Journal of Economic Behavior and Organization 224 (2024) 785–809

Contents lists available at ScienceDirect

Journal of Economic Behavior and Organization

journal homepage: www.elsevier.com/locate/jebo

Research paper

"Friends Are Thieves of Time": Heuristic attention sharing in stable friendship networks

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ARTICLE INFO

JEL classification:
A13
C72
D85

Keywords:
Networks
Heuristics
Bilateral communication links
Social value
Link strength

ABSTRACT

This paper studies a model of network formation in which agents create links following a simple heuristic — they invest their limited resources proportionally more in neighbours who have fewer links. This decision rule captures the notion that when considering social value more connected agents are on average less beneficial as neighbours and node degree is a useful proxy when payoffs are difficult to compute. The decision rule illustrates an externalities effect whereby an agent's actions also influence his neighbours' neighbours. Besides complete networks and fragmented networks with complete components, the pairwise stable networks produced by this model include many non-standard ones with characteristics observed in real life networks like clustering and irregular components. Multiple stable states can develop from the same initial structure — the stable networks could have cliques linked by intermediary agents while sometimes they have a core–periphery structure. The observed pairwise stable networks have close to optimal welfare. This limited loss of welfare is due to the fact that when a link is established, this is beneficial to the linking agents, but makes them less attractive as neighbours for others, thereby partially internalising the externalities the new connection has generated.

1. Introduction and motivation

This paper studies the decentrally emerging stable networks when all agents employ a simple decision rule and invest proportionally more in neighbours who have a lower degree. In the absence of complex optimisation at the level of every network link (cf. Brueckner (2006), Deroian (2009), Bloch and Dutta (2009), Salonen (2016), So (2016), Bourlès et al. (2017), Baumann (2021)) as opposed to the node/agent level, a common assumption in the literature has been that agents divide their limited resources equally amongst their neighbours. The Jackson and Wolinsky (1996) co-author model is a prime example of such a setup.[2] While a proportional allocation of resources, e.g. time/attention, is a fair assumption in such a context, other social interactions could be driven by a different dynamic. Take a friendship network, for example. If one is interested in developing an enduring and deep friendship, this requires big investment from both parties and people do not usually allocate their time and effort equally amongst everyone they know — they have friends they see every day but they also have acquaintances with whom they have just minute-long conversations. Since no one wants to be on the more giving end of a "one-sided" friendship where the attention given is much more than the attention one gets, a rational friend-seeker would try to give more time/attention to people who are expected to return it. Arguably, such a logic would also be important when deciding on the intensity of online interactions between friends in case of

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[1] A significant part of this project was written in Maastricht University. The quote in the title is attributed to Francis Bacon.

[2] Consider also Albornoz et al. (2019) and Harmsen-van Hout et al. (2013) which have similar assumptions. Additionally, Bala and Goyal (2000) and Galeotti et al. (2010) assume that links are of intensity either 0 or 1.

https://doi.org/10.1016/j.jebo.2024.06.005
Received 14 June 2023; Received in revised form 3 June 2024; Accepted 7 June 2024
Available online 8 July 2024

lockdown. Even within the co-author context it would be wise to expect a smaller contribution from a co-author who is involved in many projects.[3]

Harmsen-van Hout et al. (2016) show experimentally that when faced with linking choices involving increased complexity of payoffs, experimental subjects resort to simplified decision rules, i.e. heuristics.[4] The authors identify two factors that are related to payoffs only qualitatively, but have been used by the experimental subjects: whether or not the choice option involves a deviation from the status quo and the number of direct neighbours of the (potential) linking partner. They call for future models to include the human tendencies[5] "to base complex linking decisions on heuristic cues like... node degree rather than exact payoff" (Harmsen-van Hout et al., 2016), as found by their analysis.

Addressing these findings, the current paper investigates what happens if agents decide to form stronger connections with neighbours who have fewer links. More precisely, the setup assumes that link investments are proportionally greater when they are formed with nodes of a lower degree (since the connections are bilateral, in-degree is the same as out-degree). The heuristic in the current setup is intuitively appealing, since a self-interested agent would try to maximise his potential payoff and agents who are less connected have on average more resources to allocate to a specific link. The benefit is higher if the degree of the potential neighbour is lower, since he would have to spend his resources on fewer people.[6] If the agents are unaware of the exact functional form of the utility per link or if they cannot have a complete overview of the system they are part of (i.e. they have cognitive limitations), the degree gives an indication of how much one can benefit from a specific potential neighbour (on average). Using such a "rule of thumb", the model diverges from the literature on best responses to specific investment that the agents could get from their neighbours and enters the realm of heuristics-driven decisions.

1.1. Results and contribution

One of the main objectives of the current study is to show that using a heuristic different than the equal split but equally plausible and intuitively appealing results in a highly non-trivial change in the predictions of the model, bringing it closer to observed reality. Namely, the pairwise stable networks in the current setup exhibit characteristics observed in real life social networks like irregular components and clusters (Jackson and Rogers, 2007). Moreover, they can have a core–periphery structure or separate communities sometimes bridged by intermediary nodes. While networks with regular/complete components are optimal, the inefficient pairwise stable networks have close to optimal utilitarian welfare.[7] The limited loss in welfare is due to the fact that once a link is established, agents partially internalise the externalities they impose on others. That is, in addition to the benefit an agent gets from the new connection, they immediately become less attractive to all of their other neighbours and receive a smaller benefit from them. The findings in the paper suggest that heuristics can be a powerful force in social network formation.

The results of the paper are achieved at a very low cost — a myopic decision rule that does not require complicated computations or complex strategic considerations like best responses or farsightedness,[8] for instance. There are also no assumptions like homophily as all nodes are identical at the start and their potential matchings are random.[9] Here, it is crucial that a distinction is made between the social and the informational value of links as per Harmsen-van Hout et al. (2013). For instance, social value is derived from the communication and time spent with friends, irrespective of how much they can help you solve a specific professional problem. Informational value could be the valuable knowledge one gets from communicating with others. While informational value is easier to transfer, social value is difficult to transfer without a direct link. While in other contexts the term "social network" can be used to refer to many types of networks, this paper uses the term for networks in which social value is the main driving factor for network formation. In particular, the current setup assumes that *social* (direct) value is the only source of utility as would be the case in personal relationships between friends, rather than the informational (indirect) value.[10]

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[3] In the presence of synergies and assuming that time and effort are the most important factors determining the quality of a paper one would be better off investing more in co-authors who are involved in fewer projects and less in co-authors who are involved in more projects to have higher utility, ceteris paribus. In the extreme cases, if author A connects with author B who has only one project, the common project AB would be finished much faster compared to a situation in which A connects to author C who has ten projects. If author C wants to allocate any time to even one of the remaining nine projects, he would have to spend strictly less than all of his research time on the common project AC.

[4] A related finding is discussed by Kovářík et al. (2018) in the context of learning where "people facing more complex environments... seem to resort to simpler learning rules".

[5] Consider also Hämäläinen et al. (2013) who urge researchers to include behavioural effects in OR processes.

[6] A similar logic is also captured in the Horse race betting model where betting on each horse proportional to its probability of winning is log-optimal (Cover and Thomas, 2006).

[7] The existence of stable networks which have suboptimal welfare could also be connected to the observation of Staudigl and Weidenholzer (2014) that inefficient networks can form when the linking costs are low.

[8] Morbitzer et al. (2014) show through simulations that the co-author model can produce a number of irregular networks when the agents exhibit farsightedness. Moreover, the networks in the pairwise farsightedly stable set (as defined by Herings et al. (2009)) of the current model have maximal utilitarian welfare. This suggests that farsighted agents can internalise even more distant externalities than they cause.

[9] Homophily, whereby similar types of nodes are more likely to be connected than dissimilar ones, tends to characterise all social networks (McPherson et al., 2001), and produces clustering. Consider Bramoullé et al. (2012) who, building on the work of Jackson and Rogers (2007), introduce individual heterogeneity to investigate homophily as a result of biased meeting processes. Additionally, Currarini et al. (2009) and Zuckerman (2024) introduce agent types to reproduce empirically observed homophily patterns.

[10] In contrast, people wanting to be connected to the most popular nodes (in school, for example) want that for the *indirect* value (getting some of the popularity by becoming more well-known themselves), not because of the *direct* impact that would have (support in difficult times, for instance).

1.2. Related literature

With the increased ability of people in recent years to connect and communicate there has also been a growing interest in the structures and effects of (social) networks, how they are shaped by reality and how they help shape reality through their influence on human behaviour. Hence, network formation and the resulting (stable) network structures have important effects from a social and economic perspective and have been the subject of a growing number of studies, cf. Cowan and Jonard (2004), Fagiolo (2005), Baron et al. (2006), Monsuur (2007), Kirman et al. (2007), Hancock and Raeside (2010), Janssen and Monsuur (2012), Harmsen-van Hout et al. (2013), Hellmann and Staudigl (2014), Olaizola and Valenciano (2014), Rêgo and dos Santos (2019), Hellmann (2021), Griffith (2022) and Li (2023). Understanding the potential driving forces behind network formation is important because it might imply different recommendations for optimal interventions or lack thereof. It is also possible that networks which have been formed to provide social value (friendship networks), get re-purposed, for instance for job search (Granovetter, 1974).

This paper draws inspiration from the findings and approach of Harmsen-van Hout et al. (2013, 2016) and builds on them. Harmsen-van Hout et al. (2016) empirically establishes the legitimacy of heuristic decision-making in network formation. Harmsen-van Hout et al. (2013) investigates theoretically and through simulations a setup with link specificity (the more direct connections an agent has, the less attention they have per link), used to model social value, and value transferability via indirect links (for informational but not social value). They investigate analytically two specific cases of low link specificity ($\rho = \frac{1}{2}$) and high link specificity ($\rho = 1$). In the case of only social value and low link specificity, they show that stable networks can consist of complete components. For high link specificity they characterise the pairwise stable networks. They consist of equal-neighbour degree networks in which two nodes in the same component have degrees which differ by at most 1. In the case of both social and informational value being taken into account, the representative networks from their simulations are equal-neighbour degree networks and networks with regular components. Importantly, in their model agents split their resources equally between their neighbours. This allows for a direct comparison between the setup in this paper and their findings. To this end, this paper first generalises their findings for the low level of link specificity. With $\rho = \frac{1}{2}$, the stable networks have *no irregular components* for equal splitting of resources. This is in stark contrast to the findings of the current paper, where the set of pairwise stable networks is richer. Besides networks with complete components, it consists of networks with clusters, cliques linked by intermediary agents or networks with a core–periphery structure. Finally, agents within the same component can have very different degrees and neighbours with different degrees.

This model bears a crucial similarity with the co-author model (Jackson and Wolinsky, 1996) in that it depends on the degree of the agents. There, the indirect connections, the co-author's co-authors, influence the payoff of an agent because they take from the co-author's time and so impose an externality on the co-author's co-authors. The setup of the current paper captures a more elaborate model of indirect (positive) externalities.[11] In its core idea of nodes differentiating the investment to their neighbours this paper is close to Baumann (2021). She analyses a model of weighted network formation and characterises the types of equilibria that occur. However, there players can invest in others or in themselves and the equilibria are neither pairwise stable nor efficient.

The remainder of the paper starts by outlining the model, pairwise stability and discusses the externalities in more detail (Section 2). Section 3 is devoted to the analysis of the types of pairwise stable networks which are produced. First it considers networks with irregular components that emerge from simulations and then it proceeds to analyse regular and commonly used network structures that can be approached analytically within the current framework, contrasting the outcomes with the results seen when the resources are equally split amongst all neighbours. Section 4 analyses the welfare in the stable networks of the model and possible extensions. The last section concludes the discussion.

2. Model

There is a set of agents $N = \{1, 2, \ldots, n\}$, who form an undirected network. A network g is a list of connected agent pairs. Let $ij \in g$ denote that agents i and j are connected under network g. The network obtained by adding the link ij to an existing network g (keeping all other links constant) is denoted by $g + ij$. Analogously, the network obtained by removing the link ij from an existing network g is denoted by $g - ij$. The set of all possible networks is denoted by G. A connection between two nodes can be sustained only with a positive investment from both parties. The payoff from a specific link depends positively on the bilateral investment of the involved agents. For every node i of the network, if t_{ij} is the contribution that agent i has in his link with agent j, investments in links are constrained to $t_{ij} > 0$ (if investments are 0 there is no link). Additionally, if N_i is the set of node i's neighbours, $\sum_{j \in N_i} t_{ij} = 1$ for all i, one being the highest contribution a node can make on a single link. Nodes cannot invest in themselves and therefore they always invest all their resources (a node with no links will have zero utility). There are no additional explicit costs for maintaining links except the contribution to the link, which is an *opportunity* cost of not investing in others. The investment in connections with neighbours is based on the following heuristic: *links are proportionally stronger when they are formed with nodes of a lower degree*. This means that in an equal-degree network every node spreads its resource of 1 equally amongst its neighbours, but also that if a node has two neighbours, one with degree 2 and the other one with degree 1, the neighbour with degree 1 will get an investment twice as big as the other neighbour. Therefore, the investment per link is inversely proportional to the degree of the node it connects to,

[11] For example, in a situation in which A is connected to B and X, while B is connected to A and Y, if B decides to create a new connection (say, to Z), this affects not only B's contribution to A and A's investment in B. It is interesting to note that through the second effect A's investment in X will also increase. In fact, B's decision to form a new link positively influences his neighbours' neighbour, while affecting his direct neighbours negatively.

so $t_{ik} = \frac{1/d_k}{\sum_{j\in N_i} 1/d_j}$, with d_i being node i's degree. The payoff of a specific link is given by $\sqrt{t_{ij} * t_{ji}}$, which exhibits constant returns to scale.[12,13] Such a functional form ensures that spreading resources between neighbours can be more beneficial than investing in only one neighbour. Following the specified heuristic for spreading resources between links, the payoff of every node i under network g can be expressed and rearranged in the following way:

$$u_i(g) = \sum_{k\in N_i} \sqrt{\frac{\frac{1}{d_k}}{\sum\limits_{j\in N_i} \frac{1}{d_j}} * \frac{\frac{1}{d_i}}{\sum\limits_{\ell\in N_k} \frac{1}{d_\ell}}} = \frac{1}{\sqrt{\sum\limits_{j\in N_i} \frac{d_i}{d_j}}} \left(\sum_{k\in N_i} \frac{1}{\sqrt{\sum\limits_{\ell\in N_k} \frac{d_k}{d_\ell}}} \right).$$

In other words, according to the heuristic, the payoff of an agent is a function of the ratio of the degree of the agent compared to all his neighbours' degrees and the ratio between the degree of each one of the agent's neighbours and the neighbours' neighbours. In case of no ambiguity about the network $u_i(g)$ will be denoted simply by u_i.

2.1. Efficiency

As usually, a regular network refers to a network where every node has the same degree. Additionally, in this paper the term regular components/networks will only be used for components/networks with degrees greater than or equal to one, i.e. ones which have no isolated nodes. This paper uses the notion of utilitarian welfare, the sum of all agents' individual payoffs, to evaluate the stable networks which emerge from the model. It is formalised below.

Definition 1. The utilitarian welfare of a network g is $W(g) = \sum_{i\in N} u_i(g)$.

Lemma 1. *The payoff for every node in a regular component is* 1.

Proof. This follows directly from the payoff expression and the fact that the degree of all nodes is the same, i.e. $d_i = d_j = d_k = d_\ell$ and $|N_i| = |N_k|$.

$$\frac{1}{\sqrt{\sum\limits_{j\in N_i} \frac{d_i}{d_j}}} \left(\sum_{k\in N_i} \frac{1}{\sqrt{\sum\limits_{\ell\in N_k} \frac{d_k}{d_\ell}}} \right) = \frac{1}{\sqrt{|N_i|}} \frac{|N_i|}{\sqrt{|N_k|}} = 1. \quad \square$$

Proposition 1. *The maximum utilitarian welfare of a network with n nodes is n.*

Proof. It follows from Lemma 1 that the welfare is n when the network is complete. In order to get welfare of more than n, there must be at least two nodes i and j whose link provides a positive net gain. In other words, the sum of the two investments in the link t_{ij} and t_{ji} and the return to the two players $2\sqrt{t_{ij}t_{ji}}$ should be strictly positive:

$$-t_{ij} - t_{ji} + 2\sqrt{t_{ij}t_{ji}} > 0 \Leftrightarrow (\sqrt{t_{ij}} - \sqrt{t_{ji}})^2 < 0.$$

This is a contradiction and hence the maximum utilitarian welfare is n. $\square$

Corollary 1. *A network consisting of regular components has maximum utilitarian welfare.*

Utilitarian welfare is not the only notion of welfare, another prominent measure is Rawlsian welfare, expressed by the utility of the agent who is worst-off. It is formalised below.

