \in S^{\mathrm{np}} | s_i = x\}$. For a given experiment $\pi \in \Pi$, let probabilities p_j be as in Proposition 1. The solution of the following constrained optimization problem yields an optimal experiment under which sincere voting is a BNE.

Proposition 7 (Sincere BNE) *Suppose that the voters are sophisticated. An optimal experiment under which sincere voting is a BNE is a solution to*

$$\max_{p_1,\ldots,p_m} \; 1 - \lambda^0(Y)p_0 \quad \textit{subject to}$$

$$p_j \geq 0, \quad \forall j \in \{1, \ldots, m\}, \tag{5}$$

$$\sum_{j=0}^{m} p_j = 1, \tag{6}$$

$$\sum_{j \in S_x^{\mathrm{np}}(i)} p_j \leq \lambda^0(X)/\lambda^0(Y), \quad \forall i \in N. \tag{7}$$

We omit the proof as it follows directly from the proof of Proposition 1. The main difference of Proposition 7 from Proposition 1 is the obedience constraint given by inequality (7), i.e. we employ signals in $S_x^{\mathrm{np}}(i)$ *instead of* $S_x^{\min}(i)$. Since this ensures that no agent is pivotal in either state, it is guaranteed that the solution leads to a BNE.

[48] Notice that for every $T \in W^{\mathrm{np}}$ we have $T \setminus \{i\} \in W$ and not necessarily $T \setminus \{i\} \in W^{\min}$. In fact, if we require the latter, then the signals in S^{np} target coalitions such that removing *any* two agents makes them losing, which is too restrictive and not necessarily optimal. See also footnotes [1]7 and [1]9.

[49] Notice that if there is no $i, j \in R$ such that $R \setminus \{i, j\} \notin W$, then there exists $k \in R$ who is unnecessarily targeted and therefore, not targeting him weakly increases the value.

[50] Intuitively, S^{np} balances two effects. To ensure sincere BNE, the coalitions need *more votes* than a minimal winning coalition, which means that they should involve *more* agents. This *decreases* the number of coalitions. However, to guarantee a higher value the sender wants to have as *many* winning coalitions as possible at their disposal. In that sense the sender wants to have as *few* agents as possible in a coalition of non-pivotal agents, which *increases* the number of possible coalitions.

Appendix C: Proofs

Proof of Lemma 1 It readily follows that assuming π is straightforward and $\pi(\bar{x}|X) = 1$ is without loss. Define $\pi^* \in \Pi$ via π as follows:

(i) Remove all $s \in S^\pi$ with $|i \in N : s_i = x| < q$ and transfer its probability to $\bar{y}$,
(ii) for any signal not in $S^{\min}$, replace x messages of arbitrary agents with y so that it is transformed into a signal in $S^{\min}$.

For each $R \in W \setminus S^{\min}$ define $R' \subsetneq R$ such that $\chi(R') \in S^{\min}$. Thus, π^* is given by $\pi^*(\bar{x}|X) = \pi(\bar{x}|X)$ and

$$\pi^*(s|Y) = \begin{cases} \pi(\chi(R)|Y) & \text{if } s = \chi(R'), \\ \pi(s|Y) & \text{if } s \in S^{\min}, \\ \sum_{s' \notin Z_x^q(\pi)} \pi(s'|Y) & \text{if } s = \bar{y}. \end{cases}$$

Clearly, for any $i \in N$ and $s \in S^\pi$ with $s_i = x$ and $\lambda_i^s(X) \geq 1/2$, it holds that for any $s' \in S^{\pi^*}$ with $s'_i = x$ we have $\lambda_i^{s'}(X) \geq 1/2$. Therefore, $Z_x^q(\pi^*) = Z_x^q(\pi)$ and hence, $V_q^{\pi^*}(\lambda^0) = V_q^\pi(\lambda^0)$. □

Proof of Proposition 1 First, recall that under an optimal experiment $\pi^* \in \Pi$, it holds that $\pi^*(\bar{x}|X) = 1$. Second, under a straightforward experiment, all signals in state Y except for $\bar{y}$ implement x. Thus, the total probability of x being implemented in state Y is $1 - p_0$. Hence, the objective function is obtained by rewriting the value of π^*:

$$\lambda^0(X)\pi^*(\bar{x}|X) + \lambda^0(Y)(1 - p_0) = 1 - \lambda^0(Y) + \lambda^0(Y) - \lambda^0(Y)p_0 = 1 - \lambda^0(Y)p_0.$$

