J. Econ. Theory 225 (2025) 105987

Contents lists available at ScienceDirect

Journal of Economic Theory

journal homepage: www.elsevier.com/locate/jet

Feedback design in games with ambiguity-averse players ☆

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ARTICLE INFO

JEL classification:
C72
D82

Keywords:
Self-confirming equilibrium
Ambiguity aversion
Information feedback
Strategic uncertainty
Public good games
Volunteer dilemma

ABSTRACT

We use a notion of maxmin self-confirming equilibrium (MSCE) to study the design of players' information feedback about others' behavior in simultaneous-move games with ambiguity-averse players. Coarse feedback shapes strategic uncertainty and can, therefore, modify players' equilibrium strategies in an advantageous way. We characterize MSCE and study the equilibrium implications of coarse feedback in various classes of games. We show how feedback should be optimally designed to improve contributions in generalized volunteer dilemmas and public good games with strategic substitutes, strategic complements, or more general production functions. We also study games with negative externalities and strategic substitutes, such as Cournot oligopolies. In general, perfect and no feedback are suboptimal. Some results are extended to α-maxmin preferences.

1. Introduction

A standard justification for Nash equilibrium in n-player simultaneous-move games is that a Nash equilibrium strategy profile corresponds to a steady state of a collective learning process at the population level. Whenever the game is played, each player $i \in N$ playing the game is drawn from a large population of individuals assigned to the role of player i. Each individual plays the game only once, and for each population, the proportion of individuals playing each available action in this population is perfectly observed by all individuals.

The equilibrium justification above assumes perfect feedback: individuals in each population i observe the aggregate behavior of individuals in every other population $j \neq i$, ensuring their subjective theories about others' behavior are accurate at the steady state. In other words, as recently described by Spiegler (2021), a Nash equilibrium relies on players having sufficiently rich "archival access" to steady-state behavior. However, in many cases, feedback about each population's behavior–and thus such archival access–can be imperfect. For example, in public goods problems, individuals may only observe whether total or population-specific contributions in past plays reached a certain threshold. Similarly, in volunteer dilemmas, such as crowdsourcing platforms (e.g., Wikipedia editing or citizen science projects), emergency evacuations, or journal review processes, individuals often receive only coarse feedback about how many others volunteer. They might be informed that "just enough people volunteered" or that volunteers "fell short by one," without knowing the exact frequency of volunteering in each population. This coarse feedback creates uncertainty about others'

☆ We thank the editor Pierpaolo Battigalli, the three anonymous referees, Philippe Bich, Michael Greinecker, Philippe Jehiel, Tristan Tomala, and Guillaume Vigeral for their useful feedback. Frederic Koessler acknowledges the support of the ANR (ANR-19-CE26-0010-01). Marieke Pahlke acknowledges the support of the ANR (ANR-18-ORAR-0005 and ANR-19-CE26-0010-01) and DFG (Ri 1128-9-1).

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https://doi.org/10.1016/j.jet.2025.105987
Received 27 July 2023; Received in revised form 15 February 2025; Accepted 17 February 2025

Available online 19 February 2025

behavior and can influence decisions to volunteer, as individuals may incorrectly guess whether their help is needed or if others will step up.

When individuals receive coarse feedback, a whole range of subjective beliefs about the aggregate behavior in each population could be consistent with this feedback. Under the assumption of subjective expected utility maximization, if the only restriction imposed on individuals' subjective beliefs is feedback consistency, the appropriate equilibrium concept is the notion of self-confirming equilibrium (Battigalli, 1987, Fudenberg and Levine, 1993). Because actual behaviors are consistent with any feedback, a Nash equilibrium is always a self-confirming equilibrium, and the set of Nash and self-confirming equilibria coincide under perfect feedback.[1]

Under perfect feedback, steady-state beliefs are accurate: individuals in population i observe that the share of individuals playing action a_j in population j is $s_j(a_j)$. Therefore, in a steady state, they know that the probability of a player drawn from population j playing action a_j is equal to $s_j(a_j)$. Under coarse feedback, individuals do not perfectly observe the proportions of individuals playing each action in every population. Hence, there can be multiple steady-state frequencies of play consistent with the given feedback, allowing for a multiplicity of steady-state beliefs. If individuals are ambiguity averse, they take this multiplicity of beliefs into consideration when deciding how to play. In the most extreme case of ambiguity aversion, each individual only considers the worst-case scenario when deciding which strategy to play. A steady-state behavior in such a setting corresponds to what has been called a partially specified equilibrium by Lehrer (2012) or a maxmin self-confirming equilibrium by Battigalli et al. (2015), and Battigalli et al. (2016).[2]

In this paper, we study a notion of maxmin self-confirming equilibrium adapted from Lehrer (2012) and Battigalli et al. (2015, 2016) to study how equilibrium behavior of ambiguity-averse individuals is affected by coarse feedback. We study the impact of coarse feedback from a design perspective. In standard Bayesian incentive problems (e.g., Myerson, 1982) or information design settings (e.g., Kamenica and Gentzkow, 2011, Bergemann and Morris, 2016, Taneva, 2019), players know each other's strategies and the probability distribution over states, while the designer controls players' signals. In contrast, here, players do not know others' strategies, and the designer influences strategic uncertainty by managing players' feedback. Using the terminology of Spiegler (2021), information design is concerned with designing "news access," i.e., conventional signals in the Harsanyi model, whereas feedback design centers on designing "archival access" or knowledge of steady-state behavior.

There are a few other approaches in which partial feedback is used as an instrument to affect the steady state of collective learning in games. For example, Esponda (2008) studies the effect of coarse feedback in first-price auctions on the set of self-confirming equilibrium bidding strategies and revenues. Jehiel (2011) uses the notion of analogy-based expectation equilibrium to study revenues in some classes of auction formats when bidders get partial feedback about the distribution of bids submitted in earlier auctions. There, feedback is represented by analogy partitions of decision nodes from which players construct their subjective beliefs about the average behavior at these nodes. More recently, Spiegler (2021) studies games extended with payoff-irrelevant variables, where players receive partial feedback about these variables, others' behavior, and the state of nature, and then form beliefs using maximum-entropy extrapolation.

In the spirit of other data-driven equilibrium concepts like Bayesian network equilibrium (Spiegler, 2016), analogy-based expectation equilibrium (Jehiel, 2005, 2022), or Berk-Nash equilibrium (Esponda and Pouzo, 2016), players in our model do not necessarily observe others' behavior. Each player relies solely on aggregate statistics from past interactions of other players. We assume a designer (like a regulator, manager, experimental economist, training coordinator, auction house, social networking service, or online gaming platform) has full access to past interaction data. The designer can summarize and simplify this information to shape how players perceive others' behavior, thus influencing how they think their actions will affect their payoffs. More concretely, consider a designer having a large spreadsheet containing all past action profiles (with each row representing an occurrence and each column showing the action of a player in the role of player j). The designer can choose to disclose this spreadsheet to each player but may also remove or aggregate data. In a binary action game (with actions 0 or 1), the designer could add a column showing 1 if the sum of actions in a row exceeds a threshold k, and 0 otherwise, then remove all other columns. By revealing this new spreadsheet, players would know, before playing the game, how often at least k players took action 1. In our applications, this feedback is called public (all players receive the same spreadsheet) and anonymous (the spreadsheet remains the same if the original columns are shuffled). Alternatively, for each pair of players in roles i and j, the designer might reveal information about column j to player i independent of the columns $k \neq i, j$. This kind of feedback is called separable. For example, he might reveal column j entirely to player i or not at all. In this case, player i would either learn nothing about the behavior of players in role j or know exactly how often they chose actions 1 or 0. Such separable feedback based on social networks has been explored by Lipnowski and Sadler (2019) in both simultaneous-move and dynamic games.[3] A richer form of separable feedback can also be imagined: for each pair of players, only up to one-third of the action 1 entries in column j might be kept, leaving the rest blank. In this case, if players in role j chose action 1 more than one-third of the time, then player i would indeed learn that players in role j chose action 1 more than one-third of the time, but not more.

How should such a designer design feedback in order to induce favorable equilibrium behavior? For instance, in public good problems, can the designer increase voluntary contributions by exploiting individuals' aversion to ambiguity and providing only

[1] Nash and self-confirming equilibria also coincide in more general cases, see, e.g., Battigalli et al. (2015).

[2] There are alternative definitions of equilibrium under strategic uncertainty in which uncertainty is represented by capacities; see, for example, Stauber (2019) and references therein. In some models, as in Bade (2011) or Riedel and Sass (2014), players are allowed to explicitly use non-additive or ambiguous randomization devices to select their strategies. Another trend of the literature studies refinements of Nash equilibrium by imposing properties of robustness to small strategic uncertainty (see, for example, Bich, 2019).

[3] Unlike our approach, their concept of peer-confirming equilibrium does not assume worst-case beliefs based on feedback but includes a rationalizability condition to narrow down players' beliefs about others' behavior.

coarse feedback about individuals' behavior in past occurrences of the game? Is cooperation improved or deteriorated if players receive more or less ambiguous feedback, or if players are more or less ambiguity averse?

We study simultaneous-move games with a finite number of players and an infinite set of strategies S_j for each player j. For finite games with a finite action set A_j for each player j, the set of strategies S_j of player j is identified with the set of mixed actions $\Delta(A_j)$.[4] Thus, we consider both public good games with contributions represented by a number in the interval $[0, 1]$, and finite games such as volunteer dilemma games. Each player receives feedback about the strategies of his opponents. This feedback induces a partition of the opponents' strategy sets, which can potentially depend on player i's strategy. Specifically, if each player $j \neq i$ plays a strategy s_j, then player i only knows that the strategy profile s_{-i} played by the other players belongs to the partition element containing s_{-i}.

We consider ambiguity-averse players and define a notion of maxmin self-confirming equilibrium (MSCE) closely aligned with the framework of Battigalli et al. (2016). The key difference is that it accommodates feedback that directly depends on the profile of mixed actions of opponents and, more generally, on a continuum of strategies. In an MSCE, players are extremely pessimistic and maximize their worst-case expected utility with respect to any strategy profile of the opponents that they consider possible given their feedback. To assess how robust some of our results are, we also define α-maxmin self-confirming equilibrium (α-MSCE), in which players consider best- and worst-case scenarios and maximize a weighted sum of best- and worst-case expected utilities. If players receive perfect feedback, they learn the equilibrium strategy profile perfectly, and MSCE and α-MSCE are equivalent to Nash equilibrium.

First, we characterize conditions on the feedback functions that ensure the existence of an MSCE. Under standard conditions, if feedback satisfies a linearity property, then an MSCE always exists. These conditions for equilibrium existence encompass the frameworks of partially specified equilibrium by Lehrer (2012) and the existence result in Battigalli et al. (2016, Theorem 2). We further compare our notion of MSCE with those of Lehrer (2012) and Battigalli et al. (2016). By defining feedback directly on mixed actions, our framework allows for a broader range of statistics that players can learn about the behavior of other populations. Consequently, some MSCEs in our framework cannot be implemented as equilibria in the frameworks of Lehrer (2012) or Battigalli et al. (2016).

In the second part of the paper, we apply MSCE and α-MSCE to various stylized economic situations. First, we consider a generalized class of volunteer dilemmas in which each player decides whether or not to volunteer. The probability that the public good is produced increases in the number of volunteers, but volunteering is costly. If volunteering is not a strictly dominated strategy, we show that coarse feedback allows inducing MSCE outcomes with a number of volunteers higher than in any Nash equilibrium outcome. In particular, the number of volunteers that maximizes total welfare could be implemented with coarse feedback. In addition, under some assumptions, symmetric feedback could be used to induce MSCE in which all players volunteer. We also show how coarse feedback could be used to avoid coordination problems when the designer would like to incentivize only one player to volunteer: in games such as the standard volunteer dilemma game in which the public good is produced whenever at least one player volunteers, we show that there exists simple coarse feedback such that the MSCE is unique and is such that exactly one specific player volunteers. More generally, if the Nash equilibria with the fewest volunteers are strict and there is no equilibrium with zero volunteers, then there exists feedback that can implement such a Nash equilibrium as the unique MSCE.

Second, we study a class of continuous public good games with strategic substitutes, strategic complements, or more general forms of s-shaped production functions. We characterize feedback that increases cooperation and implies a higher average contribution than Nash equilibria. For public good games with strategic substitutes or s-shaped production functions, we can construct coarse feedback that induces maximal contribution to the public good or n-times the Nash-equilibrium contribution. Further, for strategic substitutes or s-shaped production functions, more ambiguous feedback increases contributions, whereas more ambiguous feedback decreases contributions for strategic complements. For public good games with strategic substitutes, we generalize our results to α-MSCE. For low α, the highest contribution that can be implemented as an α-MSCE equals the Nash equilibrium contribution. However, as α increases, an α-MSCE with a strictly higher contribution can be implemented.

Finally, we study a class of games with strategic substitutes and negative externalities with linear best responses, such as Cournot oligopolies, and provide conditions and feedback functions that allow implementing the collusive outcome as an MSCE outcome.

Organization of the paper Section 2 presents the model and defines our equilibrium concept with coarse feedback. Section 3 contains general results, examples, and comparisons with other equilibrium concepts in games with strategic ambiguity. Sections 4, 5, and 6 characterize equilibria and optimal feedback in some classes of games. The proofs not included in the main text are in the Appendix.

2. Model

Game We consider a simultaneous-move game $G = (N, S, u)$ where

- N is the finite set of players, with $|N| = n$;
- $S = \prod_{i\in N} S_i$ is the set of strategy profiles, where S_i is the set of strategies of player i;
- $u = (u_i)_{i\in N}$, where $u_i : S \to \mathbb{R}$ is the utility function of player i.

[4] Even if it does not happen in our applications, notice that an ambiguity-averse player might be strictly better off with a mixed action than any pure action. Therefore, in some games, identifying the set of strategies with the set of mixed actions implicitly assumes that players are able to irreversibly and covertly commit to arbitrary randomization over their actions.

