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ORIGINAL RESEARCH

Characterizations of the u-prenucleolus by dually-u-essential coalitions

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Abstract
We extend the theory of TU-games with utility functions, which is a generalization of TU-games with restricted cooperation, to include dual games. By using the theory of dual games, we define dually-**u**-essential coalitions and show that they characterize the **u**-prenucleolus of **u**-balanced games. Additionally, we demonstrate that the intersection of **u**-essential and dually-**u**-essential coalitions also forms a characterization set for the **u**-prenucleolus, provided that the **u**-least-core is a proper subset of the **u**-core.

Keywords TU-games · Restricted cooperation · Prenucleolus · Least-core · Characterization set · Dual game · Anti-prenucleolus · TU-games with utility functions

1 Introduction

The prenucleolus (Schmeidler, 1969) is a widely used solution concept for transferable utility cooperative games. Its practical importance lies in minimizing the dissatisfaction of the most dissatisfied coalition, making it particularly relevant for social, financial, and technical applications. Significant evidence of its value is its numerous variants and generalizations, such as the per capita prenucleolus (Grotte, 1970, 1972), the weighted prenucleolus (Derks & Haller, 1999), the q-nucleolus (Solymosi, 2019), the modified nucleolus (Sudhölter, 1996, 1997),

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the general nucleolus (Maschler et al., 1992; Potters & Tijs, 1992), and the **u**-prenucleolus (Dornai & Pintér, 2024).

From the viewpoint of this research, the most abstract generalization is the one by Potters and Tijs (1992) and Maschler et al. (1992), called the general nucleolus. The main focus of this paper is the **u**-prenucleolus (Dornai & Pintér, 2024), which is a generalization of the per capita prenucleolus and the q-prenucleolus among others and a special case of the general nucleolus.

One motivation for introducing utility functions applied to the excesses of coalitions is to model scenarios where the relationship between different gains is inherently non-linear. For instance, in economics, the concept of diminishing marginal utility illustrates how the value or satisfaction derived from an additional unit of a resource decreases as more of that resource is accumulated. By allowing for utility functions to operate on excesses, this framework can capture such diminishing or varying returns in coalition gains more accurately than linear models.

Additionally, the choice to apply utility functions specifically to the excess, rather than other aspects of the game, is grounded in the nature of the excess itself. The excess represents the actual surplus or "real" gain that a coalition can achieve beyond its assigned value. By focusing on this surplus, utility functions can provide a nuanced evaluation of coalition outcomes, reflecting not only the magnitude of gains but also the way these gains are perceived or utilized by the players involved. This perspective enriches the theoretical framework and expands the applicability of game-theoretic models to more complex and realistic settings.

The generalization of TU-games using utility functions is theoretically significant, as it enables key results-such as the generalizations of the Bondareva-Shapley theorem and two theorems by Katsev and Yanovskaya-to extend to the **u**-prenucleolus (Dornai & Pintér, 2024). According to current knowledge, these results cannot be proven for the general prenucleolus. Furthermore, the generalization of Huberman's theorem (Huberman, 1980) to TU-games with utility functions (Dornai & Pintér, 2024) allows its application to the per-capita prenucleolus or even to games with an empty core, provided that the utility functions are chosen appropriately.

Like other solution concepts, such as the Shapley value (Shapley, 1953), the prenucleolus and the **u**-prenucleolus suffer from computational difficulty. Several algorithms compute the (pre)nucleolus (Kohlberg, 1972; Solymosi, 1993; Perea & Puerto, 2013), but these algorithms generally have a complexity of $\mathcal{O}(2^n)$.

However, for some classes of games like neighbor games (Hamers et al., 2003), permutation games under certain conditions (Solymosi et al., 2005), tree games (Maschler et al., 2010), and a large class of directed acyclic graph games (Sziklai et al., 2017), the nucleolus can be computed in polynomial time based on the number of players. Additionally, some heuristics provide efficient algorithms to find an allocation close to the nucleolus (Perea & Puerto, 2019).