Definition 2. The Rawlsian welfare of a network is $W^R(g) = \min_{i\in N}\{u_i(g)\}$.

Proposition 2. *The maximum Rawlsian welfare of a network with n nodes is* 1.

Proof. In order for Rawlsian welfare to be bigger than 1, all nodes need to have payoff higher than 1, which would contradict Proposition 1. $\square$

The propositions above establish that in the current model the most efficient networks are the ones with regular components. Since this paper focuses on social interactions and they rarely if ever produce fully regular networks, the results on efficiency present merely a benchmark against which all stable networks can be measured.

[12] This corresponds to the case of *low link specificity* of Harmsen-van Hout et al. (2013), i.e. there is a greater advantage of being connected to multiple agents. However, investing more in some of one's neighbours (if they "steal " more of one's time), implies that one would have less resources for others. Such an interpretation works straightforwardly with variables as time and attention.

[13] The results are qualitatively similar for functions with other link specificity, e.g. cube root (low specificity), or taking $\rho > \frac{1}{2}$ (intermediate). A similar payoff function is used in the occupational choice model with spillovers by Albornoz et al. (2019).

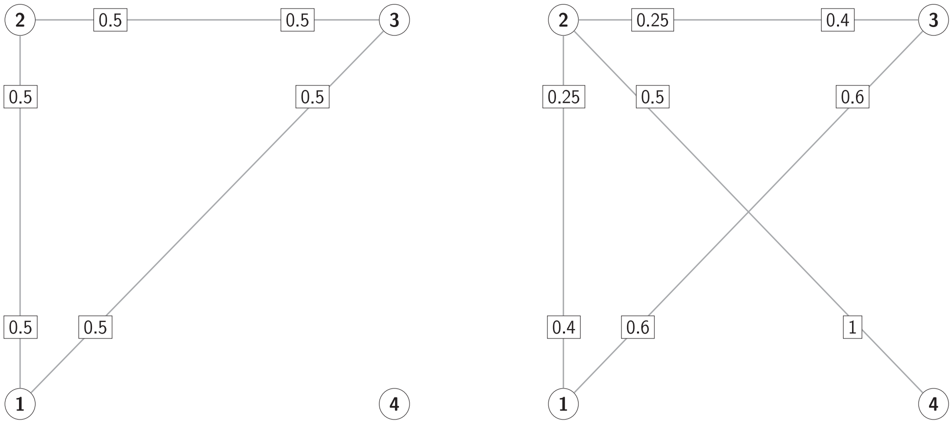


Fig. 1. Externalities effects in the model. The rectangular boxes next to the nodes specify how much each agent invests in a particular link.

2.2. Stability

To select networks, this paper uses the notion of *pairwise stability* (Jackson and Wolinsky, 1996). A network is pairwise stable if for every existing link none of the parties *strictly* benefits from the link being severed, and for links that do not exist one of the parties would be strictly worse off from the link being created even if the other party would strictly benefit from it being created. The process assumes that links can be severed unilaterally, but can form only with bilateral consent (so at least one of the nodes is *strictly* better off with the link and the other one is *at least* indifferent).

Definition 3. A network g is pairwise stable if:

(i) $\forall ij \in g$, $u_i(g) \geq u_i(g - ij)$ and $u_j(g) \geq u_j(g - ij)$; and
(ii) $\forall ij \notin g$, if $u_i(g) < u_i(g + ij)$ then $u_j(g) > u_j(g + ij)$.

In light of this definition, a network is *pairwise stable*: (i) if there is *no node* which would be better off to sever a link unilaterally; and (ii) if *no link* can be formed by bilateral consent by any two nodes (at least one strictly benefits).[14] Pairwise stability is often used as a network selection criterion, cf. Harmsen-van Hout et al. (2013), Morbitzer et al. (2014), Rêgo and dos Santos (2019) and Hellmann (2021).

2.3. Externalities effects

Employing the heuristic decision rule results in interesting externalities effects which are illustrated in Fig. 1. The first part of it consists of three connected agents and an isolated node. The isolated node 4 has zero utility, while all other nodes spread their resources equally and have utility 1. The second part of the figure shows the spread of resources after a link is formed between nodes 2 and 4. After the link is formed, there are three effects which can be identified: (i) nodes 1 and 3 immediately receive a smaller investment from node 2 (from 0.5 to 0.25) and get a lower payoff ($0.916 < 1$); (ii) nodes 2 and 4 become less desirable as neighbours, so both 1 and 3 start investing less in node 2 (from 0.5 to 0.4); (iii) as a result of (ii) nodes 1 and 3 are also affected *positively*, because they start investing more in each other (from 0.5 to 0.6). Therefore, forming a link under this setup negatively affects the linking agents' neighbours. However, the linking agents partially internalise this negative externality by becoming less desirable as neighbours themselves and receiving smaller investment by their neighbours. Finally, the neighbours' neighbours of any linking agents experience positive externalities effects. The different ways in which these effects balance each other produces a diverse set of stable networks. It is interesting to note that in case of equal splitting of resources, effects (ii) and (iii) are not observed. In the case of equal split, forming a link does not influence the linking agents' neighbours in their decision how much to invest in the two agents who form the new link. There are also no positive externalities effects.

3. Stable networks

This section analyses the stable networks which emerge from employing the heuristic. First, it introduces representative pairwise stable networks, obtained by computer simulations. After that it shows analytical results about the stability of some common network types (e.g. regular, star, wheel).

[14] One can think of the matching needed to achieve stability as a trial and error process. For instance, two agents try forming a link and if they find that they are better off with it, they keep it; otherwise, they return to the status quo.

Since the model produces a plethora of possible stable networks which are not regular and have no regular components, computer simulations were employed to see the range of potential outcomes. Cycles are possible with a specific non-random matching procedure.[15] One simple cycle that occurs under this protocol is described in Appendix E.[16] However, under random matching of potential neighbours all simulations converged to a stable network. All simulation rounds started from the empty network. Nodes were randomly matched to check if a link can be formed. This was done until exhausting all potential links. If a link was formed, the stability of other links was checked, i.e. if there are any links which can be severed. This was done sequentially, so that any addition or removal of a link from the network resulted in a resource *reallocation* according to the heuristic for all nodes (i.e. all nodes in the network reevaluated and updated their allocation in the new network). The process of random matching was terminated if there were no possible links which could be formed or cut. The process of establishing pairwise stability was systematic as all links were checked at every iteration of it, albeit in random order.[17] The resulting pairwise stable networks are investigated below with some examples which showcase the main observed trends and established patterns in all simulated stable networks.

3.1. Representative examples

Example 1 below gives an in-depth look into one of the simplest irregular stable networks of the model. Example 2 shows the variability of individual payoffs within the same stable network consisting of one component. Example 3 zooms out and shows that multiple components, regular and irregular, can form a stable network.

Example 1 (*Stability and Welfare*)**.** Consider the stable network in Fig. 2(a). It has two significant cliques — an upper and a lower one which are directly connected. The members of the upper clique all have degree 5, while the members of the lower one have degree 8. The payoff of all nodes in the upper clique is[18]:

$$\frac{1}{\sqrt{\frac{5*3}{5}+\frac{5*2}{8}}}\left(\frac{2}{\sqrt{\frac{8}{5}+\frac{8*7}{8}}}+\frac{3}{\sqrt{\frac{5*3}{5}+\frac{5*2}{8}}}\right)=1.037$$

while the nodes in the lower clique have only:

$$\frac{1}{\sqrt{\frac{8}{5}+\frac{8*7}{8}}}\left(\frac{7}{\sqrt{\frac{8*7}{8}+\frac{8}{5}}}+\frac{1}{\sqrt{\frac{5*3}{5}+\frac{5*2}{8}}}\right)=0.979$$

leading to overall welfare of 11.982. The network is stable and this is shown explicitly below. There are three types of links — between the nodes in the upper clique, between the ones in the lower clique and between the two cliques. Severing one of the links would make no agent(s) strictly better off. For example, cutting the link between a node in the upper clique and a node in the lower clique would bring the node in the upper clique:

$$\frac{1}{\sqrt{\frac{4*3}{5}+\frac{4}{8}}}\left(\frac{1}{\sqrt{\frac{8}{4}+\frac{8*6}{8}+\frac{8}{7}}}+\frac{3}{\sqrt{\frac{5*2}{5}+\frac{5}{4}+\frac{5*2}{8}}}\right)=1.025<1.037$$

and it would bring to the node in the lower clique:

$$\frac{1}{\sqrt{\frac{7*7}{8}}}\left(\frac{6}{\sqrt{\frac{8*6}{8}+\frac{8}{5}+\frac{8}{7}}}+\frac{1}{\sqrt{\frac{8*6}{8}+\frac{8}{4}+\frac{8}{7}}}\right)=0.953<0.979.$$

[15] Using the conventional ordering of the adjacency matrix, it is possible to find a cycle in the following way: first, check if a link can be formed. If a link is formed, then the stability of all possible links is tested (once again following the specified order). This procedure stops if no links (could be more than one) can be severed. After no more links can be cut, the check for adding links is implemented again. The procedure is repeated until no changes (adding or severing links) are possible.

[16] This serves as testimony that proving convergence for all cases even under random matching is not a trivial matter. The cycle illustrates that utilitarian welfare can vary in both directions as unstable networks add or sever links. This is telling since utilitarian welfare is the usual suspect for a function (similar to a potential function) à la Jackson and Watts (2001) which behaves in a predictable way, always increasing or always decreasing between two adjacent networks to show no cycles are possible in the process. Further, Hellmann and Staudigl (2014) survey different techniques which have been applied to show existence of pairwise stable networks, none of which involve properties exhibited by the current model. In particular Hellmann (2013) identifies common network properties for which there is no closed improving network cycle. One of them is ordinal convexity in own links, i.e. if a link is desirable for a player at some point, it stays desirable when adding new links. This is clearly not the case in the current setup.

[17] The simulations consisted of 2000 simulation rounds for the case $n = 20$ reported in this section. Smaller scale simulations were conducted for bigger networks to verify that the same patterns in the pairwise stable networks were observed there as well.

[18] All numbers are rounded to the third digit.

Similarly, none of the links between members of the upper clique would be cut:

$$\frac{1}{\sqrt{\frac{4*2}{5}+\frac{4*2}{8}}}\left(\frac{2}{\sqrt{\frac{8}{4}+\frac{8*7}{8}}}+\frac{2}{\sqrt{\frac{5}{5}+\frac{5*2}{4}+\frac{5*2}{8}}}\right)=0.983<1.037.$$

None of the links between members of the lower clique would be cut (in case the two nodes share a neighbour in the upper clique, the payoff is even lower than the 0.967, outlined below):

$$\frac{1}{\sqrt{\frac{7}{5}+\frac{7*6}{8}}}\left(\frac{6}{\sqrt{\frac{8*5}{8}+\frac{8}{5}+\frac{2*8}{7}}}+\frac{1}{\sqrt{\frac{5*3}{5}+\frac{5}{8}+\frac{5}{7}}}\right)=0.967<0.979.$$

Finally, the only type of link that could be formed is between the upper and lower clique. For instance, this would bring the involved node in the upper clique:

$$\frac{1}{\sqrt{\frac{6*3}{5}+\frac{6*2}{8}+\frac{6}{9}}}\left(\frac{1}{\sqrt{\frac{9}{5}+\frac{9*7}{8}+\frac{9}{6}}}+\frac{1}{\sqrt{\frac{5}{6}+\frac{5*2}{5}+\frac{5}{9}+\frac{5}{8}}}+\frac{2}{\sqrt{\frac{8}{6}+\frac{8*6}{8}+\frac{8}{9}}}+\frac{2}{\sqrt{\frac{5*2}{5}+\frac{5*2}{8}+\frac{5}{6}}}\right)=1.035<1.037$$

and therefore the link would not be formed. In contrast, the involved node in the lower clique would benefit from this link:

$$\frac{1}{\sqrt{\frac{9*7}{8}+\frac{9}{6}+\frac{9}{5}}}\left(\frac{1}{\sqrt{\frac{6*3}{5}+\frac{6*2}{8}+\frac{6}{9}}}+\frac{1}{\sqrt{\frac{5}{6}+\frac{5*2}{5}+\frac{5}{9}+\frac{5}{8}}}+\frac{2}{\sqrt{\frac{8}{9}+\frac{8}{6}+\frac{8*6}{8}}}+\frac{5}{\sqrt{\frac{8}{9}+\frac{8}{5}+\frac{8*6}{8}}}\right)=0.996>0.979.$$

Example 2 (*Degrees and Individual Payoffs*)**.** Fig. 2(b) once again has two major cliques but in each of them some members are also connected to an intermediary node and as a result they have one degree higher than the rest (degree 6 in the case of the three nodes in the upper clique and degree 13 in the case of nodes from the lower clique). In this setting the highest payoffs 1.015 are reaped by the members of the smaller clique who are not connected to the intermediary node, while the intermediary node has the smallest payoff 0.96. The overall welfare is 19.967.

Fig. 2(c) shows a very similar structure, but in it the two clearly delineated cliques also have direct connections between each other. Interestingly, the node which is connected to the upper clique directly and through the intermediary has payoff 1.019, another node has two direct connections to the upper clique and the highest payoff of 1.021, while the intermediary node has payoff 0.998. The lowest payoff in the case, 0.984, is for the members of the upper clique who are not in any way connected to the lower one. Moreover, nodes with the same degree, can have different payoffs, e.g. three different nodes in the upper clique have the same degree of 12, but one of them has 0.993, the second one has 0.984, while the third one has 0.988. The overall welfare is 19.968.

Fig. 2(d) has overall welfare of 19.995. It has 13 nodes with degrees 15, 4 with degree 11, and one with degree 12, 13 and 14.

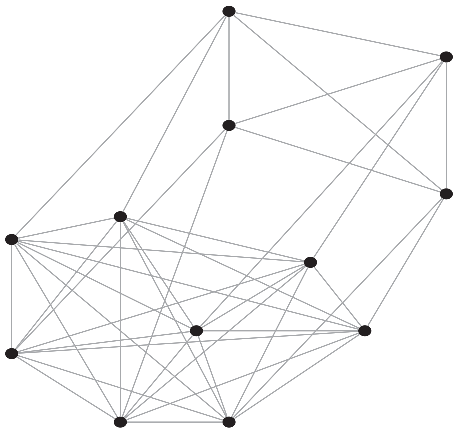
Example 3 (*Multiple Components*)**.** Fig. 3 illustrates another interesting feature of the stable networks in the model — they can consist of disconnected irregular and regular components. The irregular component in such a structure is stable on its own. However, it cannot be verified that every stable irregular component can be integrated in such a structure with multiple complete components and still stay stable. In fact Fig. 2(b) presents a counterexample. Combined with regular complete components with sizes between 6–8 and between 11–18 in one network the resulting whole is not stable. However, this is not the case for complete components with sizes between 2–5 or 9 and 10. This particular result is most likely related to the sizes of the cliques. The other subparts of the figure seem to be more resistant to such events, since combining Figs. 2(a), 2(c), 2(d) with complete regular components of sizes between 2–60 still results in stable networks.

As illustrated by Fig. 3 stable networks can consist of one irregular component and multiple regular ones. While combining any two of the irregular networks in Fig. 2 does not yield a stable network, stable networks can consist of more than one irregular component.

