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ORIGINAL PAPER

# TU-games with utilities: the prenucleolus and its characterization set

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**Abstract**
TU-games with utility functions are considered. Generalizations of the prenucleolus, essential coalitions and the core: the **u**-prenucleolus, **u**-essential coalitions and the **u**-core respectively are introduced. We show that **u**-essential coalitions form a characterisation set for the **u**-prenucleolus in case of games with nonempty **u**-core.

**Keywords** TU-games · Restricted cooperation · Prenucleolus · Core · Essential coalitions · TU-games with utility

## 1 Introduction

The (pre)nucleolus (Schmeidler 1969) is a widely used solution concept of transferable utility cooperative games. Its practical importance lies in the fact that its aim is to minimize the dissatisfaction of the most dissatisfied coailition, which makes it unequivocally interesting for social, financial and technical usages as well. A considerable evidence of its value is that it has numerous variants and generalizations, such as the percapita prenucleolus (Grotte 1970, 1972), the weighted prenucleolus (Derks and Haller 1999), the modified nucleolus (Sudhölter 1996, 1997) and the general nucleolus (Potters and Tijs 1992; Maschler et al. 1992). From the viewpoint

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of this research the most abstract generalization is the one by Potters and Tijs (1992) and Maschler et al. (1992) called the general nucleolus. The last two papers focus on the characterization of the general nucleolus, while also providing a generalization of the lexicographic center algorithm (Maschler et al. 1979), and Kohlberg's theorem (Kohlberg 1971).

In this paper, we consider the restricted cooperation case, and in connection with these games we introduce a special case of the general nucleolus, called the **u**-prenucleolus. Using this less general setting, we can also generalize the results by Katsev and Yanovskaya to the **u**-prenucleolus; namely, we give sufficient and necessary conditions for the **u**-prenucleolus to be nonempty and to be single-valued.

Moreover, the **u**-prenucleolus is a common generalization of some well known variants of the prenucleolus such as the percapita prenucleolus (Grotte 1970, 1972), the $q$-prenucleolus (Solymosi 2019) and the weighted prenucleolus (Derks and Haller 1999) among others.

All of these generalizations are common in using a modified version of the excesses. In our case, we define a **u** function on the excesses. Using this **u** function, we also define a generalization of balanced games and the core (Shapley 1955; Gillies 1959): the **u**-balanced games and the **u**-core respectively. Furthermore, we show that a game is **u**-balanced if and only if its **u**-core is nonempty, that is, among others, we generalize the Bondareva–Shapley theorem (Bondareva 1963; Shapley 1967; Faigle 1989).

Another variant of the prenucleolus, which is worth mentioning, because of its similarity to the **u**-prenucleolus, is the modified nucleolus (Sudhölter 1996, 1997). It also applies the idea of defining a modified excess vector and finding the payoff vectors, which lexicographically minimize it. However, this modification cannot be described by using an above-mentioned utility function, so it is not a special case of our approach, the **u**-prenucleolus.

Computing the prenucleolus or its variants and generalizations is time-consuming. Although it has been known that $2n-2$ coalitions are enough to characterize the prenucleolus of an $n$ player game (Brune 1983; Reijnierse and Potters 1998) – such a set of coalitions is called a characterization set –, it is not easy to find these $2n-2$ coalitions.

There are several algorithms to compute the (pre)nucleolus (Kohlberg 1972; Solymosi 1993; Perea and Puerto 2013), however these algorithms have a complexity of $\mathcal{O}(2^n)$ in general. On the other hand, for some classes of games like neighbour games (Hamers et al. 2003), permutation games under certain conditions (Solymosi et al. 2005), tree games (Maschler et al. 2010), a large class of directed acyclic graph games (Sziklai et al. 2017) the nucleolus can be computed in polynomial time in the number of players. In addition, some heuristics provide efficient algorithms for us to find an allocation close to the nucleolus (Perea and Puerto 2019).

In this paper, we consider the lexicographic center algorithm (Maschler et al. 1979), which calculates the nucleolus using $\mathcal{O}(n)$ LPs with $\mathcal{O}(n)$ variables and $\mathcal{O}(2^n)$ constraints. The number of constraints can be reduced by finding a smaller characterization set for the nucleolus than the one including all the feasible coalitions.

Huberman (1980) showed that the so-called essential coalitions give a characterization set for the nucleolus of balanced games. In certain classes of games (e.g. matching-games) the cardinality of essential coalitions is polynomial in the number of players and they are also easy to find, thereby providing a way of computing the nucleolus in polynomial time. On the other hand, Huberman's theorem, typically, cannot be applied to the variants and generalizations of the prenucleolus, neither can it be used for non-balanced games.

In this paper, we define the **u**-essential coalitions as a generalization of essential coalitions and prove that the **u**-essential coalitions give a characterization set for the **u**-prenucleolus of **u**-balanced games. Choosing our **u** function accordingly, this generalization of Huberman's theorem can be applied to give a characterization set for the prenuclolus of non-balanced games or to give a characterization set for the percapita prenucleolus of balanced games, among others; thereby solving the above-mentioned problems.

Before summarizing our main results stated in the paper, let us consider some alternative approaches for defining the **u**-excess. We define the **u**-excess by applying a utility function over the excess, however, it is also possible to take the excess of the utilities instead. This approach would also generalize the prenucleolus and the percapita prenucleolus. In addition, using the identity or the percapita utility functions, the two above-defined **u**-excesses would coincide, due to the linearity of the utility functions. However, in case of a non-linear utility function, the two approaches would differ. Taking the excess of the utililities instead of the utility of the excess is an interesting approach, however, in this paper, we stick to using the utility of the excess, since we are interested in the idea, that different coalitions might value their excesses differently.

The literature also pushes us to our choice. The general nucleolus (Potters and Tijs 1992; Maschler et al. 1992) also applies functions on the excesses. The results stated in this paper only hold in case of taking the utility of the excess and not in case of taking the excess of the utilities. This other idea can be a fuel for future work.

One of the main contributions of this paper is the notion of the **u**-prenucleolus, which is a generalization of the prenucleolus and the percapita prenucleolus. It is also a special case of the general prenucleolus. In this setup, we define the generalization of the core, least-core, balanced games: the **u**-core, the **u** -least-core and **u**-balanced games, respectively. A respectable achievement of our researches is that using these notions, we prove the generalizations of the Bondareva–Shapley theorem and two theorems by Katsev and Yanovskaya to the **u**-prenucleolus, which, according to our present knowledge, cannot be proven to the general prenucleolus. We also introduce the notion of **u**-essential coalitions, and show that the class of **u**-essential coalitions form a characterization set for the **u**-prenucleolus of **u**-balanced games. In other words, we generalize Huberman (1980)'s result to TU-games with utilities.

The setup of the paper is as follows: In Sect. 2, we discuss the basic concepts and notations used in the paper; in Sect. 3, we introduce the notions of TU-games with utilities, **u**-excess, **u**-prenucleolus and **u**-core; in Sect. 4, we define the so-called **u**-balanced games, and show that a game is **u**-balanced if and only if its **u**-core is nonempty.

In Sect. 5, we give a generalization of the lexicographic center algorithm for calculating the **u**-prenucleolus – this algorithm is a special case of the lexicographic center algorithm for calculating the general nucleolus by Maschler et al. (1992).

In Sect. 6, we generalize Katsev and Yanovskaya (2013)'s theorem, namely, we give a sufficient and necessary condition for the nonemptyness of the **u** -prenucleolus. In Sect. 7, we generalize another theorem by Katsev and Yanovskaya , namely, we give a sufficient and necessary condition for the single-valuedness of the **u**-prenucleolus.

In Sect. 8, we define the so-called **u**-essential coalitions, and generalize Huberman (1980)'s theorem by proving that the **u**-essential coalitions form a characterization set for the **u**-prenucleolus of **u**-balanced games. In Sect. 9, we describe the **u** functions for which the prenucleolus and the core coincide with the **u** -prenucleolus and the **u**-core, respectively.

In Sect. 10, we show that in case of assignment games, using the reciprocal percapita utility function, we get polynomial many **u**-essential coalitions in the number of players. Finally, the last section offers a brief conclusion.

## 2 Preliminaries

Given a nonempty finite set of players $N$ and a characteristic function $v : 2^N \to \mathbb{R}$ such that $v(\emptyset) = 0$, $v$ is called a TU-*game*. Let $\mathcal{G}^N$ denote the class of TU-games with player set $N$; additionally, let the set of coalitions be denoted by $\mathcal{P}(N) := \{S \subseteq N\}$, and the set of non-trivial coalitions be denoted by $\mathcal{P}^*(N) := \{S \subseteq N : S \neq \emptyset, S \neq N\}$. Furthermore, let $\mathcal{D}_S$ denote the class of partitions of set $S \subseteq N$ but $\{S\}$.

Let $\mathcal{A} \subseteq \mathcal{P}(N)$ be such that $\emptyset, N \in \mathcal{A}$, then $\mathcal{A}$ is called a set of feasible coalitions. In this case, the characteristic function $v : \mathcal{A} \to \mathbb{R}$ is called a TU-game with restricted cooperation. Let $\mathcal{G}^{N,\mathcal{A}}$ denote the class of TU-games (henceforth: game) with feasible coalitions $\mathcal{A}$. If $\mathcal{A} = \mathcal{P}(N)$, then $\mathcal{G}^{N,\mathcal{A}} = \mathcal{G}^N$, so every introduced concept for games with restricted cooperation is a generalization of a concept for classical TU-games.

Let $\mathcal{A}^* = \mathcal{A} \setminus \{N, \emptyset\}$ denote the set of feasible, non-trivial coalitions and let $\mathcal{D}_S^{\mathcal{A}^*} = \{B \in \mathcal{D}_S : B \subseteq \mathcal{A}^*\}$ denote the $\mathcal{A}^*$-partition set of $S \in \mathcal{A}^*$.

A solution is a set-valued mapping from a set of games with player set $N$ to $\mathbb{R}^N$. Widely used solutions in the literature are the core (Shapley 1955; Gillies 1959), the kernel (Davis and Maschler 1965), and the bargaining set (Aumann and Maschler 1964) among others. A value is a singleton valued solution; well-known values in the literature are the Shapley-value (Shapley 1953) and the (pre)nucleolus (Schmeidler 1969) among others.

Let $I(v) := \{x \in \mathbb{R}^N : \sum_{i\in N} x_i = v(N) \text{ and } x_i \geq v(\{i\}) \forall \{i\} \in \mathcal{A}\}$ and $I^*(v) := \{x \in \mathbb{R}^N : \sum_{i\in N} x_i = v(N)\}$ denote the set of imputations and preimputations of a game $v \in \mathcal{G}^{N,\mathcal{A}}$, respectively.

Given a game $v \in \mathcal{G}^{N,\mathcal{A}}$, a coalition $S \in \mathcal{A}$ and a payoff vector $x \in \mathbb{R}^N$, the *excess* of coalition $S$ by the payoff vector $x$ in the game $v$ is $e(S, x) := v(S) - x(S)$, where $x(S) := \sum_{i\in S} x_i$.