Since $p_j \geq 0$ for all $j \in \{1, \ldots, m\}$ by (1) and $\sum_{j=0}^m p_j = 1$ by (2), it follows that the solution to the optimization problem is an experiment. Since all p_j's are probabilities of signals that are contained in $S^{\min}$, an optimal experiment only assigns positive probability to minimal winning coalitions in state Y. Finally, (3) ensures that the solution is a straightforward experiment, i.e. all agents vote for x upon observing message x. □

Proof of Lemma 2 Let $j \in N$ be a veto player. Then for any $\pi \in \Pi$, and for all $s \in Z_x^q(\pi)$, it holds that $\alpha_j^\pi(s_j) = x$.

By definition, an experiment $\pi \in \Pi$ with $S^\pi \subseteq \{x, y\}^n$ is straightforward if for any $i \in N$ and $s, t \in S^\pi$ with $s_i = x$ and $t_i = y$, we have $\lambda_i^s(X) \geq 1/2$ and $\lambda_i^t(Y) > 1/2$. Hence, π is straightforward if and only if for all $i \in N$

$$\lambda^0(X)\pi_i(x|X) \geq \lambda^0(Y)\pi_i(x|Y). \tag{8}$$

Thus, $\sum_{s \in Z_x^q(\pi)} \pi(s|Y) \leq \lambda^0(X)/\lambda^0(Y)$. Therefore, the optimal value is at most $\min\{2\lambda^0(X), 1\}$. Part (ii) follows immediately. □

Proof of Proposition 3 From the symmetry of g it follows that for any $T \in W^{\min}$, it is true that $|T| = k$. Assume to the contrary that $T, T' \in W^{\min}$, $|T| = k$, and $|T'| > k$.

By symmetry, for any $T'' \subsetneq T'$ with $|T''| = k$ it holds that $T'' \in W$, a contradiction since $T' \in W^{\min}$.

Since it is optimal to target all minimal winning coalitions by Lemma 1 and by definition of symmetry, it follows that an optimal experiment π has value $V(n, k, \lambda^0)$ for behavioral voters. Part (ii) immediately follows by Lemma 4. $\square$

Proof of Proposition 4 We first state the result formally.

Proposition 4. Let g be a proper WMG that has no veto player. There exists

(i) $\pi \in \Pi$ such that $V_q^\pi(g) \geq v^{\mathrm{P}}(\varphi) = \min\{\frac{2\varphi+3}{\varphi+1}\lambda^0(X), 1\} > \underline{V}$ for $\varphi = \min\{|T \cap T'| : T, T' \in W^{\min}\}$,

(ii) $\pi \in \Pi^{\mathrm{e}}$ such that $V_q^\pi(g) \geq \hat{v}^{\mathrm{P}}(\beta) = \min\{\frac{2\beta+3}{\beta+1}\lambda^0(X), 1\}$ for $\beta = \min\{|T \cap T'| : T, T' \in W^{\mathrm{np}}\}$ for $|W^{\mathrm{np}}| \geq 2$.

We assume that $|W^{\mathrm{np}}| \geq 2$, since otherwise in case (ii) we have $\hat{v}^{\mathrm{P}}(\beta) = \underline{V}$. We only prove part (i), as the proof of part (ii) follows directly from (i).

First assume that $\lambda^0(X) \leq (\varphi+1)/(2\varphi+3)$. Let $\varphi = \min\{|T \cap T'| : T, T' \in W^{\min}\}$. In words, φ is the cardinality of the smallest intersection of two minimal winning coalitions, which allows the sender to effectively target coalitions that are constructed via the agents in the intersection. Pick $R, R' \in W^{\min}$. Since g is proper, it holds that $R \cap R' \neq \emptyset$. Assume without loss of generality that $|R \cap R'| = \varphi$. Let $R \cap R' = T$. For each $i \in T$ define $D_i = N \setminus \{i\}$. Since there is no veto player, $D_i \in W$. We show that the sender can achieve the lower bound by targeting R, R', and all D_i's in state Y. Define $\pi \in \Pi$ as

$$\pi(\tilde{s}|\omega) = \begin{cases} 1 & \text{if } \tilde{s} = \bar{x} \text{ and } \omega = X, \\ \frac{\lambda^0(X)}{(\varphi+1)\lambda^0(Y)} & \text{if } \tilde{s} \in \{\chi(R), \chi(R'), \chi(D_i)\} \text{ for all } i \in T \text{ and } \omega = Y, \\ 1 - \frac{(\varphi+2)\lambda^0(X)}{(\varphi+1)\lambda^0(Y)} & \text{if } s = \bar{y} \text{ and } \omega = Y. \end{cases}$$