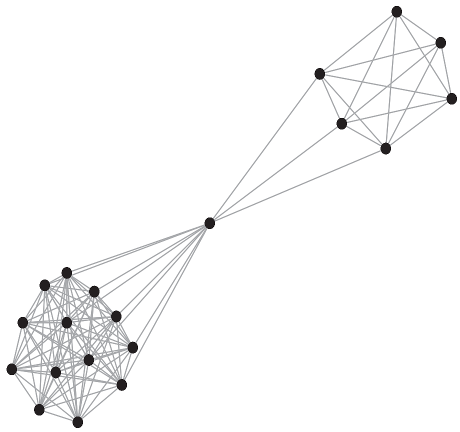
3.2. Stable network structures with irregular components

Stable networks consisting of irregular components start appearing for $n \geq 12$. These stable networks generally fall into three qualitative categories: networks with cliques and direct links between them (Figs. 2(a), 2(b)), networks with cliques and intermediaries between them (Fig. 2(c)) and networks with a core–periphery structure (Fig. 2(d)). The cliques are reminiscent of the disjoint complete components which can also be stable and are analysed below, since they are connected relatively regularly and moreover, they are never of the same (or very similar) sizes. However, the irregular stable networks occasionally have a core and some relatively peripheral nodes like in Fig. 2(d).

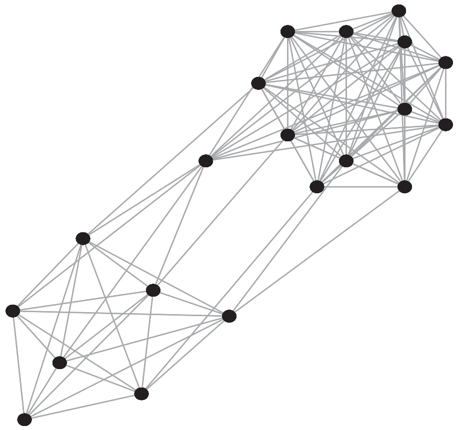
It would be extremely unusual if this simple model perfectly reproduced all stylised facts about social networks. However, qualitatively some notable features can be identified, which particularly hold for the irregular pairwise stable networks: (i) in the simulated networks the distance between pairs of nodes in components is relatively small; (ii) the networks exhibit a high degree



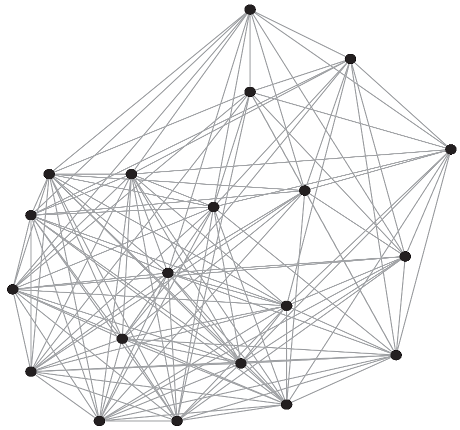
(a) The smallest simulated irregular stable network: $n = 12$



(b) Upper and lower cliques linked by an intermediary agent: $n = 20$



(c) Two cliques linked by an intermediary, but also with separate links: $n = 20$



(d) A network with a core-periphery structure: $n = 20$

Fig. 2. Irregular stable networks.

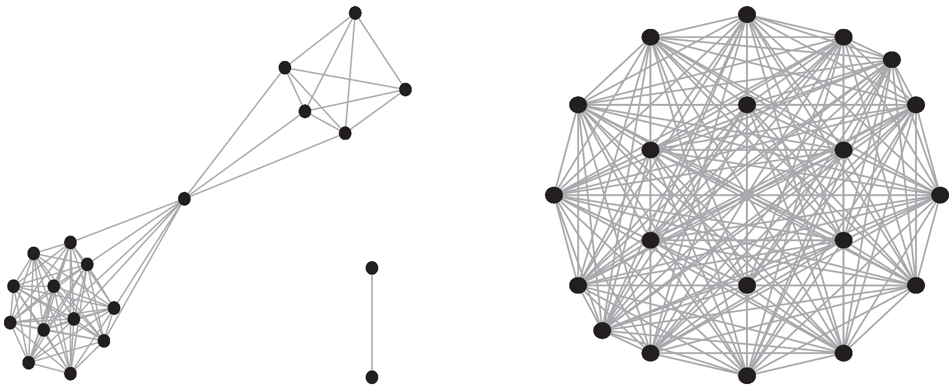


Fig. 3. A stable network with regular and irregular components: $n = 40$.

of clustering (iii) high degree nodes tend to be related to other high-degree nodes and low-degree nodes also tend to be related to lower-degree nodes; (iv) there are fewer nodes with medium degrees and relatively more with high and low degrees as compared to networks where links are formed uniformly at random. These are observed qualitative features of social networks (cf. Jackson and Rogers (2007)). In contrast, it must be noted that many models of formation of social and economic networks predict ***qualitatively different*** network structures like equal-neighbour degree networks (Harmsen-van Hout et al., 2013) or stars and wheels (e.g. Bala and Goyal (2000)), which do not exhibit the same characteristics.

This section continues by analytically investigating networks which frequently appear in the literature to identify which ones are pairwise stable in the current setup.

3.3. Complete networks

Proposition 3. *The complete network is stable.*

Proof. In a fully connected graph every node has payoff 1. In order for the complete graph to be stable, removing a link should be equally good or worse than the status quo for both nodes that are connected so that they decide not to disconnect. For a complete graph with n nodes this implies that after disconnecting the two nodes have degrees $n-2$, while the other $n-2$ nodes keep their degrees of $n-1$. The two nodes with degrees $n-2$ each have $n-2$ neighbours with degrees $n-1$, while the nodes with degrees $n-1$ have two neighbours with degrees $n-2$ and $n-3$ neighbours with degrees $n-1$. This is equivalent to:

$$\frac{1}{\sqrt{\frac{(n-2)^2}{n-1}}}\left(\frac{n-2}{\sqrt{2\frac{n-1}{n-2}+\frac{n-1}{n-1}(n-3)}}\right)\leq 1 \Leftrightarrow \frac{\sqrt{n-1}}{n-2}\left(\frac{n-2}{\sqrt{\frac{2n-2+(n-3)(n-2)}{n-2}}}\right)\leq 1 \Leftrightarrow (n-1)(n-2)\leq 2n-2+(n-3)(n-2).$$

This always holds for $n>2$. Therefore, no two nodes would decide to disconnect in a complete graph with $n>2$. If the network has $n=2$, it is clearly stable as disconnecting would bring both nodes 0. □

This establishes existence of stable networks since for any n there is a stable network that is also efficient. While the complete network is stable and it achieves maximum welfare, it might not be possible to always reach it. Fortunately, it is possible to construct paths for reaching it from the empty network as long as there is a positive probability of adding a link at any step of the process.

Proposition 4. *The complete network is reachable from the empty network.*

Proof. See Appendix A. □

The following proposition condenses what is known from the previous observations about regular networks and introduces a short discussion of the networks which are *not* pairwise stable under the current setup.

Proposition 5. *A regular network is stable if and only if it is complete.*

Proof. See Appendix A. □

Proposition 5 shows that some of the potentially welfare maximising networks, like networks with incomplete regular components, are not stable. This is an interesting result as high regularity in components is indeed *not* characteristic of social networks.[19]

3.4. Unstable incomplete networks

Proposition 5 gives the following corollary.

Corollary 2. *The circle network with $n\geq 4$ nodes is not stable.*

Proposition 6 below, just like the corollary above, shows that often used and investigated network types are all unstable in the current setting.

Proposition 6. *The line, star with $n\geq 3$ peripheral nodes, wheel with $n\geq 4$ peripheral nodes and biregular graphs are not stable.*[20]

Proof. See Appendix A. □

[19] Cf. for example the discussion in Jackson and Rogers (2007) of the relatively small distance between any pair of nodes and the fact that there are more nodes with relatively high and low degrees in social networks.

[20] Biregular graphs are bipartite graphs in which vertices in the same subset of nodes of the given bipartition have the same degree.

3.5. Networks with complete components

Besides complete networks, networks with complete components can also be pairwise stable. As observed above, such networks would also exhibit maximum welfare.

Proposition 7. *A network consisting of two disconnected complete components with sizes $m_1, m_2 \geq 2$ such that $m_1 + m_2 = n$ is stable if and only if $|m_1 - m_2| \geq 2$.*

Proof. See Appendix A. □

Corollary 3. *A network consisting of k disconnected complete components of respective sizes $m_1, \dots, m_k \geq 2$ such that $\sum_{j=1}^{k} m_j = n$ is stable if and only if $|m_j - m_{j'}| \geq 2$ for all $j, j' = 1, \dots, k$ with $j \neq j'$.*

To interpret the result above in the context of a friendship network, one can imagine that having a few good friends is as good as having many marginal friends in terms of social value. In this case they form a group in which everyone gets equal attention from their friends which is a qualitatively different result from Proposition 5. A graph with regular components is not sufficient for stability, the components need to be complete — getting equal treatment from your friends is only stable when you are a part of a tight social group.

The following statement shows that the observations made so far are the only ones needed to analyse the stable states for small networks with $n \leq 11$ nodes. After that the model also produces stable networks with irregular components as shown above. The following proposition is given without an analytical proof since it has been verified by checking all possible options with computer software.

Proposition 8. *In graphs with:*

(a) $n \leq 5$ nodes the complete network is the only stable network;
(b) $5 \leq n \leq 11$ nodes the only stable networks are the complete network and networks consisting of two disconnected complete components with sizes $m_1, m_2 \geq 2$ such that $m_1 + m_2 = n$ and $|m_1 - m_2| \geq 2$.

Proof. (a) The propositions above exclude many potential networks. Others are excluded for having a node unconnected to any other nodes. These cases are not stable because the isolated node would want to connect and the others would want to connect with it. The rest can be verified by a case-by-case check. (b) Verified on a case-by-case basis.[21] □

In other words, for $n \leq 11$ Propositions 5 and 7 describe all stable networks, since there can be no more than two fully connected disjoint parts for these cases.

At this stage it is instructive to contrast the results of the current setup with the situation in which everything is the same (in particular the payoff function) except that the heuristic applied is equal split of resources amongst all neighbours, i.e. $t_{ij} = 1/d_i$ for all $j \in N_i$.

Proposition 9. *Under equal split of resources the stable networks have no irregular components.*[22]

Proof. See Appendix B. □

Stated alternatively, under equal split the stable networks can only be the complete network or networks consisting of disjoint complete components. In light of Proposition 9, the analytically derived results described above (Propositions 3–8) mirror the ones for the case when resources are spread equally amongst the neighbours. The networks under equal split also exhibit maximum efficiency as per Corollary 1. There is, however, one major difference between the current setup and the equal split. Under equal splitting of resources there are no other stable networks but ones consisting of complete subgraphs. It is important to note that only changing the way resources are split between a node's neighbours from equal split to the current heuristic produces the much bigger and diverse set of stable networks, which were shown above.[23]

[21] The proposition has been fully verified by explicitly checking all possible networks. The data used have been taken from https://users.cecs.anu.edu.au/~bdm/data/graphs.html and for the case of $n = 11$ they were additionally generated by the software nauty (McKay and Piperno, 2014). The author of this paper owes special gratitude to Matúš Mihalák from the Department of Data Science and Knowledge Engineering at Maastricht University for his technical assistance with this task.

[22] While this proposition is not established in their work, this setup is partially investigated in Proposition 2 in Harmsen-van Hout et al. (2013), corresponding to their case of $\rho = 1/2$. They observe that regular networks with degrees $d < n - 1$ are not stable and networks consisting of *fully connected components* are stable if and only if $m \geq 4\ell - 2$ for m and ℓ being the number of nodes in every component. Moreover, they also find that the star, circle and wheel are not stable. *Proposition* 9 generalises their findings.

[23] Moreover, substituting the current heuristic for the equal split used in Jackson and Wolinsky (1996) also qualitatively changes the set of stable networks that the co-author model produces, even though the payoff function it uses is different in a few dimensions from the one in this paper. Under equal split a pairwise stable network can be partitioned into fully intraconnected components with a different number of members (as specified in Proposition 4 of Jackson and Wolinsky (1996)), while the current heuristic once again produces stable networks which can consist of irregular components. One can speculate that uniform treatment like equal splitting of resources could be an important factor for producing only regular stable networks in this case.

4. Discussion: Welfare and extensions

4.1. Welfare

While no formal results are derived for this, the price of anarchy in simulated pairwise stable networks is smaller than $\frac{n}{n-1}$, which implies that the loss of utilitarian welfare in stable networks is not big, i.e. observing clique behaviour and even apparent separation is not necessarily significantly harming utilitarian welfare and therefore might not warrant an intervention. According to the simulations the payoffs are very close to the theoretical maximum n (for utilitarian welfare), despite non-negligible differences between nodes' payoffs. One possible explanation for this occurrence is the implicit correction for externalities inherent in the current decision rule. As a node creates another link it imposes an externality on its direct neighbours but by becoming a less-desirable neighbour it immediately attracts smaller investments. As discussed already, in this way it partially internalises the externality it has imposed on others. This is not a feature of the equal split rule, where one only imposes an externality on one's current neighbours by acquiring new ones.

Another interesting observation is the fact that the networks with irregular components and suboptimal welfare will not be stable if agents exhibit farsightedness. To show this, this paper considers the concept of the *farsightedly stable set* (Herings et al., 2009). In words, a set of networks is pairwise farsightedly stable if (i) all possible *pairwise* deviations to a network outside of this set are deterred by a credible threat of ending worse off or equally well off; (ii) there exists a farsighted improving path from any network outside the set to a network in the set; (iii) this is the minimal set which satisfies the previous two conditions. The concept captures the idea that agents are willing to accept temporary setbacks (lower payoff in a current state) looking forward to a network which they expect to be reached and in which they get a better payoff. A *farsighted improving path* emerges when agents form and sever links according to the improvement the *end* network offers as compared to the current one. To add a link, both players prefer the end network to the current one, at least one of them strictly. To remove a link, an agent must prefer the end network to the current one. See Appendix C for the formal definitions and the proof of the proposition below.

Proposition 10. *The pairwise farsightedly stable networks have maximal welfare.*

Proposition 10 argues that pairwise farsightedly stable networks have regular components and hence maximal welfare (Corollary 1). Any network which does not have regular components will have at least one agent who has a payoff of less than 1. Looking forward to a network with maximal welfare and a payoff for him of 1, the agent can sever a link. This process continues until there are only singletons and regular components. After that the singleton agents can form a complete component (as in Proposition 4). Proposition 10 suggests that the far-reaching externalities which are present in this model can be internalised by farsighted agents. That is, while the heuristic decision rule allows partially internalising the externalities a new link causes to neighbours when agents are myopic, farsighted players can completely avoid the suboptimal stable networks.

4.2. Extensions

It is evident that the outcomes under the current model greatly diverge from a similar setting where agents spread their resources equally between their neighbours. In fact it is possible to construe the two setups as two extreme cases, depending on how much influence (denoted by the parameter α) the degree of the node's neighbours has on the node's investments: the equal split attributes no weight ($\alpha = 0$) while the current heuristic gives weight inversely proportional to the degree ($\alpha = 1$).[24] Taking this broader perspective for completeness, it is immediately possible to note some interesting outcomes. Networks consisting of complete components are stable. Importantly, the possible size of complete components within the same stable network varies depending on the particular α. Moreover, for any fixed n it seems to be possible to find an α close enough to 0 (but not equal) for which the stable networks would only consist of complete components. Taking the opposite perspective, for any fixed $\alpha \neq 0$ it seems to be possible to find an n after which the stable networks would not consist of only complete components.[25]

The line, star and circle networks which were shown to be unstable in Proposition 6 and Corollary 2 (for $\alpha = 1$) can also be shown to be unstable for the intermediate values of α. Finally, taking values of α sufficiently close to 1 (bigger and smaller) results in (qualitatively) similar stable networks to the ones explored with the simulations in this section.