In addition, there are ongoing efforts to enhance the algorithms for computing the nucleolus. Benedek et al. (2021) presents an improved algorithm capable of handling games of moderately large size by utilizing a refined version of Kohlberg's criterion. This refinement reduces the verification process to at most $(n-1)$ collections of coalitions for balancedness, instead of an exponentially large number. Similarly, Koenemann and Toth (2023) introduces a framework for designing efficient nucleolus computation algorithms and demonstrates that, using this approach, the nucleolus of b-matching games on graphs with bounded treewidth can be computed in polynomial time.

Although it is known that $2n-2$ coalitions are sufficient to characterize the prenucleolus of an n-player game (Brune, 1983; Reijnierse & Potters, 1998), identifying these $2n-2$

coalitions is not straightforward. For this reason, Granot et al. (1998) introduced the concept of a characterization set, a set of coalitions that determines the prenucleolus by itself. They also proved that if the number of coalitions in a characterization set is polynomial in the number of players, then the prenucleolus can be computed in polynomial time.

Huberman (1980) showed that the family of essential coalitions is a characterization set for the (pre)nucleolus in the case of balanced games. Dornai and Pintér (2024) extended Huberman's theorem, demonstrating that **u**-essential coalitions provide a characterization set for the **u**-prenucleolus of **u**-balanced games. This result also generalizes Huberman (1980)'s theorem to games with restricted cooperation.

Another characterization set for the (pre)nucleolus in balanced games was provided by Solymosi and Sziklai (2016), who showed that the family of dually essential coalitions forms a characterization set for the (pre)nucleolus. They also demonstrated that the intersection of essential and dually essential coalitions provides a characterization set for the (pre)nucleolus in balanced games if the grand coalition is vital.

In this paper, we define another variant of essential coalitions, namely the dually-**u**-essential coalitions (Definition 12). We prove a variant of Theorem 7 on p. 420 in Huberman (1980), claiming that the dually-**u**-essential coalitions form a characterization set for the **u**-prenucleolus in **u**-balanced games (Theorem 15). This result also generalizes Theorem 10 on p. 522 in Solymosi and Sziklai (2016), which states that dually essential coalitions form a characterization set for the (pre)nucleolus in balanced games.

Dually-**u**-essential coalitions are also a variant of **u**-essential coalitions (Dornai & Pintér, 2024). As in Dornai and Pintér (2024), we focus on games with restricted cooperation, adding another layer of generality to our results. In Dornai and Pintér (2024), **u**-essential coalitions are defined by considering **u**-core allocations. However, Dornai and Pintér (2024) suggests that (Remark 1) instead of the **u**-core, the **u**-least-core could be used, and the redefined **u**-essential coalitions would still form a characterization set in **u**-balanced games. Indeed, this redefined class of **u**-essential coalitions is a subset of the previously defined class. Therefore, using **u**-least-core allocations in the definition can potentially decrease the number of **u**-essential coalitions, providing a smaller characterization set. In this paper, we will use this latter definition.

To prove the theorem about dually-**u**-essential coalitions, we first define the dual of TU-games with utility functions (and with restricted cooperation) (Definition 2). We define generalizations of the anti-nucleolus, anti-prenucleolus, anti-core, least-anti-core, and dually essential coalitions, namely the **u**-anti-nucleolus, **u**-anti-prenucleolus (Definition 5), **u**-anti-core, **u**-least-anti-core (Definition 6), and dually-**u**-essential coalitions (Definition 12).

Additionally, we generalize Theorem 2.3 on p. 362 in Granot et al. (1998) about characterization sets of the prenucleolus to TU-games with utilities (and with restricted cooperation) (Theorem 20). We also show that the intersection of the set of **u**-essential and the set of dually-**u**-essential coalitions characterizes the **u**-least-core of **u**-balanced games if the **u**-least-core is a proper subset of the **u**-core (Proposition 26). With the help of these results, we generalize Theorem 11 on p. 523 in Solymosi and Sziklai (2016), demonstrating that the intersection of the set of **u**-essential and the set of dually-**u**-essential coalitions forms a characterization set for the **u**-prenucleolus in **u**-balanced games if the **u**-least-core is a proper subset of the **u**-core (Theorem 27).