The value of π is

$$V_q^\pi(g) = 1 - \lambda^0(Y)\left(1 - \frac{(\varphi+2)\lambda^0(X)}{(\varphi+1)\lambda^0(Y)}\right) = \lambda^0(X) + \frac{\varphi+2}{\varphi+1}\lambda^0(X) = \frac{2\varphi+3}{\varphi+1}\lambda^0(X).$$

Now suppose that $\lambda^0(X) > (\varphi+1)/(2\varphi+3)$. In this case, define $\pi' \in \Pi$ as

$$\pi'(\tilde{s}|\omega) = \begin{cases} 1 & \text{if } \tilde{s} = \bar{x} \text{ and } \omega = X, \\ \frac{1}{\varphi+2} & \text{if } \tilde{s} \in \{\chi(R), \chi(R'), \chi(D_i)\} \text{ for all } i \in T \text{ and } \omega = Y. \end{cases}$$

For any $i \in N$, it holds that $\pi_i'(x|Y) = (\varphi+1)/(\varphi+2)$. Since $\lambda^0(X) > (\varphi+1)/(2\varphi+3)$ and $\lambda^0(Y)(\varphi+1)/(\varphi+2) < (\varphi+1)/(2\varphi+3)$, we have $\lambda^0(X) > \lambda^0(Y)(\varphi+1)/(\varphi+2)$ so that π' is straightforward. In this case, we have $V_q^{\pi'}(g) = 1$.

Hence, the optimal value is $\min\{\frac{2\varphi+3}{\varphi+1}\lambda^0(X), 1\}$. Note that the lower bounds we provide strictly improve upon public communication, since g being proper implies that $\varphi \geq 1$. $\square$

Proof of Proposition 5 We first state the result formally.
Proposition 5. Let g be a strong WMG and $\delta = \max\{|T| : T \in W^{\min}\}$. Then there exists $\pi \in \Pi$ such that $V_q^{\pi}(g) \geq \min\{\frac{3\delta-1}{\delta}\lambda^0(X), 1\}$.

Here, δ is the maximum cardinality of a minimal winning coalition. First assume that $\lambda^0(X) \leq \delta/(3\delta-1)$. Pick $T \in W^{\min}$, fix $i \in T$, and assume that $|T| > 1$.[51] Since $T \in W^{\min}$, it holds that $T \setminus \{i\} \notin W$. Then, since g is strong, $(N \setminus T) \cup \{i\} \in W$.

Therefore, for any strong game *two types* of winning coalitions always exist: (i) the minimal coalition T and (ii) the complement of T together with *one* agent $i \in T$, (where $|T|$ many such coalitions exist). Overall, there are at least $|T| + 1$ winning coalitions, excluding the grand coalition. The sender can achieve the lower bound by targeting these coalitions in state Y. Since the variation in the minimal winning coalitions will be exploited, taking $T \in W^{\min}$ such that $|T| = \max\{|R| : R \in W^{\min}\} = \delta$, i.e. the minimal winning coalition with the most members, benefits the sender the most. Define $\pi \in \Pi$ as

$$\pi(\tilde{s}|\omega) = \begin{cases} 1 & \text{if } \tilde{s} = \bar{x} \text{ and } \omega = X, \\ \frac{\lambda^0(X)}{\delta\lambda^0(Y)} & \text{if } i \in T, \tilde{s} = \chi(\{i\} \cup N \setminus T) \text{ and } \omega = Y, \\ \frac{(\delta-1)\lambda^0(X)}{\delta\lambda^0(Y)} & \text{if } \tilde{s} = \chi(T) \text{ and } \omega = Y, \\ 1 - \frac{(2\delta-1)\lambda^0(X)}{\delta\lambda^0(Y)} & \text{if } \tilde{s} = \bar{y} \text{ and } \omega = Y. \end{cases}$$

The value of π is

$$V_q^{\pi}(g) = 1 - \lambda^0(Y)\left(1 - \frac{(2\delta-1)\lambda^0(X)}{\delta\lambda^0(Y)}\right) = \lambda^0(X) + \frac{2\delta-1}{\delta}\lambda^0(X) = \frac{3\delta-1}{\delta}\lambda^0(X).$$