A natural addition to this model is to consider the case with fixed linking costs incurred for every link one makes. Clearly, if the linking costs are relatively high, the set of stable networks is going to shrink and allow only the ones with the highest welfare be stable (e.g. networks consisting of complete components). In the opposite case, for relatively low linking costs, it is possible that for a specific n the set of stable networks is preserved (this corresponds to the findings in Staudigl and Weidenholzer (2014)). High and low linking costs are determined by the relative size of the network. A network with *more* nodes would have a *lower* threshold for the linking costs for which the set of stable networks is preserved compared to the case of no linking costs, and vice versa. This is due to the fact that in a network with more nodes an additional link would bring less on average (within a connected component) and compared to the fixed linking cost this would sometimes not justify creating the link. Intermediate linking costs would logically change the nature of the stable networks for any specific n, but it would still be possible to observe pairwise stable networks consisting or regular and irregular components.

24 Appendix D partially addresses this question analytically.

25 One intuitive way to explain this is that the proof of Proposition 9 includes strict rather than weak inequalities in its end.

5. Conclusion

Heuristics have been a relatively overlooked possibility for network formation, while being widely explored in other areas of economics. This paper takes first steps in this direction by investigating the possible outcomes of a simple heuristic, investing more in people who are deemed more likely to invest in you, based on their degree in the network. This is an intuitively appealing idea for modelling social value and it is also supported by experimental evidence. However, an important test for the credibility of such an approach is also considering the outcomes it produces. This is the main objective of this paper. It looked into the pairwise stable networks that emerge from applying such a heuristic. They are non-trivial and possess interesting properties. On the one hand one starting state can yield many possible outcomes. On the other hand, especially with larger networks, the outcomes frequently exhibit descriptive features that are observed in real-life social networks. They often have cliques, connected by several intermediaries, but also sometimes have a well connected core and distinct periphery in their structure.

It is interesting to note that while on the individual level the potential losses that are implied by following the heuristic are non-negligible, they do not result in big losses in utilitarian welfare when looking at the aggregate level, since the externalities the heuristic imposes are partially internalised. It is also worth noting that it is possible for people to be in the periphery of the network and to benefit more than the ones in more central positions — having a few good friends could be better than having many superficial acquaintances. Moreover, agents with the same degree can have different payoffs in the same network i.e. the correlation between the number of connections and payoff is not one-to-one and results in a diverse set of possible outcomes.

One of the most appealing features of the current setup is that the rich set of stable networks results from a minimal number of simple starting assumptions — the agents are not initially differentiated, they are matched randomly, use myopic decisions and do not calculate complicated best responses but still end up in particular identifiable structures. Therefore, the paper addresses the calls by Hämäläinen et al. (2013) and Harmsen-van Hout et al. (2016) and establishes that heuristic decision-making can be a prominent factor in OR processes and network formation in particular, because employing a simple heuristic goes a long way in replicating many observed features of real-life social networks.

Building on the current results, a promising possible extension of the model could be adding an informational dimension to it. This would undoubtedly change the set of stable networks depending on the specific payoffs from information and can be an additional robustness check for the networks identified in the current paper. In general, adding an indirect utility from links to the model could make it richer and capable of accounting for more reasons people form networks, e.g. to have greater access to expert opinions. Looking at an even larger set of different payoff functions for the links, e.g. functions with multiple additive elements, can give an indication to what extent the stable network structures are determined by the allocation of resources at the link level. Finally, it would be interesting to see if there are functional forms that can produce best response results (in terms of the allocation of resources) which come qualitatively close to the allocations prescribed by the heuristic in this paper.

Declaration of competing interest

The authors declare the following financial interests/personal relationships which may be considered as potential competing interests: Anastas P. Tenev reports financial support was provided by Hungarian National Research, Development and Innovation Office.

Data availability

Data will be made available on request.

Acknowledgements

This project has benefited greatly from the useful feedback at: seminar talks in Maastricht University, the Netherlands and the Otago Business School in Dunedin, New Zealand; poster presentation at the Conference in Honour of Hans Peters in Maastricht; conference talks at SING15 in Turku, Finland; the International Conference on Game Theory in Stony Brook, USA, and at the 25th CTN workshop in Barcelona, Spain. The author is especially grateful to Péter Bayer, Jean-Jacques Herings, Dominik Karos, Ronald Peeters, Frank Thuijsman and two anonymous referees for their invaluable help and comments.

Funding

This work was supported by the Hungarian National Research, Development and Innovation Office, Project Number K-143276.

Appendix A. Proofs

Proof of Proposition 4. To prove that a fully connected network is reachable starting from an empty graph with n nodes it is sufficient to show that:

(i) it can form a complete subgraph with $n \geq m \geq 2$ nodes;
(ii) that a new node could be connected to it;
(iii) when a new node is added to the complete subgraph it triggers a process leading to a new complete (sub)graph with $m+1$ nodes.
(iv) the process is repeated until $m = n$.

Step (i) follows directly for $m = 2$ since any two loose nodes prefer to be connected to being isolated. Step (ii) implicitly includes two conditions:

(a) at least one node from the complete subgraph wants to connect to one of the isolated nodes;
(b) the isolated node should also be willing to create the link.

As noted above, isolated nodes are always willing to form a link as any link brings them more than 0. Condition (a) is captured in (1) below.

$$\frac{1}{\sqrt{m\frac{m-1}{m-1}+\frac{m}{1}}}\left(\frac{m-1}{\sqrt{\frac{m-1}{m}+(m-2)\frac{m-1}{m-1}}}+\frac{1}{\sqrt{\frac{1}{m}}}\right)>1 \tag{1}$$

Simplified:

$$\frac{1}{\sqrt{2m}}\left(\frac{m-1}{\sqrt{\frac{m-1+m^2-2m}{m}}}+\sqrt{m}\right)>1 \Leftrightarrow \frac{m-1}{\sqrt{2(m^2-m-1)}}+\frac{1}{\sqrt{2}}>1$$

$$m-1+\sqrt{m^2-m-1}>\sqrt{2(m^2-m-1)} \Leftrightarrow m-1>\sqrt{m^2-m-1}(\sqrt{2}-1)$$

$$m^2-2m+1>(m^2-m-1)(3-2\sqrt{2}) \Leftrightarrow m^2(2\sqrt{2}-2)+m(1-2\sqrt{2})-2\sqrt{2}+4>0$$

This is always true for $m \geq 2$ because the discriminant of the left-hand side is negative. Hence, step (ii) is always possible.

Step (iii) also includes two parts:

(a) at least one node from the *formerly* complete subgraph of m nodes wants to connect to the *formerly* isolated node. This is captured in condition (3) below where c refers to the number of nodes in the complete graph that are connected to the *formerly* isolated node with $1 \leq c < m$.
(b) the *formerly* isolated node should also be willing to create the link. This is captured in condition (2) below.

$$\frac{1}{\sqrt{\frac{(c+1)(c+1)}{m}}}\frac{c+1}{\sqrt{\frac{m}{c+1}+\frac{m(m-c-1)}{m-1}+\frac{(c+1-1)m}{m}}}-\frac{1}{\sqrt{\frac{cc}{m}}}\frac{c}{\sqrt{\frac{m}{c}+\frac{m(m-c)}{m-1}+\frac{(c-1)m}{m}}}\geq 0 \tag{2}$$

Simplifies to:

$$\frac{1}{\sqrt{\frac{(m-c-1)}{m-1}+\frac{1}{c+1}+\frac{c}{m}}}-\frac{1}{\sqrt{\frac{m-c}{m-1}+\frac{1}{c}+\frac{c-1}{m}}}\geq 0 \Leftrightarrow \frac{m-c}{m-1}+\frac{1}{c}+\frac{c-1}{m}\geq\frac{(m-c-1)}{m-1}+\frac{1}{c+1}+\frac{c}{m} \Leftrightarrow \frac{1}{m-1}-\frac{1}{m}+\frac{1}{c}-\frac{1}{c+1}\geq 0$$

This holds for $m \geq 2, c \geq 1, c < m$.

Regarding (a) the following needs to hold:

$$\frac{1}{\sqrt{\frac{m(m-c-1)}{m-1}+\frac{m}{c+1}+\frac{m}{m}c}}\left(\frac{1}{\sqrt{\frac{(c+1)^2}{m}}}+\frac{c}{\sqrt{\frac{m(m-c-1)}{m-1}+\frac{m}{c+1}+\frac{m}{m}c}}+\frac{m-c-1}{\sqrt{\frac{(c+1)(m-1)}{m}+\frac{m-1}{m-1}(m-c-2)}}\right)-$$
$$\frac{1}{\sqrt{\frac{c(m-1)}{m}+\frac{m-1}{m-1}(m-c-1)}}\left(\frac{c}{\sqrt{\frac{m(m-c)}{m-1}+\frac{m}{c}+\frac{m}{m}(c-1)}}+\frac{m-c-1}{\sqrt{\frac{c(m-1)}{m}+\frac{m-1}{m-1}(m-c-1)}}\right)>0 \tag{3}$$

In order to show that (3) holds, it is separated in two parts. The first part (first line of (3)) will be shown to always be bigger than or equal to 1 (captured by condition (4) which follows) while the second part (second line of (3)) is always strictly smaller than 1 (captured by (11) below).

$$\frac{1}{\sqrt{\frac{m(m-c-1)}{m-1}+\frac{m}{c+1}+c}}\left(\frac{1}{\sqrt{\frac{(c+1)(c+1)}{m}}}+\frac{c}{\sqrt{\frac{m(m-c-1)}{m-1}+\frac{m}{c+1}+c}}+\frac{m-c-1}{\sqrt{m-c-2+\frac{(m-1)(c+1)}{m}}}\right)\geq 1 \tag{4}$$
$$\frac{\sqrt{m}}{(c+1)\sqrt{\frac{m(m-c-1)}{m-1}+\frac{m}{c+1}+c}}+\frac{c}{\frac{m(m-c-1)}{m-1}+\frac{m}{c+1}+c}+\frac{m-c-1}{\sqrt{(m-c-2+\frac{(m-1)(c+1)}{m})(\frac{m(m-c-1)}{m-1}+\frac{m}{c+1}+c)}}\geq 1$$

$$\sqrt{m}\sqrt{\left(\frac{m(m-c-1)}{m-1}+\frac{m}{c+1}+c\right)\left(m-c-2+\frac{(m-1)(c+1)}{m}\right)}+$$
$$c(c+1)\sqrt{m-c-2+\frac{(m-1)(c+1)}{m}}+(m-c-1)(c+1)\sqrt{\frac{m(m-c-1)}{m-1}+\frac{m}{c+1}+c}\geq$$
$$(c+1)\left(\frac{m(m-c-1)}{m-1}+\frac{m}{c+1}+c\right)\sqrt{m-c-2+\frac{(m-1)(c+1)}{m}}$$

For the conditions for the terms under the square root, see (9) and (10).

$$\sqrt{m}\sqrt{\frac{m^2(c+2)-m(c+2)-c^2-c}{(m-1)(c+1)}}\sqrt{\frac{m^2-m-c-1}{m}}+$$
$$c(c+1)\sqrt{\frac{m^2-m-c-1}{m}}+(m-c-1)(c+1)\sqrt{\frac{m^2(c+2)-m(c+2)-c^2-c}{(m-1)(c+1)}}\geq$$
$$(c+1)\left(\frac{m^2(c+2)-m(c+2)-c^2-c}{(m-1)(c+1)}\right)\sqrt{\frac{m^2-m-c-1}{m}}$$

Group the first and third term together and the second and last term together.

$$\sqrt{\frac{m^2(c+2)-m(c+2)-c^2-c}{(m-1)(c+1)}}\left(\sqrt{m^2-m-c-1}+(m-c-1)(c+1)\right)\geq$$
$$(c+1)\sqrt{\frac{m^2-m-c-1}{m}}\left(\frac{m^2(c+2)-m(c+2)-c^2-c}{(m-1)(c+1)}-c\right)$$

Simplify the right-hand side.

$$\sqrt{\frac{m^2(c+2)-m(c+2)-c^2-c}{(m-1)(c+1)}}\left(\sqrt{m^2-m-c-1}+(m-c-1)(c+1)\right)\geq\quad\sqrt{m(m^2-m-c-1)}\left(\frac{m(c+2)-(c^2+2c+2)}{m-1}\right)$$

Remove the denominators.

$$\sqrt{(m-1)\left(m^2(c+2)-m(c+2)-c^2-c\right)}\left(\sqrt{m^2-m-c-1}+(c+1)(m-c-1)\right)$$
$$\geq\sqrt{m(c+1)\left(m^2-m-c-1\right)}\left(m(c+2)-(c^2+2c+2)\right)\tag{5}$$

(5) holds if both parts (under the square root and what is in the brackets) of the left-hand side of the inequality are bigger than the corresponding parts of the right-hand side. This is checked separately — first the square root parts (see (6) below) of both sides and then the rest (see (8)).

$$\sqrt{(m-1)\left(m^2(c+2)-m(c+2)-c^2-c\right)}\geq\sqrt{m(c+1)\left(m^2-m-c-1\right)}\tag{6}$$

$$m^3(c+2)-m^2(c+2)-m(c^2+c)-m^2(c+2)+m(c+2)+c^2+c\geq m^3(c+1)-m^2(c+1)-mc(c+1)-m(c+1)$$

$$m^3-m^2(2c+4-c-1)-m(c^2+c-c-2-c^2-c-c-1)+c^2+c\geq 0$$

$$m^3(c+1)-m^2(c+3)+m(2c+3)+c^2+c\geq 0\tag{7}$$

(7) is true if $m^3(c+1)-m^2(c+3)\geq 0$ which is equivalent to $m\geq\frac{c+3}{c+1}$. $\frac{c+3}{c+1}$ decreases as c increases, its highest value for $c\geq 1$ being 2 and hence $m\geq 2$ always.

Now the second part of (5):

$$\sqrt{m^2-m-c-1}+(c+1)(m-c-1)\geq m(c+2)-(c^2+2c+2)\tag{8}$$

$$\sqrt{m^2-m-c-1}+m(c+1)-(c+1)^2\geq m(c+2)-(c+1)^2-1$$

$$\sqrt{m^2-m-c-1}\geq m-1\Leftrightarrow m^2-m-c-1\geq m^2-2m+1\Leftrightarrow m\geq c+2$$

Therefore (6) and (8) hold for $m\geq c+2$ and so does (5). What is left is to check is if (5) also holds for $m=c+1$. Substituting $c=m-1$ in (5) yields:

$$\sqrt{(m-1)\left(m^2(m+1)-m(m+1)-m(m-1)\right)}\left(\sqrt{m^2-m-m}+m(m-m)\right)\quad\geq\sqrt{m^2\left(m^2-m-m\right)}\left(m(m+1)-((m-1)(m+1)+2)\right)$$

$$\sqrt{m^2(m-1)^2}\sqrt{m^2-2m}\geq\sqrt{m^3(m-2)}(m-1)$$

This holds with equality. So, (5) and hence (4) holds also for $m = c + 1$. Therefore, (4) is true for $m \geq c + 1$. Here,

$$m - c - 2 + \frac{(m-1)(c+1)}{m} > 0 \tag{9}$$

$$(\frac{m}{m-1})(m - c - 1) + \frac{m}{c+1} + c > 0 \tag{10}$$

need to hold. (10) always holds for $m \geq c + 1$, $c \geq 1$ and $m \geq 2$. Now consider (9).

$$m^2 - mc - 2m + mc - c + m - 1 \geq 0 \Leftrightarrow m^2 - m - c - 1 \geq 0$$

which holds if $m \geq 2, c \geq 1$ when $m \geq \frac{1+\sqrt{4c+5}}{2}$ which is always true if $m \geq c + 1$. Therefore (4) is true for $m \geq c + 1$, $c \geq 1$ and $m \geq 2$.