In summary, we significantly generalize the results by Huberman (1980), Granot et al. (1998), Solymosi and Sziklai (2016), and Dornai and Pinter (2024). Our contributions extend these findings in two key ways. First, we broaden the scope of the results to include games with restricted cooperation. Second, we apply our generalizations to transferable utility (TU) games with utility functions. Although a TU-game with restricted cooperation is formally a

TU-game with utility functions, we emphasize these two layers to highlight the depth of our generalizations. These enhancements lead to a deeper understanding of the (pre)nucleolus, particularly regarding its computational properties.

The paper is structured as follows: In Sect. 2, we discuss the basic concepts and notations used in the paper. In Sect. 3, we define the notions of TU-games with utilities and introduce the dual of TU-games with utilities and related notions such as the **u**-anti-nucleolus, **u**-anti-prenucleolus, **u**-anti-core, and **u**-least-anti-core.

In Sect. 4, we discuss a generalization of the lexicographic center algorithm for calculating the **u**-prenucleolus-this algorithm is a special case of the lexicographic center algorithm for calculating the general nucleolus by Maschler et al. (1992), and show a modification of this algorithm, which can be used to calculate the **u**-anti-prenucleolus.

In Sect. 5, we define the so-called dually-**u**-essential coalitions and generalize a theorem of Solymosi and Sziklai (2016), proving that the dually-**u**-essential coalitions form a characterization set for the **u**-prenucleolus of **u**-balanced games.

In Sect. 6, we generalize a theorem by Granot et al. (1998) about characterization sets. Using this theorem, we further generalize a result of Solymosi and Sziklai (2016), demonstrating that the intersection of the set of **u**-essential and the set of dually-**u**-essential coalitions forms a characterization set for the **u**-prenucleolus of **u**-balanced games if the **u**-least-core is a proper subset of the **u**-core.

All technical lemmata and their proofs are provided in the Appendix. Finally, in Appendix 10, Table 1 summarizes the various notions considered in this paper and their relationships to each other.

2 Preliminaries

Given a nonempty finite set of the players N and a function $v\colon 2^N \to \mathbb{R}$ such that $v(\emptyset) = 0$; then v is called a TU-*game* (henceforth game for short). Let $\mathcal{G}^N$ denote the class of games with player set N, moreover, let the set of coalitions be denoted by $\mathcal{P}(N) := \{S\colon S \subseteq N\}$ and the set of non-trivial coalitions be denoted by $\hat{\mathcal{P}}(N) := \{S \subseteq N\colon S \neq \emptyset,\ S \neq N\}$. Let $\mathcal{D}_S$ denote the class of partitions of set $S \subseteq N$ except $\{S\}$.

Let $\mathcal{A} \subseteq \mathcal{P}(N)$ be such that $\emptyset, N \in \mathcal{A}$, then $\mathcal{A}$ is called a set of feasible coalitions. In this case, the function $v\colon \mathcal{A} \to \mathbb{R}$ is called a game with restricted cooperation. Let $\mathcal{G}^{N,\mathcal{A}}$ denote the class of games with restricted cooperation on $\mathcal{A}$. If $\mathcal{A} = \mathcal{P}(N)$ then $\mathcal{G}^{N,\mathcal{A}} = \mathcal{G}^N$, therefore each introduced concept for games with restricted cooperation is a generalization of the related concept for classical games.

Let $\hat{\mathcal{A}} := \mathcal{A} \setminus \{N, \emptyset\}$ denote the set of non-trivial feasible coalitions, and let $\mathcal{D}_S^{\hat{\mathcal{A}}} := \{B \in \mathcal{D}_S\colon B \subseteq \hat{\mathcal{A}}\}$ denote the non-trivial $\hat{\mathcal{A}}$-partitions of set $S \in \hat{\mathcal{A}}$.

A solution is a set-valued mapping from a set of games with player set N to $\mathbb{R}^N$. For example, the core (Shapley, 1955; Gillies, 1959), the kernel (Davis & Maschler, 1965), and the bargaining set (Aumann & Maschler, 1964). A value is a singleton valued solution, for example the Shapley-value (Shapley, 1953) and the (pre)nucleolus (Schmeidler, 1969).