Now suppose that $\lambda^0(X) > \delta/(3\delta-1)$. Define $\pi' \in \Pi$ as

$$\pi(\tilde{s}|\omega) = \begin{cases} 1 & \text{if } \tilde{s} = \bar{x} \text{ and } \omega = X, \\ \frac{1}{2\delta} & \text{if } i \in T, \tilde{s} = \chi(\{i\} \cup N \setminus T) \text{ and } \omega = Y, \\ \frac{\delta-1}{2\delta} & \text{if } \tilde{s} = \chi(T) \text{ and } \omega = Y. \end{cases}$$

It follows easily that π' is straightforward and thus $V_q^{\pi'}(g) = 1$. Hence, the optimal value is $\min\{\frac{3\delta-1}{\delta}\lambda^0(X), 1\}$. In particular, the sender can implement x with probability 1 if $\lambda^0(X) \geq 2/5$. □

Proof of Proposition 6 We first state the result formally.
Proposition 6. Let g be a WMG and $\gamma = \max\{|R| : R \subseteq W \text{ s.t. } i \notin C_1 \cap C_2, \forall i \in N \text{ and } C_1, C_2 \in R\}$. If g is improper, then there exists

(i) $\pi \in \Pi$ such that $V_q^{\pi}(g) \geq \min\{(\gamma+1)\lambda^0(X), 1\} > \underline{V}$,

[51] Note that if $|T| = 1$, then either the game has a dictator or the game is improper. For the former optimal communication is public (which gives the lower bound) and for the latter we provide a lower bound in Proposition 6.

(ii) $\pi \in \Pi^{e}$ such that $V_q^{\pi}(g) \geq \max\{\underline{V}, \min\{V(\gamma, 2, \lambda^0)\}\}$.

Suppose that g is improper. Let $Q = \{T \in W : T \cap R = \emptyset, \forall R \in W \backslash T\}$. Let $\pi \in \Pi$ be given by $\pi(\bar{x}|X) = 1$, $\pi(\chi(R)|Y) = \min\{\lambda^0(X)/\lambda^0(Y), 1/\gamma\}$ for any $R \in Q$, and $\pi(\bar{y}|Y) = \max\{1 - \gamma\lambda^0(X)/\lambda^0(Y), 0\}$. Thus, $V_q^{\pi}(g) = 1 - \lambda^0(Y)(1 - \gamma\lambda^0(X)/\lambda^0(Y)) = \min\{(\gamma + 1)\lambda^0(X), 1\}$.

In words, γ is the number of *mutually exclusive winning coalitions* the sender can target.[52] Part (i) implies that even when there are only two disjoint winning coalitions ($\gamma = 2$) and $\lambda^0(X) \geq 1/3$, the sender can implement her preferred outcome with probability 1. It is easy to see that when the sender targets more than one mutually exclusive coalitions in a signal, no agent is pivotal. Using all pairs of these coalitions provides the lower bound of the value in equilibrium, given in part (ii). Here, it must be noted that if $\gamma = 2$, the lower bound will be $\underline{V}$. However, in all other cases (i.e. $\gamma > 2$) the lower bound in equilibrium improves upon public communication. □

Proof of Theorem 1 First, suppose g is such that there is a veto player. By Lemma 2 it follows that π^{pub} is optimal.

Now, suppose that π^{pub} is optimal. We need to prove that the WMG has a veto player. Assume to the contrary that g has no veto player. Then, there are two options:

(i) g is proper. Then g is both proper and has no veto player and Proposition 4 applies.
(ii) g is improper. Then Proposition 6 applies.

Hence, the sender can always improve upon public communication and π^{pub} is not optimal, a contradiction. □

Proof of Theorem 2 We first start by giving the definition of pseudo-anonymity and proving a lemma.