The core of a game $v \in \mathcal{G}^{N,\mathcal{A}}$ is the set of preimputations for which the excess of any feasible coalition is non-positive:

$$\text{core}(v) := \left\{ x \in \mathbb{R}^N : x(N) = v(N) \text{ and } e(S,x) \leq 0, \forall S \in \mathcal{A}^* \right\}.$$

In case the core of a game is nonempty, we say that the game is balanced.

Given a game $v \in \mathcal{G}^{N,\mathcal{A}}$ and a payoff vector $x \in \mathbb{R}^N$, the excess vector by the payoff $x$ in the game $v$ is the vector containing all the excesses in non-increasing order, that is $E(x) := (e(S,x))_{S \in \mathcal{A}^*} \in \mathbb{R}^{|\mathcal{A}^*|}$, where $E(x)_i \geq E(x)_j$ if $i \leq j$.

The lexicographical ordering between $x, y \in \mathbb{R}^n$ is the following: we say that $x \leq_L y$ if $x = y$ or if there exists a $k$, such that $x_k < y_k$ and $x_i = y_i$ for every $i < k$.

The nucleolus is the set of imputations which lexicographically minimize the excess vector over the set of imputations, that is, $N(v) = \{x \in I(v) : E(x) \leq_L E(y) \ \forall y \in I(v)\}$; and the prenucleolus is the set of preimputations which lexicographically minimize the excess vector over the set of preimputations, that is, $N^*(v) = \{x \in I^*(v) : E(x) \leq_L E(y) \ \forall y \in I^*(v)\}$.

A set of coalitions $\mathcal{S} \subseteq \mathcal{A}$ is a balanced set system if there exists a balancing weight system $\lambda_S \in \mathbb{R}_+$, $S \in \mathcal{S}$ such that

$$\sum_{S \in \mathcal{S}} \lambda_S \chi_S = \chi_N,$$

where $\chi_T \in \mathbb{R}^N$ is the characteristic vector of set $T$.

## 3 Games with utility functions

A well-known variant of the prenucleolus is the percapita prenucleolus (Grotte 1970, 1972). The percapita prenucleolus differs from the preucleolus in a way that instead of using the excesses, it uses the so-called percapita excesses. The percapita excess of a coalition $S \in \mathcal{A}^*$ of a game $v \in \mathcal{G}^{N,\mathcal{A}}$ with a payoff vector $x \in \mathbb{R}^N$ is $\frac{e(S,x)}{|S|}$. Similarly, the percapita excess vector is $E_{pc}(x) := (\frac{e(S,x)}{|S|})_{S \in \mathcal{A}^*} \in \mathbb{R}^{|\mathcal{A}^*|}$, where $E_{pc}(x)_i \geq E_{pc}(x)_j$ if $i \leq j$. Accordingly, the percapita prenucleolus is defined as follows: $N^*_{pc} = \{x \in I^*(v) : E_{pc}(x) \leq_L E_{pc}(y) \ \forall y \in I^*(v)\}$.

Solymosi (2019) considers a further generalization of the percapita prenucleolus, where, instead of dividing the excess by the cardinality of $S$, it is divided by $q(S)$, where $q$ is a real valued function over the feasible coalitions. Thereby, Solymosi (2019) introduced the notion of the $q$-nucleolus $N_q$, which is defined as follows: $N^*_q = \{x \in I^*(v) : E_q(x) \leq_L E_q(y) \ \forall y \in I^*(v)\}$, where the $q$-excess vector is defined as $E_q(x) := (\frac{e(S,x)}{q(S)})_{S \in \mathcal{A}^*}$, where $E_q(x)_i \geq E_q(x)_j$ if $i \leq j$.

Partially inspired by the above generalizations of the prenucleolus we generalize the prenucleolus further by introducing functions, called utility functions, applied to the excesses. Formally, see the following definition.

**Definition 1** A utility function $\mathbf{u} : \mathcal{A}^* \times \mathbb{R} \to \mathbb{R}$ is a family of functions $(u_S)_{S\in\mathcal{A}^*}$ such that $u_S : \mathbb{R} \to \mathbb{R}$ is strictly monotone increasing, continuous, and its domain is $\mathbb{R}$. Moreover, the ranges of $u_S$ and $u_T$ are the same for every $S, T \in \mathcal{A}^*$; let $R_{\mathbf{u}}$ denote the common range.

Let the **u**-excess of a coalition $S \in \mathcal{A}$ by the payoff vector $x \in \mathbb{R}$ in the game $v$ be as follows: $u_S \circ e(S,x) = u_S(v(S) - x(S))$. Moreover, let the **u**-excess vector be defined as $E(x) := (u_S(e(S,x)))_{S\in\mathcal{A}^*} \in \mathbb{R}^{|\mathcal{A}^*|}$, where $E(x)_i \geq E(x)_j$ if $i \leq j$.

We can now define the **u**-prenucleolus similarly to the percapita prenucleolus.

**Definition 2** The **u**-prenucleolus is the set of preimputations, which lexicographically minimizes the **u**-excess vectors over the set of preimputations. Formally,

$$N^*_{\mathbf{u}}(v) := \{x \in I^*(v) : E_{\mathbf{u}}(x) \leq_L E_{\mathbf{u}}(y)\ \forall y \in I^*(v)\}.$$

***Example 1*** Some examples of utility functions:

- If **u** is the identity function, then the **u**-penucleolius is the prenucleolus.
- If **u** is defined for all $S \in \mathcal{A}^*$ as $u_S(t) = \frac{t}{|S|}$, then the **u**-prenucleolus is the percapita prenucleolus.
- We can also define **u** as a shift by a constant $c$. In this case $u_S(t) = t + c$, and for any game $v \in \mathcal{G}^{N,\mathcal{A}}$ the **u**-prenucleolus is the prenucleolus of the game $v'$, where $v'(S) = v(S) + c$ for all $S \in \mathcal{A}^*$, and $v'(N) = v(N)$. Since the prenucleolus is invariant for shifting, in this case the prenucleolus and the **u**-prenucleolus of a game coincide.
- Note that **u** is not necessarily a family of linear functions. For example $u_S(t) = \arctan(t)$ for all $S \in \mathcal{A}^*$ can also be a utility function.

Next, we introduce a generalization of the core (Shapley 1955; Gillies 1959):

**Definition 3** Given a utility function **u**, the **u**-core of a game $v \in \mathcal{G}^{N,\mathcal{A}}$ is defined as follows:

$$\mathbf{u}\text{-core}(v) := \left\{x \in \mathbb{R}^N : x(N) = v(N) and u_S \circ e(S,x) \leq 0, \forall S \in \mathcal{A}^*\right\}.$$

Notice that, if $\mathcal{A} = \mathcal{P}(N)$ and **u** is the identity function, then the **u**-core is the core.

## 4 u-balanced games

Let $\mathfrak{B}$ denote the class of balanced set systems of $\mathcal{A}$.

**Definition 4** Given a game $v \in \mathcal{G}^{N,\mathcal{A}}$ and a utility function **u**, the game $v$ is **u**-balanced, if either $R_{\mathbf{u}} \subseteq \mathbb{R}_- \setminus \{0\}$ or if $0 \in R_{\mathbf{u}}$ and

$$\max_{\mathcal{B}\in\mathfrak{B}}\left(\lambda_N v(N) + \sum_{S\in\mathcal{B}\setminus\{N\}} \lambda_S \left(v(S) - \mathbf{u}_S^{-1}(0)\right)\right) \leq v(N), \tag{1}$$

where $(\lambda_S)_{S\in\mathcal{B}}$ is the balancing weight system of the balanced coalition system $\mathcal{B}$.

Notice that, if $\mathcal{A} = \mathcal{P}(N)$ and $\mathbf{u}$ is the identity function, then the $\mathbf{u}$-balancedness reverts to balancedness.

The following theorem is a generalization of the Bondareva–Shapley theorem (Bondareva 1963; Shapley 1967; Faigle 1989) to games with utilities.

**Theorem 1** *Given a game $v \in \mathcal{G}^{N,\mathcal{A}}$ and a utility function $\mathbf{u}$, the $\mathbf{u}$-core$(v) \neq \emptyset$ if and only if $v$ is $\mathbf{u}$-balanced.*

***Proof*** We consider three cases:

Case 1: $R_{\mathbf{u}} \subseteq \mathbb{R}_{-}\setminus\{0\}$. In case of such a utility function, the $\mathbf{u}$-core of a game is always non-empty. The reason for this is that $u_S \circ e(S,x) < 0$ for all $x \in I^*(v)$ and $S \in \mathcal{A}^*$. Therefore, the $\mathbf{u}$-core of the game is not empty, even more, $\mathbf{u}$-core$(v) = I^*(v)$
.

Case 2: $R_{\mathbf{u}} \subseteq \mathbb{R}_{+}\setminus\{0\}$. In case of such a utility function the $\mathbf{u}$-core of a game is always empty. The reason behind it is that $u_S \circ e(S,x) > 0$ for all $x \in I^*(v)$ and $S \in \mathcal{A}^*$; therefore, the $\mathbf{u}$-core of the game is empty.

Case 3: Otherwise, that is, $0 \in R_{\mathbf{u}}$. First, consider the following problem

$$\begin{array}{lll} & x(N) \to \min & \\ s.t. & u_S \circ e(S,x) \leq 0 & \forall S \in \mathcal{A}^* \\ & e(N,x) \leq 0 & \\ & x \in \mathbb{R}^N & \end{array} \tag{2}$$

The problem (2) is equivalent to the following LP (here we use that $0 \in R_u$)

$$\begin{array}{lll} & x(N) \to \min & \\ s.t. & e(S,x) \leq u_S^{-1}(0) & \forall S \in \mathcal{A}^* \\ & e(N,x) \leq 0 & \\ & x \in \mathbb{R}^N & \end{array} \tag{3}$$

The LP (3) is equivalent to the following LP

$$\begin{array}{lll} & x(N) \to \min & \\ s.t. & x(S) \geq v(S) - u_S^{-1}(0) & \forall S \in \mathcal{A}^* \\ & x(N) \geq v(N) & \\ & x \in \mathbb{R}^N & \end{array} \tag{4}$$

It is easy to see that LP (4) always has a feasible solution. Moreover, since its objective function is bounded from below ($x(N) \geq v(N)$) it always has an optimal solution. Let $x^*$ denote the optimal solution of (4). Then the $\mathbf{u}$-core is nonempty if and only if $x^*(N) = v(N)$.

The dual of (4) is the following:

$$\begin{array}{ll} & \lambda_N v(N) + \sum_{S \in \mathcal{A}^*} \lambda_S (v(S) - u_S^{-1}(0)) \to \max \\ s.t. & \sum_{S \in \mathcal{A}^* \cup \{N\}} \lambda_S \chi_S = \chi_N \\ & \lambda_S \geq 0 \qquad \forall S \in \mathcal{A}^* \cup \{N\} \end{array}$$

By the strong duality theorem of LPs we know that the optimum of the primal LP equals the optimum of the dual LP.