Consider the second part of (3), transformed into condition (11):

$$\frac{1}{\sqrt{\frac{m-1}{m-1}(m-c-1) + \frac{c(m-1)}{m}}}\left(\frac{c}{\sqrt{\frac{m(m-c)}{m-1} + \frac{m}{m}(c-1) + \frac{m}{c}}} + \frac{m-c-1}{\sqrt{\frac{m-1}{m-1}(m-c-1) + \frac{(m-1)c}{m}}}\right) < 1 \tag{11}$$

$$\frac{m-c-1}{m-1-\frac{c}{m}} + \frac{c}{\sqrt{(\frac{m(m-c)}{m-1} + c - 1 + \frac{m}{c})(m-1-\frac{c}{m})}} < 1$$

$$\frac{m-c-1-m+1+\frac{c}{m}}{m-1-\frac{c}{m}} + \frac{c}{\sqrt{(\frac{m(m-c)}{m-1} + c - 1 + \frac{m}{c})(m-1-\frac{c}{m})}} < 0$$

$$\frac{\frac{c}{m}-c}{m-1-\frac{c}{m}} + \frac{c}{\sqrt{(\frac{m(m-c)}{m-1} + c - 1 + \frac{m}{c})(m-1-\frac{c}{m})}} < 0$$

Simplify by dividing by $c > 0$ and multiplying by $\sqrt{m-1-\frac{c}{m}}$.

$$\frac{\frac{1}{m}-1}{\sqrt{m-1-\frac{c}{m}}} + \frac{1}{\sqrt{\frac{m(m-c)}{m-1} + c - 1 + \frac{m}{c}}} < 0 \Leftrightarrow \frac{\frac{1-m}{m}}{\sqrt{\frac{m^2-m-c}{m}}} + \frac{1}{\sqrt{\frac{m(m-c)}{m-1} + c - 1 + \frac{m}{c}}} < 0$$

$$\frac{1}{\sqrt{\frac{m^2(c+1)-m(c+1)-c^2+c}{(m-1)c}}} < \frac{\frac{m-1}{m}}{\sqrt{\frac{m^2-m-c}{m}}} \Leftrightarrow \frac{\sqrt{(m-1)c}}{\sqrt{m^2(c+1)-m(c+1)-c^2+c}} < \frac{\frac{m-1}{m}\sqrt{m}}{\sqrt{m^2-m-c}}$$

$$\frac{\sqrt{c}}{\sqrt{m^2(c+1)-m(c+1)-c^2+c}} < \frac{\sqrt{m-1}}{\sqrt{m(m^2-m-c)}}$$

This is equivalent to:

$$\sqrt{\frac{cm(m^2-m-c)}{(m-1)(m^2(c+1)-m(c+1)-c^2+c)}} < 1$$

which holds if the numerator is smaller than the denominator, or:

$$cm(m^2 - m - c) < (m-1)(m^2(c+1) - m(c+1) - c^2 + c)$$

$$cm^3 - cm^2 - c^2m < m^3(c+1) - m^2(c+1) - mc^2 + mc - m^2(c+1) + m(c+1) + c^2 - c$$

$$0 < m^3 - m^2(c+2) + m(2c+1) + c^2 - c$$

This always holds true for $m \geq c + 2$. What is left is to check for $m = c + 1, m \geq 2$. Then the expression becomes:

$$m^3 - m^2(m+1) + m(2m-2+1) + (m-1)(m-2) > 0$$

$$m^3 - m^3 - m^2 + 2m^2 - m + m^2 - m - 2m + 2 > 0 \Leftrightarrow (m-1)^2 > 0$$

This is correct for $m \geq 2$. So, both (4) and (11) hold. Therefore, (3) is always true. Hence steps (iii) and (iv) are always possible. ☐

Proof of Proposition 5. To prove the statement it is sufficient to show that in any regular network with degree k which is not complete two of its nodes would want to form a connection. This is equivalent to the condition:

$$\frac{1}{\sqrt{\frac{k+1}{k+1} + k\frac{k+1}{k}}}\left(\frac{1}{\sqrt{\frac{k+1}{k+1} + k\frac{k+1}{k}}} + \frac{\alpha}{\sqrt{2\frac{k}{k+1} + (k-2)\frac{k}{k}}} + \frac{k-\alpha}{\sqrt{\frac{k}{k+1} + (k-1)\frac{k}{k}}}\right) > 1$$

where $\alpha \geq 1$ is the number of agents that are mutual neighbours of the two connecting nodes. The condition can be simplified.

$$\frac{1}{\sqrt{k+2}}\left(\frac{1}{\sqrt{k+2}}+\frac{\alpha\sqrt{k+1}}{\sqrt{k^2+k-2}}+\frac{(k-\alpha)\sqrt{k+1}}{\sqrt{k^2+k-1}}\right)>1 \tag{12}$$

Since

$$\frac{\sqrt{k+1}}{\sqrt{k^2+k-2}} \geq \frac{\sqrt{k+1}}{\sqrt{k^2+k-1}}$$

holds for every $k \geq 1$, taking $\alpha = 0$ presents the worst-case scenario for Inequality (12), i.e. the case in which its left-hand side has the lowest possible value, presenting the lowest incentive for the two nodes to connect. This leaves:

$$\frac{1}{\sqrt{k+2}}\left(\frac{1}{\sqrt{k+2}}+\frac{k\sqrt{k+1}}{\sqrt{k^2+k-1}}\right)>1 \Leftrightarrow \frac{1}{k+2}+\frac{k\sqrt{k+1}}{\sqrt{(k^2+k-1)(k+2)}}>1 \Leftrightarrow \frac{k\sqrt{k+1}}{\sqrt{(k^2+k-1)(k+2)}}>\frac{k+1}{k+2}$$

$$k\sqrt{k+2} > \sqrt{(k+1)(k^2+k-1)} \Leftrightarrow k^3+2k^2 > k^3+k^2-k+k^2+k-1$$

which is always true for $k \geq 3$. □

Proof of Proposition 6. This proof contains four parts.

(A) Line: The two ends would want to connect, since:

$$\frac{1}{\sqrt{\frac{1}{2}}}\frac{1}{\sqrt{\frac{2}{1}+\frac{2}{1}}}<1;\ \frac{1}{\sqrt{\frac{1}{2}}}\frac{1}{\sqrt{\frac{2}{1}+\frac{2}{2}}}<1$$

The first inequality refers to a line of length 3, while the second one covers all cases of longer lines.

(B) Star: It is sufficient to show that two of the periphery nodes of a star network with $n \geq 3$ would want to form a link. The condition is:

$$\frac{1}{\sqrt{\frac{2}{n-1}+\frac{2}{2}}}\left(\frac{1}{\sqrt{\frac{n-1}{2}2+\frac{n-1}{1}(n-3)}}+\frac{1}{\sqrt{\frac{2}{n-1}+\frac{2}{2}}}\right) \geq \frac{1}{\sqrt{\frac{1}{n-1}}}\frac{1}{\sqrt{(n-1)\frac{n-1}{1}}}$$

Simplified:

$$\frac{1}{\sqrt{\frac{2}{n-1}+1}}\left(\frac{1}{\sqrt{n-1+(n-1)(n-3)}}+\frac{1}{\sqrt{\frac{2}{n-1}+1}}\right) \geq \frac{1}{\sqrt{n-1}}$$

$$\frac{1}{\sqrt{\frac{n+1}{n-1}}}\left(\frac{1}{\sqrt{(n-1)(n-2)}}+\frac{1}{\sqrt{\frac{n+1}{n-1}}}\right) \geq \frac{1}{\sqrt{n-1}} \Leftrightarrow \frac{1}{\sqrt{(n+1)(n-2)}}+\frac{n-1}{n+1} \geq \frac{1}{\sqrt{n-1}}$$

$$\frac{\sqrt{n-1}}{\sqrt{(n+1)(n-2)}}+\frac{(n-1)\sqrt{n-1}}{n+1} \geq 1 \tag{13}$$

$\frac{(n-1)\sqrt{n-1}}{n+1}$ is bigger than 1 for $n \geq 4$ and it has a positive first derivative. The first term of (13) is always positive. Finally, a specific check for $n = 3$ shows that (13) holds for integers $n \geq 3$.

(C) Wheel: Here it is sufficient to prove that at least one node would want to *disconnect* from the centre. Taking a peripheral node the following inequality should hold:

$$\frac{1}{\sqrt{2*\frac{3}{3}+\frac{3}{n}}}\left(\frac{2}{\sqrt{2*\frac{3}{3}+\frac{3}{n}}}+\frac{1}{\sqrt{n\frac{n}{3}}}\right) < \frac{1}{\sqrt{2*\frac{2}{3}}}\frac{2}{\sqrt{\frac{3}{2}+\frac{3}{3}+\frac{3}{n}}} \tag{14}$$

where the right-hand side expresses the payoff of disconnecting from the centre. Simplifying:

$$\frac{1}{\sqrt{\frac{2n+3}{n}}}\left(\frac{2}{\sqrt{\frac{2n+3}{n}}}+\frac{\sqrt{3}}{n}\right) < \frac{\sqrt{3}}{\sqrt{\frac{5}{2}+\frac{3}{n}}} \Leftrightarrow \frac{1}{\sqrt{\frac{2n+3}{n}}}\left(\frac{2}{\sqrt{\frac{2n+3}{n}}}+\frac{\sqrt{3}}{n}\right) < \frac{\sqrt{3}}{\sqrt{\frac{5n+6}{2n}}}$$

$$\frac{2n}{2n+3}+\frac{\sqrt{3}}{\sqrt{n(2n+3)}} < \sqrt{\frac{6n}{5n+6}} \tag{15}$$

The right-hand side of condition (15) is greater than or equal to 1 for $n \geq 6$. The left-hand side is strictly smaller than 1 in this range of values for n:

$$\frac{2n}{2n+3}+\frac{\sqrt{3}}{\sqrt{n(2n+3)}}<1 \Leftrightarrow \frac{\sqrt{3}}{\sqrt{n(2n+3)}}<\frac{3}{2n+3} \Leftrightarrow \frac{1}{\sqrt{n}}<\frac{\sqrt{3}}{\sqrt{2n+3}} \Leftrightarrow 2n+3<3n$$

Therefore, condition (14) holds for $n \geq 6$ and in these cases the wheel is not stable. To show that when $n \in \{4,5\}$ the wheel is also not stable, the conditions for two peripheral nodes to *connect* are checked separately. For $n=4$:

$$\frac{1}{\sqrt{2*\frac{3}{3}+\frac{3}{n}}}\left(\frac{2}{\sqrt{2*\frac{3}{3}+\frac{3}{n}}}+\frac{1}{\sqrt{n\frac{n}{3}}}\right)<\frac{1}{\sqrt{\frac{4}{n}+2*\frac{4}{3}+\frac{4}{4}}}\left(\frac{1}{\sqrt{\frac{4}{n}+2*\frac{4}{3}+\frac{4}{4}}}+\frac{1}{\sqrt{2*\frac{n}{4}+(n-2)\frac{n}{3}}}+\frac{2}{\sqrt{2*\frac{3}{4}+\frac{3}{n}}}\right)$$

And for $n=5$:

$$\frac{1}{\sqrt{2*\frac{3}{3}+\frac{3}{n}}}\left(\frac{2}{\sqrt{2*\frac{3}{3}+\frac{3}{n}}}+\frac{1}{\sqrt{n\frac{n}{3}}}\right)<\frac{1}{\sqrt{\frac{4}{n}+2*\frac{4}{3}+\frac{4}{4}}}\left(\frac{1}{\sqrt{\frac{4}{n}+2*\frac{4}{3}+\frac{4}{4}}}+\frac{1}{\sqrt{2*\frac{n}{4}+(n-2)\frac{n}{3}}}+\frac{1}{\sqrt{2*\frac{3}{4}+\frac{3}{n}}}+\frac{1}{\sqrt{\frac{3}{n}+\frac{3}{4}+\frac{3}{3}}}\right)$$

Both conditions hold for the specific values of n. Therefore, the wheel with $n \geq 4$ in the periphery is not stable.

(D) This proof will be presented in two parts — (D1) which deals with complete biregular graphs and (D2) which deal with incomplete biregular graphs.

(D1) In a bipartite graph with m and $k=n-m$ nodes in the two sets such that $m \leq k$ at least one set of nodes would want to connect and therefore the network would not be stable, since the payoff of a connection between two nodes in the m-set would be always preferred to the status quo:

$$\frac{1}{\sqrt{\frac{m^2}{k}}}*\frac{m}{\sqrt{\frac{k^2}{m}}} \leq \frac{1}{\sqrt{\frac{m(m+1)}{k}+\frac{m+1}{m+1}}}\left(\frac{1}{\sqrt{\frac{m(m+1)}{k}+\frac{m+1}{m+1}}}+\frac{m}{\sqrt{\frac{k(k-2)}{m}+\frac{2k}{m+1}}}\right)$$

Simplify:

$$\frac{1}{\sqrt{\frac{(m+1)m+n}{k}}}\left(\frac{1}{\sqrt{\frac{(m+1)m+k}{k}}}+\frac{m}{\sqrt{\frac{k(k-2)(m+1)+2\ mk}{m(m+1)}}}\right) \geq \sqrt{\frac{m}{k}} \tag{16}$$

$$\frac{k}{(m+1)m+k}+\frac{m\sqrt{m(m+1)}}{\sqrt{((m+1)m+k)((k-2)(m+1)+2\ m)}} \geq \sqrt{\frac{m}{k}}$$

$$n\sqrt{k((k-2)(m+1)+2\ m)}+m\sqrt{mk(m+1)((m+1)m+k)} \geq ((m+1)m+k)\sqrt{m((k-2)(m+1)+2\ m)}$$

$$k\sqrt{k(mk+k-2)}+m\sqrt{mk(m+1)(m^2+m+k)} \geq (m^2+m+n)\sqrt{m(mk+k-2)}$$

$$k\sqrt{k(mk+k-2)} \geq \sqrt{m^2+m+k}(\sqrt{m(m^2+m+k)(mk+k-2)}-m\sqrt{mk(m+1)})$$

Take the first part of the left-hand side and the second part of the right-hand side.

$$k\sqrt{k} \geq \sqrt{m(m^2+m+k)(mk+k-2)}-m\sqrt{mk(m+1)}$$

Rearrange:

$$k\sqrt{k}+m\sqrt{mk(m+1)} \geq \sqrt{m(m^2+m+k)(mk+k-2)}$$

Square both sides of the condition.

$$k^3+2k^2m\sqrt{m(m+1)}+m^3k(m+1) \geq (m^2(m+1)+km)(k(m+1)-2)$$

$$k^3+2k^2m\sqrt{m(m+1)}+m^3k(m+1) \geq km^2(m+1)^2+k^2m(m+1)-2m^2(m+1)-2km$$

$$k^3+2k^2m\sqrt{m(m+1)}+m^2k(m+1)(m-m-1) \geq k^2m(m+1)-2m^2(m+1)-2km$$

$$k^3+2m^2(m+1)+2km+2k^2m\sqrt{m(m+1)}-m^2k(m+1)-k^2m(m+1) \geq 0$$

Consider only the last three terms (the first of them is split) $k^2m\sqrt{m(m+1)}-m^2k(m+1)+k^2m\sqrt{m(m+1)}-k^2m(m+1)$. Each of them is positive separately, except for $k=m+1$. This requires a separate check.

Now take the second part of the left-hand side and the first part of the right-hand side.

$$\sqrt{mk+k-2} \geq \sqrt{m^2+m+k} \tag{17}$$

Inequality (17) is true for $k \geq m+1+\frac{2}{m}$, so what is left is to see that condition (16) holds for $m \in \{k, k-1, k-2\}$. A manual check shows this is also true. Therefore a complete bipartite graph is not stable in this setting.