A *permutation* is a bijection defined by $b : N \to N$. Denote the set of all permutations by B. We consider a restricted set of permutations $\tilde{B} \subsetneq B$, which we define as the set of all permutations that keeps the apex player fixed; that is, any $b \in \tilde{B}$ only permutes the remaining $n-1$ minor players. Given $\pi \in \Pi$, for each $s \in S^{\pi}$ and $b \in \tilde{B}$, let s^b be such that $s_i^b = s_{b(i)}$. We call $\pi \in \Pi$ *pseudo-anonymous* if $\pi(s|\omega) = \pi(s^b|\omega)$ for all $b \in \tilde{B}$ and $\omega \in \Omega$.[53] □

Lemma 5 *Let v be a general apex game and let $\hat{\pi} \in \Pi$. There exists a pseudo-anonymous $\pi \in \Pi$ such that $V_q^{\pi}(g) = V_q^{\hat{\pi}}(g)$.*

Proof For each $\omega \in \Omega$, $s \in S$, and $b \in \tilde{B}$, define π^b by $\pi^b(s|\omega) = \hat{\pi}(s^b|\omega)$, so $S^{\pi^b} = \left\{s^{b^{-1}} : s \in S^{\hat{\pi}}\right\}$. Let $\pi \in \Pi$ be defined by

$$\pi\,(s|\omega) = \frac{1}{(n-1)!}\sum_{b \in B} \pi^b\,(s|\omega)\,, \quad \omega \in \Omega, s \in S.$$

[52] If a WMG is proper, then $\gamma = 0$, since the intersection of any two winning coalitions is nonempty.

[53] Note that pseudo-anonymity can be related to the concept of symmetric players.

In words, given the signals in $S^{\hat{\pi}}$, S^{π} includes all permutations of those signals (in which the apex player is kept fixed) and π assigns equal probability to them. Clearly, π is pseudo-anonymous.

All that is left to show is that π is straightforward: if all agents who observe x continue to vote for x, the value of π will be the same as the value of $\hat{\pi}$ since the total probability of signals that implement x will remain the same. Fix $i \in N$. Then

$$\begin{aligned}\pi_i(x|X) &= \sum_{s\in S^{\pi}:s_i=x} \pi(s|X) = \sum_{s\in S^{\pi}:s_i=x} \frac{1}{(n-1)!}\sum_{b\in B}\pi^b(s|X)\\ &= \frac{1}{(n-1)!}\sum_{b\in B}\sum_{s\in S^{\pi}:s_i=x}\hat{\pi}\left(s^b|X\right) = \frac{1}{(n-1)!}\sum_{b\in B}\sum_{s\in S^{\pi}:s_{b(i)}=x}\hat{\pi}(s|X)\\ &= \frac{1}{(n-1)!}\sum_{b\in B}\hat{\pi}_{b(i)}(x|X).\end{aligned}$$

Since $\hat{\pi}$ is straightforward, one has $\lambda^0(X)\hat{\pi}_{b(i)}(x|X) \geq \lambda^0(Y)\hat{\pi}_{b(i)}(x|Y)$ for all $b \in B$. Therefore,

$$\begin{aligned}\lambda^0(X)\pi_i(x|X) = \lambda^0(X)\frac{1}{(n-1)!}\sum_{b\in B}\hat{\pi}_{b(i)}(x|X) \geq \\ \lambda^0(Y)\frac{1}{(n-1)!}\sum_{b\in B}\hat{\pi}_{b(i)}(x|Y) = \lambda^0(Y)\pi_i(x|Y).\end{aligned}$$

Thus, π satisfies (8). Therefore, π is straightforward. It immediately follows that $V_q^{\pi}(g) = V_q^{\hat{\pi}}(g)$. □

Now we prove the theorem. Let $\pi^* \in \Pi$. By Lemma 1 and Lemma 5, it is without loss of generality to assume that $\pi^*(\bar{x}|X) = 1$ and that π^* is pseudo-anonymous. This implies that it is optimal for the sender to assign positive probability to all signals that target minimal winning coalitions in state Y.

First, note that there are $\binom{n-1}{c_a}$ signals in S^a. Since all signals in S^a target the apex player, the total probability of these signals must add up to $\lambda^0(X)/\lambda^0(Y)$, so that obedience constraint (8) is satisfied (and is binding) for the apex player. Thus, for any $s \in S^a$ let $\pi^*(s|Y) = \frac{\lambda^0(X)}{\lambda^0(Y)}\binom{n-1}{c_a}^{-1}$.