Let $I(v) := \{x \in \mathbb{R}^N\colon \sum_{i\in N} x_i = v(N) \text{ and } x_i \geq v(\{i\}) \forall \{i\} \in \mathcal{A}\}$ and $PI(v) := \{x \in \mathbb{R}^N\colon \sum_{i\in N} x_i = v(N)\}$ denote the set of imputations and preimputations of a game $v \in \mathcal{G}^{N,\mathcal{A}}$ respectively.

Given a game with restricted cooperation $v \in \mathcal{G}^{N,\mathcal{A}}$, a coalition $S \in \mathcal{A}$ and a payoff a vector $x \in \mathbb{R}^N$, the *excess* of coalition S by the payoff vector x in the game v is $e_v(S, x) := v(S) - x(S)$, where $x(S) := \sum_{i\in S} x_i$.

The core of a game with restricted cooperation v is the set of preimputations for which the excess of every feasible coalition is non-positive:

$$\mathrm{core}(v) := \left\{ x \in \mathbb{R}^N : x(N) = v(N) \text{ and } e_v(S, x) \leq 0, \ \forall S \in \hat{\mathcal{A}} \right\}.$$

In case, the core is nonempty, we say the game is balanced.

The ε-core of the game is the set of preimputations, for which the maximal excess is at most ε, that is

$$\mathrm{core}_\varepsilon(v) := \{x \in PI(v) : \max_{S \in \hat{\mathcal{A}}} e_v(S, x) \leq \varepsilon\}.$$

Moreover, the least-core of the game is its smallest nonempty ε-core.

The vector $E_v(x) := [\ldots \geq e_v(S, x) \geq \ldots : S \in \hat{\mathcal{A}}]$ is called excess vector. It consists of all the excesses in non-increasing order. The lexicographical ordering between $x, y \in \mathbb{R}^n$ is the following: $x \leq_L y$ if $x = y$ or if there exists k such that $x_k < y_k$ and for every $i < k$ it holds that $x_i = y_i$. The nucleolus is the set of imputations which lexicographically minimize the excess vectors over the set of imputations, that is,

$$N(v) := \{x \in I(v) : E_v(x) \leq_L E_v(y), \ \forall y \in I(v)\}.$$

Moreover, the prenucleolus is the set of preimputations which lexicographically minimize the excess vectors over the set of preimputations, that is,

$$PN(v) := \{x \in PI(v) : E_v(x) \leq_L E_v(y), \ \forall y \in PI(v)\}.$$

A set of coalitions $\mathcal{S} \subseteq \mathcal{A}$ is a balanced set system if there exist $\lambda_S \in \mathbb{R}_+$, $S \in \mathcal{S}$, called balancing weight system, such that

$$\sum_{S \in \mathcal{S}} \lambda_S \chi_S = \chi_N,$$

where $\chi_S \in \mathbb{R}^N$ is the characteristic vector of set S.

Given a game $v \in \mathcal{G}^N$, then its dual is the game $v^* : 2^N \to \mathbb{R}$ such that $v^*(S) = v(N) - v(N \backslash S)$, for all $S \in \mathcal{P}(N)$.

Let $\mathcal{A}^{\complement}$ denote the set of the complements of the sets from $\mathcal{A}$, that is, $\mathcal{A}^{\complement} := \{N \backslash S : S \in \mathcal{A}\}$. Given a game with restricted cooperation $v \in \mathcal{G}^{N,\mathcal{A}}$, then its dual is the game with restricted cooperation $v^* : 2^{\mathcal{A}^{\complement}} \to \mathbb{R}$, such that $v^*(S) = v(N) - v(N \backslash S)$ for all $S \in \mathcal{A}^{\complement}$. Moreover, let anti-$I(v^*) := \{x \in PI(v^*) : v^*(N \backslash \{i\}) \geq x(N \backslash \{i\}), \ \forall N \backslash \{i\} \in \mathcal{A}^{\complement}\}$ denote the set of anti-imputations of the dual game v^*. Notice that $I(v) = $ anti-$I(v^*)$.