We assume that for every player i, S_i is a nonempty, compact, and convex subset of a Euclidean space, and u_i is continuous and quasi-concave on S_i. Our primary interpretation of S_i is as the set of mixed actions available to player i in a finite simultaneous-move game (N, A, u), where $A = \prod_{i\in N} A_i$ and A_i is the finite set of actions of player i. We identify S_i with the set of mixed actions of player i, i.e., $S_i = \Delta(A_i)$, and utility functions are extended to mixed actions in the usual way:

$$u_i(s) = \sum_{a\in A} s(a)u_i(a), \text{ where } s(a) = \prod_i s_i(a_i).$$

The game (N, S, u) is then called the mixed extension of (N, A, u). In some applications, such as the public goods games in Section 5 and the Cournot games in Section 6, s_i is a real number representing quantities.

Feedback Each player i gets feedback through a feedback function that depends on the steady-state strategy profile $s \in S$. Formally, a *feedback function* for player i is given by a function

$$\phi_i : S \to Y_i,$$

where Y_i is a nonempty subset of a Euclidian space. Player i's feedback is *separable* if player i receives separable feedback about each opponent's strategy, i.e., if there is a profile of pair-specific feedback functions $\phi_{ij} : S_j \to Y_{ij}$, $j \neq i$, such that $\phi_i(s) = (\phi_{ij}(s_j))_{j\neq i}$. Let $\phi = (\phi_i)_{i\in N}$. A game with feedback is denoted by (G, ϕ). Notice that when strategy sets are sets of mixed actions, a feedback function ϕ_i depends directly on profiles of mixed actions. We show in Section 3.2 how ϕ_i can be derived from feedback on profiles of pure actions as in Battigalli et al. (2016), and how separable feedback can be derived from partially specified probabilities in Lehrer (2012).

The set of strategy profiles of players other than i that are consistent with player i's feedback given the strategy profile s is given by

$$\Sigma_i(s) = \{\tilde{s}_{-i} \in S_{-i} : \phi_i(s_i, \tilde{s}_{-i}) = \phi_i(s)\}.$$

That is, the set $\Sigma_i(s)$ is the set of strategy profiles of players other than i deemed as possible by player i according to his feedback if the actual strategy profile is s. The correspondence $\Sigma_i : S \rightrightarrows S_{-i}$ is called the *identification correspondence* of player i. Implicit in the formulation is the assumption that player i knows the steady-state strategy s_i of his own population.[5] Notice that, for every s, we have $s_{-i} \in \Sigma_i(s)$.

We say that player i gets *perfect feedback* if $\Sigma_i(s) = \{s_{-i}\}$ for every s, i.e., if $\phi_i(s) \neq \phi_i(s_i, s'_{-i})$ for every s_i and $s_{-i} \neq s'_{-i}$. Player i gets *no feedback* if $\Sigma_i(s) = S_{-i}$ for every s, i.e., if $\phi_i(s) = \phi_i(s_i, s'_{-i})$ for every s_i, s_{-i}, and s'_{-i}. We say that the feedback of player i satisfies *own-strategy independence* if the identification correspondence does not depend on the strategy of player i, i.e., $\Sigma_i(s_i, s_{-i}) = \Sigma_i(s'_i, s_{-i})$ for every s_i, s'_i, and s_{-i}. Under separable feedback we have $\Sigma_i(s) = \times_{j\neq i}\Sigma_{ij}(s_j)$, with

$$\Sigma_{ij}(s_j) = \{\tilde{s}_j \in S_j : \phi_{ij}(\tilde{s}_j) = \phi_{ij}(s_j)\}.$$

Hence, separable feedback implies own-strategy independence, but own-strategy independence does not imply separability. For example, if players receive feedback about the sum of players' strategies in a group $\tilde{N} \subseteq N$ (e.g., $\phi_i(s) = \sum_{j\in\tilde{N}} s_j$), then own-strategy independence is satisfied, but the feedback is not necessarily separable.

Equilibrium with coarse feedback and ambiguity aversion When player i does not get perfect feedback, then $\Sigma_i(s)$ may not be reduced to a singleton, i.e., player i may not know the strategy played by the other players. In that case, when evaluating his payoff from some strategy $\tilde{s}_i$, we assume that player i is considering the worst-case scenario given his feedback: he evaluates the payoff from $\tilde{s}_i$ with $\inf_{\tilde{s}_{-i}\in\Sigma_i(s)} u_i(\tilde{s}_i, \tilde{s}_{-i})$, the lowest utility he could possibly get by playing $\tilde{s}_i$.[6] A strategy profile s^* is then a steady-state equilibrium if, for each player i, given the set $\Sigma_i(s^*)$ of opponents' strategy profiles consistent with player i's feedback, s_i^* maximizes player i's worst-case payoff. Such an equilibrium is called a maxmin self-confirming equilibrium:

Definition 1. A strategy profile s^* is a *maxmin self-confirming equilibrium* (MSCE) of (G, ϕ) iff for every $i \in N$ and every $s_i \in S_i$ we have

$$\inf_{s_{-i}\in\Sigma_i(s^*)} u_i(s_i^*, s_{-i}) \geq \inf_{s_{-i}\in\Sigma_i(s^*)} u_i(s_i, s_{-i}).$$

Hence, in a maxmin self-confirming equilibrium s^*, each player i considers, for each of his strategies s_i, the subjective theory about others' behavior consistent with his feedback, s_{-i} in $\Sigma_i(s^*)$, that is, the most pessimistic theory given s_i. If $S_i = \Delta(A_i)$ is the set of mixed actions of a finite game (N, A, u), then an MSCE corresponds to an equilibrium with extreme ambiguity aversion.

To check the robustness of some of our results, we also consider the case where each player assigns weight $\alpha \in [0, 1]$ on the worst-case scenario and weight $1 - \alpha$ on the best-case scenario, which leads to the following definition:

[5] This is consistent with how the identification correspondence is defined in Battigalli et al. (2016); see Section 3.2. However, note that we do not assume players have payoff knowledge of their population.

[6] Despite the assumed continuity of u_i, the infimum is used here instead of the minimum because the set $\Sigma_i(s)$ is not necessarily a closed set.

Definition 2. A strategy profile s^* is an *α-maxmin self-confirming equilibrium* (α-MSCE) of (G, ϕ) iff for every $i \in N$ we have

$$s_i^* \in \arg\max_{s_i \in S_i} \left(\alpha \inf_{s_{-i} \in \Sigma_i(s_{-i}^*)} u_i(s_i, s_{-i}) + (1-\alpha) \sup_{s_{-i} \in \Sigma_i(s_{-i}^*)} u_i(s_i, s_{-i}) \right).$$

For $\alpha = 1$, an α-MSCE is an MSCE, and the case $\alpha = 0$ can be interpreted as a self-confirming equilibrium when players are extremely optimistic or wishful thinkers. Under perfect feedback we have $\Sigma_i(s_{-i}) = \{s_{-i}\}$, so the definition of self-confirming equilibrium of (G, ϕ) coincides with the definition of Nash equilibrium of G. The following simple example illustrates the difference between Nash equilibrium and α-MSCE and shows that players could benefit from coarse feedback. In the example, there is a unique Nash equilibrium, Pareto dominated by the unique α-MSCE for α large enough when player 1 gets no feedback and player 2 gets perfect feedback.

Example 1. There are two players. Player 1 plays top (T) or bottom (B), and player 2 left (L) or right (R).

	L	R
T	2,4	3,7
B	-1,3	5,2

There is only one Nash Equilibrium: player 1 plays T with probability $\frac{1}{4}$ and player 2 plays L with probability $\frac{2}{5}$. The Nash equilibrium expected payoff is $2\frac{3}{5}$ for player 1 and $3\frac{1}{4}$ for player 2. Further, (T, R) is the unique MSCE if player 1 receives no feedback and player 2 receives perfect feedback. The MSCE (T, R) Pareto dominates the unique Nash equilibrium of this game. (T, R) is also the unique α-MSCE as long as $\alpha > \frac{2}{5}$. If $\alpha < \frac{2}{5}$, then the unique α-MSCE is (B, L). ⋄

3. Existence and comparison with other concepts

This section characterizes sufficient conditions on feedback functions to ensure the existence of MSCE for games with feedback. We also compare and illustrate the difference between MSCE and the notions of partially specified equilibrium of Lehrer (2012) and maxmin self-confirming equilibrium of Battigalli et al. (2016).

3.1. Existence

We first show that, under own-strategy independence, an MSCE can be defined as a standard Nash equilibrium of an auxiliary game. Consider a game with feedback (G, ϕ). Under own-strategy independence, the identification correspondence of each player i can be rewritten as $\Sigma_i : S_{-i} \rightrightarrows S_{-i}$. For every i, define the function $v_i : S \to \mathbb{R}$ as follows for every $s \in S$:

$$v_i(s) = \inf_{\tilde{s}_{-i} \in \Sigma_i(s_{-i})} u_i(s_i, \tilde{s}_{-i}).$$

Let $v = (v_i)_{i \in N}$. By definition, s^* is an MSCE of (G, ϕ) iff s^* is a Nash equilibrium of the auxiliary game (N, S, v). Hence, the existence of an MSCE of (G, ϕ) is equivalent to the existence of a Nash equilibrium of (N, S, v).

The usual assumptions for the existence of a Nash equilibrium of (N, S, v) are satisfied if, for every i, v_i is continuous and quasi-concave on S_i. The next lemma shows that v_i is quasi-concave on S_i.

Lemma 1. *For every i, v_i is quasi-concave on S_i.*

Proof. See Appendix A.1. ■

To get continuity of v_i, we need additional assumptions on the identification correspondences Σ_i. Precisely, v_i is continuous if for every i the identification correspondence Σ_i of player i is continuous (i.e., both upper and lower hemicontinuous) and compact-valued. Combined with Lemma 1, we get the following proposition:

Proposition 1. *If feedback satisfies own-strategy independence and the identification correspondence Σ_i is continuous and compact-valued for every player i, then the game with feedback (G, ϕ) has at least one MSCE.*

Proof. From Berge's theorem, v_i is continuous because u_i is continuous, and the identification correspondence Σ_i is continuous and compact-valued. From Lemma 1, v_i is quasi-concave on S_i. Hence, the auxiliary game (N, S, v) has at least one Nash equilibrium, i.e., the game with feedback (G, ϕ) has a least one MSCE. ■

An α-MSCE may not exist even if the identification correspondences are continuous and compact-valued, because the quasi-concavity of v_i under α-maxmin might fail for $\alpha < 1$.

If the feedback function ϕ_i is continuous, then the identification correspondence Σ_i of player i is upper hemicontinuous and compact-valued. However, the continuity of the feedback functions is not sufficient for the identification correspondences to be lower hemicontinuous, which is crucial for the existence of an MSCE. In the following example, we consider a (mixed extension of a) 2-player 2-action game with own-strategy independence of feedback and continuous feedback functions in which the identification correspondence of player 1 is not lower hemicontinuous, and there is no MSCE.

Example 2. Consider again the game of Example 1. Let s_1 be the probability that player 1 plays T and s_2 the probability that player 2 plays L. Player 2 gets perfect feedback, and the identification correspondence of player 1 is:

$$\Sigma_1(s_2) = \begin{cases} [\underline{s}_2, \overline{s}_2] & \text{if } s_2 \in [\underline{s}_2, \overline{s}_2] \\ \{s_2\} & \text{if } s_2 \notin [\underline{s}_2, \overline{s}_2], \end{cases}$$

where $0 < \underline{s}_2 < \frac{2}{5} < \overline{s}_2$. This identification correspondence is upper but not lower hemicontinuous and can be induced by the following continuous feedback function:

$$\phi_1(s_2) = \begin{cases} s_2 & \text{if } s_2 \leq \underline{s}_2 \\ \underline{s}_2 & \text{if } s_2 \in [\underline{s}_2, \overline{s}_2] \\ s_2 - (\overline{s}_2 - \underline{s}_2) & \text{if } s_2 \geq \overline{s}_2. \end{cases}$$

Assume that (s_1, s_2) is an MSCE. Then we have:

- If $s_2 > \overline{s}_2$, then player 1 best responds to s_2 with $s_1 = 1$. Thus, player 2 best responds with $s_2 = 0$, a contradiction.
- If $s_2 < \underline{s}_2$, then player 1 best responds to s_2 with $s_1 = 0$. Thus, player 2 best responds with $s_2 = 1$, a contradiction.
- If $s_2 \in [\underline{s}_2, \overline{s}_2]$, then player 1 best responds to $\overline{s}_2$ with $s_1 = 1$. Thus, player 2 best responds with $s_2 = 0$, a contradiction.

Hence, there is no MSCE. Note that the existence of an MSCE continues to fail even if we replace Σ_1 by the following identification correspondence, which induces a finite partition of S_2:

$$\Sigma_1(s_2) = \begin{cases} [0, \underline{s}_2) & \text{if } s_2 < \underline{s}_2 \\ [\underline{s}_2, \overline{s}_2] & \text{if } s_2 \in [\underline{s}_2, \overline{s}_2] \\ (\overline{s}_2, 1] & \text{if } s_2 > \overline{s}_2. \quad \diamond \end{cases}$$

To get lower hemicontinuous identification correspondences, we need a stronger condition on feedback functions. We introduce such a condition below, which generalizes the framework of partially specified equilibrium of Lehrer (2012) and covers the class of separable feedback functions used to get existence of a maxmin self-confirming equilibrium in Battigalli et al. (2016, Theorem 2) (see Section 3.2).