Suppose that $x^*$ and $\lambda^*$ are optimal solutions of the primal and the dual LPs, respectively. Notice, that $\lambda_N^* v(N) + \sum_{S \in \mathcal{A}^*} \lambda_S^* (v(S) - u_S^{-1}(0))$ equals the left-hand side of (1). Due to the strong duality theorem, it is less or equal than $v(N)$ if and only if $x^*(N)$ is less than or equal to $v(N)$, which is equivalent to the **u**-core being nonempty. □

## 5 A lexicographic center approach for the u-prenucleolus

In this section, we introduce a modification of the lexicographic center algorithm (Kopelowitz 1967; Maschler et al. 1979) for the **u**-prenucleolus. More precisely, we show how the idea behind the lexicographic center algorithm can be applied for the **u**-prenucleolus.

The lexicographic center algorithm works by solving a series of LPs. Note that in our case the optimization problems are not necessarily linear. We do not provide any algorithm to solve the non-linear problems, but we introduce a condition to decide whether the problems have optimums or not. The following lemma provides the considered condition:

**Lemma 1** *Let $v \in \mathcal{G}^{N,\mathcal{A}}$ be a game,* **u** *be a utility function and $X \subseteq I^*(v)$. Then*

$$\begin{array}{lll} & k \to \min & \\ s.t. & e(S,x) \leq k & S \in \mathcal{A}^* \\ & x \in X & \end{array} \tag{5}$$

*has an optimal solution, if and only if*

$$\begin{array}{lll} & k \to \min & \\ s.t. & u_S \circ e(S,x) \leq k & S \in \mathcal{A}^* \\ & x \in X & \end{array} \tag{6}$$

*has an optimal solution.*

***Proof*** If $X = \emptyset$, then neither problem (5) nor problem (6) has an optimal solution. Therefore, w.l.o.g. we can assume that $X \neq \emptyset$.

Since $X$ is nonempty, if problem (5) does not have an optimal solution, then for every $k \in \mathbb{R}$ there exists an $x_k \in X$ such that $\max_{S \in \mathcal{A}^*} e(S, x_k) \leq k$.

Let us define the following sequence: let $k_1 \in \mathbb{R}$ be an arbitrary number and $x_1 \in X$ a payoff vector such that $\max_{S \in \mathcal{A}^*} e(S, x_1) \leq k_1$.

Let $k_2 := \min_{S \in \mathcal{A}^*} e(S, x_1) - 1$, and $x_2$ be such that $\max_{S \in \mathcal{A}^*} e(S, x_2) \leq k_2$.

For $i > 2$ let $k_i := \min_{S\in\mathcal{A}^*} e(S,x_{i-1}) - 1$, and $x_i \in X$ be such that $\max_{S\in\mathcal{A}^*} e(S,x_i) \le k_i$.

Then for every $n \in \mathbb{N}^+$ $\min_{S\in\mathcal{A}^*} e(S,x_n) > \max_{S\in\mathcal{A}^*} e(S,x_{n+1})$, therefore, for all $S \in \mathcal{A}^*$ we have that $e(S,x_n) > e(S,x_{n+1})$. Since $u_S$ is strictly monotone increasing for every $S \in \mathcal{A}^*$, it follows that for every $S \in \mathcal{A}^*$ it holds that $u_S \circ e(S,x_n) > u_S \circ e(S,x_{n+1})$. Therefore, $\max_{S\in\mathcal{A}^*} u_S \circ e(S,x_n) > \max_{S\in\mathcal{A}^*} u_S \circ e(S,x_{n+1})$.

So for every $x \in X$ there exists an $x' \in X$ such that

$$\max_{S\in\mathcal{A}^*} u_S \circ e(S,x) > \max_{S\in\mathcal{A}^*} u_S \circ e(S,x').$$

Therefore, (6) does not have an optimal solution.

Now suppose that (6) does not have an optimal solution. Since $X$ is nonempty and $D_{u_S} = \mathbb{R}$ (where $D_f$ is the domain of $f$) for every $S \in \mathcal{A}^*$ we have that problem (6) has a feasible solution. Let $R_{\mathbf{u}} = (a,b)$ denote the range of $\mathbf{u}$, where $a,b \in \mathbb{R} \cup \{\infty, -\infty\}$ .

If problem (6) does not have an optimal solution, then $\inf\{t : u_S \circ e(S,x) \le t \; \forall S \in \mathcal{A}^*, x \in X\} = a$. It means that for every $k \in (a,b) \cap \mathbb{R}$ there exists $x_k \in X$ such that $\max_{S\in\mathcal{A}^*} u_S \circ e(S,x_k) \le k$. Furthermore, it is clear that $\min_{S\in\mathcal{A}^*} u_S \circ e(S,x) \in (a,b) \cap \mathbb{R}$ for every $x \in X$.

Let us define the following sequence: let $k_1 \in (a,b) \cap \mathbb{R}$ be an arbitrary number and $x_1 \in X$ be such that $\max_{S\in\mathcal{A}^*} u_S \circ e(S,x_1) \le k_1$.

Since $\min_{S\in\mathcal{A}^*} u_S \circ e(S,x_1) \in (a,b)$, there exists $\varepsilon > 0$ such that $\min_{S\in\mathcal{A}^*} u_S \circ e(S,x_1) - \varepsilon > a$. Let $k_2 := \min_{S\in\mathcal{A}^*} u_S \circ e(S,x_1) - \varepsilon$ and $x_2 \in X$ be such that $\max_{S\in\mathcal{A}^*} u_S \circ e(S,x_2) \le k_2$.

For any $n > 2$: let $\varepsilon_n > 0$ be such that $\min_{S\in\mathcal{A}^*} u_S \circ e(S,x_{n-1}) - \varepsilon_n > a$. Let $k_n := \min_{S\in\mathcal{A}^*} u_S \circ e(S,x_{n-1}) - \varepsilon_n$ and $x_n \in X$ be such that $\max_{S\in\mathcal{A}^*} u_S \circ e(S,x_n) \le k_n$.

Then, for every $n \in \mathbb{N}^+$ $\min_{S\in\mathcal{A}^*} u_S \circ e(S,x_n) > \max_{S\in\mathcal{A}^*} u_S \circ e(S,x_{n+1})$, furthermore, for each $S \in \mathcal{A}^*$ it holds that $u_S \circ e(S,x_n) > u_S \circ e(S,x_{n+1})$. Since $u_S$ is strictly monotone increasing, it holds that $e(S,x_n) > e(S,x_{n+1})$, for all $S \in \mathcal{A}^*$. Meaning that $\max_{S\in\mathcal{A}^*} e(S,x_n) > \max_{S\in\mathcal{A}^*} e(S,x_{n+1})$.

So, for every $x \in X$ there exists an $x' \in X$ such that

$$\max_{S\in\mathcal{A}^*} e(S,x) > \max_{S\in\mathcal{A}^*} e(S,x').$$

Therefore, problem (5) does not have an optimal solution either. □

The following lemma is a direct corollary of Lemma 1.

**Lemma 2** *Let $v \in \mathcal{G}^{N,\mathcal{A}}$ be a game, and $\mathbf{u^1}, \mathbf{u^2}$ be utility functions. Let $X \subseteq I^*(v)$, then*

$$\min_{x\in X} \max_{S\in\mathcal{A}^*} u^1_S \circ (v(S) - x(S))$$

*exists if and only if*

$$\min_{x\in X} \max_{S\in\mathcal{A}^*} u^2_S \circ (v(S) - x(S))$$

*exists*.

Next, we introduce a variant of the lexicographic center algorithm (Kopelowitz 1967; Maschler et al. 1979), which can be used for calculating the **u**-prenucleolus of a game.

Let $v \in \mathcal{G}^{N,\mathcal{A}}$ be a game and **u** be a utility function. Consider the following problem:

$$\begin{array}{ll} & t \to \min \\ s.t. & u_S \circ e(S,x) \le t, \quad S \in \mathcal{A}^* \\ & x \in I^*(v) \\ & t \in R_{\mathbf{u}} \end{array} \tag{7}$$

If problem (7) has an optimal solution, let $t_1$ denote the optimum of (7).

Let $X_1$ be defined as follows:

$$X_1 = \{x \in I^*(v) : u_S \circ e(S,x) \le t_1,\ \forall S \in \mathcal{A}^*\}.$$

Furthermore, let

$$W_1 = \{S \in \mathcal{A}^* : \exists c_S \in \mathbb{R},\ \text{such that } u_S \circ e(S,x) = c_S,\ \forall x \in X_1\}.$$

Let $k \ge 2$, and let us consider the following problem:

$$\begin{array}{ll} & t \to \min \\ s.t. & u_S \circ e(S,x) \le t, \quad S \in \mathcal{A}^* \setminus (\cup_{r=1}^{k-1} W_r) \\ & x \in X_{k-1} \\ & t \in \mathbb{R} \end{array} \tag{8}$$

If (8) has an optimal solution, let $t_k$ denote the optimum of (8).

Let $X_k$ be defined as follows

$$X_k = \{x \in X_{k-1} : u_S \circ e(S,x) \le t_k,\ \forall S \in \mathcal{A}^* \setminus (\cup_{r=1}^{k-1} W_r)\}.$$

Furthermore, let

$$W_k = \{S \in \mathcal{A}^* : \exists c_S \in \mathbb{R},\ \text{such that } u_S \circ e(S,x) = c_S,\ \forall x \in X_k\}.$$

It is easy to see that $t_k \ge t_{k+1}$ and $X_k \supseteq X_{k+1}$ for all $k \in \mathbb{N}_+$, and there exists $k^*$ such that for all $l \ge k^*$ it holds that $X_l = X_{k^*}$, and $X_{k^*} \neq \emptyset$.

Maschler et al. (1992) proved that a more general version of the above algorithm - the lexicographical center algorithm for finding the general prenucleolus - returns with the general prenucleolus. The **u**-prenucleolus is a special case of the general prenucleolus, hence the result by Maschler et al. (1992) implies the following theorem:

**Theorem 2** *For every game $v \in \mathcal{G}^{N,\mathcal{A}}$ and utility function* **u** *it holds that*

$$N^*_{\mathbf{u}}(v) = X_{k^*}.$$

## 6 The nonemptyness of the u-prenucleolus

In case of classical TU-games ($\mathcal{A} = \mathcal{P}(N)$ and $u_S$ is the identity function for all $S \in \mathcal{A}^*$), the prenucleolus always consists of exactly one point (payoff vector) (Schmeidler 1967). However, this does not hold in case of TU-games with restricted cooperation, where the prenucleolus is a set – not necessarily a singleton set – of payoff vectors. Katsev and Yanovskaya (2013) showed that the prenucleolus of a game is nonempty if and only if the set of feasible coalitions is a balanced set of coalitions. In this section we are going to prove that this statement holds for the **u** -prenucleolus of a game as well. The proof relies on the generalization of Kohlberg's theorem (Theorem 4) and on Lemma 2.