Consider a game $v \in \mathcal{G}^{N,\mathcal{A}}$, a coalition $S \in \mathcal{A}^{\complement}$ and a payoff vector $x \in \mathbb{R}^N$. Then the satisfaction of coalition S by the payoff vector x in the dual game v^* is $f_{v^*}(S, x) := x(S) - v^*(S)$. Notice that if $x \in PI(v)$ then $f_{v^*}(S, x) = e_v(N \backslash S, x)$. Let $F_{v^*}(x) := [\ldots \geq f_{v^*}(S, x) \geq \ldots : S \in \hat{\mathcal{A}}^{\complement}]$ denote the satisfaction vector of the dual game v^*. It consists of all the satisfactions in non-increasing order. The anti-nucleolus of the dual game is the set of anti-imputations which lexicographically minimize the satisfaction vectors over the set of anti-imputations, that is,

$$\text{anti-N}(v^*) := \{x \in \text{anti-}I(v^*) : F_{v^*}(x) \leq_L F_{v^*}(y), \ \forall y \in \text{anti-}I(v^*)\}.$$

Moreover, the anti-prenucleolus of the dual game is the set of preimputations which lexicographically minimize the satisfaction vectors over the set of preimputations, that is,

$$\text{anti-}PN(v^*) := \{x \in PI(v^*) : F_{v^*}(x) \leq_L F_{v^*}(y), \ \forall y \in PI(v^*)\}.$$

The anti-core of the dual of the game v is the set of preimputations for which the satisfaction of any coalition from the complementer set of the set of feasible coalitions is non-positive, that is,

$$\text{anti-core}(v^*) := \{x \in PI(v^*) \colon f_{v^*}(S, x) \leq 0, \ \forall S \in \hat{\mathcal{A}}^{\complement}\} .$$

3 Games with utility functions

In this paper, we focus on TU-games with utility functions. This notion was introduced by Dornai and Pintér (2024). Each coalition has a utility function, and it is applied on the excess by the coalition. Then solutions and values modify to **u**-solutions and **u**-values, respectively, where **u** denotes the considered family of utility functions. Thereby, we get generalizations of the nucleolus, prenucleolus, core and least-core, namely, the **u**-nucleolus, **u**-prenucleolus, **u**-core and **u**-least-core respectively.

Next, we introduce the notions related to games with utility functions (for a more detailed discussion of the notion please consult with Dornai and Pintér (2024)).

Consider a game with restricted cooperation $v \in \mathcal{G}^{N,\mathcal{A}}$. A utility function $\mathbf{u} \colon \hat{\mathcal{A}} \times \mathbb{R} \to \mathbb{R}$ is a family of functions u_S, $S \in \hat{\mathcal{A}}$, such that u_S is strictly monotone increasing, continuous, the domain of u_S is $\mathbb{R}$ and the ranges of u_S and u_T are the same for all $S, T \in \hat{\mathcal{A}}$. This common range is denoted by $R_{\mathbf{u}}$.

The **u**-excess of coalition $S \in \mathcal{A}$ by the payoff vector $x \in \mathbb{R}$ in the game v is $u_S \circ e_v(S, x) := u_S(v(S) - x(S))$ and the **u**-excess vector is $E_v^{\mathbf{u}}(x) := [\cdots \geq u_S \circ e_v(S, x) \geq \ldots, S \in \hat{\mathcal{A}}]$.