We say that feedback is *linear* if every function ϕ_i is a linear map. Clearly, linearity implies own-strategy independence, but it does not imply separability. For example, if the feedback is the sum of players' strategies, then linearity is satisfied but not separability. The next lemma shows that in games with linear feedback and in which the set of strategies of each player is a finite dimensional simplex, the identification correspondences are lower hemicontinuous.[7]

Lemma 2. *Assume that S_i is a finite dimensional simplex. If ϕ_i is a linear map, then the identification correspondence Σ_i of player i is lower hemicontinuous.*

Proof. See Appendix A.1. ■

Therefore, we get the following proposition:

Proposition 2. *For every game with linear feedback (G, ϕ) where S_i is a finite dimensional simplex for every $i \in N$, there exists an MSCE.*

Proof. Directly from Proposition 1 and Lemma 2. ■

3.2. Partially specified equilibrium and other notions of self-confirming equilibria

In the standard definition of maxmin self-confirming equilibrium in the literature (see, e.g., Battigalli et al., 2016), a feedback function is defined on the set of terminal nodes of a finite game in extensive form. Restricting attention to our class of simultaneous-move games, such feedback function is then simply defined on the set of action profiles, not directly on the set of mixed action profiles

[7] Another sufficient condition for the identification correspondences to be lower hemicontinuous is to assume that feedback functions are open mappings. Then, by Theorem 17.7 in Aliprantis and Border (2006), inverse correspondences are lower hemicontinuous.

or on a continuum of strategy profiles as in our setting. Precisely, given a finite game (N, A, u), a *pure feedback function* for player i is given by a mapping

$$F_i : A \to M,$$

where M is a finite set of messages. A mixed action profile $s \in \prod_{j\in N} \Delta(A_j)$ induces a probability distribution $\hat{F}_i(s) \in \Delta(M)$, with $\hat{F}_i(m \mid s) = \sum_{a\in F_i^{-1}(m)} \prod_{j\in N} s_j(a_j)$ for every $m \in M$.

The set of mixed action profiles consistent with the feedback of player i given s is then given by

$$\Sigma_i(s) = \{\tilde{s}_{-i} \in S_{-i} : \hat{F}_i(s_i, \tilde{s}_{-i}) = \hat{F}_i(s)\}.$$

Therefore, we get this setting with pure feedback as a particular case of our model by letting $\phi_i : S \to \Delta(M)$ and $\phi_i = \hat{F}_i$.[8]

The next example illustrates that equilibria in our setting cannot always be induced by pure feedback functions.

Example 3. Consider the following 2-player game. Player 2 has a strictly dominant strategy ($s_2 = \delta_L$), so he plays L in any equilibrium.

	L	R
T	3,1	-3,0
M	2,1	0,0
B	1,1	1,0

In the setting of Battigalli et al. (2016), there are only two types of pure feedback functions for player 1 because player 2 has only two actions: for each action a_1 of player 1, either $F_1(a_1, L) \neq F_1(a_1, R)$, or $F_1(a_1, L) = F_1(a_1, R)$. In the first case, the unique best response for player 1 is T; in the second case, the unique best response for player 1 is B. Hence, there is no MSCE with pure feedback functions in which player 1 plays M.

Consider now the following class of feedback functions for player 1, where $\underline{s} \in [0, 1]$:

$$\phi_1(s_2) = \begin{cases} s_2(L) & \text{if } s_2(L) \le \underline{s} \\ \underline{s} & \text{if } s_2(L) \ge \underline{s}. \end{cases}$$

That is, player 1 learns the relative frequency of action L if it is below the threshold $\underline{s}$. The set of strategies of player 2 consistent with player 1's feedback is

$$\Sigma_1(s_2) = \begin{cases} \{s_2\} & \text{if } s_2(L) < \underline{s} \\ \{\tilde{s}_2 : \tilde{s}_2(L) \ge \underline{s}\} & \text{if } s_2(L) \ge \underline{s}. \end{cases}$$

Because in equilibrium we have $s_2^*(L) = 1$, player 1 knows that player 2's strategy belongs to $\Sigma_1(s_2^*) = \{s_2 : s_2(L) \ge \underline{s}\}$, and we have

$$\inf_{s_2\in\Sigma_1(s_2^*)} u_1(a_1, s_2) = u_1(a_1, \underline{s}) = \begin{cases} 6\underline{s} - 3 & \text{if } a_1 = T \\ 2\underline{s} & \text{if } a_1 = M \\ 1 & \text{if } a_1 = B. \end{cases}$$

Hence, for $\underline{s} \in (\frac{1}{2}, \frac{3}{4})$, the unique MSCE is (M, L), which is not an MSCE with any pure feedback function. ⋄

Lehrer (2012) introduced a concept similar to MSCE, called partially specified equilibrium, in which for each player i and $j \neq i$, player i observes the expected values of random variables $(X_{ij}^k)_k$ with $X_{ij}^k : A_j \to \mathbb{R}$. Such feedback is obtained as a special case in our setting by considering separable feedback functions and letting

$$\phi_{ij}(s_j) = (E_{s_j}(X_{ij}^k))_k.$$

As in the setting with pure feedback, if player j has only two actions, say L and R, such feedback reduces to either perfect feedback (if there exists k such that $X_{ij}^k(R) \neq X_{ij}^k(L)$) or no feedback (if $X_{ij}^k(R) = X_{ij}^k(L)$ for every k). Hence, whatever the feedback of player 1 in Example 3, (M, L) is never a partially specified equilibrium.

Our Proposition 2 implies that a partially specified equilibrium always exists because $\phi_{ij}(s_j) = (E_{s_j}(X_{ij}^k))_k$ is linear in s_j. Hence, it also covers the separable case of Battigalli et al. (2016, Theorem 2).[9]

Riedel and Sass (2014) use a different approach to induce strategic ambiguity. They define Ellsberg equilibria by allowing every player to use an ambiguous randomization device, described as a set of mixed actions. In the previous version of the paper (Koessler

[8] We assume here that player i only observes $\hat{F}_i(s_i, s_{-i})$, not $\hat{F}_i(\delta_{a_i}, s_{-i})$ for every $a_i \in \text{supp}[s_i]$. In this later case, we would have $\Sigma_i(s) = \{\tilde{s}_{-i} \in S_{-i} : \forall a_i \in \text{supp}[s_i], \hat{F}_i(\delta_{a_i}, \tilde{s}_{-i}) = \hat{F}_i(\delta_{a_i}, s_{-i})\}$. This formulation is considered in Battigalli et al. (2016), and is also obtained as a special case of our setting by letting $\phi_i(s) = (\mathbb{1}_{a_i\in \text{supp} s_i} \hat{F}_i(\delta_{a_i}, s_{-i}))_{a_i\in A_i}$.

[9] See Battigalli et al. (2016) for details on how pure and separable feedback in Battigalli et al. (2016) induce partially specified probabilities as in Lehrer (2012).

and Pahlke, 2023), we showed that for every strategy profile s^* in the support of an Ellsberg equilibrium, we can construct separable feedback such that s^* is an MSCE. However, there could exist MSCE strategy profiles that are not in the support of any Ellsberg equilibrium.

4. Generalized volunteer dilemmas

In this section, we consider the following class of generalized symmetric volunteer dilemmas. The set of players $N = \{1, \dots, n\}$ is the set of potential volunteers and beneficiaries of a public good. The set of actions of each player i is $A_i = \{0, 1\}$, where action $a_i = 0$ corresponds to not volunteering, action $a_i = 1$ corresponds to volunteering by exerting a costly effort, and $s_i \in [0, 1]$ corresponds to the probability that player i volunteers. The probability $h(k) \in [0, 1]$ that the public good is produced is weakly increasing in the number $k = 0, 1, \dots, n$ of players who volunteer. We normalize the probability of producing the public good to 0 when there is no volunteer, i.e., $h(0) = 0$. The value of the public good is the same for each player and is normalized to 1. The individual cost of effort is $c \in (0, 1)$. The utility function of each player i is therefore given by

$$u_i(a) = h\Big(\sum_{j \in N} a_j \Big) - a_i c.$$

In the standard volunteer dilemma (Diekmann, 1985), the public good is produced if and only if at least one player volunteers, so $h(k) = 1$ for every $k \geq 1$. The more general case where the public good is produced if and only if at least $\underline{k} \geq 1$ players volunteers, i.e., $h(k) = 0$ for every $k < \underline{k}$ and $h(k) = 1$ for every $k \geq \underline{k}$, is a binary threshold public good game that has been studied extensively in problems of collective decisions such as lobbying, petitioning, getting vaccinated, and voting (see, e.g., Dziuda et al., 2021, Ginzburg, 2022, and references therein), as well as in the experimental literature (see, e.g., Goeree et al., 2017). The general class of volunteer dilemmas introduced above also covers the case in which the public good is produced with some exogenous probability if the number of volunteers exceeds the threshold. For example, if the probability that a legislature supports a petition is equal to q if the minimum number of signatures required for the petition to be forwarded is $\underline{k}$, then $h(k) = 0$ for every $k < \underline{k}$ and $h(k) = q$ for every $k \geq \underline{k}$. As another example, consider the case in which the public good is produced if and only if at least one player contributes, but each player who decides to volunteer fails to contribute with probability ε, then $h(0) = 0$ and $h(k) = 1 - \varepsilon^k$ for every $k \geq 1$.

The total number of volunteers, k, corresponds to a (pure strategy) Nash equilibrium outcome (i.e., an equilibrium such that $\sum_{j \in N} a_j = k$) iff

$$\begin{aligned} h(1) &\leq c, && \text{for } k = 0, \\ h(k+1) - h(k) \leq c &\leq h(k) - h(k-1), && \text{for } k \in \{1, \dots, n-1\}, \\ c &\leq h(n) - h(n-1), && \text{for } k = n. \end{aligned}$$

Observe that a pure strategy Nash equilibrium always exists, but the equilibrium number of volunteers is not necessarily unique, and mixed strategies equilibria might also exist. Also, observe that if

$$c \leq h(k) - h(k-1) \text{ for some } k \in \{1, \dots, n\}, \tag{1}$$

then there exists at least one pure strategy Nash equilibrium with at least one volunteer. Otherwise, volunteering is a strictly dominated strategy, so players' feedback does not impact players' behavior. Therefore, we assume that condition (1) is satisfied in the rest of this section and denote by $\underline{k}^{NE} \geq 1$ the smallest number of volunteers among the set of pure strategy Nash equilibrium outcomes with at least one volunteer.

Consider the case in which a designer would like to induce a total number of volunteers equal to k^*. For example, if the designer's objective is to maximize total welfare, then k^* maximizes $nh(k^*) - k^* c$. It is immediate to check that, in this case, if k^{NE} is a Nash equilibrium outcome that does not maximize total welfare, then we necessarily have $k^{NE} < k^*$. Intuitively, there cannot be a Nash equilibrium that induces a higher number of volunteers compared to the social optimum because if the personal gain from volunteering is higher than the individual cost, it is also higher for the collective gain.

In the remaining part of this section, we show how feedback can be used to generate specific equilibria. Feedback is *public* if $\phi_i = \phi_j$ for every i and j. For games with $S_i = S_j$ for all players $i, j \in N$, we call feedback *anonymous* if $\phi_i(s_i, s_{-i}) = \phi_i(s_i, \tilde{s}_{-i})$ for every permutation $\tilde{s}_{-i}$ of s_{-i}.

In the next proposition, we show that for every number k^* greater than or equal to some pure strategy Nash equilibrium outcome with at least one volunteer (which exists under condition (1)), there exist two simple and appealing classes of feedback functions for which there is an MSCE with exactly k^* volunteers. A first class of feedback, (i), is separable, and for each pair of players (i, j), $i \neq j$, player i either gets perfect or no feedback from player j. A second class of feedback, (ii), is public and anonymous but is not separable: each player simply learns whether at least $\underline{k}^{NE}$ players volunteer or not. The idea is rather simple: such feedback ensures that if there are at least $\underline{k}^{NE}$ players who volunteer, then every player's strategy and worst-case belief coincide with the best response and belief of some player in a Nash equilibrium with $\underline{k}^{NE}$ volunteers.

Proposition 3. *Let $k^* \geq \underline{k}^{NE}$. There exist (i) separable feedback ((ii) public and anonymous feedback, respectively) and an MSCE such that the total number of volunteers is k^*.*

Proof. Consider the following strategy profile s^*: every player $i \leq k^*$ volunteers and every player $i \geq k^*+1$ does not volunteer, i.e., $s_i^* = 1$ if $i \leq k^*$, and $s_i^* = 0$ if $i \geq k^*+1$.

(i) Consider the following separable feedback: every player $i \leq k^*$ gets perfect feedback about the strategies of all players in $\{k^*+1, \ldots, n\}$, perfect feedback about the strategies of $\underline{k}^{NE}-1$ players (other than i) in $\{1, \ldots, k^*\}$, and no feedback from the other players. Every player $i \geq k^*+1$ gets perfect feedback about the strategy of every player in $\{1, \ldots, \underline{k}^{NE}\}$ and no feedback from every other player. A maxmin player getting no feedback from player $j \neq i$ behaves as if player j does not volunteer. Hence, every player $i \leq k^*$ believes that exactly $\underline{k}^{NE}-1$ other players volunteer. Because $\underline{k}^{NE}$ is a Nash equilibrium outcome, it is a best response for these players to volunteer. Similarly, every player $i \geq k^*+1$ believes that exactly $\underline{k}^{NE}$ other players volunteer, so it is a best response for these players not to volunteer. We conclude that s^* is an MSCE with the above feedback.