Next, note that a minor player observes x $\binom{n-2}{c_a-1}$ times in all signals in S^a. So, the probability of observing x for messages profiles in S^a is

$$\binom{n-2}{c_a-1}\frac{\lambda^0(X)}{\lambda^0(Y)}\binom{n-1}{c_a}^{-1} = \frac{c_a}{n-1}\frac{\lambda^0(X)}{\lambda^0(Y)}.$$

Thus, in order to satisfy (8), a minor player can observe x in signals in S^m with probability at most

$$\frac{\lambda^0(X)}{\lambda^0(Y)} - \frac{c_a}{n-1}\frac{\lambda^0(X)}{\lambda^0(Y)} = \frac{\lambda^0(X)}{\lambda^0(Y)}\left(1 - \frac{c_a}{n-1}\right).$$

Moreover, note that a minor player observes x in signals in S^m $\binom{n-2}{c_m-1}$ many times. Hence, for any $t \in S^m$, let $\pi^*(t|Y) = \frac{\lambda^0(X)}{\lambda^0(Y)}\left(1 - \frac{c_a}{n-1}\right)\binom{n-2}{c_m-1}^{-1}$.

So, the total probability of signals in S^a is $\lambda^0(X)/\lambda^0(Y)$ and the total probability of signals in S^m is $\frac{\lambda^0(X)}{\lambda^0(Y)}\left(1 - \frac{c_a}{n-1}\right)\binom{n-2}{c_m-1}^{-1}\binom{n-1}{c_m}$. Subtracting the sum of all signals in $S^a \cup S^m$ yields $\pi^*(\bar{y}|Y)$, as given in the statement of the theorem.

Finally, the value of π^* is given by

$$\begin{aligned} 1 - \lambda^0(Y)\left(1 - \frac{\lambda^0(X)}{\lambda^0(Y)}\frac{n-1-c_a+c_m}{c_m}\right) &= \lambda^0(X) + \lambda^0(X)\left(\frac{n-1-c_a+c_m}{c_m}\right) \\ &= \lambda^0(X)\frac{n-1-c_a+2c_m}{c_m}. \end{aligned}$$

Proof of Corollary 2 We first state the result formally.

Corollary 2. Let $q \geq \tau/2$. For any weight profile $(w_i)_{i\in N}$, if $|\{i \in N : w_i = 1\}| \geq q$ then the optimal value is $\overline{V}$.

Let $N^1 = \{i \in N | w_i = 1\}$, $N^h = \{i \in N | w_i > 1\}$, and $S = \{x, y\}^n$. Moreover, let $|N^1| = k$. Define

$$R = \left\{s \in S' : \forall i \in N^h, s_i = x \text{ and } \left|\left\{j \in N^1 : s_j = x\right\}\right| = q - (\tau - k)\right\},$$

which is the set of signals in which all receivers with weight higher than 1 and $q-\tau+k$ receivers with weight equal to 1 observe x. Similarly, define

$$T = \left\{t \in S' : \forall i \in N^h, t_i = y \text{ and } \left|\left\{j \in N^1 : t_j = x\right\}\right| = q\right\},$$

which is the set of signals in which all receivers with weight higher than 1 and $k-q$ receivers with weight equal to 1 observe y, while q receivers with weight equal to 1 observe x. Let $\lambda^0(Y)/\lambda^0(X) = \ell$. It follows from the proof of Proposition 4.3 in Kerman and Tenev (2021), that an optimal experiment is given by

$$\pi(s|\omega) = \begin{cases} 1 & \text{if } s = \bar{x} \text{ and } \omega = X, \\ 1 - \frac{\tau}{q\ell} & \text{if } s = \bar{y} \text{ and } \omega = Y, \\ \frac{\tau-q}{k\ell}\binom{k-1}{q-1}^{-1} & \text{if } s \in T \text{ and } \omega = Y, \\ \frac{1}{\binom{k}{q-\tau+k}\ell} & \text{if } s \in R \text{ and } \omega = Y. \end{cases}$$

□

Proof of Lemma 3 Let $P = (N, \lambda^0, q, (w_i)_{i\in N}, (\tilde{u}_i)_{i\in N}, U)$ be a voting problem. Let $\pi \in \Pi$, fix $i \in N$ and let $s \in S^\pi$ be such that $s_i = x$. An agent $i \in N$ votes in favor of x upon observing x if the expected utility of choosing x is higher than choosing y, i.e.

$$\lambda_i^s(X) \cdot a_i + (1 - \lambda_i^s(X)) \cdot c_i \geq \lambda_i^s(X) \cdot d_i + (1 - \lambda_i^s(X)) \cdot b_i,$$

which simplifies to

$$\lambda_i^s(X) = \frac{\lambda^0(X)}{\lambda^0(X) + \lambda^0(Y)\pi_i(x|Y)} \geq \frac{b_i - c_i}{a_i + b_i - c_i - d_i}.$$

We can rewrite the above inequality in terms of $\lambda^0(X)$:

$$\lambda^0(X) \geq \frac{(b_i - c_i)\pi_i(x|Y)}{a_i - d_i + (b_i - c_i)\pi_i(x|Y)}. \tag{9}$$

Note that the right-hand side of the inequality is less than 1.