The **u**-(pre)nucleolus is the set of **u**-(pre)imputations, which lexicographically minimize the **u**-excess vectors over the set of **u**-(pre)imputations, that is

$$N_{\mathbf{u}}(v) := \{x \in \mathbf{u}\text{-}I(v) \colon E_v^{\mathbf{u}}(x) \leq_L E_v^{\mathbf{u}}(y) \ \forall y \in \mathbf{u}\text{-}I(v)\} ,$$

where $\mathbf{u}\text{-}I(v) := \{x \in PI(v) \colon u_{\{i\}} \circ e_v(\{i\}, x) \leq 0, \ \forall \{i\} \in \mathcal{A}\}$, and

$$PN_{\mathbf{u}}(v) := \{x \in PI(v) \colon E_v^{\mathbf{u}}(x) \leq_L E_v^{\mathbf{u}}(y) \ \forall y \in PI(v)\} .$$

Similarly, the **u**-core is defined as follows:

$$\mathbf{u}\text{-core}(v) := \left\{x \in \mathbb{R}^N : x(N) = v(N) \text{ and } u_S \circ e_v(S, x) \leq 0, \forall S \in \hat{\mathcal{A}}\right\} .$$

If the **u**-core of a game is not empty then we say that the game is **u**-balanced.

Example 1 $N = \{1, 2, 3\}$, $\mathcal{A} = \mathcal{P}(N)$ and

$$v(S) = \begin{cases} 0, & \text{if } |S| = 1 \text{ or } S = \{1, 2\}, \\ 4, & \text{if } S = \{1, 3\}, \\ -1, & \text{if } S = \{2, 3\}, \\ 2, & \text{if } S = N. \end{cases}$$

Let the utility function be a shift by -1, that is $u_S(t) = t - 1$ for all $S \in \hat{\mathcal{A}}$. Then the **u**-excess of coalition S by the payoff vector x is $v(S) - x(S) - 1$ and the **u**-prenucleolus of the game is $(2, -1, 1)$, which coincides with the prenucleolus. However, the **u**-core of the game is the non-empty set: $\{x \in PI(v) \colon x_i \geq -1, x_1 + x_2 \geq -1, x_1 + x_3 \geq 3, x_2 + x_3 \geq -2\}$, while the core of the game is empty.

The idea of applying utility functions to games can be extended to dual games. Since the feasible coalitions of the dual game are the complementer set of the feasible coalitions of the primal game, we define the **u**-satisfaction of the dual game using the u_S functions corresponding to their complementers.

Definition 2 The dual of a game $v \in \mathcal{G}^{N,\mathcal{A}}$ with utility function $\mathbf{u}$ is the game $v^* \in \mathcal{G}^{N,\mathcal{A}^{\complement}}$ with utility function $\mathbf{u}^* := [u_{N\setminus S}]_{S\in\hat{\mathcal{A}}^{\complement}}$.

Notice that the dual of the dual game with utility function is the primal game with utility function, because $v^{**} = v$ and $\mathbf{u}^{**} = ([u_{N\setminus S}]_{S\in\hat{\mathcal{A}}^{\complement}})^* = [u_{N\setminus(N\setminus S)}]_{S\in(\hat{\mathcal{A}}^{\complement})^{\complement}} = [u_S]_{S\in\hat{\mathcal{A}}}$.

Example 3 Consider the following game with the percapita utility function: $N = \{1, 2, 3\}$,

$$v(S) = \begin{cases} 1, & \text{if } |S| = 1, \\ 2, & \text{if } |S| = 2, \\ 6, & \text{if } S = N \end{cases}$$

and $\hat{\mathcal{A}} = \{\{1\}, \{2\}, \{1, 2\}, \{1, 3\}\}$.

Then $\hat{\mathcal{A}}^{\complement} = \{\{2\}, \{3\}, \{1, 3\}, \{2, 3\}\}$, $v^*(\{2\}) = v^*(\{2\}) = 6 - 2 = 4$, $v^*(\{1, 3\}) = v^*(\{2, 3\}) = 6 - 1 = 5$ and $v^*(N) = 6 - 0 = 6$. Moreover, let $\mathbf{u}$ be the percapita utility function, that is $u_S(t) = \frac{t}{|S|}$ for all $S \in \hat{\mathcal{A}}$. Then $u^*_{\{3\}}(t) = u^*_{\{2\}}(t) = \frac{t}{2}$. Similarly, $u^*_{\{1,3\}}(t) = u^*_{\{2,3\}}(t) = \frac{t}{1} = t$.