(D2) In a biregular graph which is not complete at least two different nodes which are not in the same subgroup of nodes (with degrees k and ℓ) would want to connect.

$$\frac{1}{\sqrt{\frac{k+1}{\ell+1}+\frac{(k+1)k}{\ell}}}\left(\frac{1}{\sqrt{\frac{\ell+1}{k+1}+\frac{(\ell+1)\ell}{k}}}+\frac{k}{\sqrt{\frac{\ell(\ell-1)}{k}+\frac{\ell}{k+1}}}\right) \geq \frac{1}{\sqrt{\frac{k*k}{\ell}}}\frac{k}{\sqrt{\frac{\ell*\ell}{k}}}$$

Simplify:

$$\frac{1}{\sqrt{\frac{(k+1)(\ell+k\ell+k)}{(\ell+1)\ell}}}\left(\frac{1}{\sqrt{\frac{(\ell+1)(\ell+k\ell+k)}{(k+1)k}}}+\frac{k}{\sqrt{\frac{\ell(k\ell+\ell-1)}{k(k+1)}}}\right) \geq \sqrt{\frac{k}{\ell}} \Leftrightarrow \frac{\sqrt{\ell k}}{\ell+k\ell+k}+\frac{k\sqrt{k(\ell+1)}}{\sqrt{(\ell+k\ell+k)(k\ell+\ell-1)}} \geq \sqrt{\frac{k}{\ell}}$$

$$\frac{\ell}{\ell+k\ell+k}+\frac{k\sqrt{\ell(\ell+1)}}{\sqrt{(\ell+k\ell+k)(k\ell+\ell-1)}} \geq 1 \Leftrightarrow \frac{k\sqrt{\ell(\ell+1)}}{\sqrt{(\ell+k\ell+k)(k\ell+\ell-1)}} \geq \frac{k\ell+k}{\ell+k\ell+k}$$

$$\frac{\sqrt{\ell}}{\sqrt{k\ell+\ell-1}} \geq \frac{\sqrt{\ell+1}}{\sqrt{\ell+k\ell+k}} \Leftrightarrow \frac{\ell}{k\ell+\ell-1} \geq \frac{\ell+1}{\ell+k\ell+k}$$

$$\ell(\ell+k\ell+k) \geq \ell(k\ell+\ell-1)+k\ell+\ell-1 \Leftrightarrow \ell(k+1) \geq k\ell+\ell-1$$

This is always true. □

Proof of Proposition 7. As already showed in Proposition 3 no node in a complete graph would disconnect from the rest. Therefore, in order to prove the statement it is sufficient to show that the two subgraphs would not form links for $m_1 \geq m_2+2$ (here it is assumed that $m_1 > m_2$ without loss of generality). The conditions for two nodes from the subgraphs to want to form a link are:

$$\frac{1}{\sqrt{\frac{m_2}{m_1}+\frac{(m_2-1)m_2}{m_2-1}}}\left(\frac{1}{\sqrt{\frac{m_1}{m_2}+\frac{(m_1-1)m_1}{m_1-1}}}+\frac{m_2-1}{\sqrt{\frac{m_2-1}{m_2}+\frac{(m_2-2)(m_2-1)}{m_2-1}}}\right) \geq 1$$

$$\frac{1}{\sqrt{\frac{m_1}{m_2}+\frac{(m_1-1)m_1}{m_1-1}}}\left(\frac{1}{\sqrt{\frac{m_2}{m_1}+\frac{(m_2-1)m_2}{m_2-1}}}+\frac{m_1-1}{\sqrt{\frac{m_1-1}{m_1}+\frac{(m_1-2)(m_1-1)}{m_1-1}}}\right) \geq 1$$

where one of them needs to hold strictly. They could be simplified to:

$$\frac{1}{\sqrt{\frac{m_2(m_1+1)}{m_1}}}\left(\frac{1}{\sqrt{\frac{m_1(m_2+1)}{m_2}}}+\frac{m_2-1}{\sqrt{\frac{m_2^2-m_2-1}{m_2}}}\right) \geq 1 \tag{18}$$

$$\frac{1}{\sqrt{\frac{m_1(m_2+1)}{m_2}}}\left(\frac{1}{\sqrt{\frac{m_2(m_1+1)}{m_1}}}+\frac{m_1-1}{\sqrt{\frac{m_1^2-m_1-1}{m_1}}}\right) \geq 1 \tag{19}$$

The first parts of the expressions on the left-hand side of the inequalities, $\frac{1}{\sqrt{\frac{m_1(m_2+1)}{m_2}}}\frac{1}{\sqrt{\frac{m_2(m_1+1)}{m_1}}}$, are the same. Comparing the second parts, it is true that:

$$\frac{(m_2-1)\sqrt{m_1}}{\sqrt{(m_1+1)(m_2^2-m_2-1)}} \geq \frac{(m_1-1)\sqrt{m_2}}{\sqrt{(m_2+1)(m_1^2-m_1-1)}}$$

because, for $m_1 \geq m_2$, it always holds that:

$$\frac{\sqrt{m_1(m_1^2-m_1-1)}}{(m_1-1)\sqrt{m_1+1}} \geq \frac{\sqrt{m_2(m_2^2-m_2-1)}}{(m_2-1)\sqrt{m_2+1}}.$$

Therefore, condition (19) is binding and it is sufficient to show that it holds for both (18) and (19) to hold. For $m_1 \geq 4$, condition (19) is equivalent to:

$$\sqrt{m_1^2 - m_1 - 1} + (m_1 - 1)\sqrt{m_2(m_1+1)} \geq \sqrt{(m_1+1)(m_2+1)(m_1^2 - m_1 - 1)}$$

$$m_1^2 - m_1 - 1 + m_2(m_1-1)^2(m_1+1) + 2(m_1-1)\sqrt{m_2(m_1+1)(m_1^2 - m_1 - 1)} \geq (m_1+1)(m_2+1)(m_1^2 - m_1 - 1)$$

$$m_2(m_1^3 - m_1^2 - m_1 - 1 - m_1^3 + 2m_1 + 1) + 2\sqrt{m_2(m_1+1)(m_1^2 - m_1 - 1)}(m_1 - 1) - m_1(m_1^2 - m_1 - 1) \geq 0$$

This yields: $\frac{m_1^2 - m_1 - 1}{m_1+1} \leq m_2 \leq \frac{m_1\sqrt{(m_1+1)(m_1^2-m_1-1)}}{(m_1-2)(m_1+1)}$, but since $m_1 \geq m_2$, it simplifies to $\frac{m_1^2-m_1-1}{m_1+1} \leq m_2 \leq m_1$ or $m_1 - 1 - \frac{m_1}{m_1+1} \leq m_2 \leq m_1$. Therefore, conditions (18) and (19) hold for only $m_2 \in \{m_1 - 1, m\}$. □

Appendix B. Equal split

Proof of Proposition 9. It is sufficient to show that any other networks would have (at least) two nodes which would always want to make a connection. The strategy in this proof is excluding all networks in which it is clear that there are two nodes which want to form a connection and considering what this implies for all other networks.

The payoff of a node i with N_i being the set of its neighbours and d_i being its degree can be expressed and rearranged in the following way:

$$u_i = \sum_{j\in N_i} \sqrt{\frac{1}{d_i} * \frac{1}{d_j}} = \sum_{j\in N_i} \frac{1}{\sqrt{d_i * d_j}} = \frac{1}{\sqrt{d_i}}\left(\sum_{j\in N_i} \frac{1}{\sqrt{d_j}}\right).$$

In order for two nodes i and k to want to connect the following inequalities should hold with at least one of them being strict.

$$\frac{1}{\sqrt{d_i}}\left(\sum_{j\in N_i}\frac{1}{\sqrt{d_j}}\right) \leq \frac{1}{\sqrt{d_i+1}}\left(\sum_{j\in N_i}\frac{1}{\sqrt{d_j}} + \frac{1}{\sqrt{d_k+1}}\right) \tag{20}$$

$$\frac{1}{\sqrt{d_k}}\left(\sum_{j\in N_k}\frac{1}{\sqrt{d_j}}\right) \leq \frac{1}{\sqrt{d_k+1}}\left(\sum_{j\in N_k}\frac{1}{\sqrt{d_j}} + \frac{1}{\sqrt{d_i+1}}\right) \tag{21}$$

Note that d_i and N_i refer to the situation before a connection has been made and so $k \notin N_i$. Inequality (20) expresses that agent i would be better off connecting to agent k, because his current payoff (left-hand side) is smaller than the payoff he would have if agent k was his direct neighbour. In this case (right-hand side) he would split his resources in $d_i + 1$ equal parts and get $1/(d_k + 1)$ of k's resources as investment.[26] Consider inequality (20), which can be rewritten as:

$$\left(\frac{\sqrt{d_i+1} - \sqrt{d_i}}{\sqrt{d_i} * \sqrt{d_i+1}}\right)\left(\sum_{j\in N_i}\frac{1}{\sqrt{d_j}}\right) = \left(\frac{1}{\sqrt{d_i}} - \frac{1}{\sqrt{d_i+1}}\right)\left(\sum_{j\in N_i}\frac{1}{\sqrt{d_j}}\right) \leq \frac{1}{\sqrt{d_i+1}} * \frac{1}{\sqrt{d_k+1}}$$

$$\Leftrightarrow \sum_{j\in N_i}\sqrt{\frac{d_k+1}{d_j}} \leq \frac{\sqrt{d_i}}{\sqrt{d_i+1} - \sqrt{d_i}} \Leftrightarrow \sum_{j\in N_i}\sqrt{\frac{d_k+1}{d_j}} \leq \sqrt{d_i}(\sqrt{d_i+1} + \sqrt{d_i}) \Leftrightarrow$$

$$\sum_{j\in N_i}\sqrt{\frac{d_k+1}{d_j}} \leq d_i + \sqrt{d_i(d_i+1)} \tag{22}$$

Clearly, inequality (21) can be rewritten in a similar fashion. □

Lemma 2. *In a stable network if nodes i, x and k are such that $d_i \leq d_k, d_x \leq d_k$ and $i \in N_k$, then node i wants to connect to node x.*

Proof of Lemma 2. In a stable network if i is connected to k this implies that it also wants to be connected to it, otherwise the network would not be stable. Here it is useful to distinguish two cases: (i) $d_x < d_k$ and (ii) $d_x = d_k$.

Since the network is stable i does not want to delete the link with k, so:

$$\sum_{j\in N_i}\sqrt{\frac{d_k}{d_j}} \leq d_i - 1 + \sqrt{d_i(d_i-1)} \tag{23}$$

In case $d_x < d_k$, Inequality (23) implies:

$$\sum_{j\in N_i}\sqrt{\frac{d_x+1}{d_j}} \leq \sum_{j\in N_i}\sqrt{\frac{d_k}{d_j}} \leq d_i - 1 + \sqrt{d_i(d_i-1)}$$

[26] Inequality (21) expresses the analogous idea for agent k connecting to agent i.

Excluding the middle part and adding $\sqrt{\frac{d_x+1}{d_k}} \leq 1$ on both sides implies:[27]

$$\sum_{j\in N_i}\sqrt{\frac{d_x+1}{d_j}}+\sqrt{\frac{d_x+1}{d_k}} \leq d_i-1+\sqrt{d_i(d_i-1)}+\sqrt{\frac{d_x+1}{d_k}} < d_i+\sqrt{d_i(d_i+1)}$$

$$\sum_{j\in N_i\cup\{k\}}\sqrt{\frac{d_x+1}{d_j}} < d_i+\sqrt{d_i(d_i+1)}$$

Therefore, i is willing to connect to x.

In case $d_x = d_k$, multiply both sides of Inequality (23) with $\sqrt{\frac{d_k+1}{d_k}} > 1$ to get:

$$\sum_{j\in N_i}\sqrt{\frac{d_k+1}{d_j}} \leq d_i\sqrt{\frac{d_k+1}{d_k}}-\sqrt{\frac{d_k+1}{d_k}}+\sqrt{\frac{d_i(d_i-1)(d_k+1)}{d_k}} \Leftrightarrow \sum_{j\in N_i}\sqrt{\frac{d_k+1}{d_j}}+\sqrt{\frac{d_k+1}{d_k}} \leq d_i\sqrt{\frac{d_k+1}{d_k}}+\sqrt{\frac{d_i(d_i-1)(d_k+1)}{d_k}}$$

$$\sum_{j\in N_i\cup\{k\}}\sqrt{\frac{d_x+1}{d_j}} = \sum_{j\in N_i}\sqrt{\frac{d_k+1}{d_j}}+\sqrt{\frac{d_k+1}{d_k}} \leq d_i\sqrt{\frac{d_k+1}{d_k}}+\sqrt{\frac{d_i(d_i-1)(d_k+1)}{d_k}} \tag{24}$$

In order for i to want to connect to x, the following is sufficient for the right-hand side:

$$d_i\sqrt{\frac{d_k+1}{d_k}}+\sqrt{\frac{d_i(d_i-1)(d_k+1)}{d_k}} < d_i+\sqrt{d_i(d_i+1)} \Leftrightarrow \sqrt{d_i}\sqrt{\frac{d_k+1}{d_k}}(\sqrt{d_i}+\sqrt{d_i-1}) < \sqrt{d_i}(\sqrt{d_i}+\sqrt{d_i+1})$$

$$(\sqrt{d_i}+\sqrt{d_i-1})\sqrt{d_k+1} < \sqrt{d_k}(\sqrt{d_i}+\sqrt{d_i+1}) \Leftrightarrow \sqrt{d_i(d_k+1)}+\sqrt{(d_i-1)(d_k+1)} < \sqrt{d_k(d_i+1)}+\sqrt{d_i d_k}$$

$$\sqrt{d_i d_k+d_i}+\sqrt{d_i d_k-d_k+d_i-1} < \sqrt{d_i d_k+d_k}+\sqrt{d_i d_k}$$

The condition holds since $d_k \geq d_i \geq 1$ (no node wants to stay singleton with a zero payoff and everyone would want to have a link with a singleton). Therefore, Inequality (24) is equivalent to Condition (22) and i wants to connect to x in both cases outlined above. □

Lemma 3. *In a stable network if nodes i, x and k are such that $d_i \leq d_k, d_x \leq d_k, i \in N_k$ and $x \in N_k$, then $x \in N_i$.*

Proof of Lemma 3. By Lemma 2 if k has a neighbour i with $d_i \leq d_k$, that means i is willing to connect to all nodes with degrees $\leq d_k$, which includes x. By the same token, x wants to be connected to all nodes with degrees $\leq d_k$, which includes i. Therefore, in a stable network all neighbours x of k with $d_x \leq d_k$ will be also i's neighbours. □

By Lemma 3 in a stable network all neighbours of the node with the highest degree within a component, say h, form a clique, i.e. they are fully interconnected and connected to h. Hence, they must all have degrees $\geq d_h$. They cannot have degrees strictly bigger than d_h, because that would contradict the assumption that d_h has the highest degree in the component. Therefore, they must have equal degrees. In other words, in a stable network any component forms a regular subgraph. Moreover, the subgraphs are complete as every node is connected to all other nodes in the component as per Lemma 3. □

Appendix C. Farsightedly stable networks

Following Herings et al. (2009), farsightedly stable networks are defined via *farsighted improving paths*.

Definition 4. Farsighted Improving Path] A farsighted improving path from a network g to a network $g' \neq g$ is a finite sequence of graphs $g_1, \dots, g_K$, with $g_1 = g$ and $g_K = g'$ such that for any $k \in \{1, \dots, K-1\}$ either:

(i) $g_{k+1} = g_k - ij$ for some ij such that $u_i(g_K) > u_i(g_k)$ or $u_j(g_K) > u_j(g_k)$, or
(ii) $g_{k+1} = g_k + ij$ for some ij such that $u_i(g_K) > u_i(g_k)$ and $u_j(g_K) \geq u_j(g_k)$

For a given network g, let $F(g) = \{g' \in G | g \rightarrow g'\}$ be the set of networks that can be reached by a farsighted improving path from g. Now *pairwise farsighted stability* can be defined.

Definition 5 (*Pairwise Farsightedly Stable Networks*)**.** A set of networks G^F is pairwise farsightedly stable if:

(i) $\forall g \in G^F$,

  (a) $\forall ij \notin g$ such that $g+ij \notin G^F, \exists g' \in F(g+ij) \cap G^F$ such that $(u_i(g'), u_j(g')) = (u_i(g), u_j(g))$ or $u_i(g') < u_i(g)$ or $u_j(g') < u_j(g)$.

[27] The strict inequality comes from the change of signs under the square root.

(b) $\forall ij \in g$, such that $g - ij \notin G^F, \exists g', g'' \in F(g - ij) \cap G^F$ such that $u_i(g') \leq u_i(g)$ and $u_j(g'') \leq u(g)$.

(ii) $\forall g' \in G \setminus G^F, F(g') \cap G^F \neq \emptyset$.
(iii) $\nexists G' \subsetneq G^F$ such that G' satisfies conditions $(ia), (ib)$ and (ii).

In words, a set of networks is pairwise farsightedly stable if (i) all possible *pairwise* deviations to a network outside of this set are deterred by a credible threat of ending worse off or equally well off; (ii) there exists a farsighted improving path from any network outside the set to a network in the set; (iii) this is the minimal set which satisfies the previous two conditions. Observe that, as Herings et al. (2009) show, G^F always exists.