(ii) Consider the following public and anonymous feedback: every player i learns whether or not the total number of players who volunteer is at least $\underline{k}^{NE}$. For example, $\phi_i(s) = 1$ if $|\{j \in N : s_j = 1\}| \geq \underline{k}^{NE}$ and $\phi_i(s) = 0$ if $|\{j \in N : s_j = 1\}| < \underline{k}^{NE}$. As in (i), in the worst-case scenario, every player $i \leq k^*$ believes that exactly $\underline{k}^{NE}-1$ other players volunteer, and every player $i \geq k^*+1$ believes that exactly $\underline{k}^{NE}$ other players volunteer, so s^* is an MSCE with the above feedback. ■

Remark 1 *(Feedback satisfying "observable payoffs")*. In general, the feedback functions studied in this paper, like the ones used in Proposition 3, do not satisfy the property called "observable payoffs" in Battigalli et al. (2016). This property requires that if a player's utility differs for two strategy profiles, then that player should also receive different feedback for these strategy profiles. In the underlying learning interpretation of the equilibrium concept, this property would be satisfied if, for every population, each individual in that population is perfectly informed about the payoffs of individuals in that population or if each individual plays the game multiple times instead of only once. We do not require this property in our paper, but in the classical class of volunteer dilemmas, where the public good is produced if and only if at least $\underline{k}$ players volunteer, we can construct public and anonymous feedback functions satisfying "observable payoffs" such that the total number of volunteers is k^* in an MSCE. More generally, if there exists a Nash equilibrium outcome k^{NE} such that $h(k^{NE}) = h(k^*)$ for $k^* \geq k^{NE}$, then there exists public and anonymous feedback satisfying "observable payoffs" such that k^* is an MSCE. For example:

$$\phi_i(s) = \begin{cases} s & \text{if } |\{j : s_j = 1\}| \leq k^{NE}-1 \\ \text{"ok"} & \text{if } |\{j : s_j = 1\}| \in \{k^{NE}, \ldots, k^*\} \\ s & \text{if } |\{j : s_j = 1\}| \geq k^*+1. \quad \diamond \end{cases} \tag{2}$$

To illustrate Proposition 3, consider the standard volunteer dilemma game but assume each player who volunteers fails to contribute to the public good with probability $\varepsilon \in (0, 1-c)$. So, even if a player chooses to volunteer, he only contributes to the public good with probability $1-\varepsilon$. Hence, $h(0) = 0$ and $h(k) = 1 - \varepsilon^k$ for every $k \geq 1$. The number k^{NE} of volunteers in a pure strategy Nash equilibrium is generically unique: if $c < \varepsilon^{n-1}(1-\varepsilon)$, then $k^{NE} = n$; otherwise, if $c > \varepsilon^{n-1}(1-\varepsilon)$ it is equal to the value $k \in \{1, \ldots, n-1\}$ such that $\varepsilon^k(1-\varepsilon) < c < \varepsilon^{k-1}(1-\varepsilon)$. Note that when the number of equilibrium volunteers is interior, it does not change as the total number of players, n, increases. On the contrary, the socially optimal number of volunteers $k^* \geq k^{NE}$ that maximizes $nh(k) - kc$ in k is increasing in n and is strictly higher than k^{NE} for n large enough. From Proposition 3, separable (or public and anonymous, respectively) feedback functions always exist such that a socially optimal number of volunteers constitutes an MSCE.

For example, assume that there are $n = 10$ players, and the probability that each player fails to contribute to the public good is $\varepsilon = \frac{1}{2}$. If $c = \frac{1}{3}$, the Nash equilibrium number of volunteers is $k^{NE} = 1$ (because $\varepsilon^1(1-\varepsilon) = \frac{1}{4} < c = \frac{1}{3} < \varepsilon^0(1-\varepsilon) = \frac{1}{2}$), as in the standard volunteer dilemma with $\varepsilon = 0$, but the socially optimal number of volunteers is $k^* = 4$. An MSCE with $k^* = 4$ volunteers is obtained with public and anonymous feedback by simply revealing to all players whether or not there is at least one volunteer. An MSCE with $k^* = 4$ volunteers is obtained with separable feedback by providing (i) four players (the volunteers) no feedback about the strategies of the other volunteers, and perfect feedback about the remaining players, and (ii) perfect feedback from one of the volunteers and no feedback from the other players to the six remaining players who do not volunteer. If $c = \frac{1}{10}$, the Nash equilibrium number of volunteers is $k^{NE} = 3$ (because $\varepsilon^3(1-\varepsilon) = \frac{1}{16} < c = \frac{1}{10} < \varepsilon^2(1-\varepsilon) = \frac{1}{8}$), but the socially optimal number of volunteers is $k^* = 6$. An MSCE with $k^* = 6$ volunteers is obtained with public and anonymous feedback by simply revealing to all players whether or not there are at least three volunteers. An MSCE with $k^* = 6$ volunteers is obtained with separable feedback by providing (i) six players (the six volunteers) perfect feedback from two of them, perfect feedback from the four players not volunteering, and no feedback from the other players and (ii) perfect feedback from three of the volunteers and no feedback from the other players to the four remaining players not volunteering.

Proposition 3 can be extended to α-MSCE for α large enough, but the exact values of α allowing to sustain an equilibrium with k^* volunteers depend on k^* and the shape of the function h. For instance, in the previous example, there is an α-MSCE in which k^* players volunteer with the separable feedback used above iff

$$\alpha h(k^{NE}) + (1-\alpha)h(k^*) - c \geq \alpha h(k^{NE}-1) + (1-\alpha)h(k^*-1),$$

i.e.,

$$\alpha \geq \frac{c - (h(k^*) - h(k^*-1))}{(h(k^{NE}) - h(k^{NE}-1)) - (h(k^*) - h(k^*-1))}.$$

Observe that this condition becomes stronger (i.e., requires more aversion to ambiguity) when the number of volunteers the designer would like to induce, k^*, increases. Going back to the numerical examples of the previous paragraph, if $c = \frac{1}{3}$, then there is an α-MSCE in which $k^* = 4$ players volunteer with the above separable feedback iff $\alpha \geq \frac{13}{21}$, and if $c = \frac{1}{10}$, then there is an α-MSCE in which $k^* = 6$ players volunteer with the above separable feedback iff $\alpha \geq \frac{27}{35}$.

The previous proposition implies that there always exist separable (or public and anonymous, respectively) feedback and an MSCE in which all players volunteer (i.e., $k^* = n$). However, the constructed separable feedback is asymmetric if there is not already a Nash equilibrium in which all players volunteer. For games with $S_i = S_j$ for all players $i, j \in N$, separable feedback is called *symmetric* if two conditions are satisfied. First, every player i has the same feedback function about every opponent, i.e., $\phi_{ij} = \phi_{ik}$ for every j and k. Second, two distinct players i and k have the same feedback function about every third player j, $\phi_{ij} = \phi_{kj}$ for every j and every i and k different from j. Under an additional condition, the next proposition shows that separable feedback functions can also be chosen symmetric to get an MSCE in which all players volunteer. In general, to construct such an MSCE, players must get partial (mixed) feedback from each other player.

Similarly to the feedback analyzed in Example 3, consider a class of separable threshold feedback functions that induce an identification correspondence with the following property for every i and $j \neq i$, where the strategy s_j of player j is identified with the probability that player j volunteers and $\underline{s} \in [0, 1]$:

$$\Sigma_{ij}(s_j) = [\underline{s}, 1], \text{ for every } s_j \geq \underline{s} \text{ and}$$

$$\Sigma_{ij}(s_j) = \{s_j\}, \text{ for every } s_j < \underline{s}.$$

That is, if player j volunteers with probability s_j higher than $\underline{s}$, then player i only learns that player j volunteers with a probability higher than $\underline{s}$.

Proposition 4. *Assume that a symmetric Nash equilibrium exists in which each player volunteers with strictly positive probability. Then, there exist symmetric separable threshold feedback functions and a symmetric MSCE such that all players volunteer with probability one.*

Proof. Assume that there exists a symmetric Nash equilibrium in which each player i volunteers with probability $s_i = s^M > 0$. Consider the symmetric separable threshold feedback functions such that $\Sigma_{ij}(s_j) = [s^M, 1]$ for every $s_j \geq s^M$. Consider the strategy profile s^* in which all players volunteer with probability 1, i.e., $s_i^* = 1$ for every i. Given the feedback above, every player believes that every other player volunteers with probability s^M. Because $s_i = s^M$ for every i is a Nash equilibrium in which all players volunteer with strictly positive probability, it is indeed a best response for every player to volunteer when believing that each other player volunteers with probability s^M. Hence, s^* is an MSCE with the above feedback. More generally, the construction above works for every symmetric separable threshold feedback function such that volunteering is a best response when all the other players volunteer with probability $\underline{s}$. ■

The existence of a symmetric Nash equilibrium in which each player volunteers with strictly positive probability is not guaranteed in every generalized volunteer dilemma. However, such an equilibrium is guaranteed to exist whenever $c < h(1)$. Indeed, in that case, not volunteering at all ($s_i = 0$ for every i) is not a Nash equilibrium, but a symmetric equilibrium must exist because the game is finite and symmetric. Hence, it should be a symmetric Nash equilibrium in which each player i volunteers with strictly positive probability. The standard volunteer dilemma ($h(k) = 1$ for every $k \geq 1$) satisfies this condition, and the symmetric Nash equilibrium is unique and given by

$$s_i = s^M = 1 - c^{\frac{1}{n-1}}, \text{for every } i.$$

Then, every symmetric separable threshold feedback function with $\underline{s} \leq s^M$ induces an MSCE in which all players volunteer. Note that this MSCE, even if not socially efficient, leads to the same social welfare as the expected social welfare in the symmetric Nash equilibrium. Hence, under symmetry, a designer who would like to maximize social welfare but is risk-averse would strictly prefer to provide such coarse feedback instead of perfect feedback.

We have shown that coarse feedback allows us to induce new equilibria with more volunteers. We now show that coarse feedback may also be a relevant "full implementation" tool by avoiding coordination problems and adversarial equilibrium selection. For instance, in the standard volunteer dilemma, there are socially efficient Nash equilibria, but many other inefficient equilibria exist. The next proposition provides such "full implementation" results for socially efficient equilibria in such a game and, more generally, for the lowest Nash equilibrium if it is strict and has exactly $k^* > 0$ volunteers. Proposition 5 (i) shows that separable feedback functions exist such that the induced MSCE is unique (among all pure and mixed MSCE) and is such that exactly k^* players volunteer with probability 1. Proposition 5 (ii) shows that public and anonymous feedback functions exist such that in every induced MSCE the number of volunteers is k^* with probability 1. Proposition 5 (i) has a slightly stronger notion of uniqueness because the induced equilibrium is unique, but feedback functions are asymmetric and non-anonymous. On the other hand, in Proposition 5 (ii) feedback functions are public and anonymous, the number of volunteers is k^* in every equilibrium, but the equilibrium is not unique.

Proposition 5. *Let k^* be the lowest Nash equilibrium number of volunteers, and assume that the equilibrium is strict and $k^* > 0$, i.e.,*

$$c > h(k^* + 1) - h(k^*), \tag{3}$$

$$c < h(k + 1) - h(k), \text{ for every } k = 0, 1, \dots, k^* - 1. \tag{4}$$

(i) There exist separable feedback functions such that the MSCE is unique and is such that exactly k^* players volunteer.
(ii) There exist public and anonymous feedback functions such that in every MSCE exactly k^* players volunteer.

Proof. (i) We consider the following separable feedback and show that the corresponding MSCE in which each player $j \in \{1, \dots, k^*\}$ volunteers and each other player $j \in \{k^* + 1, \dots, n\}$ does not volunteer is the unique MSCE.

Each player $j \in \{1, \dots, k^*\}$ receives perfect feedback from players in $\{1, \dots, k^*\} \backslash \{j\}$, and no feedback from players in $\{k^* + 1, \dots, n\}$. Hence, each player $j \in \{1, \dots, k^*\}$ knows the strategies of players in $\{1, \dots, k^*\}$ and believes that players in $\{k^* + 1, \dots, n\}$ do not volunteer. Thus, given condition (4), the unique best response of players in $\{1, \dots, k^*\}$ is to volunteer with probability 1 whatever the (pure and mixed) strategies of the other players.

Each player $j \in \{k^* + 1, \dots, n\}$ receives perfect feedback from players in $\{1, \dots, k^*\}$, and no feedback from players in $\{k^* + 1, \dots, n\} \backslash \{j\}$. Hence, in every MSCE, each player $j \in \{k^* + 1, \dots, n\}$ believes that each player in $\{1, \dots, k^*\}$ volunteers and that each player in $\{k^* + 1, \dots, n\} \backslash \{j\}$ does not volunteer. Thus, given condition (3), the unique best response of players in $\{k^* + 1, \dots, n\}$ is not to volunteer.

(ii) Consider the following public and anonymous feedback for every i, and let $s \in S$ be an MSCE:

$$\phi_i(s) = \phi(s) = \begin{cases} 2 & \text{if } |\{j \in N : s_j = 1\}| \geq k^* + 1 \\ 1 & \text{if } |\{j \in N : s_j = 1\}| = k^* \\ 0 & \text{otherwise.} \end{cases} \tag{5}$$

If $\phi(s) = 0$ and $s_i \neq 1$ for some i, then player i believes (in his worst-case scenario) that every other player volunteers with probability 0, so by (4) ($h(1) > c$) i's unique best response is to volunteer, a contradiction. If $\phi(s) = 0$ and $s_i = 1$ for some i, then there is another player j such that $s_j \neq 1$, so j's unique best response is to volunteer, and we get a contradiction as above.

If $\phi(s) = 2$, then every player i such that $s_i = 1$ knows that k^* other players volunteer with probability 1 and believes that the rest are volunteering with probability 0. Hence, by (3), i's unique best response is not to volunteer, a contradiction.

If $\phi(s) = 1$, then every player i such that $s_i = 1$ knows that $k^* - 1$ other players volunteer with probability 1 and believes that the rest are volunteering with probability 0. Hence, by (4) we have $h(k^*) - h(k^* - 1) > c$, so i's unique best response is to volunteer. Every player i such that $s_i \neq 1$ knows that k^* other players volunteer with probability 1 and believes that the rest are volunteering with probability 0. Hence, by (3), i's unique best response is not to volunteer.

We conclude that the set of MSCE is the set of strategy profiles s such that $\phi(s) = 1$ (i.e., $|\{j \in N : s_j = 1\}| = k^*$) and $s_i = 0$ if player i is not volunteering with probability 1, i.e., if $i \notin \{j \in N : s_j = 1\}$. ■

In this section, we have shown how coarse feedback could improve equilibrium contributions in a general class of public good games with binary actions. In the next section, we show that similar insights are obtained in a general class of public good games with a continuous range of possible contributions for each player.