Kohlberg's theorem (Kohlberg (1971)) for classical TU-games is as follows:

**Theorem 3** (Kohlberg's theorem) *Given a game $v \in \mathcal{G}^N$ and a payoff vector $x \in I^*(v)$, $x$ is the prenucleous if and only if for every $\alpha \in \mathbb{R}$ it holds that either $\mathcal{D}(\alpha, x^*) := \{S \in \mathcal{P}^*(N) : e(S, x) \geq \alpha\} = \emptyset$, or $\mathcal{D}(\alpha, x^*)$ is a balanced set of coalitions.*

Maschler et al. (1992) proposed a generalisation of Kohlberg 's theorem for the general prencucleolus. The **u**-prenucleolus is a special case of the general prenucleolus, therefore the following generalization of Kohlberg's theorem is a corollary of the result by Maschler et al. (1992):

**Theorem 4** (Generalization of Kohlberg's theorem) *Given a game $v \in \mathcal{G}^{N,\mathcal{A}}$, a utility function $\mathbf{u}$, and $x \in I^*(v)$: $x$ is an element of the $\mathbf{u}$-prenucleolus ($x \in N^*_{\mathbf{u}}(v)$) if and only if $\mathcal{D}_{\mathbf{u}}(\alpha, x)$ is a balanced set of coalitions for every $\alpha$ such that $\mathcal{D}_{\mathbf{u}}(\alpha, x) \neq \emptyset$, where $\mathcal{D}_{\mathbf{u}}(\alpha, x^*) := \{S \in \mathcal{A}^* : u_S \circ e(S, x^*) \geq \alpha\}$.*

The following proposition, which generalizes Theorem 2. on page 58 of Katsev and Yanovskaya (2013), is a corollary of Theorem 4.

**Proposition 1** *Let $v \in \mathcal{G}^{N,\mathcal{A}}$ be a game and $\mathbf{u}$ be a utility function. If the $\mathbf{u}$-prenucleolus of the game is nonempty, then $\mathcal{A}^*$ is a balanced set of coalitions.*

***Proof*** Since the **u**-prenucleolus is nonempty, there exists $x \in N^*_{\mathbf{u}}(v)$. Then by Theorem 4 it holds that $\mathcal{D}_{\mathbf{u}}(\alpha, x)$ is a balanced set of coalitions for every $\alpha$ such that $\mathcal{D}_{\mathbf{u}}(\alpha, x) \neq \emptyset$.

Take $\alpha^*$ such that it is the smallest element of $E_{\mathbf{u}}(x)$, that is, let $\alpha^* := E_{\mathbf{u}}(x)_{|\mathcal{A}^*|}$. Then, $\mathcal{D}_{\mathbf{u}}(\alpha^*, x) = \mathcal{A}^*$, hence, by Theorem 4, $\mathcal{A}^*$ is a balanced set of coalitions. □

**Proposition 2** *Let $v \in \mathcal{G}^{N,\mathcal{A}}$ be a game and $\mathbf{u}$ be a utility function. If $\mathcal{A}^*$ is a balanced set of coalitions, then the $\mathbf{u}$-prenucleolus of the game is nonempty.*

***Proof*** The **u**-prenucleolus of the game is nonempty, if the considered optimization problem attains an optimal solution in every iteration of the generalized lexicographic center approach discussed in Sect. 5. We are going to show that if $\mathcal{A}^*$ is a balanced set of coalitions, then in every iteration of the generalized lexicographic center approach the considered optimization problem attains an optimal solution.

Take an arbitrary step in the generalized lexicographic center approach (see section 5). Then, we have to solve the following minimization problem:

$$\begin{array}{ll} & t \to \min \\ s.t. & u_S \circ e(S,x) \leq t, \quad S \in \mathcal{A}^* \setminus W \\ & u_S \circ e(S,x) = c_S, \quad S \in W \\ & x \in I^*(v) \\ & t \in R_{\mathbf{u}}, \end{array} \tag{9}$$

where $W = \cup_{i=1}^{k} W_i^u$ for some $k$.

By Lemma 2, problem (9) has an optimal solution if and only if the following LP has:

$$\begin{array}{ll} & t \to \min \\ s.t. & v(S) - x(S) \leq t, \quad S \in \mathcal{A}^* \setminus W \\ & v(S) - x(S) = u_S^{-1}(c_S), \quad S \in W \\ & x(N) = v(N) \\ & t \in R_{\mathbf{u}} \end{array} \tag{10}$$

Since the LP (10) has a feasible solution (every solution of the optimization problem in the previous iteration is a feasible solution here, or, if we are at the first iteration, that is $W = \emptyset$, then it is easy to see that LP (10) has a feasible solution), it is enough to prove that the objective function in (10) is bounded from below.

Let $\{\lambda_S\}_{S \in \mathcal{A}^*}$ be a balancing weight system, then

$$\sum_{S \in \mathcal{A}^* \setminus W} \lambda_S (v(S) - t) + \sum_{S \in W} \lambda_S (v(S) - u_S^{-1}(c_S)) \leq \sum_{S \in \mathcal{A}^*} \lambda_S \chi_S^\top = v(N)$$
$$\sum_{S \in \mathcal{A}^* \setminus W} \lambda_S v(S) + \sum_{S \in W} \lambda_S (v(S) - u_S^{-1}(c_S)) - v(N) \leq \sum_{S \in \mathcal{A}^* \setminus W} \lambda_S t,$$

where $\chi_S$ denotes the characteristic vector of a set $S \subseteq N$.

Since $0 < \lambda_S$, $\forall S \in \mathcal{A}^*$ it holds that $\sum_{S \in \mathcal{A}^* \setminus W} \lambda_S > 0$, hence

$$\frac{\sum_{S \in \mathcal{A}^* \setminus W} \lambda_S v(S) + \sum_{S \in W} \lambda_S (v(S) - u_S^{-1}(c_S)) - v(N)}{\sum_{S \in \mathcal{A}^* \setminus W} \lambda_S} \leq t$$

The left-hand side of the inequality is a constant, hence it gives a lower bound for the right-hand side, which is the objective function of problem (10). Therefore, it is bounded from below meaning that problem (10) has an optimal solution. □

The following theorem, which generalizes Theorem 2. on page 58 of Katsev and Yanovskaya (2013) comes from Propositions 1 and 2:

**Theorem 5** *Let $v \in \mathcal{G}^{N,\mathcal{A}}$ be a game and* $\mathbf{u}$ *be a utility function. Then $\mathcal{A}^*$ is a balanced set of coalitions, if and only if the* $\mathbf{u}$*-prenucleolus of the game is nonempty.*

## 7 The cardinality of the u-prenucleolus

In this section, we consider the size of the $\mathbf{u}$-prenucleolus. First, we introduce a notation. For a family of coalitions $\mathcal{A} \subseteq \mathcal{P}(N)$ let $X(\mathcal{A})$ denote the $|\mathcal{A}| \times |N|$ dimensional matrix, where its row vectors are the characteristic vectors of the sets from $\mathcal{A}$.

The following theorem is a generalization of Theorem 3. on page 59 of Katsev and Yanovskaya (2013).

**Theorem 6** *Given a game $v \in \mathcal{G}^{N,\mathcal{A}}$, where $\mathcal{A}$ is a balanced set of coalitions, and a utility function* $\mathbf{u}$, *the* $\mathbf{u}$*-prenucleolus of the game $v$ is a singleton if and only if $rank(X(\mathcal{A})) = |N|$.*

***Proof*** Only if: Let $x \in N^*_{\mathbf{u}}(v)$ be the only element of the $\mathbf{u}$-prenucleolus. Suppose for contradiction that $\text{rank}(X(\mathcal{A})) < |N|$. Consider the following system of linear equations:

$$\begin{cases} y(S) = x(S), \ \forall S \in \mathcal{A}^* \\ y(N) = v(N) \end{cases} \tag{11}$$

Then (11) can be rewritten as follows:

$$X(\mathcal{A})y = e(x),$$

where $e(x) = (0, \ldots, x(S), \ldots, v(N))^{\top}, S \in \mathcal{A}$.

Since $\text{rank}(X(\mathcal{A})) < |N|$, the system (11) has an infinite many solutions and all of those belong to the $\mathbf{u}$-prenucleolus, which is a contradiction.

If: Suppose for contradiction that $\text{rank}(X(\mathcal{A})) = |N|$ and there exist $x, y \in N^*_{\mathbf{u}}(v)$ such that $x \neq y$. This means that $E_{\mathbf{u}}(x) = E_{\mathbf{u}}(y)$.

Notice that for all $S \in \mathcal{A}^*$, such that $x(S) = y(S)$ we have the following:

$$u_S \circ \left( v(S) - \frac{x+y}{2}(S) \right) = u_S \circ (v(S) - x(S)) = u_S \circ (v(S) - y(S)).$$

If there exists a coalition $S \in \mathcal{A}^*$ such that $x(S) \neq y(S)$ (without loss of generality we can suppose that $x(S) > y(S)$), then

$$u_S \circ (v(S) - x(S)) < u_S \circ \left( v(S) - \frac{x+y}{2}(S) \right) < u_S \circ (v(S) - y(S)).$$

Then let $T_i$ be the first coalition according to the order by vector $E^{\mathbf{u}}_v(\frac{x+y}{2})$ for which $x(T_i) \neq y(T_i)$. Then either $u_{T_i} \circ (v(T_i) - \frac{x+y}{2}(T_i)) < u_{T_i} \circ (v(T_i) - y(T_i))$ or $u_{T_i} \circ (v(T_i) - \frac{x+y}{2}(T_i)) < u_{T_i} \circ (v(T_i) - x(T_i))$.

Since the **u**-excesses are in non-increasing order in $E_v^{\mathbf{u}}$ we have that $E_v^{\mathbf{u}}(\frac{x+y}{2}) <_L E_v^{\mathbf{u}}(x) = E_v^{\mathbf{u}}(y)$. Therefore, $x(S) = y(S)$ for all $S \in \mathcal{A}$, and $y$ is a solution of the following system of linear equations:

$$\begin{cases} z(S) = x(S), \forall S \in \mathcal{A}^* \\ z(N) = v(N) \end{cases} \tag{12}$$

which can be rewritten as

$$X(\mathcal{A})z = e(x).$$

Since $rank(X(\mathcal{A})) = |N|$, this system has a unique solution $y = x$, which is a contradiction. □

## 8 The u-essential coalitions

Huberman (1980) showed that the so-called essential coalitions give a characterization set for the nucleolus of balanced TU-games. Since in case of balanced games, the nucleolus and the prenucleolus coincide, the essential coalitions also give a characterization set for the prenucleolus. First, consider the definition of essential coalitions used by Huberman (1980).

**Definition 5** Let $v \in \mathcal{G}^N$ be a game. Then, a coalition $S \in \mathcal{P}^*(N)$ is *essential*, if either $|S| = 1$, or

$$v(S) > \max_{\mathcal{B} \in \mathcal{D}_S} \sum_{T \in \mathcal{B}} v(T).$$

Let $\mathcal{E}_v$ denote the class of essential coalitions of the $v$ game.

Here is Huberman (1980)'s theorem:

**Theorem 7** (Huberman (1980)) *Let $v \in \mathcal{G}^N$ be a balanced game. Then $\mathcal{E}_v$ is a characterization set for the nucleolus, that is, the values $(v(S))_{S \in \mathcal{E}_v}$ determine the nucleolus of the game $v$.*

When generalizing Huberman's theorem, we need to "redefine" the essential coalitions.