Proof of Proposition 10. The proof of this statement follows the general structure of the proof of Proposition 1 in Herings et al. (2009). The proof shows that for every $g \in G \setminus G^F, \exists g^F \subseteq G^F$ such that $g^F \in F(g)$. This is done in two parts.

(i) For any $g \notin G^F$, there is always a node who wants to delete a link looking forward to some $g^F \in G^F$. This enables building a sequence of networks, where at each step there is a node which is looking forward to some $g^F \in G^F$ and is deleting a link as a consequence of that. This process continues until a network with *only* regular components and singletons is reached.
(ii) Starting from this network, it is possible to build a sequence of networks towards $g^F \in G^F$, so that at each step links are only added and nodes who are adding links are strictly better off compared to the current network. This step is essentially repeating parts of the proof of Proposition 4, which shows that there is a path from the empty to the complete network.

Taken together, the two steps imply that from any possible network which does not have only regular components, there exists a farsighted improving path to a network with only regular components.

To show (i), take any g such that $W(g) \neq n$. There exists a player $i \in N$ with $u_i(g) < 1$ (cf. Lemma 1). Looking forward to some $g^F \in G^F$ (in which $u_i(g^F) = 1$), the player is willing to delete a link. Take a non-singleton non-regular part of the network. It is true that $\exists j$ s.t. $u_j < 1$. Then this node wants to sever links looking forward to $g^F \in G^F$. This process continues until the component is irregular (i.e. until it becomes regular). Once a regular component is reached, there is no longer a player in the component who wants to sever links. The procedure can be repeated until a network with regular components and singletons is reached.

To show (ii), consider a network with only singletons and regular components. All agents in the regular components have a utility of 1. Now it is useful to distinguish between two cases:

(a) There are multiple singleton agents. In this case, the construction of Proposition 4 is useful. That is, it is possible to construct a sequence of subnetworks *only* using the singleton agents, such that the sequence ends with a complete subnetwork and the agents improve at every step. For this it needs to be the case that at every step the two nodes which build a link prefer the end network to the current one (at least one of them strictly). In case there are only two singletons, this is clearly the case. If there are more, first they form a pair, i.e. a complete component. Now, every new agent (singleton) who is added to this should prefer the end network (where he gets a payoff of 1) to the current one. In other words the second part of condition (2) should be smaller than 1.

$$\frac{1}{\sqrt{\frac{cc}{m}}}\frac{c}{\sqrt{\frac{m}{c}+\frac{m(m-c)}{m-1}+\frac{(c-1)m}{m}}}<1 \iff \frac{1}{\sqrt{\frac{m-c}{m-1}+\frac{1}{c}+\frac{c-1}{m}}}<1 \iff \frac{m-c}{m-1}+\frac{1}{c}+\frac{c-1}{m}>1$$

This always holds for $m \geq 2, c \geq 1, c < m$. Additionally, every agent who is in the (formerly) complete component and not yet connected to the singleton should (at least weakly) prefer their future payoff (looking forward to the network with a complete component) to their current one. For the first step of the process, when there is a complete component and a singleton which will be integrated in it, the agent in the complete component weakly prefers the end payoff to their current payoff of 1. This does not impede the construction, because the other agent strictly prefers their end payoff to the current one (0). Once a link is established between a complete component and a singleton, every next step is equivalent to condition (11), which is proven above.

(b) There is only one singleton agent i. In this case the computations from the first case are the same with the provision that m is not the number of agents in the component. Instead, the degree of all agents in this regular component is $m-1$ (analogous to the case of a complete component with m nodes). After that all computations follow. The singleton agent can first "destabilise" the regular component, making it irregular and in this way it can produce at least one more singleton. That is, when agent i forms its first link with agent j in the regular component, a neighbour of j, say $k \in N_j$ will have a utility $u_k < 1$ (once again following condition (11)). In this case, looking forward to a network with regular components, he can sever the link to j. However, looking forward to a network with regular components, agent i can now sever the link he just made with k. This is the case, because when $t_{ji} < 1$, the utility of agent i is always $u_i = \sqrt{t_{ij} * t_{ji}} = \sqrt{t_{ji}} < 1$ (agent i invests everything in the link with j). As a result, the starting regular component is not regular anymore and agent i is still a singleton. Therefore, as in (i) above, the irregular component will produce at least one additional singleton. With at least two singletons, case (a) above applies.

Therefore, from every network which does not have maximal welfare there is a farsighted improving path to a network with regular components, which has maximal welfare. So, the networks in the farsighted stable set have maximal utilitarian welfare. □

Appendix D. Extensions

One can view the heuristic investigated in this paper and the equal split of resources between different neighbours as extreme cases of a common model, which just varies the weight that a node puts of the degree of its neighbours. In the case of equal split the degree of a neighbour plays no role, while for the heuristic used in this paper it is inversely proportional to the resources allocation. A more general version of the payoff could be expressed in the following way:

$$u_i = \sum_{n\in N_i} \sqrt{ \frac{\frac{1}{d_n^\alpha}}{\sum_{j\in N_i}\frac{1}{d_j^\alpha}} \frac{\frac{1}{d_i^\alpha}}{\sum_{l\in N_n}\frac{1}{d_l^\alpha}} } = \frac{1}{\sqrt{\sum_{j\in N_i}\left(\frac{d_i}{d_j}\right)^\alpha}} \left(\sum_{n\in N_i}\frac{1}{\sqrt{\sum_{l\in N_n}\left(\frac{d_n}{d_l}\right)^\alpha}}\right)$$

where $\alpha = 0$ corresponds to the case in which the resources are spread equally amongst all neighbours and $\alpha = 1$ is the current heuristic. For $0 \le \alpha \le 1$ Propositions 3 and 5 hold.

Proof of Proposition 3EXT. In a fully connected graph every node has payoff 1. In order for the complete graph to be stable, removing a link should be equally good or worse than the status quo for both nodes that are connected so that they decide not to disconnect. For a complete graph with n nodes, given that $1 \ge \alpha \ge 0$ this is equivalent to:

$$\frac{1}{\sqrt{(n-2)\frac{(n-2)^\alpha}{(n-1)^\alpha}}}\left(\frac{n-2}{\sqrt{2\frac{(n-1)^\alpha}{(n-2)^\alpha}+\frac{(n-1)^\alpha}{(n-1)\alpha}(n-3)}}\right) \le 1 \Leftrightarrow \frac{n-2}{\sqrt{\frac{(n-2)(2(n-1)^\alpha+(n-3)(n-2)^\alpha)}{(n-1)^\alpha}}} \le 1$$

$$(n-2)(n-1)^\alpha \le 2(n-1)^\alpha + (n-3)(n-2)^\alpha \Leftrightarrow (n-4)(n-1)^\alpha \le (n-3)(n-2)^\alpha$$

$$\left(\frac{n-1}{n-2}\right)^\alpha \le \frac{n-3}{n-4}$$

Both parts of the inequality are bigger than 1 for all positive $n > 4$. Moreover, as α comes closer to 1, the left-hand side grows. Therefore, the biggest value for the left-hand side is $\frac{n-1}{n-2}$. In this case the inequality still holds for $n > 4$. What is left is to check if this holds for $n \in \{3, 4\}$. For $n = 3$:

$$\frac{1}{\sqrt{1\frac{1^\alpha}{2^\alpha}}}\frac{1}{\sqrt{2\frac{2^\alpha}{1^\alpha}}} \le 1$$

For $n = 4$:

$$\frac{1}{\sqrt{2\frac{2^\alpha}{3^\alpha}}}\frac{2}{\sqrt{2\frac{3^\alpha}{2^\alpha}+1}} \le 1$$

Both conditions hold. Therefore, no two nodes would decide to disconnect in a complete graph. □

Proof of Proposition 5EXT. 5EXT To prove the statement it is sufficient to show that in any regular network with degrees n and $1 \ge \alpha \ge 0$ which is not complete two of its nodes would want to form a connection. This is equivalent to the condition:

$$\frac{1}{\sqrt{\frac{(n+1)^\alpha}{(n+1)^\alpha}+n\frac{(n+1)^\alpha}{n^\alpha}}}\left(\frac{1}{\sqrt{\frac{(n+1)^\alpha}{(n+1)^\alpha}+n\frac{(n+1)^\alpha}{n^\alpha}}}+\frac{\beta}{\sqrt{2\frac{n^\alpha}{(n+1)^\alpha}+(n-2)\frac{n^\alpha}{n^\alpha}}}+\frac{n-\beta}{\sqrt{\frac{n^\alpha}{(n+1)^\alpha}+(n-1)\frac{n^\alpha}{n^\alpha}}}\right) > 1$$

where $\beta \ge 0$ is the number of agents that are mutual neighbours of the two connecting nodes. The condition can be simplified.

$$\frac{\sqrt{n^\alpha}}{\sqrt{n^\alpha+n(n+1)^\alpha}}\left(\frac{\sqrt{n^\alpha}}{\sqrt{n^\alpha+n(n+1)^\alpha}}+\frac{\beta\sqrt{(n+1)^\alpha}}{\sqrt{2n^\alpha+(n-2)(n+1)^\alpha}}+\frac{(n-\beta)\sqrt{(n+1)^\alpha}}{\sqrt{n^\alpha+(n-1)(n+1)^\alpha}}\right) > 1 \tag{25}$$

Since

$$\frac{\sqrt{(n+1)^\alpha}}{\sqrt{2n^\alpha+(n-2)(n+1)^\alpha}} \ge \frac{\sqrt{(n+1)^\alpha}}{\sqrt{n^\alpha+(n-1)(n+1)^\alpha}}$$

holds for every $n \ge 1, 1 \ge \alpha \ge 0$, taking $\beta = 0$ presents the worst-case scenario for Inequality (25), i.e. the case in which its left-hand side has the lowest possible value, presenting the lowest incentive for the two nodes to connect. This leaves:

$$\frac{\sqrt{n^\alpha}}{\sqrt{n^\alpha+n(n+1)^\alpha}}\left(\frac{\sqrt{n^\alpha}}{\sqrt{n^\alpha+n(n+1)^\alpha}}+\frac{n\sqrt{(n+1)^\alpha}}{\sqrt{n^\alpha+(n-1)(n+1)^\alpha}}\right) > 1 \Leftrightarrow \frac{n^\alpha}{n^\alpha+n(n+1)^\alpha}+\frac{n\sqrt{n^\alpha(n+1)^\alpha}}{\sqrt{(n^\alpha+(n-1)(n+1)^\alpha)(n^\alpha+n(n+1)^\alpha)}} > 1$$

$$\frac{n\sqrt{n^\alpha(n+1)^\alpha}}{\sqrt{(n^\alpha+(n-1)(n+1)^\alpha)(n^\alpha+n(n+1)^\alpha)}} > \frac{n(n+1)^\alpha}{n^\alpha+n(n+1)^\alpha} \Leftrightarrow \frac{\sqrt{n^\alpha}}{\sqrt{n^\alpha+(n-1)(n+1)^\alpha}} > \frac{\sqrt{(n+1)^\alpha}}{\sqrt{n^\alpha+n(n+1)^\alpha}}$$

$$n^{2\alpha} + n^{\alpha+1}(n+1)^{\alpha} > n^{\alpha}(n+1)^{\alpha} + (n-1)(n+1)^{2\alpha}$$

$$\left(\frac{n}{n+1}\right)^{2\alpha} + \left(\frac{n}{n+1}\right)^{\alpha}(n-1) - (n-1) > 0$$

This expression is at its lowest for α as big as possible. Taking $\alpha = 1$, the expression is always true for $n \geq 3$. □

Proof of Proposition 6EXT. This proof looks at the first two parts of Proposition 6.

(A) Line: The two ends would want to connect, since:

$$\frac{1}{\sqrt{\frac{1^{\alpha}}{2^{\alpha}}}} \frac{1}{\sqrt{\frac{2^{\alpha}}{1^{\alpha}} + \frac{2^{\alpha}}{1^{\alpha}}}} < 1; \frac{1}{\sqrt{\frac{1^{\alpha}}{2^{\alpha}}}} \frac{1}{\sqrt{\frac{2^{\alpha}}{1^{\alpha}} + \frac{2^{\alpha}}{2^{\alpha}}}} < 1$$

The first inequality refers to a line of length 3, while the second one covers all other cases.

(B) Star: It is sufficient to show that two of the periphery nodes of a star network with $n \geq 3$ would want to form a link. The condition is:

$$\frac{1}{\sqrt{\frac{2^{\alpha}}{(n-1)^{\alpha}} + \frac{2^{\alpha}}{2^{\alpha}}}} \left(\frac{1}{\sqrt{\frac{(n-1)^{\alpha}}{2^{\alpha}}2 + \frac{(n-1)^{\alpha}}{1^{\alpha}}(n-3)}} + \frac{1}{\sqrt{\frac{2^{\alpha}}{(n-1)^{\alpha}} + \frac{2^{\alpha}}{2^{\alpha}}}} \right) \geq \frac{1}{\sqrt{\frac{1^{\alpha}}{(n-1)^{\alpha}}}} \frac{1}{\sqrt{(n-1)\frac{(n-1)^{\alpha}}{1^{\alpha}}}}$$

Simplified:

$$\frac{1}{\sqrt{\frac{2^{\alpha}+(n-1)^{\alpha}}{(n-1)^{\alpha}}}} \left(\frac{1}{\sqrt{(n-1)^{\alpha}\left(\frac{2+2^{\alpha}(n-3)}{2^{\alpha}}\right)}} + \frac{1}{\sqrt{\frac{2^{\alpha}+(n-1)^{\alpha}}{(n-1)^{\alpha}}}} \right) \geq \frac{1}{\sqrt{n-1}} \Leftrightarrow \frac{2^{\alpha}}{\sqrt{(2^{\alpha}+(n-1)^{\alpha})(2+2^{\alpha}(n-3))}} + \frac{(n-1)^{\alpha}}{2^{\alpha}+(n-1)^{\alpha}} \geq \frac{1}{\sqrt{n-1}}$$

Taking the second part:

$$\frac{(n-1)^{\alpha}}{2^{\alpha}+(n-1)^{\alpha}} \geq \frac{1}{\sqrt{n-1}} \tag{26}$$

$\frac{(n-1)^{\alpha}}{2^{\alpha}+(n-1)^{\alpha}}$ is bigger than $1/2$ for $n = 4$ and it has a positive first derivative w.r.t. n so as n increases the term becomes bigger. The derivative is:

$$\frac{\alpha(n-1)^{\alpha}}{2^{\alpha}+(n-1)^{\alpha}} - \frac{\alpha(n-1)^{2\alpha}}{(2^{\alpha}+(n-1)^{\alpha})^2}$$

The first term of (26) is always positive. Finally, a specific check for $n = 3$ shows that (26) holds for integers $n \geq 3$. □

Appendix E. Cycle with fixed order

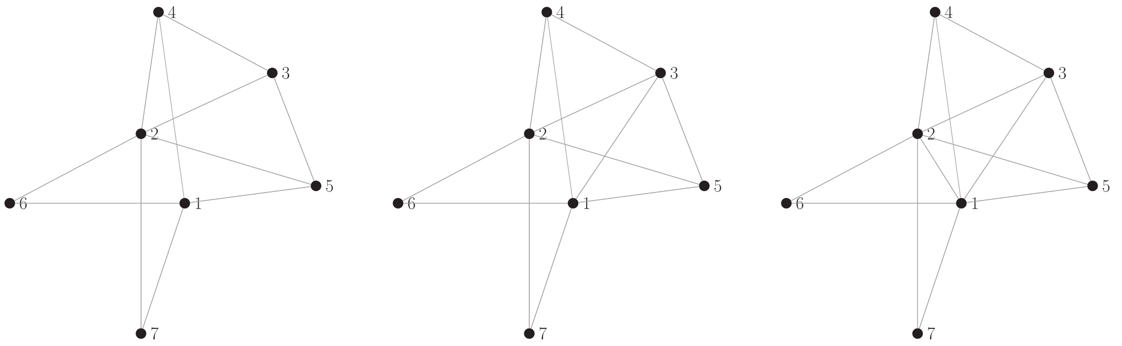
This is a description of a short cycle that can occur within this setup if a specific order of operations is followed. There are two procedures — addition and deletion of a link. The addition checks if *one* link could be added. The deletion checks if *any number* of links could be gradually removed. First is the addition procedure after which the deletion procedure begins. The deletion cuts links step by step (restarting the checking procedure after every change) until no links can be removed and only then can the addition procedure start again. The deletion and addition procedures alternate until no links can be added or removed and then the process stops.