Now consider a voting problem with heterogeneous beliefs and identical preferences, i.e. $P' = (N, \lambda^0, \lambda_R^0, q, (w_i)_{i\in N}, (\bar{u}_i)_{i\in N}, U)$. Agent i votes in favor of x upon observing x if

$$\lambda_i^0(X) \geq \frac{(b - c)\pi_i(x|Y)}{a - d + (b - c)\pi_i(x|Y)}. \tag{10}$$

Hence, for any $\pi \in \Pi$, $a_i, b_i, c_i, d_i \in \mathbb{R}$ with $a_i > d_i$ and $b_i > c_i$, and $\lambda^0 \in \Delta(\Omega)$ that satisfies (9) one can find $a, b, c, d \in \mathbb{R}$ with $a > d$ and $b > c$, and $\lambda_i^0 \in \Delta(\Omega)$ for each $i \in N$ such that (10) is satisfied. Therefore, $V[G(P, \pi)] = V[G(P', \pi)]$. □

Proof of Theorem 3 First, assume that g contains a veto player. By Lemma 2 it follows that π^{pub} is optimal.

Now suppose that π^{pub} is optimal. We start with a trivial observation, which we provide without proof as it is straightforward. □

Lemma 6 *Let $\pi \in \Pi$ be a public experiment and $s \in S^\pi$. If $\alpha_i^\pi(s) = x$, then $\alpha_j^\pi(s) = x$ for any $j \in N$ with $\lambda_j^0(X) \geq \lambda_i^0(X)$.*

In words, if an agent with a lower prior belief is persuaded under a public experiment, then all agents with a weakly higher prior are also persuaded. Therefore, the optimal public value depends on the "cutoff" receiver's prior, which we denote by $\lambda_c^0(X)$. More precisely, $\lambda_c^0(X)$ is the highest possible prior belief such that the coalition consisting of the cutoff receiver and receivers with weakly higher priors is a winning coalition.

Recall that λ_R^0 is the vector of heterogeneous priors. Let the value of the optimal public experiment when the sender has prior λ^0 and the receivers have heterogeneous priors be denote by $V^{\text{pub}}(\lambda^0, \lambda_R^0)$. So, the sender's expected utility under the optimal public experiment is

$$V^{\text{pub}}(\lambda^0, \lambda_R^0) = \min\left\{1, \lambda^0(X) + \lambda^0(Y)\frac{\lambda_c^0(X)}{\lambda_c^0(Y)}\right\}.$$ [54]

[54] Note that the value of the optimal public experiment might be higher or lower than in the common prior case, $\min\{2\lambda^0(X), 1\}$, depending on whether the cutoff receiver has a lower or higher prior than the sender.

Let $\lambda^0_\ell(X) = \min\{\lambda^0_1(X), \lambda^0_2(X), \dots, \lambda^0_n(X)\}$. Assume to the contrary that g has no veto player. We consider two cases.

Case 1. $\lambda^0_c(X) = \lambda^0_\ell(X)$. Since all receivers vote in favor of x whenever the cutoff receiver does by Lemma 6, the sender may use private communication by treating λ^0_ℓ as the common prior, in which case all receivers vote for x upon observing x under the optimal experiment. Since there is no veto player, private communication improves upon public by Theorem 1, a contradiction.

Case 2. $\lambda^0_c(X) > \lambda^0_\ell(X)$. Since the optimal communication is public, there exists a coalition $R \subsetneq N$ such that $R \in W$, i.e. there is a winning coalition which is not the grand coalition. In particular, agents in R have priors that are weakly greater than $\lambda^0_c(X)$. The public experiment in this case has the following form.[55]

$$\pi^{\text{pub}}(\tilde{s}|\omega) = \begin{cases} 1 & \text{if } \tilde{s} = \bar{x} \text{ and } \omega = X, \\ \frac{\lambda^0_c(X)}{\lambda^0_c(Y)} & \text{if } \tilde{s} = \bar{x} \text{ and } \omega = Y, \\ 1 - \frac{\lambda^0_c(X)}{\lambda^0_c(Y)} & \text{if } s = \bar{y} \text{ and } \omega = Y. \end{cases}$$

We will construct a strategy which modifies π^{pub} and increases the value beyond $V^{\text{pub}}(\lambda^0, \lambda^0_R)$. More precisely, the sender can effectively target coalitions that are constructed via the agents in R. We consider two subcases.