**Definition 6** Given a game $v \in \mathcal{G}^{N,\mathcal{A}}$ and utility function **u**, a coalition $S \in \mathcal{A}^*$ is **u** *-essential*, if either $\mathcal{D}_S^{\mathcal{A}^*} = \emptyset$ or if $\exists x \in$ **u**-core($v$) such that

$$u_S \circ e(S, x) > \max_{\mathcal{B} \in \mathcal{D}_S^{\mathcal{A}^*}} \sum_{T \in \mathcal{B}} u_T \circ e(T, x).$$

Let $\mathcal{E}_v^{\mathbf{u}}$ denote the class of **u**-essential coalitions of the game $v$.

Note, that Definition 6 uses $x$, while Definition 5 seemingly does not. However, if $\mathbf{u}$ is the identity function, $x$ is cancelled out from the inequality. Moreover, the idea of the proof of Huberman's theorem is that the excesses of the essential coalitions exceed the excesses of the other coalitions.

Notice, that if $\mathbf{u}$ is the identity function and $\mathcal{A} = \mathcal{P}(N)$; then, the $\mathbf{u}$-essential coalitions are the essential coalitions, hence the $\mathbf{u}$-essential coalitions are generalizations of the essential coalitions.

In addition, one could consider using '$\forall x \in \mathbf{u}$-core$(v)$' instead of '$\exists x \in \mathbf{u}$-core$(v)$' in Definition 6, as both lead to reasonable generalizations of essential coalitions. By using the alternative definition ('$\forall x \in \mathbf{u}$-core$(v)$'), one would obtain a subset of our class of $\mathbf{u}$-essential coalitions. However, our proofs would not hold in that case, as Lemma 3 would only establish that there exists $x \in \mathbf{u}$-core$(v)$ for which the lemma's statements apply. Consequently, in the proof of Theorem 8, when applying Lemma 3, we would not be able to demonstrate that the lemma's statements hold for $y^*$.

We should also note, that the inclusion of $x$ in Definition 6 raises the question which $x$ must be considered. A straightforward option would be that every element of the preimputations, as it is stated in the definition of the $\mathbf{u}$-prenucleolus, have to be considered. However, it turns out that not all preimputations must be considered, in Definition 6 we choose the elements of the $\mathbf{u}$-core. What is more, even a smaller set, the $\mathbf{u}$-least-core could be applied in the definition of $\mathbf{u}$-essential coalitions.

***Example 2*** Consider the following game, which is a modification of the game examined in Example 3 in Solymosi (2019): $v(N) = 12, v(\{1,2\}) = v(\{3,4\}) = v(\{2,3,4\}) = 6, v(\{1,4\}) = 4, v(\{4\}) = 3, v(\{1,2,3\}) = 9$ and for every other coalition $S \in \mathcal{P}(N)$ let $v(S) = 0$.

Let the utility function be the percapita-utility function, that is $u_S(t) = \frac{t}{|S|}$ for all $S \in \mathcal{P}^*(N)$.

The percapita-core coincides with the core, so $\mathbf{u}-$core$(v) = \{(t, 6-t, 3, 3) : 1 \leq t \leq 6\}$.

The percapita-prenucleolus of $v$ is (3, 3, 3, 3). The first iteration in the lexicographic center algorithm (see (7)) gives $t_1 = 0$ and

$$W_1 = \{\{3\},\{4\},\{1,2\},\{3,4\},\{1,2,3\},\{1,2,4\}\}.$$

The second iteration (see (8)) gives $t_2 = -1$ and

$$W_2 = \{\{1\},\{2\},\{1,3\},\{1,4\},\{2,3\},\{2,4\},\{1,3,4\},\{2,3,4\}\}.$$

Therefore, the $\mathbf{u}$-prenucleolus is (3, 3, 3, 3).

However, if we consider only the essential coalitions in the calculation of the percapita-prenucleolus of the game, we get $(4+\frac{1}{3}, 1+\frac{2}{3}, 3, 3)$. The essential coalitions of $v$ are:

$$\mathcal{E}_v = \{\{1\},\{2\},\{3\},\{4\},\{1,2\},\{3,4\},\{1,4\},\{1,2,3\}\}.$$

The first iteration gives $t_1 = 0$ and

$$W_1 = \{\{3\}, \{4\}, \{1,2\}, \{3,4\}, \{1,2,3\}\}.$$

The second iteration gives: $t_2 = -(1 + \frac{2}{3})$ and

$$W_2 = \{\{1\}, \{2\}, \{1,4\}\}.$$

Therefore, the **u**-prenucleolus of the game with restricted coalitions $\mathcal{E}_v$ is $(4 + \frac{1}{3}, 1 + \frac{2}{3}, 3, 3)$.

However, if we consider the **u**-essential coalitions when calculating the percapita-prenucleolus, we get (3, 3, 3, 3), which coincides with the percapita-prenucleolus. Indeed, the **u**-essential coalitions are

$$\begin{aligned}\mathcal{E}_v^{\mathbf{u}}(v) = &\{\{1\}, \{2\}, \{3\}, \{4\}, \{1,2\}, \{1,3\}, \{1,4\}, \{2,3\}, \{2,4\}, \{3,4\},\\ &\{1,2,3\}, \{1,3,4\}, \{2,3,4\}\}.\end{aligned}$$

Notice, that not all coalitions are **u**-essential. Coalition $\{1,2,4\}$ is not **u** -essential, since for all $x \in \mathbf{u}\text{-core}(v)$ $u_{\{1,2,4\}} \circ e_v(\{1,2,4\}, x) = \frac{0-9}{3} = -3$ and $u_{\{1,2\}} \circ e_v(\{1,2\}, x) + u_{\{4\}} \circ e_v(\{4\}, x) = \frac{6-6}{2} + 3 - 3 = 0$, hence, $u_{\{1,2,4\}} \circ e_v(\{1,2,4\}, x) < u_{\{1,2\}} \circ e_v(\{1,2\}, x) + u_{\{4\}} \circ e_v(\{4\}, x)$.

Considering only the **u**-essential coalitions the first iteration gives $t_1 = 0$ and

$$W_1 = \{\{3\}, \{4\}, \{1,2\}, \{3,4\}, \{1,2,3\}\}.$$

The second iteration gives $t_2 = -1$ and

$$W_2 = \{\{1\}, \{2\}, \{1,3\}, \{1,4\}, \{2,3\}, \{2,4\}, \{1,3,4\}, \{2,3,4\}\}.$$

Therefore, the per-capita prenucleolus of the game with restricted coalitions $\mathcal{E}_v^{\mathbf{u}}$ is (3, 3, 3, 3), which coincides with the percapita-prenucleolus of the original game.

In the following, we show that the **u**-essential coalitions form a characterization set for the **u**-prenucleolus in case of **u**-balanced games. First, consider the following two optimization problems:

$$\begin{aligned} & t \to \min \\ s.t. \quad & u_S \circ e(S,x) \leq t, \quad S \in \mathcal{A}^* \\ & x \in I^*(v) \\ & t \in R_{\mathbf{u}} \end{aligned} \tag{13}$$

and

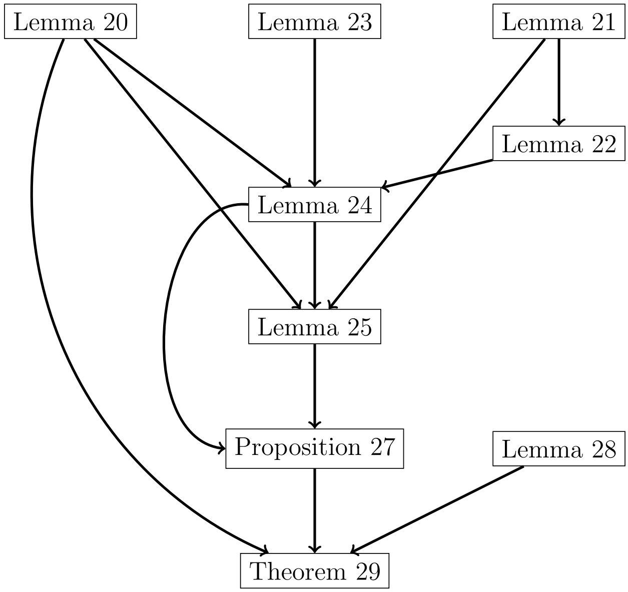
$$\begin{aligned} & t \to \min \\ s.t. \quad & u_S \circ e(S,x) \leq t, \quad S \in \mathcal{E}_v^{\mathbf{u}} \\ & x \in I^*(v) \\ & t \in R_{\mathbf{u}} \end{aligned} \tag{14}$$

Let $t_1$ be the optimum of problem (13) and $X_1$ be the set of optimal solutions of problem (13) except from t, that is, $X_1 = \{x \in I^*(v) : u_S \circ e_v(S,x) \leq t_1 \ \forall S \in \mathcal{A}^*\}$.

Similarly, let $t_1'$ be the optimum of problem (14) and $X_1$ be the set of optimal solutions of problem (14) except from $t$, that is, $X_1' = \{x \in I^*(v) : u_S \circ e_v(S,x) \leq t_1 \ \forall S \in \mathcal{E}_v^{\mathbf{u}}\}$.

Next, we consider some lemmata which are needed for showing that $X_1 = X_1'$ (Proposition 3). Then, by these results we show that the **u**-essential coalitions characterize the **u**-prenucleolus of **u**-balanced games (Theorem 8).

In order to help the reader in following the interdependence of the upcoming results, we have constructed the following graph:




**Lemma 3** *Let $S \in \mathcal{A}^* \setminus \mathcal{E}_v^{\mathbf{u}}$. Then for every $x \in \mathbf{u}$-core$(v)$ there exists $\mathcal{B}^* \in \mathcal{D}_S^{\mathcal{A}^*}$ such that $u_S \circ e(S,x) \leq \sum_{T \in \mathcal{B}^*} u_T \circ e(T,x)$ and $\mathcal{B}^* \subseteq \mathcal{E}_v^{\mathbf{u}}$.*

***Proof*** Since $S$ is not **u**-essential, $\forall x \in \mathbf{u}$-core$(v)$ there exists $\mathcal{B} \in \mathcal{D}_S^{\mathcal{A}^*}$ such that $u_S \circ e(S,x) \leq \sum_{T \in \mathcal{B}} u_T \circ e(T,x)$ by definition. For all $x \in \mathbf{u}$-core$(v)$ let $\mathbf{B}(x) := \{\mathcal{B} \in \mathcal{D}_S^{\mathcal{A}^*} : u_S \circ e(S,x) \leq \sum_{T \in \mathcal{B}} u_S \circ e(S,x)\}$ and let $\mathcal{B}^* \in \mathbf{B}(x)$ be such that for every partition $\mathcal{B} \in \mathbf{B}(x)$ it holds that $|\mathcal{B}^*| \geq |\mathcal{B}|$.