5. Public goods

We study a class of symmetric public good games in which the contribution of each player i is $s_i \in [0, 1] = S_i$, and the utility of player i depends on the average contribution of all players and the cost cs_i of his own contribution:

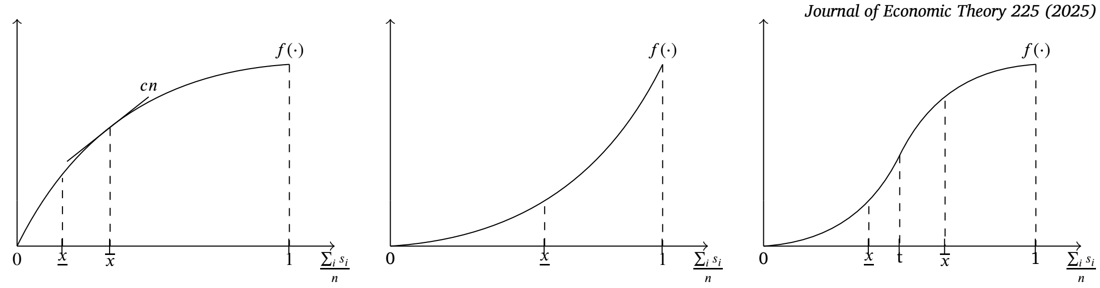
$$u_i(s_i, s_{-i}) = f\left(\frac{\sum_{j \in N} s_j}{n}\right) - cs_i,$$

where $f : [0, 1] \to \mathbb{R}$ is a production function satisfying the following assumption.

Assumption 1. (i) $f(0) = 0$; (ii) f is strictly increasing; (iii) f is continuous and differentiable; (iv) $f(1) > c$ and $f'(1) \geq c$; (v) f is a sigmoid function: there exists a $t \in [0, 1]$ such that $f(x)$ is strictly convex for all $x < t$ and strictly concave for all $x > t$.

Assumption 1 (i)–(iii) are standard assumptions. For simplicity, we assume that the total welfare is maximized if each player contributes 1 (Assumption 1 (iv)). With Assumption 1 (v), we consider s-shaped, convex, and concave production functions. If $t = 0$, the production function f is concave, and G is a public good game with strategic substitutes. If $t = 1$, the production function f is convex, and G is a public good game with strategic complements. Fig. 1 depicts convex, concave, and s-shaped production functions.

Depending on the production function, there exist at most two symmetric Nash equilibria: one with a positive contribution and one with no contribution. In this section, we are mostly interested in characterizing feedback that implies a symmetric MSCE with a contribution of 1 for each player–the social optimum–or at least a higher contribution than the symmetric Nash equilibrium with a



(a) concave: $t = 0$ (b) convex: $\overline{x} = t = 1$ (c) s-shaped

Fig. 1. Production functions for $t = 0$, $t = 1$ and $t \in (0, 1)$.

positive contribution. Thus, under multiplicity, we focus mainly on a designer-favorable equilibrium selection. However, if players coordinate on the MSCE, which is worse for the designer, they would coordinate on the no-contribution equilibrium, which is also the worst equilibrium for the players.

We can always characterize the symmetric Nash equilibrium with positive contribution by $\overline{x}$ with

$$\overline{x} := \begin{cases} x & \text{if } x \in [t,1] \text{ and } f'(x) = cn, \\ 1 & \text{if } f'(x) \geq cn \, \forall x \in [t,1], \\ t & \text{if } f'(x) \leq cn \, \forall x \in [t,1]. \end{cases}$$

For concave parts of the production function, $\overline{x}$ solves the first-order condition or equals a corner solution. For convex production functions, this definition simplifies to $\overline{x} = 1$. To ensure the existence of equilibria with positive contributions, we assume that costs are not too high:

Assumption 2. $c \leq \frac{1}{n} f'\left(\max\left\{0, \overline{x} - \frac{1}{n}\right\}\right)$.

The next lemma characterizes the best response function for any production function satisfying Assumption 1 and Assumption 2. For convex parts of the production function, players only choose a positive contribution if they know that the average contribution is high enough. We denote this threshold by $\underline{x}$. In Appendix A.2, we characterize and show the uniqueness of $\underline{x}$.

Lemma 3. *There exists a threshold $\underline{x} \leq \overline{x}$ such that player i's best response function is given by*

$$BR_i(s_{-i}) = \begin{cases} 0 & \text{if } \sum_{j \neq i} s_j \leq n\underline{x} - 1, \\ 1 & \text{if } n\underline{x} - 1 \leq \sum_{j \neq i} s_j < n\bar{x} - 1, \\ n\bar{x} - \sum_{j \neq i} s_j & \text{if } n\bar{x} - 1 \leq \sum_{j \neq i} s_j < n\bar{x}, \\ 0 & \text{if } n\bar{x} \leq \sum_{j \neq i} s_j \end{cases} \tag{6}$$

Proof. See Appendix A.2. ■

5.1. Nash equilibrium and MSCE

First, we show that the average contribution of any Nash equilibrium is indeed at most $\bar{x}$. Thus, $s_i^{NE} = \overline{x}$ is the symmetric Nash equilibrium with the highest average contribution.

Lemma 4. *The average contribution of every Nash equilibrium is weakly smaller than $\bar{x}$.*

Proof. If f is convex, $\bar{x} = 1$ by definition, and the proof is trivial. Suppose f is concave or s-shaped and $(s_i^*)_{i \in N}$ is a Nash equilibrium with $\frac{\sum_{i \in N} s_i^*}{n} > \bar{x}$. Then, there exists a player i with $s_i^* > 0$ and $0 \leq \bar{s}_i < s_i^*$ such that $\frac{\sum_{j \neq i} s_j^* + s_i}{n} \geq \bar{x}$ for all $s_i \in [\bar{s}_i, s_i^*]$. By the definition of $\bar{x}$, $f(x)$ is concave for all $x \geq \bar{x}$ and

$$\frac{1}{n} f'\left(\frac{\sum_{j \neq i} s_j^* + s_i}{n}\right) \leq c$$

for all $s_i \in [\bar{s}_i, s_i^*]$. Thus, player i would prefer to deviate to $\bar{s}_i$, which contradicts the Nash equilibrium assumption. ■

We consider a similar class of separable feedback functions as in the previous section. Separable feedback of player i about j is called *interval feedback* if there exist $\underline{s}$ and $\overline{s}$ such that the identification correspondence is given by

$$\Sigma_{ij}(s_j) = \begin{cases} [\underline{s}, \overline{s}] & \text{if } s_j \in [\underline{s}, \overline{s}] \\ \{s_j\} & \text{if } s_j \notin [\underline{s}, \overline{s}]. \end{cases}$$

For example, this identification correspondence can be obtained with the following continuous feedback function

$$\phi_{ij}(s_j) = \begin{cases} s_j & \text{if } s_j \leq \underline{s} \\ \underline{s} & \text{if } s_j \in [\underline{s}, \overline{s}] \\ s_j - (\overline{s} - \underline{s}) & \text{if } s_j \geq \overline{s}. \end{cases} \tag{7}$$

The public good game is symmetric, and u_i decreases in s_j for $j \neq i$. Therefore, for any symmetric MSCE $(s_i^*)_i$ of a public good game with separable feedback functions, we can construct symmetric interval feedback such that $(s_i^*)_i$ is an MSCE given the symmetric interval feedback. Thus, instead of considering arbitrary separable feedback functions, we can focus without loss of generality on symmetric interval feedback. Further, the worst case of player i is the minimal contribution consistent with the identification correspondence, and for MSCE, we can set $\overline{s} = 1$ without loss of generality.

Note that all results of this section can also be obtained with public and anonymous feedback. Consider the following feedback function

$$\phi(s) = \begin{cases} \frac{\sum_{j\in N} s_j}{n} & \text{if } \frac{\sum_{j\in N} s_j}{n} \leq \underline{s} \\ \underline{s} & \text{if } \frac{\sum_{j\in N} s_j}{n} \in [\underline{s}, \overline{s}] \\ \frac{\sum_{j\in N} s_j}{n} - (\overline{s} - \underline{s}) & \text{if } \frac{\sum_{j\in N} s_j}{n} \geq \overline{s}. \end{cases}$$

This feedback function assumes that players receive information only about the average contribution of all players. That is, each player learns the correct average contribution of all players if it is above $\overline{s}$ or below $\underline{s}$. If the average contribution is in the interval $[\underline{s}, \overline{s}]$, each player only learns that the average contribution belongs to the interval $[\underline{s}, \overline{s}]$. Since the utility of a player only depends on the average contribution of all players and his own contribution, receiving feedback about the average contribution of all players is equivalent to receiving feedback about each opponent. Then, the characterization of symmetric MSCE of Lemma 5 and, therefore, all further results of Section 5 can also be obtained with the above public and anonymous feedback.

Lemma 5. *For every public good game with symmetric interval feedback (G, ϕ) with $\overline{s} = 1$, there exists a symmetric MSCE, and all symmetric MSCE are characterized as follows:*

i) *Suppose f is convex. Then, one symmetric MSCE is*

$$s_i^* = \begin{cases} 1 & \text{if } \underline{s} \geq \frac{n\underline{x}-1}{(n-1)}, \\ 0 & \text{if } \underline{s} \leq \frac{n\underline{x}-1}{(n-1)}. \end{cases}$$

Further, if $f(\frac{1}{n}) < c$, then there exists a second symmetric MSCE with $s_i^* = 0$ for any symmetric interval feedback.

ii) *Suppose f is concave. Then, the unique symmetric MSCE is*

$$s_i^* = \begin{cases} 1 & \text{if } \underline{s} < \frac{n\bar{x}-1}{(n-1)}, \\ n\bar{x} - (n-1)\underline{s} & \text{if } \underline{s} \in \left[\frac{n\bar{x}-1}{(n-1)}, \bar{x}\right], \\ \bar{x} & \text{if } \underline{s} \geq \bar{x}. \end{cases}$$

iii) *Suppose f is s-shaped. Then, one symmetric MSCE is*

$$s_i^* = \begin{cases} 1 & \text{if } \underline{s} \in \left[\frac{n\underline{x}-1}{(n-1)}, \frac{n\bar{x}-1}{n-1}\right], \\ n\bar{x} - (n-1)\underline{s} & \text{if } \underline{s} \in \left[\frac{n\bar{x}-1}{(n-1)}, \bar{x}\right], \\ \bar{x} & \text{if } \underline{s} \geq \bar{x}. \end{cases}$$

Further, if $f(\frac{1}{n}) < c$, then there exists a second symmetric MSCE with $s_i^* = 0$ for any symmetric interval feedback.

Proof. See Appendix A.2. ■

Remember that for convex and s-shaped production functions, players only contribute to the public good if they know that the average contribution is high enough. We denoted this threshold by $\underline{x}$. For convex and s-shaped production functions, we show in Appendix A.2 that $f(\frac{1}{n}) \geq c$ is equivalent to $\underline{x} < \frac{1}{n}$. Therefore, $s_i^* = 0$ is only a symmetric MSCE if $f(\frac{1}{n}) < c$.

The next proposition compares the contribution in symmetric MSCE and Nash equilibria. Remember that full contribution is the first best solution for any production function satisfying Assumption 1. Therefore, we consider a designer who tries to implement the highest possible contribution.

Due to Assumption 2, full contribution is always a Nash equilibrium for convex production function. Thus, no feedback exists that implements an MSCE with a strictly larger contribution than the best Nash equilibrium.

However, for concave and s-shaped production functions, we can find symmetric interval feedback such that the contribution of the induced MSCE is strictly larger than the contribution of Nash equilibria. First, if the total contribution of the unique symmetric Nash equilibrium is below one, i.e., if $s_i^{NE} = \overline{x} < \frac{1}{n}$, we cannot implement full contribution. Even if player i believes that all opponents contribute zero, the best response would be smaller than one. Therefore, it is impossible to implement full contribution as an MSCE in this case. However, Proposition 6 shows that there still exist feedback functions such that the contribution of the MSCE is strictly larger than the symmetric Nash equilibrium. If the total Nash equilibrium contribution exceeds one, i.e., $s_i^{NE} = \overline{x} \geq \frac{1}{n}$, we can always construct feedback that implements full contribution and, therefore, the first best solution.

Proposition 6. *Consider the public good game with interval feedback (G, ϕ) and suppose f is concave or s-shaped. Then, in any MSCE s^*, $s_i^* \leq \min\{1, n\bar{x}\}$, that is, the individual contribution is weakly smaller than n times the maximal average contribution of a Nash equilibrium. This upper bound is reached*

i) *by the unique symmetric MSCE with no feedback if $n\bar{x} < 1$,*
ii) *by a symmetric MSCE with interval feedback characterized by $\underline{s} \in [\frac{n\underline{x}-1}{(n-1)}, \frac{n\bar{x}-1}{(n-1)}]$ if $n\bar{x} \geq 1$. This MSCE is the unique symmetric MSCE if f is concave or s-shaped and $f(\frac{1}{n}) \geq c$.*

Proof. First, let $n\bar{x} < 1$. Then, $s_i^* \leq n\bar{x}$ follows from the best response in Lemma 3. If players receive no feedback, we have $\Sigma_{ij}(s_j^*) = [0,1]$ for all $i, j \in N$. Thus, $\min_{s_j \in \Sigma_{ij}(s_j^*)} s_j = 0$ and $s_i^* = n\bar{x} - \sum_{j \neq i} \min_{s_j \in \Sigma_{ij}(s_j^*)} s_j = n\bar{x}$ for all $i \in N$.

Second, let $n\bar{x} \geq 1$. Then, $s_i^* \leq n\bar{x}$ is trivial. By Lemma 5 for concave and s-shaped f, the strategy profile $s_i^* = 1$ for all $i \in N$ is an MSCE if $(n-1)\underline{s} \in [n\underline{x} - 1, n\bar{x} - 1]$. Note that for concave f, $n\underline{x} < 1$, and therefore $n\underline{x} - 1 < 0$. Further, the uniqueness conditions directly follow from Lemma 5. ■

5.2. More ambiguous feedback

For interval feedback, we can define a simple notion of more ambiguous feedback. Remember that the identification correspondence of interval feedback is

$$\Sigma_{ij}(s_j) = \begin{cases} [\underline{s}, \overline{s}] & \text{if } s_j \in [\underline{s}, \overline{s}], \\ \{s_j\} & \text{if } s_j \notin [\underline{s}, \overline{s}]. \end{cases}$$

Further, since MSCE only considers the worst case, we can assume that $\overline{s} = 1$ without loss of generality. If $s_j < \underline{s}$, player i receives perfect feedback about s_j. Thus, an interval feedback function is called more ambiguous if it induces a larger interval $[\underline{s}, 1]$.[10]

Definition 3. Let ϕ_1 and ϕ_2 be two interval feedback functions with $\overline{s}_1 = \overline{s}_2 = 1$. We call ϕ_1 *more ambiguous than* ϕ_2 if $\underline{s}_1 \leq \underline{s}_2$.