Case 2.1. $|R| > 1$. For each $i \in R$ define $D_i = N \setminus \{i\}$. Since there is no veto player, $D_i \in W$ for each $i \in N$. We show that the sender can improve upon public communication by targeting R and all D_i's in state Y. Define $\pi \in \Pi$ as

$$\pi(\tilde{s}|\omega) = \begin{cases} 1 & \text{if } \tilde{s} = \bar{x} \text{ and } \omega = X, \\ \frac{\lambda^0_c(X)}{\lambda^0_c(Y)} - \varepsilon & \text{if } \tilde{s} = \chi(R) \text{ and } \omega = Y, \\ \frac{\varepsilon}{|R|-1} & \text{if } \tilde{s} = \chi(D_i) \text{ for all } i \in R \text{ and } \omega = Y, \\ 1 - \frac{\lambda^0_c(X)}{\lambda^0_c(Y)} + \varepsilon - |R|\frac{\varepsilon}{|R|-1} & \text{if } s = \bar{y} \text{ and } \omega = Y. \end{cases}$$

Note that while all $j \in N$ with $\lambda^0_j(X) \in [\lambda^0_\ell(X), \lambda^0_c(X))$ do not vote for x upon observing x under π^{pub}, they do vote for x upon observing x under π since

$$\frac{\lambda^0_j(X) \cdot 1}{\lambda^0_j(X) \cdot 1 + \lambda^0_j(Y)|R|\frac{\varepsilon}{|R|-1}} \geq \frac{1}{2},$$

for sufficiently small ε. Moreover, all receivers with higher priors than $\lambda^0_c(X)$ still vote for x as in π^{pub}. Thus, the value of π is

$$V^\pi_q(g) = 1 - \lambda^0(Y)\left(1 - \frac{\lambda^0_c(X)}{\lambda^0_c(Y)} + \varepsilon - |R|\frac{\varepsilon}{|R|-1}\right)$$

[55] When the realization is $\bar{x}$, only receivers who have a prior of at least $\lambda^0_c(X)$ vote for x. Hence, the optimal public experiment is not straightforward.

$$= 1 - \lambda^0(Y)\left(1 - \frac{\lambda_c^0(X)}{\lambda_c^0(Y)} - \frac{\varepsilon}{|R|-1}\right) = \lambda^0(X) + \lambda^0(Y)\frac{\lambda_c^0(X)}{\lambda_c^0(Y)} + \lambda^0(Y)\frac{\varepsilon}{|R|-1}$$
$$> V^{\text{pub}}(\lambda^0, \lambda_R^0).$$

Hence, when $|R| > 1$ the sender can always improve upon public communication and thus π^{pub} is not optimal, a contradiction.

Case 2.2. $|R| = 1$. Let $R = \{j\}$. By the same argument as in Case 2.1, $D_i = N \setminus \{i\} \in W$ for all $i \in N$, as otherwise i is a veto player. However, here we need to take into account the fact that $D_i \cap R = \emptyset$ for all $i \neq j$, thus the construction we use is different than in Case 2.1. Define $\pi \in \Pi$ as

$$\pi(\tilde{s}|\omega) = \begin{cases} 1 & \text{if } \tilde{s} = \bar{x} \text{ and } \omega = X, \\ \frac{\lambda_c^0(X)}{\lambda_c^0(Y)} & \text{if } \tilde{s} = \chi(R) \text{ and } \omega = Y, \\ \varepsilon & \text{if } \tilde{s} = \chi(D_i) \text{ for all } i \in R \text{ and } \omega = Y, \\ 1 - \frac{\lambda_c^0(X)}{\lambda_c^0(Y)} - \varepsilon & \text{if } s = \bar{y} \text{ and } \omega = Y. \end{cases}$$

Observe that there always exists a sufficiently small ε that can be taken from $\bar{y}$ in π^{pub} and reallocated to $\tilde{s} = \chi(D_i)$, as long as $\lambda_c^0(Y) > \lambda_c^0(X)$, which holds by assumption.

We can make a similar check as above and conclude that when $|R| = 1$ the sender can always improve upon public communication and thus π^{pub} is not optimal, a contradiction. This concludes the proof.

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