Assuming, for the purpose of deriving a contradiction, that a coalition $T^* \in \mathcal{B}^*$ is not **u**-essential. Since $T^*$ is not **u**-essential by Definition 6 $\exists \mathcal{B}' \in \mathcal{D}_{T^*}^{\mathcal{A}^*}$ such that $u_{T^*} \circ e(T^*,x) \leq \sum_{T' \in \mathcal{B}'} u_{T'} \circ e(T',x)$, therefore,

$$u_S \circ e(S,x) \leq \sum_{T' \in (\mathcal{B}^* \setminus \{T^*\}) \cup \mathcal{B}'} u_{T'} \circ e(T',x),$$

and $|(\mathcal{B}^* \setminus \{T\}) \cup \mathcal{B}'| > |\mathcal{B}^*|$, which is a contradiction. $\square$

Next, we introduce the following notion: for a class of coalitions $\mathcal{S} \subseteq \mathcal{A}^*$ and $t \in \mathbb{R}$ let $X(\mathcal{S}, t) := \{x \in I^*(v) : u_S \circ e_v(S, x) \leq t, \forall S \in \mathcal{S}\}$.

**Lemma 4** *Given a game $v \in \mathcal{G}^{N,\mathcal{A}}$ and $t', t'' \in \mathbb{R}$ such that $t' \leq t''$, then $X(\mathcal{S}, t') \subseteq X(\mathcal{S}, t'')$.*

***Proof*** For any $x \in X(\mathcal{S}, t')$ it holds that $x \in I^*(v)$, and for all $S \in \mathcal{S}$:

$$u_S \circ e_v(S, x) \leq t' \leq t''$$

Therefore, $x \in X(\mathcal{S}, t'')$. □

**Lemma 5** *Given a utility function* **u** *and a* **u**-*balanced game $v \in \mathcal{G}^{N,\mathcal{A}}$, for every $t_1 \leq t$ it holds that both $X(\mathcal{A}^*, t)$ and $X(\mathcal{E}_v^{\mathbf{u}}, t)$ are nonempty, convex and closed.*

***Proof*** By Lemma $X(\mathcal{A}^*, t) \supseteq X(\mathcal{A}^*, t_1) =$ **u**-least-core$(v)$. We know that **u**-least-core$(v) \neq \emptyset$, and $X(\mathcal{A}^*, t) \subseteq X(\mathcal{E}_v^{\mathbf{u}}, t)$, because $\mathcal{E}_v^{\mathbf{u}} \subseteq \mathcal{A}^*$. Therefore, $X(\mathcal{E}_v^{\mathbf{u}}, t) \neq \emptyset$ as well.

Let $H_S := \{x \in \mathbb{R}^N : u_S \circ e_v(S, x) \leq t\}$ for all $S \in \mathcal{A}^*$. Then $H_S = \{x \in \mathbb{R}^N : e_v(S, x) \leq u_S^{-1}(t)\}$, hence $H_S$ is a closed half-space, therefore, it is convex and closed. $X_0 = I^*(v) = \{x \in \mathbb{R}^N : x(N) = v(N)\}$ is a hyperplane, therefore it is convex and closed. Finally,

$$X(\mathcal{A}^*, t) = \{x \in I^*(v) : u_S \circ e_v(S, x) \leq t \ \forall S \in \mathcal{A}^*\} = \cap_{S \in \mathcal{A}^*} H_S \cap X_0$$

hence, $X(\mathcal{A}^*, t)$ is an intersection of finitely many convex, closed sets; therefore, it is convex and closed.

Similarly, $X(\mathcal{E}_v^{\mathbf{u}}, t) = \cap_{S \in \mathcal{E}_v^{\mathbf{u}}} H_S \cap X_0$ is an intersection of finitely many convex, closed sets, hence, it is convex and closed. □

**Lemma 6** *Given a utility function* **u** *and a game $v \in \mathcal{G}^{N,\mathcal{A}}$, take $x_1, x_2 \in \mathbb{R}^N$ and $S \in \mathcal{A}$. If $u_S \circ e_v(S, x_1) < u_S \circ e_v(S, x_2)$, then for every $\lambda \in (0, 1)$ it holds that*

$$u_S \circ e_v(S, x_1) < u_S \circ e_v(S, \lambda x_1 + (1 - \lambda)x_2) < u_S \circ e_v(S, x_2).$$

***Proof*** Let $\lambda \in (0, 1)$, then

$$(\lambda x_1 + (1 - \lambda)x_2)(S) = \lambda x_1(S) + (1 - \lambda)x_2(S).$$

Therefore,

$$\begin{aligned} e_v(S, \lambda x_1 + (1 - \lambda)x_2) &= v(S) - (\lambda x_1 + (1 - \lambda)x_2)(S) \\ &= \lambda v(S) + (1 - \lambda)v(S) - (\lambda x_1(S) + (1 - \lambda)x_2(S)) \\ &= \lambda e_v(S, x_1) + (1 - \lambda)e_v(S, x_2). \end{aligned}$$

Since $u_S$ is strictly monotone increasing we have that

$$u_S \circ e_v(S, x_1) < u_S \circ e_v(S, \lambda x_1 + (1 - \lambda)x_2) < u_S \circ e_v(S, x_2).$$

□

**Lemma 7** *Given a utility function* $\mathbf{u}$ *and a game* $v \in \mathcal{G}^{N,\mathcal{A}}$, *if* $v$ *is* $\mathbf{u}$-*balanced, then* $X(\mathcal{A}^*, t_1) = X(\mathcal{E}_v^{\mathbf{u}}, t_1)$.

***Proof*** Since $\mathcal{E}_v^{\mathbf{u}} \subseteq \mathcal{A}^*$, it holds that $X(\mathcal{A}^*, t_1) \subseteq X(\mathcal{E}_v^{\mathbf{u}}, t_1)$,.

Indirectly assume that $\exists x^1 \in X(\mathcal{E}_v^{\mathbf{u}}, t_1) \setminus X(\mathcal{A}^*, t_1)$. This means that $\exists S \in \mathcal{A}^*$ such that

$$u_S \circ e_v(S, x^1) > t_1. \tag{15}$$

Let $\mathcal{S}_{x^1} = \{S \in \mathcal{A}^* : u_S \circ e_v(S, x^1) > t_1\}$. Then, for all $S \in \mathcal{S}_{x^1}$ it holds that $S \notin \mathcal{E}_v^{\mathbf{u}}$.

Let $x^* \in X(\mathcal{A}^*, t_1)$ be the closest point of set $X(\mathcal{A}^*, t_1)$ to point $x^1$. It is clear that such $x^*$ exists, since $X(\mathcal{A}^*, t_1)$ is nonempty and closed according to Lemma 5.

Since $X(\mathcal{E}_v^{\mathbf{u}}, t_1)$ is a convex set, for every $\lambda \in [0, 1]$ it holds that $\lambda x^* + (1 - \lambda)x^1 \in X(\mathcal{E}_v^{\mathbf{u}}, t_1)$.

By Lemma 3 we have that for each $S \in \mathcal{S}_{x^1}$ $\exists \mathcal{B}_S^* \subseteq \mathcal{E}_v^{\mathbf{u}}$, $\mathcal{B}_S^* \in \mathcal{D}_{S^1}^{\mathcal{A}^*}$ such that

$$u_S \circ e_v(S, x^*) \leq \sum_{T \in \mathcal{B}_S^*} u_T \circ e_v(T, x^*). \tag{16}$$

Since $t_1 \leq 0$ and for all $x \in X(\mathcal{E}_v^{\mathbf{u}}, t_1)$ and $T \in \mathcal{E}_v^{\mathbf{u}}$ it holds that $u_T \circ e_v(T, x) \leq t_1$, therefore for all $S \in \mathcal{S}_{x^1}$

$$u_S \circ e_v(S, x^*) \leq \sum_{T \in \mathcal{B}_S^*} u_T \circ e_v(T, x^*) \leq t_1 \tag{17}$$

By (15), (17) and the continuity of $u_S$, for every $S \in \mathcal{S}_{x^1}$ there exists $\lambda_S \in [0, 1]$ such that

$$u_S \circ e_v(S, \lambda_S x^* + (1 - \lambda_S)x^1) = t_1$$

Let $S^1 \in \operatorname{argmin}_{S \in \mathcal{S}_{x^1}} ||x^* - (\lambda_S x^* + (1 - \lambda_S)x^1))||$, and let $x^2 = \lambda_{S^1} x^* + (1 - \lambda_{S^1})x^1$.

Then,

$$u_{S^1} \circ e_v(S^1, x^2) \geq \sum_{T \in \mathcal{B}_{S^1}^*} u_T \circ e_v(T, x^2),$$

because $t_1 \leq 0$ and $u_T \circ e_v(T, x^2) \leq t_1$ for all $T \in \mathcal{B}_{S^1}^*$.

Then, there are two cases:

Case 1:

$$u_{S^1} \circ e_v(S^1, x^2) = \sum_{T \in \mathcal{B}_{S^1}^*} u_T \circ e_v(T, x^2).$$

In this case, $t_1 = u_{S^1} \circ e_v(S^1, x^2) = \sum_{T \in \mathcal{B}_{S^1}^*} u_T \circ e_v(T, x^2) \leq |\mathcal{B}_{S^1}^*| t_1 \leq t_1$; therefore, $t_1 = 0$.

Then, $\sum_{T\in\mathcal{B}^*_{S^1}} u_T \circ e_v(T,x^2) = 0$, hence, for all $T \in \mathcal{B}^*_{S^1}$ it holds that $u_T \circ e_v(T,x^2) = 0$.

We know that $u_{S^1} \circ e_v(S^1,x^1) > t_1 \geq \sum_{T\in\mathcal{B}^*_{S^1}} u_T \circ e_v(T,x^1)$ and that $x^2(S^1) = \sum_{T\in\mathcal{B}^*_{S^1}} x^2(T)$ and $x^1(S^1) = \sum_{T\in\mathcal{B}^*_{S^1}} x^1(T)$. Then, $x^1(S^1) < x^2(S^1)$, but then, $\exists T' \in \mathcal{B}^*_{S^1}$ such that $x^1(T') < x^2(T')$. However, $u_{T'}$ is a strictly monotone increasing function; hence, we have that $u_{T'} \circ e_v(T',x^1) > t_1$, which is a contradiction, because $x^1 \in X(\mathcal{E}^{\mathbf{u}}_v, t_1)$.

Case 2:

$$u_{S^1} \circ e_v(S^1,x^2) > \sum_{T\in\mathcal{B}^*_{S^1}} u_T \circ e_v(T,x^2) \tag{18}$$

By the choice of $x^2$ and Lemma 6, for all $S \in \mathcal{S}_{x^1}$

$$u_S \circ e_v(S,x^2) \leq t_1$$

Since for all $S \in \mathcal{A}^* \setminus \mathcal{S}_{x_1}$ it holds that $u_S \circ e_v(S,x^1) \leq t_1$, by Lemma 6

$$u_S \circ e_v(S,x^2) \leq t_1$$

This means that for all $S \in \mathcal{A}^*$ it holds that $u_S \circ e_v(S,x^2) \leq t_1$; hence, $x^2 \in X(\mathcal{A}^*,t_1)$.

Since $x^* \in X(\mathcal{A}^*,t_1)$ is the closest point of set $X(\mathcal{A}^*,t_1)$ to point $x^1$, we have that $x^2 = x^*$. However, then, (18) contradicts (16). □

**Lemma 8** *Given a utility function* $\mathbf{u}$, *a* $\mathbf{u}$*-balanced game* $v \in \mathcal{G}^{N,\mathcal{A}}$ *and* $t \in \mathbb{R}$, *if* $t < t_1$, *then* $X(\mathcal{A}^*,t) = X(\mathcal{E}^{\mathbf{u}}_v,t) = \emptyset$.