Given the notion of more ambiguous feedback, we can study the impact of ambiguity on MSCE. As the following proposition shows, for concave and s-shaped production functions, more ambiguous feedback always induces a higher MSCE contribution.

Proposition 7. *Let ϕ_1 be more ambiguous than ϕ_2 and consider (G, ϕ^k) with $\phi_{ij}^k = \phi_k$ for all $i, j \in N$ and $k = 1, 2$.*

i) *Suppose f is concave, or s-shaped and $f(\frac{1}{n}) \geq c$. Then, the average contribution of the unique MSCE of (G, ϕ^1) is higher than the average contribution of the unique MSCE of (G, ϕ^2).*
ii) *Suppose f is convex. Then, the highest average contribution of an MSCE of (G, ϕ^1) is lower than the highest average contribution of an MSCE of (G, ϕ^2)*

Proof. Consider a concave or s-shaped production function f. $f(\frac{1}{n}) \geq c$ implies $\underline{x} \leq \frac{1}{n}$. Therefore, Lemma 5 shows that the unique symmetric MSCE of a public good game with symmetric interval feedback (G, ϕ) with $\underline{s}$ is s_i^* for all $i \in N$ with

[10] For the public and anonymous feedback function discussed above, we can define a similar notion of more ambiguous feedback. Players learn the correct average contribution if the average contribution is below $\underline{s}$. If the average contribution is above $\underline{s}$, players only learn that the average contribution is above $\underline{s}$. Thus, a smaller $\underline{s}$ implies more ambiguity.

$$s_i^* = \begin{cases} 1 & \text{if } (n-1)\underline{s} < n\bar{x} - 1 \\ n\bar{x} - (n-1)\underline{s} & \text{if } n\bar{x} - 1 \le (n-1)\underline{s} < (n-1)\bar{x} \\ \bar{x} & \text{if } \underline{s} \ge \bar{x} \end{cases}$$

Further, $n\bar{x} - (n-1)\underline{s} \in [\bar{x}, 1]$ and $n\bar{x} - (n-1)\underline{s}$ is strictly decreasing in $\underline{s}$ for all $\underline{s} \in \left[\frac{n\bar{x}-1}{n-1}, \bar{x}\right]$. Thus, the result follows since $\underline{s}_1 \le \underline{s}_2$.

For convex production functions, Lemma 5 shows that for $\underline{x} \le \frac{1}{n}$ the unique symmetric MSCE is $s_i^* = 1$ for all $i \in N$ for any threshold feedback. Thus, if $\underline{x} < \frac{1}{n}$, the unique MSCE of (G, ϕ_1) equals the unique MSCE of (G, ϕ_2). If $\underline{x} > \frac{1}{n}$, the highest contribution of a symmetric MSCE of (G, ϕ) with symmetric interval feedback ϕ is

$$s_i^* = \begin{cases} 1 & \text{if } (n-1)\underline{s} \ge n\underline{x} - 1 \\ 0 & \text{if } (n-1)\underline{s} < n\underline{x} - 1. \end{cases}$$

Then, the result follows from $\underline{s}_1 \le \underline{s}_2$. ■

Note that the assumption $f(\frac{1}{n}) \ge c$ ensures the uniqueness of the MSCE for s-shaped production functions. However, even if we allow for multiple MSCE and compare the MSCE with the highest average contribution, Proposition 7 i) only holds if $f(\frac{1}{n}) \ge c$. Suppose $f(\frac{1}{n}) < c$, then $\underline{x} > \frac{1}{n}$ and we can find $\underline{s}_1$ and $\underline{s}_2$ such that $(n-1)\underline{s}_1 < n\underline{x} - 1 \le (n-1)\underline{s}_2$. By Lemma 5, the unique MSCE of (G, ϕ_1) is $\hat{s}_i = 0$ for all $i \in N$. Further, (G, ϕ_2) has two MSCE: $s_i^* = 0$ and $\tilde{s}_i = 1$ for all $i \in N$. Thus, even if ϕ_1 is more ambiguous than ϕ_2, the highest average contribution in an MSCE of (G, ϕ_1) is strictly smaller than the highest average contribution in an MSCE of (G, ϕ_2).

Proposition 7 is in line with the results of Eichberger and Kelsey (2002). Eichberger and Kelsey (2002) study the impact of ambiguity in symmetric games with aggregated externalities. In contrast to our setting, they assume that players' beliefs about opponents' strategies are represented by capacities, i.e., non-additive probability measures. Similar to Proposition 7, they show that in settings with positive externalities and strategic substitutes, more ambiguous beliefs induce a larger equilibrium strategy. Further, they study a public good setting with a concave production function, n players, and a finite set of pure strategies. They show that maxmin preferences imply an equilibrium contribution of at least $n\tilde{s}$, where $\tilde{s}$ is the lowest strategy played in a symmetric Nash equilibrium. In our setting, if f is concave and $\overline{x} < \frac{1}{n}$, then there exists a unique symmetric Nash equilibrium with $\tilde{s} = \overline{x}$. Further, by Proposition 6, there exists an MSCE with contribution $n\overline{x}$.

5.3. α-maxmin self-confirming equilibrium

For simplicity, we restrict the analysis of α-MSCE to concave production functions and cases where full contribution is implementable as an MSCE. By Lemma 5, an MSCE with full contribution exists if $f(\frac{1}{n}) \ge cn$, i.e., $\bar{x} \ge \frac{1}{n}$. Then, full contribution can be implemented as an α-MSCE if α is large enough.

Proposition 8. *There exist feedback such that full contribution, i.e., $s_i^* = 1$ for all $i \in N$, is an α-MSCE iff $\alpha \ge \frac{cn - f'(1)}{f'\left(\frac{1}{n}\right) - f'(1)}$. Such feedback functions are characterized by an interval feedback function with $\overline{s} = 1$ and $\underline{s} \le \tilde{s}_\alpha$ where $\tilde{s}_\alpha \ge 0$ satisfies*

$$f'\left(\frac{(n-1)\tilde{s}_\alpha + 1}{n}\right) = \frac{cn - (1-\alpha)f'(1)}{\alpha}. \tag{8}$$

Proof. Since f is concave, $s_i^* = 1$ is a best response to symmetric interval feedback with $\overline{s} = 1$ and $\underline{s}$ iff

$$\alpha f'\left(\frac{(n-1)\underline{s} + 1}{n}\right) + (1-\alpha)f'(1) \ge cn. \tag{9}$$

Thus, by the definition of $\tilde{s}_\alpha$ and since f is concave, Equation (9) is satisfied for all $\underline{s} \le \tilde{s}_\alpha$. Further, $\tilde{s}_\alpha$ is increasing in α and the assumption $\alpha \ge \frac{cn - f'(1)}{f'\left(\frac{1}{n}\right) - f'(1)}$ ensures that $\tilde{s}_\alpha \ge 0$. ■

If $f'(1) \ge cn$, the unique Nash equilibrium is $s_i^* = 1$ for all $i \in N$, and $\frac{cn - f'(1)}{f'\left(\frac{1}{n}\right) - f'(1)} < 0$. Then, for any $\alpha \in [0, 1]$, there exists an α-MSCE with full contribution. Further, $\tilde{s}_\alpha$ increases in α if full contribution is not a Nash equilibrium. Thus, for higher α, there exist more feedback functions implementing full contribution.

If $\alpha < \frac{cn - f'(1)}{f'\left(\frac{1}{n}\right) - f'(1)}$, full contribution cannot be implemented as α-MSCE. However, we can still find the symmetric α-MSCE that maximizes the contribution. These α-MSCE are induced by interval feedback with $\underline{s} = 0$ and $\overline{s}$ such that the symmetric α-MSCE is $s_i = \overline{s}$ for all $i \in N$ where $\overline{s}$ satisfies

$$\alpha f'\left(\frac{\overline{s}}{n}\right)+(1-\alpha)f'\left(\overline{s}\right)=cn. \tag{10}$$

Since f is concave, $f'(x)$ is decreasing in x and there exists a unique $\overline{s}$ satisfying Equation (10).

Further, $\overline{s}$ increases as α increases. Thus, higher α leads to higher contribution in the α-MSCE. For $\alpha=0$, players maximize their best-case expected utility. Then, $\overline{s}=\overline{x}$ and the α-MSCE with $\alpha=0$ coincides with the Nash equilibrium. For $\alpha=\frac{cn-f'(1)}{f'\left(\frac{1}{n}\right)-f'(1)}$, Equation (10) with $\overline{s}=1$ and Equation (8) with $\tilde{s}_\alpha=0$ are equivalent. Thus, as α increases from 0 to $\frac{cn-f'(1)}{f'\left(\frac{1}{n}\right)-f'(1)}$, the contribution in the α-MSCE increases from the Nash equilibrium contribution to full contribution.

Similarly, we can extend the analysis of convex production functions to α-MSCE. For convex production functions, full contribution is always a Nash equilibrium and an MSCE if the interval feedback satisfies $\underline{s}\geq\frac{n\underline{x}-1}{(n-1)}$ (Lemma 5). It can be shown that for α-MSCE, there exists a similar threshold for $\underline{s}$. However, this threshold is decreasing in α. Thus, for lower α, more ambiguous feedback still implements full contribution.

6. Strategic substitutes and negative externalities

In this section, we consider a class of symmetric games with strategic substitutes, negative externalities, and linear best responses, such as Cournot oligopolies with linear demand and symmetric constant marginal costs. Specifically, let $N=\{1,\ldots,n\}$, $S_i=[0,1]$, and

$$u_i(s)=\gamma s_i-s_i^2-d s_i\sum_{j\neq i}s_j,$$

where $\gamma,d>0$. To avoid corner solutions, we assume $\gamma<1+d(n-1)$. The best response function of player i is

$$BR_i(s_{-i})=\frac{\gamma-d\sum_{j\neq i}s_j}{2},$$

and the unique Nash equilibrium is symmetric and given by

$$s_i^{NE}=\frac{\gamma}{2+d(n-1)},\quad i\in N.$$

The symmetric collusive solution, which maximizes the sum of players' utilities, is given by

$$s_i^{C}=\frac{\gamma}{2+2d(n-1)},\quad i\in N.$$

Similar to the public good setting, the game is symmetric, and u_i is decreasing in s_j for $j\neq i$. Therefore, we can focus on separable and symmetric interval feedback functions, i.e., for every i and $j\neq i$:

$$\Sigma_{ij}(s_j)=\begin{cases}[\underline{s},\overline{s}] & \text{if } s_j\in[\underline{s},\overline{s}]\\ \{s_j\} & \text{if } s_j\notin[\underline{s},\overline{s}].\end{cases}$$

Moreover, the best response function depends only on the sum of the opponents' strategies, $\sum_{j\neq i}s_j$. Thus, instead of separable and symmetric interval feedback, we can obtain the same results with the following public and anonymous feedback: players learn the correct $\sum_{i\in N}s_i$ if $\sum_{i\in N}s_i\notin[n\underline{s},n\overline{s}]$. If $\sum_{i\in N}s_i\in[n\underline{s},n\overline{s}]$, players only learn that $\sum_{i\in N}s_i\in[n\underline{s},n\overline{s}]$.

The next proposition shows that interval feedback always exists such that the collusive outcome is an MSCE. The following proposition shows that this result also holds (with different interval feedback functions) for α-MSCE, provided α is sufficiently large.

Proposition 9. *There exist symmetric feedback functions such that the collusive strategy profile $(s_i^C)_i$ is an MSCE. Such feedback functions are given by interval feedback functions with $\underline{s}\leq s_i^C=\frac{\gamma}{2+2d(n-1)}$ and $\overline{s}=2s_i^C=\frac{\gamma}{1+d(n-1)}$. In addition, if $\underline{s}=0$ and $\overline{s}=2s_i^C$ the collusive strategy profile $(s_i^C)_i$ is the unique MSCE.*

Proof. The first part follows immediately from the fact that the best response of player i to $2s_{-i}^C$ is s_i^C. For the uniqueness, suppose $(s_i^*)_i$ is an MSCE. First, consider the case $s_j^*\in[\underline{s},\overline{s}]$ for all $j\neq i$. Then, in the worst case, player i believes that all players $j\neq i$ play $s_j=\overline{s}=2s_i^c$ and $s_i^*=s_i^C$ is the unique best response. Thus, there cannot exist an MSCE in which only one player is playing a strategy $s_i^*\notin[\underline{s},\overline{s}]$. Now, suppose there exist $N^1\subseteq N$ with $|N^1|=n^1\geq 2$ such that $s_j^*\notin[\underline{s},\overline{s}]$ for all $j\in N^1$ and $s_j^*\in[\underline{s},\overline{s}]$ for all $j\notin N^1$. Then, since $\underline{s}=0$, it follows that $s_j^*>\overline{s}=2s_i^C$ for all $j\in N^1$. Consider an arbitrary player $i\in N^1$. In the worst case, player i believes that

$$\sum_{j\neq i}s_j=\sum_{j\notin N^1}s_j+\sum_{j\in N^1\setminus\{i\}}s_j=(n-n^1)\overline{s}+\sum_{j\in N^1\setminus\{i\}}s_j^*>(n-n^1)\overline{s}+(n^1-1)\overline{s}=(n-1)\overline{s}=(n-1)2s_i^C,$$

where the strict inequality follows from $s_j^*>\overline{s}$ for $j\in N^1$. The best response function is decreasing in s_{-i}. Thus, $BR_i(s_i^*)<s_i^C\in[\underline{s},\overline{s}]$ which contradicts $i\in N^1$. Therefore, there cannot exist an MSCE with two or more players playing a strategy $s_i^*\notin[\underline{s},\overline{s}]$. Thus, in

every MSCE, every player $i \in N$ plays a strategy $s_i^* \in [\underline{s}, \overline{s}]$. However, if $s_i^* \in [\underline{s}, \overline{s}]$ for all $i \in N$, the collusive strategy is the unique best response and $(s_i^C)_i$ is the unique MSCE. ■