Importantly, there is a fixed order followed for every check after the adjacency matrix has been changed (this includes every time a link has been severed) — always starting from the cell $(1,1)$, continuing along the row and going to the next row after checking the whole row. Since the adjacency matrix is symmetric, only the values above the main diagonal are checked.

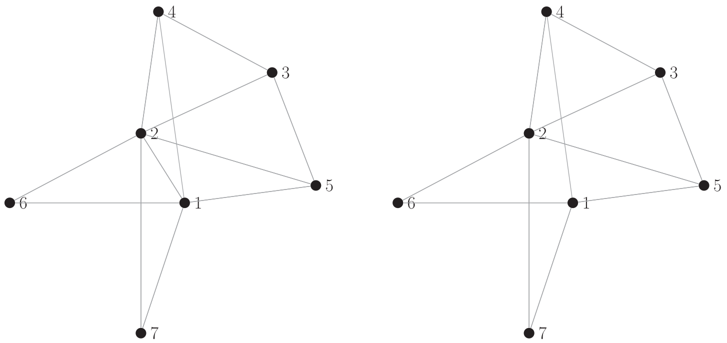
Consider Fig. 4(a). The overall welfare is $W = 6.813$. The contributions of each node to each node and the payoffs at each node are: (i) for 1: $(\frac{1}{5}, \frac{1}{5}, \frac{3}{10}, \frac{3}{10})$, payoff 1.322; (ii) for 2: $(\frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{4}, \frac{1}{4})$, payoff 1.275; (iii) for 3: $\frac{3}{13}, \frac{5}{13}, \frac{5}{13}$, payoff 1.005; (iv) for 4, 5: $\frac{15}{47}, \frac{12}{47}, \frac{20}{47}$, payoff 0.863; (v) for 6, 7: $\frac{5}{9}, \frac{4}{9}$, payoff 0.742.

In Fig. 4(b) overall welfare is $W = 6.752$ and the difference to (a) is that nodes 1 and 3 are connected. To see why consider the corresponding payoffs: (i) for 1, 2: $\frac{1}{\sqrt{\frac{5\,2}{3}+\frac{5}{4}+\frac{5\,2}{2}}}\left(\frac{2}{\sqrt{\frac{3}{4}+\frac{2\,3}{5}}} + \frac{1}{\sqrt{\frac{4\,2}{3}+\frac{2\,4}{5}}} + \frac{2}{\sqrt{\frac{2\,2}{5}}}\right) = 1.341$; (ii) for 3: $\frac{1}{\sqrt{\frac{4\,2}{3}+\frac{4\,2}{5}}}\left(\frac{2}{\sqrt{\frac{3}{4}+\frac{2\,3}{5}}} + \frac{2}{\sqrt{\frac{5}{4}+\frac{2\,5}{3}+\frac{5\,2}{2}}}\right) = 1.006$; (iii) for 4, 5: $\frac{1}{\sqrt{\frac{3}{4}+\frac{3\,2}{5}}}\left(\frac{1}{\sqrt{\frac{4\,2}{3}+\frac{2\,4}{5}}} + \frac{2}{\sqrt{\frac{5}{4}+\frac{2\,5}{3}+\frac{5\,2}{2}}}\right) = 0.809$; (iv) for 6, 7: $\frac{1}{\sqrt{\frac{2\,2}{5}}}\frac{2}{\sqrt{\frac{5}{4}+\frac{2\,5}{3}+\frac{5\,2}{2}}} = 0.722$.

A connection between 1 and 2 would make 1 worse off (cf. Fig. 4(d)), since the payoff of 1 would be $\frac{1}{\sqrt{\frac{5\,2}{2}+\frac{5}{6}+\frac{5\,2}{3}}}\left(\frac{2}{\sqrt{\frac{2}{5}+\frac{2}{6}}} + \frac{2}{\sqrt{\frac{3}{3}+\frac{3}{5}+\frac{3}{6}}} + \frac{1}{\sqrt{\frac{6\,2}{2}+\frac{6}{5}+\frac{6\,3}{3}}}\right) = 1.318$. And as a result nodes 1 and 3 connect, since both are better off connected. However, now no two nodes should be willing to sever their link. Clearly, 1 and 3 would not want to disconnect.



(a) Initial stage (b) 1 and 3 connect (c) 1 and 2 connect



Node	1	2	3	W
(a/e)	1.322	1.275	1.005	6.813
(b)	1.341	1.341	1.006	6.752
(c)	1.342	1.342	1.039	6.719
(d)	1.318	1.288	1.047	6.776

(d) 1 and 3 disconnect (e) 1 and 2 disconnect ⇒ (a). (f) Node payoffs and welfare (W)

Fig. 4. Cycle with fixed order for $n = 7$.

If 1 and 4 disconnect their corresponding payoffs would be: (i) for 1: $\frac{1}{\sqrt{\frac{4}{4}+\frac{4}{3}+\frac{4\,2}{2}}}\left(\frac{1}{\sqrt{\frac{3}{5}+\frac{2\,3}{4}}}+\frac{1}{\sqrt{\frac{4}{2}+\frac{4}{4}+\frac{4}{3}+\frac{4}{5}}}+\frac{2}{\sqrt{\frac{2}{5}+\frac{2}{4}}}\right)=1.287$; (ii) for 4: $\frac{1}{\sqrt{\frac{2}{4}+\frac{2}{5}}}\left(\frac{1}{\sqrt{\frac{4}{2}+\frac{4}{4}+\frac{4}{3}+\frac{4}{5}}}+\frac{1}{\sqrt{\frac{5}{4}+\frac{5}{3}+\frac{5\,3}{2}}}\right)=0.792$. So, they would not disconnect. Same reasoning holds for 1 and 5.

If 1 and 6 disconnect their corresponding payoffs would be: (i) for 1: $\frac{1}{\sqrt{\frac{4\,2}{3}+\frac{4}{4}+\frac{4}{2}}}\left(\frac{1}{\sqrt{\frac{2}{4}+\frac{2}{5}}}+\frac{1}{\sqrt{\frac{4}{4}+\frac{4\,2}{3}+\frac{4}{5}}}+\frac{2}{\sqrt{\frac{3}{5}+\frac{2\,3}{4}}}\right)=1.221$; (ii) for 6: $\frac{1}{\sqrt{\frac{1}{5}}\sqrt{\frac{5}{4}+\frac{2\,5}{3}+\frac{5}{2}+5\frac{1}{1}}}=0.643$ So, they would not disconnect. Same reasoning holds for 1 and 7 and 2 and all its neighbours. If 3 and 4 disconnect their corresponding payoffs would be: (i) for 3: $\frac{1}{\sqrt{\frac{3}{3}+\frac{3\,2}{5}}}\left(\frac{1}{\sqrt{\frac{3}{3}+\frac{2\,3}{5}}}+\frac{2}{\sqrt{\frac{3\,5}{2}+\frac{2\,5}{2}}}\right)=0.836$; (ii) for 4: $\frac{2}{\sqrt{\frac{4\,2}{5}}\sqrt{\frac{3\,5}{2}+\frac{2\,5}{3}}}=0.480$. So, they would not disconnect. Same reasoning holds for 3 and 5. Therefore, no nodes would want to disconnect in Fig. 4(b). This leads to the next check if additional nodes want to form a connection.

In Fig. 4(c) the overall welfare is $W=6.719$ and nodes 1 and 2 have a link. To see why, consider the corresponding payoffs of 1, 2 which are $\frac{1}{\sqrt{\frac{6\,2}{2}+\frac{6}{6}+\frac{6}{4}+\frac{6\,2}{3}}}\left(\frac{1}{\sqrt{\frac{4\,2}{6}+\frac{4\,2}{3}}}+\frac{1}{\sqrt{\frac{6\,2}{2}+\frac{6}{6}+\frac{6}{4}+\frac{6\,2}{3}}}+\frac{2}{\sqrt{\frac{3}{4}+\frac{3\,2}{6}}}+\frac{2}{\sqrt{\frac{2\,2}{6}}}\right)=1.342$. So, they would connect since it improves both of them strictly.

At this point, 3 would choose to disconnect from 1 (Fig. 4(d)) since it gets a payoff of $\frac{1}{\sqrt{\frac{2\,3}{3}+\frac{3}{5}}}\left(\frac{2}{\sqrt{\frac{3}{3}+\frac{3}{6}+\frac{3}{5}}}+\frac{1}{\sqrt{\frac{3\,6}{6}+\frac{6}{5}+\frac{6}{2}}}\right)=1.087$ as compared to the payoff in (c), $\frac{1}{\sqrt{\frac{2\,4}{3}+\frac{4\,2}{6}}}\left(\frac{2}{\sqrt{\frac{3\,2}{6}+\frac{3}{4}}}+\frac{2}{\sqrt{\frac{6\,2}{2}+\frac{6}{6}+\frac{6}{4}+\frac{6\,2}{3}}}\right)=1.039$. There the overall welfare is $W=6.776$.

In Fig. 4(e), 2 would choose to disconnect from 1 since it gets a payoff of 1.341 (same as in (a)) as compared to the payoff in (d), $\frac{1}{\sqrt{\frac{5\,2}{2}+\frac{5}{6}+\frac{5\,2}{3}}}\left(\frac{2}{\sqrt{\frac{2}{5}+\frac{2}{6}}}+\frac{2}{\sqrt{\frac{3}{3}+\frac{3}{5}+\frac{3}{6}}}+\frac{1}{\sqrt{\frac{6\,2}{2}+\frac{6}{5}+\frac{6\,3}{3}}}\right)=1.318$. This completes the cycle. It must be noted that the overall welfare in the different instances first decreases and then increases.

References

Albornoz, F., Cabrales, A., Hauk, E., 2019. Occupational choice with endogenous spillovers. Econ. J. 129 (621), 1953–1970.
Bala, V., Goyal, S., 2000. A noncooperative model of network formation. Econometrica 68 (5), 1181–1229.
Baron, R., Durieu, J., Haller, H., Solal, P., 2006. Complexity and stochastic evolution of dyadic networks. Comput. Oper. Res. 33 (2), 312–327.
Baumann, L., 2021. A model of weighted network formation. Theor. Econ. 16 (1), 1–23.

Bloch, F., Dutta, B., 2009. Communication networks with endogenous link strength. Games Econom. Behav. 66 (1), 39–56.
Bourlès, R., Bramoullé, Y., Perez-Richet, E., 2017. Altruism in networks. Econometrica 85 (2), 675–689.
Bramoullé, Y., Currarini, S., Jackson, M.O., Pin, P., Rogers, B.W., 2012. Homophily and long-run integration in social networks. J. Econom. Theory 147 (5), 1754–1786.
Brueckner, J.K., 2006. Friendship networks. J. Reg. Sci. 46 (5), 847–865.
Cover, T.M., Thomas, J.A., 2006. Elements of Information Theory, second ed. In: A Wiley-Interscience Publication, Wiley-Interscience, Hoboken, N.J..
Cowan, R., Jonard, N., 2004. Network structure and the diffusion of knowledge. J. Econom. Dynam. Control 28 (8), 1557–1575.
Currarini, S., Jackson, M.O., Pin, P., 2009. An economic model of friendship: Homophily, minorities, and segregation. Econometrica 77 (4), 1003–1045.
Deroian, F., 2009. Endogenous link strength in directed communication networks. Math. Social Sci. 57 (1), 110–116.
Fagiolo, G., 2005. Endogenous neighborhood formation in a local coordination model with negative network externalities. J. Econom. Dynam. Control 29 (1–2), 297–319.
Galeotti, A., Goyal, S., Jackson, M.O., Vega-Redondo, F., Yariv, L., 2010. Network games. Rev. Econ. Stud. 77 (1), 218–244.
Granovetter, M.S., 1974. Getting a Job : A Study of Contacts and Careers. Harvard University Press, Cambridge, Mass.
Griffith, A., 2022. A continuous model of strong and weak ties. J. Public Econ. Theory 24 (6), 1519–1563.
Hämäläinen, R.P., Luoma, J., Saarinen, E., 2013. On the importance of behavioral operational research: The case of understanding and communicating about dynamic systems. European J. Oper. Res. 228 (3), 623–634.
Hancock, P.G., Raeside, R., 2010. Analysing communication in a complex service process: An application of social network analysis in the scottish prison service. J. Oper. Res. Soc. 61 (2), 265–274.
Hellmann, T., 2013. On the existence and uniqueness of pairwise stable networks. Int. J. Game Theory 42 (1), 211–237.
Hellmann, T., 2021. Pairwise stable networks in homogeneous societies with weak link externalities. European J. Oper. Res. 291 (3), 1164–1179.
Hellmann, T., Staudigl, M., 2014. Evolution of social networks. European J. Oper. Res. 234 (3), 583–596.
Herings, P.J.-J., Mauleon, A., Vannetelbosch, V., 2009. Farsightedly stable networks. Games Econom. Behav. 67 (2), 526–541.
Harmsen-van Hout, M.J.W., Dellaert, B.G.C., Herings, P.J.-J., 2016. Heuristic decision making in network linking. European J. Oper. Res. 251 (1), 158–170.
Harmsen-van Hout, M.J.W., Herings, P.J.-J., Dellaert, B.G.C., 2013. Communication network formation with link specificity and value transferability. European J. Oper. Res. 229 (1), 199–211.
Jackson, M.O., Rogers, B.W., 2007. Meeting strangers and friends of friends: How random are social networks? Am. Econ. Rev. 97 (3), 890–915.
Jackson, M.O., Watts, A., 2001. The existence of pairwise stable networks. Seoul J. Econ. 14 (3).
Jackson, M.O., Wolinsky, A., 1996. A strategic model of social and economic networks. J. Econom. Theory 71 (1), 44–74.
Janssen, R.H.P., Monsuur, H., 2012. Stable network topologies using the notion of covering. European J. Oper. Res. 218 (3), 755–763.
Kirman, A., Markose, S., Giansante, S., Pin, P., 2007. Marginal contribution, reciprocity and equity in segregated groups: Bounded rationality and self-organization in social networks. J. Econom. Dynam. Control 31 (6), 2085–2107.
Kovářík, J., Mengel, F., Romero, J.G., 2018. Learning in network games. Quant. Econ. 9 (1), 85–139.
Li, X., 2023. Designing weighted and directed networks under complementarities. Games Econom. Behav. 140, 556–574.
McKay, B.D., Piperno, A., 2014. Practical graph isomorphism, {II}. J. Symbolic Comput. 60, 94–112.
McPherson, M., Smith-Lovin, L., Cook, J.M., 2001. Birds of a feather: Homophily in social networks. Annu. Rev. Sociol. 27 (1), 415–444.
Monsuur, H., 2007. Stable and emergent network topologies: A structural approach. European J. Oper. Res. 183 (1), 432–441.
Morbitzer, D., Buskens, V., Rosenkranz, S., Raub, W., 2014. How farsightedness affects network formation. Anal. Kritik 36 (1), 103.
Olaizola, N., Valenciano, F., 2014. Asymmetric flow networks. European J. Oper. Res. 237 (2), 566–579.
Rêgo, L.C., dos Santos, A.M., 2019. Co-authorship model with link strength. European J. Oper. Res. 272 (2), 587–594.
Salonen, H., 2016. Reciprocal equilibria in link formation games. Czech Econ. Rev. 9 (3), 169–183.
So, C., 2016. Network formation with endogenous link strength and decreasing returns to investment. Games 7 (4).
Staudigl, M., Weidenholzer, S., 2014. Constrained interactions and social coordination. J. Econom. Theory 152, 41–63.
Zuckerman, D., 2024. Multidimensional homophily. J. Econ. Behav. Organ. 218, 486–513.