***Proof*** By the definition of $t_1$, we have that $X(\mathcal{A}^*,t) = \emptyset$ and we know that $X(\mathcal{A}^*,t) \subseteq X(\mathcal{E}^{\mathbf{u}}_v,t)$.

By Lemma , since $t < t_1$, we have that $X(\mathcal{E}^{\mathbf{u}}_v,t) \subseteq X(\mathcal{E}^{\mathbf{u}}_v,t_1)$. Furthermore, by Lemma 7 it holds that $X(\mathcal{E}^{\mathbf{u}}_v,t_1) = X(\mathcal{A}^*,t_1)$. Since $t_1 \leq 0$, it holds that $X(\mathcal{A}^*,t_1) \subseteq X(\mathcal{A}^*,0) = \mathbf{u}\text{-core}(v)$. Therefore,

$$X(\mathcal{E}^{\mathbf{u}}_v,t) \subseteq X(\mathcal{E}^{\mathbf{u}}_v,t_1) = X(\mathcal{A}^*,t_1) \subseteq X(\mathcal{A}^*,0) = \mathbf{u}\text{-core}(v)$$

Then, for every $x \in X(\mathcal{E}^{\mathbf{u}}_v,t)$ and for every $S \in \mathcal{A}^* \setminus \mathcal{E}^{\mathbf{u}}_v$ by Lemma 3 and by the non-positivity of $t$ it holds that $\exists \mathcal{B} \in \mathcal{D}^{\mathcal{A}^*}_S$, $\mathcal{B} \subseteq \mathcal{E}^{\mathbf{u}}_v$ such that

$$u_S \circ e_v(S,x) \leq \sum_{T\in\mathcal{B}} u_T \circ e_v(T,x) \leq t$$

Then, $x \in X(\mathcal{A}^*,t)$, that is, $X(\mathcal{E}^{\mathbf{u}}_v,t) \subseteq X(\mathcal{A}^*,t)$. Summing up, we can can conclude that $X(\mathcal{E}^{\mathbf{u}}_v,t) = X(\mathcal{A}^*,t) = \emptyset$. □

***Remark 1*** Notice, that all the results we have discussed so far hold if one defines $\mathbf{u}$-essentiality by the elements of the $\mathbf{u}$-least core instead of the elements of the $\mathbf{u}$-core.

The following proposition is a consequence of Lemmata 7 and 8.

**Proposition 3** *The following holds*: $t_1 = t_1'$ *and* $X_1 = X_1'$.

***Proof*** By definition $t_1' = \min\{t : X(\mathcal{E}_v^{\mathbf{u}}, t) \neq \emptyset\}$. We know, that $t_1' \leq t_1$; therefore, by Lemmata 7 and 8 we have that $X_1' = X(\mathcal{E}_v^{\mathbf{u}}, t_1') = X(\mathcal{A}^*, t_1')$. However, $X(\mathcal{A}^*, t_1') \neq \emptyset$ if and only if $t_1' \geq t_1$, hence $t_1 = t_1'$ and $X_1 = X_1'$. □

**Lemma 9** *Let* $k$ *be a positive integer,* $S \in \mathcal{A}^* \setminus \cup_{r=1}^{k-1} W_r$ *be such that* $\mathcal{D}_S^{\mathcal{A}^*} \neq \emptyset$*, and* $\mathcal{B}^* \in \mathcal{D}_S^{\mathcal{A}^*}$. *Then, there exists a coalition* $T^* \in \mathcal{B}^*$ *such that* $T^* \notin \cup_{r=1}^{k-1} W_r$.

***Proof*** If $k = 1$, then $\cup_{r=1}^{k-1} W_r = \emptyset$, hence, $T^* \notin \cup_{r=1}^{k-1} W_r$.

If $k \geq 2$, then indirectly assume that $\mathcal{B}^* \subseteq \cup_{r=1}^{k-1} W_r$. Then for every $x \in X_{k-1}$

$$u_S \circ e(S, x) = u_S(v(S) - x(S)) = u_S\left( v(S) - \sum_{T \in \mathcal{B}^*} x(T) \right)$$

$$= u_S\left( v(S) - \sum_{T \in \mathcal{B}^*} (v(T) - u_T^{-1}(c_T)) \right)$$

Therefore, for each $x, x' \in X_{k-1}$ it holds that $u_S \circ e(S, x) = u_S \circ e(S, x')$, meaning that $S \in \cup_{r=0}^{k-1} W_r$, which is a contradiction. □

The following theorem generalizes Huberman (1980)'s theorem (Theorem 7 on page 420 of Huberman (1980)):

**Theorem 8** *Consider a* $\mathbf{u}$*-balanced game* $v \in \mathcal{G}^{N,\mathcal{A}}$, *and let*

$$Y_1 = \{x \in I^*(v) : u_S \circ e(S, x) \leq t_1, \ \forall S \in \mathcal{E}_v^{\mathbf{u}}\},$$

*and for all* $k \geq 2$ *let* $Y_k$ *be defined as follows*:

$$Y_k = \{x \in X_{k-1} : u_S \circ e(S, x) \leq t_k, \ \forall S \in \mathcal{E}_v^{\mathbf{u}} \setminus (\cup_{r=1}^{k-1} W_r)\}.$$

*Then,* $X_k = Y_k$ *for all* $k \geq 1$.

In other words, Theorem 8 claims that the $\mathbf{u}$-essential coalitions give a characterization set for the $\mathbf{u}$-prenucleolus of $\mathbf{u}$-balanced games.

***Proof*** First, notice that $X_k \subseteq Y_k$ holds for all $k$ by definition.

By Proposition 3, we have that $X_1 = Y_1$; therefore, $X_1, Y_1 \subseteq \mathbf{u}\text{-core}(v)$.

Suppose for contradiction that there exists $k > 1$ such that $X_k \not\supseteq Y_k$, which means that there exist $y^* \in Y_k$ and $S \in \mathcal{A}^* \setminus (\mathcal{E}_v^{\mathbf{u}} \cup (\cup_{r=1}^{k-1} W_r))$ such that $u_S \circ e(S, y^*) > t_k$.

By Lemma 3 for each $x \in X_1$ there exists $\mathcal{B}_x \in \mathcal{D}_S^{\mathcal{A}^*} \cap \mathcal{E}_v^{\mathbf{u}}$ such that $u_S \circ e(S, x) \leq \sum_{T \in \mathcal{B}_x} u_T \circ e(T, x)$.

Then

$$u_S \circ e(S, y^*) \leq \sum_{T \in \mathcal{B}_{y^*}} u_T \circ e(T, y^*)$$

Moreover, by definition $u_S \circ e(S, x) \leq 0$ for all $S \in \mathcal{A}^*, x \in X_{k-1}$. Therefore, for any coalition $T \in \mathcal{B}_{y^*}$ we have that $u_S \circ e(S, y^*) \leq u_T \circ e(T, y^*)$. By Lemma 9, there exists $T^* \in \mathcal{B}_{y^*}$ such that $T^* \notin \cup_{r=1}^{k-1} W_r$. Therefore, $u_S \circ e(S, y^*) \leq u_{T^*} \circ e(T^*, y^*) \leq t_k$, which is a contradiction. □

In words, Theorem 8 gives a characterization set for the **u**-prenucleolus in case of **u**-balanced games. As a direct corollary of this theorem, we can find a characterization set for the percapita prenucleolus in case of balanced games (notice, that the core and the percapita core coincide), or we can shift the values of the non-trivial coalitions uniformly so that the **u**-core (**u** here is the shift) of the game becomes nonempty, while the **u**-prenucleolus coincides with the prenucleolus. We discuss these applications in more detail in Sect. 9.

## 9 Two invariance results

There are **u** functions such that the **u**-prenucleolus of a game is the same as its prenucleolus, while the **u**-core of the game is different from its core. These utility functions can be useful in finding a characterization set for the prenucleolus of non-balanced games.

In this section, we characterize the classes of **u**-functions under which the **u**-prenucleolus and the **u**-core are the same as the prenucleolus and the core, respectively.

**Lemma 10** *Given a game $v \in \mathcal{G}^{N,\mathcal{A}}$, and utility functions $\mathbf{u}^1$ and $\mathbf{u}^2$, the $\mathbf{u}^1$-prenucleolus coincides with the $\mathbf{u}^2$-prenucleolus if for every $S, T \in \mathcal{A}^*$ and $x \in I^*(v)$*

$$u^1_S \circ e(S, x) \leq u^1_T \circ e(T, x) \tag{19}$$

*if and only if*

$$u^2_S \circ e(S, x) \leq u^2_T \circ e(T, x). \tag{20}$$

***Proof*** Notice, that w.l.o.g. we can assume that $\mathbf{u}^1 = \mathbf{id}$. Then let $\mathbf{u}$ denote $\mathbf{u}^2$.

Let $x, y \in I^*(v)$ be such that $E(x) \leq_L E(y)$, where

$$\begin{aligned} E(x) &:= [e(S_1, x), e(S_2, x), \dots, e(S_{|\mathcal{A}^*|}, x)] \\ E(y) &:= [e(T_1, y), e(T_2, y), \dots, e(T_{|\mathcal{A}^*|}, y)]. \end{aligned}$$

Case 1: $E(x) = E(y)$: By (19) and (20) $e(S_n, x) = e(T_n, y)$ for all $1 \leq n \leq |\mathcal{A}^*|$ is equivalent with $u_{S_n} \circ e(S_n, x) = u_{T_n} \circ e(T_n, y)$ for all $1 \leq n \leq |\mathcal{A}^*|$. Meaning $E_{\mathbf{u}}(x) = E_{\mathbf{u}}(y)$.

Case 2: $E(x) \neq E(y)$: Then, there exists $k$ such that

$$\begin{aligned} e(S_n, x) &= e(T_n, y) \ \forall n < k, \\ e(S_k, x) &< e(T_k, y). \end{aligned} \tag{21}$$

By (19) and (20) we have that (21) is equivalent with

$$\begin{aligned} u_{S_n} \circ e(S_n, x) &= u_{T_n} \circ e(T_n, y) \ \forall n < k, \\ u_{S_k} \circ e(S_k, x) &< u_{T_k} \circ e(T_k, y). \end{aligned} \tag{22}$$

This proves that for each $x, y \in I^*(v)$ $E(x) \leq_L E(y)$ if and only if $E_{\mathbf{u}}(x) \leq_L E_{\mathbf{u}}(y)$. Therefore, $x \in N^*(v)$ if and only if $x \in N^*_{\mathbf{u}}(v)$. □

For example, if $\mathbf{u}$ is such that $u_S = u_T$ for all $S, T \in \mathcal{A}^*$, then the $\mathbf{u}$-prenucleolus coincides with the prenucleolus.

***Example 3*** In this example, we show how to find a characterization set for the prenucleolus of non-balanced games using Lemma 10.