Proposition 10. *There exist feedback functions such that the collusive strategy profile $(s_i^C)_i$ is an α-MSCE iff $\frac{\alpha}{1+\alpha} \geq s_i^C = \frac{\gamma}{2+2d(n-1)}$. Such feedback functions are characterized by interval feedback functions with $\underline{s} = s_i^C = \frac{\gamma}{2+2d(n-1)}$ and $\overline{s} = \frac{1+\alpha}{\alpha} s_i^C$.*

Proof. The collusive strategy profile is an α-MSCE iff for every $i \neq j$ the strategy of player j as perceived by player i is $\alpha\overline{s} + (1-\alpha)\underline{s} = 2s_i^C$. Such an equilibrium exists only if it exists for $\underline{s} = s_i^C$, and therefore $\overline{s} = \frac{1+\alpha}{\alpha} s_i^C$. Since we must have $\overline{s} \leq 1$, the collusive outcome is an α-MSCE iff $\frac{\alpha}{1+\alpha} \geq s_i^C$. ■

Now assume that α is too small (players are too optimistic) for the collusive outcome to be implementable as an α-MSCE. That is, assume that $\frac{\alpha}{1+\alpha} < s_i^C$, i.e., $\alpha < \frac{s_i^C}{1-s_i^C} = \frac{\gamma}{2+2d(n-1)-\gamma}$. The best (symmetric) outcome is then obtained with the feedback that minimizes the equilibrium strategy s_i, i.e., $\overline{s} = 1$ and $\underline{s} = s_i$. The α-MSCE $(s_i)_i$ is then given by $s_i = BR_i(\tilde{s}_{-i})$ with $\tilde{s}_j = \alpha\overline{s} + (1-\alpha)\underline{s} = \alpha + (1-\alpha)s_j$, so in equilibrium

$$\begin{aligned} & s_i = \frac{\gamma - d(n-1)(\alpha + (1-\alpha)s_i)}{2} \\ \Leftrightarrow \quad & s_i = \frac{\gamma - d(n-1)\alpha}{2 + d(n-1)(1-\alpha)}. \end{aligned}$$

It is immediate to check that the equilibrium strategy s_i (and hence, each player's actual equilibrium utility) decreases in α. For $\alpha = 0$, we get $s_i = s_i^{NE}$, and the best feedback to maximize players' welfare is to give perfect feedback. For $\alpha = \frac{s_i^C}{1-s_i^C}$ we get the first best $s_i = s_i^C$ as for $\alpha \geq \frac{s_i^C}{1-s_i^C}$, with $\underline{s} = s_i^C$ and $\overline{s} = 1$.

To conclude, for $\alpha \geq \frac{s_i^C}{1-s_i^C}$ we get the collusive outcome $s_i = s_i^C$ as an α-MSCE with the interval feedback $\underline{s} = s_i^C$ and $\overline{s} = \frac{1+\alpha}{\alpha} s_i^C$. For $\alpha < \frac{s_i^C}{1-s_i^C}$ the collusive outcome is not implementable as an α-MSCE, and the best α-MSCE is obtained with the interval feedback $\underline{s} = s_i$ and $\overline{s} = 1$. In that case, the best α-MSCE equilibrium strategy is decreasing in α (going from s_i^{NE} when $\alpha = 0$ to s_i^C when $\alpha = \frac{s_i^C}{1-s_i^C}$).

So far, we have studied feedback functions that implement the collusive outcome, i.e., in the Cournot oligopoly context, MSCE that maximize producers' surplus. In contrast, we could study MSCE that increase consumer surplus relative to the Nash equilibrium. However, there does not exist feedback and MSCE that induce a higher consumer surplus than the Nash equilibrium. Maximizing the consumer surplus is equivalent to maximizing the equilibrium strategies s_i^*. Suppose there exist feedback functions and a symmetric MSCE with $s_i^* > s_i^{NE}$. Remember that the true strategy profile of the opponents is always one of the strategy profiles that player i considers, i.e., $s_{-i}^* \in \Sigma_i(s_{-i}^*)$. Further, the worst-case belief of player i is the highest possible production of his opponents. Thus, in the worst case, player i believes that his opponents play a strategy greater than or equal to $s_j^* > s_j^{NE}$. The best response function is decreasing in the strategies of the opponents. Therefore, the best response of player i has to be smaller than the Nash equilibrium strategy. Thus, no symmetric MSCE exists with strategies higher than the Nash equilibrium strategy. These results align with Eichberger and Kelsey (2002), who showed that increased ambiguity leads to lower equilibrium strategies in settings with strategic substitutes and negative externalities.

CRediT authorship contribution statement

Frederic Koessler: Writing – review & editing, Writing – original draft, Investigation, Formal analysis. **Marieke Pahlke:** Writing – review & editing, Writing – original draft, Investigation, Formal analysis.

Declaration of competing interest

The authors declare that they have no relevant or material financial interests that relate to the research described in this paper.

Appendix A

A.1. Proofs of Section 3

Proof of Lemma 1. Fix any $i \in N$, $s_{-i} \in S_{-i}$ and $y \in \mathbb{R}$. To prove that v_i is quasi-concave on S_i, we must show that

$$\{s_i \in S_i : v_i(s_i, s_{-i}) \geq y\},$$

is convex. That is, we must show that for every $s_i, s_i' \in S_i$ and $\beta \in [0,1]$,

$$v_i(s_i, s_{-i}) \geq y \text{ and } v_i(s_i', s_{-i}) \geq y \tag{11}$$

implies

$$v_i(\beta s_i + (1-\beta)s_i', s_{-i}) \geq y. \tag{12}$$

(11) is equivalent to

$$\inf_{\tilde{s}_{-i} \in \Sigma_i(s_{-i})} u_i(s_i, \tilde{s}_{-i}) \geq y \text{ and } \inf_{\tilde{s}_{-i} \in \Sigma_i(s_{-i})} u_i(s_i', \tilde{s}_{-i}) \geq y,$$

i.e.,

$$u_i(s_i, \tilde{s}_{-i}) \geq y \text{ and } u_i(s_i', \tilde{s}_{-i}) \geq y, \text{ for every } \tilde{s}_{-i} \in \Sigma_i(s_{-i}).$$

From the assumption that u_i is quasi-concave on S_i we get

$$u_i(\beta s_i + (1-\beta)s_i', \tilde{s}_{-i}) \geq y \text{ for every } \tilde{s}_{-i} \in \Sigma_i(s_{-i}).$$

Hence,

$$\inf_{\tilde{s}_{-i} \in \Sigma_i(s_{-i})} u_i(\beta s_i + (1-\beta)s_i', \tilde{s}_{-i}) \geq y,$$

which establishes (12). ■

Proof of Lemma 2. Because the linearity of ϕ_i implies own-strategy independence, i.e., $\Sigma_i(s)$ does not depend of s_i, we omit s_i in the notation. We must show that for all sequences (s_{-i}^k) in S_{-i} with $s_{-i}^k \to s_{-i}$, and for all $\tilde{s}_{-i} \in \Sigma_i(s_{-i})$, there exist a subsequence $(s_{-i}^{k(m)})$ and a sequence $(\tilde{s}_{-i}^m)$ in S_{-i} such that $\tilde{s}_{-i}^m \to \tilde{s}_{-i}$ and $\phi_i(\tilde{s}_{-i}^m) = \phi_i(s_{-i}^{k(m)})$ for all m. For $m = 1, 2, \dots$ and all $j \neq i$, let

$$s_j^m := (1 - \frac{1}{m})\tilde{s}_j + \frac{1}{m}s_j.$$

Observe that $s_j^m \to \tilde{s}_j$ as $m \to \infty$, and therefore $s_{-i}^m \to \tilde{s}_{-i}^m$. Further, by linearity of ϕ_i we have $\phi_i(s_{-i}^m) = \phi_i(\tilde{s}_{-i}) = \phi_i(s_{-i})$ for all m. Let

$$z_j^{m,k} := s_j^k - s_j + s_j^m,$$

and $z_{-i}^{m,k} = (z_j^{m,k})_{j\neq i}$. By the linearity of ϕ_i, we have $\phi_i(z_{-i}^{m,k}) = \phi_i(s_{-i}^k)$ for all m and k.

Claim 1. *For every m, and for k large enough, we have $z_{-i}^{m,k} \in S_{-i}$.*

Proof. We show that $z_j^{m,k} \in S_j$ for all $j \neq i$. For every $z \in S_j$, denote by $z[l]$ the lth coordinate of z. First, observe that

$$\sum_l z_j^{m,k}[l] = \sum_l s_j^{m,k}[l] - \sum_l s_j[l] + \sum_l s_j^m[l] = 1 - 1 + 1 = 1,$$

because $s_j^k, s_j, s_j^m \in S_j$. It remains to show that $z_j^{m,k}[l] \geq 0$ for all l and for k large enough. If $s_j^m[l] > 0$, then we have $z_j^{m,k}[l] = s_j^k[l] - s_j[l] + s_j^m[l] > 0$ for k large enough because $s_j^k[l] \to s_j[l]$. If $s_j^m[l] = 0$, then $s_j[l] = 0$, so $z_j^{m,k}[l] = s_j^k[l] \geq 0$. This completes the proof of the claim. ■

Hence, for all m, there is $k(m)$ such that $\tilde{s}_{-i}^m := z_{-i}^{m,k(m)} \in S_{-i}$, $\phi_i(\tilde{s}_{-i}^m) = \phi_i(s_{-i}^{k(m)})$ for all m, and $\tilde{s}_{-i}^m \to \tilde{s}_{-i}$. This completes the proof of the lemma. ■

A.2. Proofs of Section 5

Proof of Lemma 3. Suppose f is concave, i.e., $t = 0$. We set $\underline{x} = 0$ and Equation (6) reduces to

$$BR_i(s_{-i}) = \begin{cases} 1 & \text{if } \sum_{j\neq i} s_j < n\bar{x} - 1, \\ n\bar{x} - \sum_{j\neq i} s_j & \text{if } n\bar{x} - 1 \leq \sum_{j\neq i} s_j < n\bar{x}, \\ 0 & \text{if } n\bar{x} \leq \sum_{j\neq i} s_j. \end{cases}$$

By the definition of $\bar{x}$ and Assumption 1, if $\sum_{j\neq i} s_j < n\bar{x} - 1$ we have

$$f'\left(\frac{\sum_{j\neq i} s_j + s_i}{n}\right) \geq cn$$

for all $s_i \in [0,1]$. Thus $BR_i(s_{-i}) = 1$. Similarly, if $n\bar{x} \le \sum_{j\neq i} s_j$ we have

$$f'\left(\frac{\sum_{j\neq i} s_j + s_i}{n}\right) \le cn$$

for all $s_i \in [0,1]$ and $BR_i(s_{-i}) = 0$. If $n\bar{x} - 1 \le \sum_{j\neq i} s_j < n\bar{x}$, then there exists a $BR_i(s_{-i}) \in [0,1]$ that solves the FOC

$$\begin{aligned} & f'\left(\frac{\sum_{j\neq i} s_j + BR_i(s_{-i})}{n}\right) = cn \\ \Leftrightarrow \quad & \frac{\sum_{j\neq i} s_j + BR_i(s_{-i})}{n} = \bar{x} \\ \Leftrightarrow \quad & BR_i(s_{-i}) = n\bar{x} - \sum_{j\neq i} s_j. \end{aligned}$$

Suppose f is convex, i.e., $t = 1$. By definition of $\bar{x}$ we have $\bar{x} = 1$ and the best response function reduces to

$$BR_i(s_{-i}) = \begin{cases} 0 & \text{if } \sum_{j\neq i} s_j < n\underline{x} - 1, \\ 1 & \text{if } n\underline{x} - 1 \le \sum_{j\neq i} s_j, \end{cases}$$

where

$$\underline{x} := \begin{cases} 0 & \text{if } f(x) - f\left(x - \frac{1}{n}\right) > c, \ \forall x \in [0,1], \\ 1 & \text{if } f(x) - f\left(x - \frac{1}{n}\right) < c, \ \forall x \in [0,1], \\ x & \text{if } x \in [0,1] \text{ and } f(x) - f\left(x - \frac{1}{n}\right) = c. \end{cases}$$

Note that $\underline{x}$ is unique and well defined for convex production functions f because $g(x) := f(x) - f\left(x - \frac{1}{n}\right)$ is increasing in x. Further, by the monotonicity of $g(x)$, the best response is $BR_i(s_{-i}) = 1$ if and only if

$$\begin{aligned} & f\left(\frac{\sum_{j\neq i} s_j + 1}{n}\right) - c \ge f\left(\frac{\sum_{j\neq i} s_j}{n}\right) \\ \Leftrightarrow \quad & f\left(\frac{\sum_{j\neq i} s_j + 1}{n}\right) - f\left(\frac{\sum_{j\neq i} s_j}{n}\right) \ge c \\ \Leftrightarrow \quad & \frac{\sum_{j\neq i} s_j + 1}{n} \ge \underline{x} \\ \Leftrightarrow \quad & \sum_{j\neq i} s_j \ge n\underline{x} - 1. \end{aligned}$$

Now, suppose f is s-shaped, i.e. $t \in (0,1)$. If $\sum_{j\neq i} s_j \in [0, nt-1]$ (or $\sum_{j\neq i} s_j \in [nt, n-1]$), then f is convex (or concave) for all $s_i \in [0,1]$ and the best response function can be derived as above. Therefore, consider a strategy profile s_{-i} such that $\sum_{j\neq i} s_j \in [nt-1, nt]$.

Case 1. Suppose $nt \le n\bar{x} - 1$. Then, $\sum_{j\neq i} s_j \in [nt-1, nt]$ implies that $\sum_{j\neq i} s_j \le n\bar{x} - 1$. Therefore, there are two possible solutions of the maximization problem: $BR_i(s_{-i}) = 0$ and $BR_i(s_{-i}) = 1$. $BR_i(s_{-i}) = 1$ is a global maximum if and only if

$$f\left(\frac{\sum_{j\neq i} s_j + 1}{n}\right) - c \ge f\left(\frac{\sum_{j\neq i} s_j}{n}\right).$$

Define an auxiliary function $g(x) : \left[t, t + \frac{1}{n}\right] \to \mathbb{R}$ with $g(x) = f(x) - f\left(x - \frac{1}{n}\right)$. Then, g is concave[11] and

$$\begin{aligned} & g(x) \ge c, \ \forall x \in \left[t, t + \frac{1}{n}\right] \\ \Leftrightarrow \quad & \min_{x \in \left[t, t + \frac{1}{n}\right]} g(x) \ge c \\ \Leftrightarrow \quad & \min\left\{ g(t), g\left(t + \frac{1}{n}\right) \right\} \ge c. \end{aligned}$$

Since $f(x)$ is concave for all $x \in [t,1]$ and $t + \frac{1}{n} \le \bar{x}$, we have $g\left(t + \frac{1}{n}\right) = f\left(t + \frac{1}{n}\right) - f(t) \ge c$.

[11] Note that g is not strictly increasing, since $f(x)$ is concave and $f\left(x - \frac{1}{n}\right)$ is convex for all $x \in \left[t, t + \frac{1}{n}\right]$.

Further, $g(t) = f(t) - f\left(t - \frac{1}{n}\right)$. Define $\underline{x}$ as

$$\underline{x} = \begin{cases} 0 & \text{if } f(x) - f\left(x - \frac{1}{n}\right) > c, \forall x \in \left[0, t + \frac{1}{n}\right], \\ x & \text{if } x \in \left[0, t + \frac{1}{n}\right] \text{ and } f(x) - f\left(x - \frac{1}{n}\right) = c. \end{cases}$$

If $\underline{x} \le t$ uniqueness follows similarly as for convex production functions. Suppose $\underline{x} > t$. Then, since g is concave and $g\left(t + \frac{1}{n}\right) \ge c$ there exists an unique $\underline{x} \in \left[t, t + \frac{1}{n}\right]$ such that $g(\underline{x}) = f(\underline{x}) - f\left(\underline{x} - \frac{1}{n}\right) = c$ and $g(x) > c$ iff $x > \underline{x}$. Thus, $\underline{x}$ is well-defined and unique.

Further, if $\underline{x} \le t$, we have $g(t) = f(t) - f\left(t - \frac{1}{n}\right) \ge c$ and $g(x) \ge c$ for all $x \in \left[t, t + \frac{1}{n}\right]$. Then, $BR_i(s_{-i}) = 1$ for all $\sum_{j \ne i} s_j \in [nt - 1, nt]$. If $\underline{x} > t$, then we have $g(x) < c$ for $x < \underline{x}$ and $g(x) \ge c$ for $x \ge \underline{x}$. Thus, $BR_i(s_{-i}) = 1$ iff $\sum_{j \ne i} s_j \ge n\underline{x} - 1$.

Case 2. Suppose $nt > n\bar{x} - 1$. First, consider $\bar{x} \le \frac{1}{n}$ and $\sum_{j \ne i} s_j \in [0, nt]$. Define $\underline{x} := 0$. By Assumption 2, we have $f'(0) \ge cn$. Further, by definition of $\bar{x}$ we have $f'(\bar{x}) \ge cn$. Then, convexity of f on $[0, t]$ and concavity on $[t, 1]$ implies that $f'(x) \ge cn$ for all $x \in [0, \bar{x}]$. Thus,

$$\frac{f(\bar{x}) - f\left(\frac{\sum_{j \ne i} s_j}{n}\right)}{\frac{n\bar{x} - \sum_{j \ne i} s_j}{n}} \ge cn$$

$$\Leftrightarrow \qquad f(\bar{x}) - c\left(n\bar{x} - \sum_{j \ne i} s_j\right) \ge f\left(\frac{\sum_{j \ne i} s_j}{n}\right)$$

and $BR_i(s_{-i}) = n\bar{x} - \sum_{j \ne i} s_j$ is the best response for all s_{-i} such that $\sum_{j \ne i} s_j \in [0, nt]$.

Now suppose $\bar{x} > \frac{1}{n}$. We consider two subcases.

Case 2.a. Suppose $\sum_{j \ne i} s_j \in [n\bar{x} - 1, nt]$.

Now $BR_i(s_{-i}) = 1$ can never be a best response. Instead we have to compare the candidates $BR_i(s_{-i}) = 0$ and $BR_i(s_{-i}) = n\bar{x} - \sum_{j \ne i} s_j$. Then, $BR_i(s_{-i}) = n\bar{x} - \sum_{j \ne i} s_j$ iff

$$f(\bar{x}) - c\left(n\bar{x} - \sum_{j \ne i} s_j\right) \ge f\left(\frac{\sum_{j \ne i} s_j}{n}\right).$$

Define $h : [n\bar{x} - 1, nt] \to \mathbb{R}$ as $h(x) := f\left(\frac{x}{n}\right) + c(n\bar{x} - x)$. Then, $h(x)$ is convex for all $x \in [n\bar{x} - 1, nt]$ and $h(x) \le f(\bar{x})$ for all $x \in [n\bar{x} - 1, nt]$ iff

$$\max\{h(n\bar{x} - 1), h(nt)\} \le f(\bar{x}).$$

Further, since f is concave for $x \in [t, \bar{x}]$ and $f'(\bar{x}) = cn$, we have $f(t) + cn(\bar{x} - t) \le f(\bar{x})$. Therefore, $h(nt) = f(t) + cn(\bar{x} - t) \le f(\bar{x})$.

By Assumption 2, we have that $f'(\bar{x} - \frac{1}{n}) \ge cn$ and by definition of $\bar{x}$ we have $f'(\overline{x}) = cn$. Therefore,

$$\frac{f(\bar{x}) - f\left(\bar{x} - \frac{1}{n}\right)}{\frac{1}{n}} \ge cn$$

$$\Leftrightarrow \qquad f(\bar{x}) - f\left(\bar{x} - \frac{1}{n}\right) \ge c$$

$$\Leftrightarrow \qquad f(\bar{x}) \ge f\left(\bar{x} - \frac{1}{n}\right) + c = h(n\bar{x} - 1)$$

Thus, $h(x) \le f(\bar{x})$ for all $x \in [n\bar{x} - 1, nt]$ and therefore, $BR_i(s_{-i}) = n\bar{x} - \sum_{j \ne i} s_j$ for all s_{-i} such that $\sum_{j \ne i} s_j \in [n\bar{x} - 1, nt]$. Thus, $\underline{x}$ does not influence the best response for s_{-i} with $\sum_{j \ne i} s_j \in [n\bar{x} - 1, nt]$.

Case 2.b. Suppose $\sum_{j \ne i} s_j \in [nt - 1, n\bar{x} - 1]$.

Similar to Case 1 there exist two candidates $BR_i(s_{-i}) = 0$ or $BR_i(s_{-i}) = 1$. Define $g(x)$ as in Case 1 with $x \in \left[t, \overline{x}\right]$. Analogously to above it follows that g is concave and $g(x) \ge c$ for all $x \in [t, \bar{x}]$ iff

$$\min\{g(t), g(\bar{x})\} \ge c.$$

Then, Case 2.a. implies $g(\bar{x}) = f(\bar{x}) - f\left(\frac{n\bar{x} - 1}{n}\right) \ge c$ and the definition, existence and uniqueness of $\underline{x}$ follows analogously to Case 1. ■

Proof of Lemma 5. For any $\underline{s} \in [0, 1]$, we construct all symmetric MSCE, thereby establishing existence.

Part i) Suppose f is convex. Then, $\bar{x} = 1$ and the best response function of player i for a symmetric strategy profile s^*_{-i} reduces to

$$BR_i(s^*_{-i}) = \begin{cases} 0 & \text{if } \sum_{j \neq i} \min_{s_j \in \Sigma_{ij}(s^*_j)} s_j < n\underline{x} - 1, \\ 1 & \text{if } n\underline{x} - 1 \leq \sum_{j \neq i} \min_{s_j \in \Sigma_{ij}(s^*_j)} s_j \leq n - 1, \end{cases}$$

$$= \begin{cases} 0 & \text{if } (n-1) \min_{s_j \in \Sigma_{ij}(s^*_j)} s_j < n\underline{x} - 1, \\ 1 & \text{if } n\underline{x} - 1 \leq (n-1) \min_{s_j \in \Sigma_{ij}(s^*_j)} s_j \leq n - 1. \end{cases}$$

If $\underline{x} > \frac{1}{n}$, we have $n\underline{x} - 1 > 0$. Further, if $s^*_j = 0$ for all $j \neq i$, $\min_{s_j \in \Sigma_{ij}(s^*_j)} s_j = 0$ for all $j \neq i$. Thus, the best response of player i is $s^*_i = 0$ for all $\underline{s} \in [0, 1]$.

Suppose that $(n-1)\underline{s} \geq n\underline{x} - 1$ and that $s^*_j = 1$ for all $j \neq i$. Then, $(n-1) \min_{s_j \in \Sigma_{ij}(s^*_j)} s_j = (n-1)\underline{s}$ and the best response of player i is $s^*_i = 1$. Thus, $s^*_i = 1$ for all $i \in N$ is a symmetric MSCE.

Part ii) Suppose f is concave. Then, the best response function is

$$BR_i(s^*_{-i}) = \begin{cases} 1 & \text{if } (n-1) \min_{s_j \in \Sigma_{ij}(s^*_j)} s_j < n\bar{x} - 1, \\ n\bar{x} - (n-1) \min_{s_j \in \Sigma_{ij}(s^*_j)} s_j & \text{if } n\bar{x} - 1 \leq (n-1) \min_{s_j \in \Sigma_{ij}(s^*_j)} s_j < n\bar{x}, \\ 0 & \text{if } n\bar{x} \leq (n-1) \min_{s_j \in \Sigma_{ij}(s^*_j)} s_j. \end{cases}$$

If $(n-1)\underline{s} < n\bar{x} - 1$, we have $(n-1) \min_{s_j \in \Sigma_{ij}(s^*_j)} s_j < n\bar{x} - 1$ for any s^*_j and therefore $s^*_i = 1$ is the unique MSCE.

If $n\bar{x} - 1 \leq (n-1)\underline{s} < (n-1)\bar{x}$, we have $(n-1) \min_{s_j \in \Sigma_{ij}(s^*_j)} s_j < n\bar{x}$ for any s^*_j and the last case of the best response function can never occur. Further, suppose $s^*_j = 1$ for all $j \neq i$. Then, $s^*_j \geq \underline{s}$ and $(n-1) \min_{s_j \in \Sigma_{ij}(s^*_j)} s_j = (n-1)\underline{s} \in [n\bar{x} - 1, n\bar{x})$. Thus, any symmetric MSCE must satisfy $s^*_i = n\bar{x} - (n-1) \min_{s_j \in \Sigma_{ij}(s^*_j)} s_j$ for all $i \in n$. Remember that

$$\min_{s_j \in \Sigma_{ij}(s^*_j)} s_j = \begin{cases} s^*_j & \text{if } s^*_j < \underline{s}, \\ \underline{s} & \text{if } s^*_j \geq \underline{s}. \end{cases}$$

Suppose $s^*_j < \underline{s}$, then $s^*_i = n\bar{x} - (n-1)s^*_j$. By symmetry, we have $s^*_i = s^*_j = \bar{x}$. But this contradicts, the assumption that $(n-1)\underline{s} < (n-1)\bar{x}$. Hence, the unique MSCE has to satisfy $s^*_i = n\bar{x} - (n-1)\underline{s}$ for all $i \in n$.

For $(n-1)\bar{x} \leq (n-1)\underline{s} < n\bar{x}$, it follows analogously to above that the unique MSCE has to satisfy $s^*_i = n\bar{x} - (n-1) \min_{s_j \in \Sigma_{ij}(s^*_j)} s_j$ for all $i \in n$. Now, since $\bar{x} < \underline{s}$, we have $\min_{s_j \in \Sigma_{ij}(s^*_j)} s_j = s^*_j = \bar{x}$ and $s^*_i = \bar{x}$ for all $i \in N$ is the unique MSCE.

If $(n-1)\underline{s} \geq n\bar{x}$, the unique symmetric MSCE equals the symmetric Nash equilibrium $s^*_i = \bar{x}$. Suppose $\bar{x} \in (0, 1)$. Then, the best response to $s^*_j = 1$ for all $j \neq i$ is $s^*_i = 0$ and the best response to $s^*_j = 0$ for all $j \neq i$ is $s^*_i > 0$. Thus, $s^*_i = 0$ for all $i \in N$ or $s^*_i = 1$ for all $i \in N$ cannot be symmetric MSCE, and any MSCE has to satisfy the second case of the best response function. For the second case of the best response function, we have $(n-1) \min_{s_j \in \Sigma_{ij}(s^*_j)} s_j < n\bar{x} \leq (n-1)\underline{s}$. Thus, $\min_{s_j \in \Sigma_{ij}(s^*_j)} s_j < \underline{s}$ and $\min_{s_j \in \Sigma_{ij}(s^*_j)} s_j = s^*_j$. Therefore, the unique MSCE equals the Nash equilibrium $s^*_i = \bar{x}$ for all $i \in N$. If $\bar{x} = 0$, the best response function reduces to the last case, and $s^*_i = 0 = \bar{x}$ is the unique MSCE. Similarly, if $\bar{x} = 1$, the strategy profile $s^*_i = 1 = \bar{x}$ for all $i \in N$ is the unique MSCE. Hence, for all $\bar{x} \in [0, 1]$ the unique MSCE is given by $s^*_i = \bar{x}$ for all $i \in N$.

Part iii) If f is s-shaped, all cases follow analogously to one of the cases for convex or concave f.

Thus, by construction, there exists at least one symmetric MSCE for any production function and any $\underline{s} \in [0, 1]$. Note that for s-shaped (or convex) production functions, $f(\frac{1}{n}) > c$ is equivalent to $\underline{x} < \frac{1}{n}$. Then, we have $n\underline{x} - 1 < 0$ and $s^*_i = 1$ is a symmetric MSCE for all $\underline{s}$ with $(n-1)\underline{s} < n\bar{x} - 1$ (or $\underline{s} \leq 1$). ■

Data availability

No data was used for the research described in the article.

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