Let $v \in \mathcal{G}^{N,\mathcal{A}}$ be a game and let $\mathbf{u}$ be the utility function, such that $u_S \circ e(S, x) := e(S, x) - \varepsilon^*$ for all $S \in \mathcal{A}^*$, where $\varepsilon^*$ is the optimum of the following LP:

$$\begin{aligned} & \varepsilon \to \min \\ s.t. \quad & e(S, x) \leq \varepsilon, \quad S \in \mathcal{A}^* \\ & x \in I^*(v) \\ & \varepsilon \in \mathbb{R}. \end{aligned} \tag{23}$$

In other words, $\varepsilon^*$ is such that the least core is the $\varepsilon^*$-core.

Then, by Lemma 10 the $\mathbf{u}$-prenucleolus of the game coincides with the prenucleolus. In addition, the game is $\mathbf{u}$-balanced because the least core is never empty.

Therefore, by Theorem 8, in the case of this $\mathbf{u}$ function ($u_S(t) = t - \varepsilon^*$, $S \in \mathcal{A}^*$), the $\mathbf{u}$-essential coalitions form a characterization set for the prenucleolus of $v$ even if $v$ is not balanced.

In the following we consider the equivalence of the core and the $\mathbf{u}$-core.

**Lemma 11** *Given a game $v \in \mathcal{G}^{N,\mathcal{A}}$, and utility functions $\mathbf{u}^1$ and $\mathbf{u}^2$, the $\mathbf{u}^1$-core coincides with the $\mathbf{u}^2$-core if for every $S \in \mathcal{A}^*$ and $x \in I^*(v)$*

$$u^1_S \circ e(S, x) \leq 0 \tag{24}$$

*if and only if*

$$u^2_S \circ e(S, x) \leq 0. \tag{25}$$

***Proof*** $\mathbf{u}^1$-core$(v) = \{x \in I^*(v) : u^1_S \circ e(S, x) \leq 0 \ \forall S \in \mathcal{A}^*\}$. Due to the equivalence of (24) and (25) this equals $\{x \in I^*(v) : u^2_S \circ e(S, x) \leq 0 \ \forall S \in \mathcal{A}^*\} = \mathbf{u}^2$-core$(v)$. □

For example, if $\mathbf{u}$ is such that $u_S(0) = 0$ for all $S \in \mathcal{A}^*$, then the $\mathbf{u}$-core coincides with the core; therefore a game is balanced if and only if it is $\mathbf{u}$-balanced.

***Example 4*** In this example, we show how to find a characterization set for the percapita prenucleolus of balanced games.

Let $v \in \mathcal{G}^{N,\mathcal{A}}$ be a game and let $\mathbf{u}$ be the percapita function, that is, $u_S \circ e(S,x) = \frac{e(S,x)}{|S|}$ for all $S \in \mathcal{A}^*$.

By Lemma 11, the $\mathbf{u}$-core coincides with the core.

Then, a coalition $S \in \mathcal{A}^*$ is $\mathbf{u}$-essential (see Definition 6), if either $\mathcal{D}_S^{\mathcal{A}^*} = \emptyset$, (if $\mathcal{A} = \mathcal{P}(N)$, these coalitions are the singletons) or if there exists $x \in \text{core}(v)$ such that

$$\frac{e(S,x)}{|S|} > \max_{\mathcal{B} \in \mathcal{D}_S^{\mathcal{A}^*}} \sum_{T \in \mathcal{B}} \frac{e(T,x)}{|T|}.$$

By Theorem 8, the $\mathbf{u}$-essential (percapita-essential) coalitions form a characterization set for the percapita prenucleolus in case of balanced games.

## 10 An example

There are certain classes of games and utility functions for which there are only polynomial many $\mathbf{u}$-essential coalitions in the number of players. For example, consider the class of assignment games with the reciprocal percapita utility function $\mathbf{u}$. Our definition of the the reciprocal percapita utility function $\mathbf{u}$ is that for all $v \in \mathcal{G}^N, S \in \mathcal{P}^*(N)$ we have that $u_S \circ e(S,x) = |S| e(S,x)$. The reciprocal percapita utility function can be "interpreted" in a way that the value of a coalition is a public value for the members of the coalition. Therefore, each player's utility is the excess of the coalition; hence, the total utility of the excess of the coalition is the excess of the coalition multiplied by the size of the coalition.

In case of an assignment game, there are sellers ($M'$) and buyers ($M$). Each seller $j \in M'$ has a reservation value of $c_j \geq 0$ and each buyer $i \in M$ values the object of seller $j$ to $h_{i,j} \geq 0$. If a buyer and a seller trade, they make a joint profit of $a_{i,j} = \max\{0, h_{i,j} - c_j\}$. These joint profits can be displayed in an assignment matrix $A$:

$$A = \begin{bmatrix} a_{1,1} & a_{1,2} & \dots & a_{1,m'} \\ a_{2,1} & a_{2,2} & \dots & a_{2,m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m,1} & a_{m,2} & \dots & a_{m,m'} \end{bmatrix}$$

A matching $\mu'$ is a subset of $M \times M'$, where each agent appears in at most one pair. Let $\mathcal{M}(M, M')$ be the set of matchings.

An assignment game has the set of players $M \cup M'$ and a characteristic function $w_A$ defined as: for all $S \subseteq M, T \subseteq M'$

$$w_A(S \cup T) = \max\{ \sum_{(i,j)\in\mu} a_{i,j} \ : \ \mu \in \mathcal{M}(S,T)\}.$$

The core can be described using only the matchings and the singletons in the following way:

$$\text{core}(w_A) = \Big\{(x,y) \in \mathbb{R}^M \times \mathbb{R}^{M'} \ : \ \sum_{i\in M} x_i + \sum_{j\in M'} y_j = w_A(M \cup M'),$$
$$x_i + y_j \geq a_{i,j} \ \forall (i,j) \in M \times M', x_i \geq 0 \ \forall i \in M, y_j \geq 0 \ \forall j \in M' \Big\}.$$

Moreover, the core of an assignment game is non-empty (Shapley and Shubik 1972). By Lemma 11, the core of an assignment game coincides with the **u**-core of the game in case of the reciprocal percapita utility function.

Find the **u**-essential coalitions in case of assignment games. The singletons are **u**-essential by definition. The matchings are **u**-essential, but the other pairs are not **u**-essential. Indeed, let $i,j \in N$, $\{i,j\} \notin \mathcal{M}(M,M')$, then for each $x \in$ **u**-core$(w_A)$ we have that $2(w_A(\{i,j\}) - x_i - x_j) \leq -x_i - x_j$, because $0 \leq x_i + x_j$.

Consider a coalition $S \in \mathcal{P}^*(N)$ with cardinality larger than two. Let $x \in$ **u**-core$(w_A)$ and $\mu^*$ be an optimal matching for $S$. The left-hand side of the inequality in Definition 6 is:

$$\begin{aligned} u_S \circ e(S,x) &= |S|(w_A(S) - x(S)) \\ &= |S| \left( \sum_{(i,j)\in\mu^*} (a_{i,j} - x_i - x_j) - \sum_{\substack{k \in N, \\ (k,\cdot) \notin \mu^*, \\ (\cdot,k) \notin \mu^*}} x_k \right). \end{aligned}$$

Moreover, for $\mathcal{B}^* = \mu^* \cup \{\{k\}\}_{k\in N,(k,\cdot)\notin\mu^*,(\cdot,k)\notin\mu^*}$ the right-hand side of the inequality in Definition 6 is

$$\sum_{T\in\mathcal{B}^*} u_T \circ u(T,x) = 2\left( \sum_{(i,j)\in\mu^*} (a_{i,j} - x_i - x_j) \right) - \sum_{\substack{k \in N, \\ (k,\cdot) \notin \mu^*, \\ (\cdot,k) \notin \mu^*}} x_k \, . \tag{26}$$

Subtract $2(\sum_{(i,j)\in\mu^*}(a_{i,j} - x_i - x_j)) - |S| \sum_{\substack{k \in N, \\ (k,\cdot) \notin \mu^*, \\ (\cdot,k) \notin \mu^*}} x_k$ from both sides. Then we get

$$(|S| - 2) \sum_{(i,j) \in \mu^*} a_{i,j}$$

on the left-hand side, and

$$(|S| - 2) \sum_{(i,j) \in \mu^*} (x_i + x_j) + (|S| - 1) \sum_{\substack{k \in N, \\ (k,\cdot) \notin \mu^*, \\ (\cdot,k) \notin \mu^*}} x_k$$

on the right-hand side. Since $x \in \mathbf{u}\text{-core}(w_A)$, it holds that $\sum_{(i,j)\in\mu^*}(x_i + x_j) \geq \sum_{(i,j)\in\mu^*} a_{i,j}$ and $\sum_{k \in N, (k,\cdot) \notin \mu^*, (\cdot,k) \notin \mu^*} x_k \geq 0$. Then, it follows that

$$u_S \circ e(S, x) \leq \sum_{T \in \mathcal{B}^*} u_T \circ u(T, x) \, .$$

It means that there is no $x \in \mathbf{u}\text{-core}(w_A)$ such that the left-hand side would be strictly larger than the right-hand side, hence $S$ is not **u**-essential.

In conclusion, only the singletons and the matchings are **u**-essential in case of assignment games, hence, there are only polynomial many **u**-essential coalitions in the number of players.

## 11 Conclusion

We have introduced a generalization of the prenucleolus using utility functions, namely the **u**-prenucleolus. This generalization also generalizes the percapita prenucleolus (Grotte 1970, 1972) and the $q$-nucleolus (Solymosi 2019). On the other hand, the **u**-prenucleolus is a special case of the general prenucleolus (Potters and Tijs 1992; Maschler et al. 1992).

We have considered TU-games with restricted cooperation. For such games, some of the original properties of the prenucleolus change: for example, the prenucleolus is no longer a single-valued solution. Katsev and Yanovskaya (2013) gave necessary and sufficient conditions for the prenucleolus to be non-empty and to be single-valued, respectively. We have generalized these results to the **u**-prenucleolus.

Using the idea of utility functions, we have also introduced generalizations of the core, least core, balanced games and essential coalitions: the **u**-core, **u**-least-core, **u**-balanced games and **u**-essential coalitions, respectively. We have generalized the Bondareva–Shapley theorem (Bondareva 1963; Shapley 1967; Faigle 1989) by showing that a game is **u**-balanced if and only if its **u**-core is not empty. We have also generalized Huberman's theorem (Huberman 1980); we have shown that **u**-essential coalitions form a characterization set of the **u**-prenucleolus in case of **u**-balanced games.

We have given sufficient conditions on the utility functions for the **u**-prenucleolus and the **u**-core to be invariant. Finally, we have discussed a class of games and a utility function, where a game has polynomial many **u**-essential coalitions.

Future work can focus on other generalizations of TU-games, such as those where the utility function is applied not to the excesses, but to the coalitional values and/or the preimputations. In other words, one could define TU-games with utility functions by considering the excess of utilities rather than the utility of excesses. This approach is also interesting, as it would generalize both the prenucleolus and the per capita prenucleolus.

Another possible direction of future research is the deeper understanding of the **u**-prenucleolus by considering dual games.

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