Definition 4 Consider a game $v \in \mathcal{G}^{N,\mathcal{A}}$ and a utility function $\mathbf{u}$. Then the **u**-satisfaction of a coalition $S \in \mathcal{A}^{\complement}$ by the payoff vector $x \in \mathbb{R}^N$ in the dual game v^* is $u_{N\setminus S} \circ f_{v^*}(S, x)$, and the **u**-satisfaction vector is $F^{\mathbf{u}}_{v^*}(x) := [\ldots \geq u_{N\setminus S} \circ f_{v^*}(S, x) \geq \ldots : S \in \hat{\mathcal{A}}^{\complement}]$.

The **u**-anti-nucleolus and **u**-anti-prenucleolus is defined with the **u**-satisfaction vectors:

Definition 5 Consider a game $v \in \mathcal{G}^{N,\mathcal{A}}$ and a utility function $\mathbf{u}$. Then the **u**-anti-nucleolus of v^* is the set of **u**-anti-imputations, which lexicographically minimize the **u**-satisfaction vectors over the set of **u**-anti-imputations, that is,

$$\text{anti-N}_{\mathbf{u}}(v^*) = \{x \in \mathbf{u}\text{-anti-}I(v^*) \colon F^{\mathbf{u}}_{v^*}(x) \leq_L F^{\mathbf{u}}_{v^*}(y),\ \forall y \in \mathbf{u}\text{-anti-}I(v^*)\},$$

where $\mathbf{u}\text{-anti-}I(v^*) := \{x \in PI(v^*) \colon u_{\{i\}} \circ f_{v^*}(N\setminus\{i\}, x) \leq 0,\ \forall N\setminus\{i\} \in \mathcal{A}^{\complement}\}$.

Moreover, the **u**-anti-prenucleolus of v^* is the set of preimputations which lexicographically minimize the **u**-satisfaction vectors over the set of preimputations, that is,

$$\text{anti-}PN_{\mathbf{u}}(v^*) = \{x \in PI(v^*) \colon F^{\mathbf{u}}_{v^*}(x) \leq_L F^{\mathbf{u}}_{v^*}(y),\ \forall y \in PI(v^*)\}.$$

Similarly, the **u**-anti-core of the dual game is also defined with the **u**-satisfactions:

Definition 6 Consider a game $v \in \mathcal{G}^{N,\mathcal{A}}$ and a utility function $\mathbf{u}$. Then the **u**-anti-core of the dual game v^* is the set of preimputations for which the **u**-satisfactions are non-positive:

$$\mathbf{u}\text{-anti-core}(v^*) = \{x \in PI(v^*) \colon u_{N\setminus S} \circ f_{v^*}(S, x) \leq 0,\ \forall S \in \hat{\mathcal{A}}^{\complement}\}.$$

If the **u**-anti-core of a game is not empty then we say that the game is **u**-anti-balanced.

Example 7 Consider the game described in Example 3. $\mathbf{u}\text{-anti-core}(v^*) = \{x \in \mathbb{R}^3 \colon x_1 + x_2 + x_3 = 6, \frac{x_2-4}{2} \leq 0, \frac{x_3-4}{2} \leq 0, \frac{x_2+x_3-5}{1} \leq 0, \frac{x_1+x_3-5}{1} \leq 0\} = \{x \in \mathbb{R}^3 \colon x_1 + x_2 + x_3 = 6, x_2 \leq 4, x_3 \leq 4, x_2 + x_3 \leq 5, x_1 + x_3 \leq 5\}$.

It is well-known that primal and dual games are related to each other as follows:

$$\begin{aligned} N(v) &= \text{anti-N}(v^*)\,, \\ PN(v) &= \text{anti-}PN(v^*)\,, \\ \text{core}(v) &= \text{anti-core}(v^*)\,, \\ \text{least-core}(v) &= \text{least-anti-core}(v^*). \end{aligned}$$

Similar relations hold in case of games with utility functions:

Theorem 8 *Consider a game $v \in \mathcal{G}^{N,\mathcal{A}}$ and a utility function $\mathbf{u}$. Then the following hold:*

1. $N_{\mathbf{u}}(v) = \textit{anti-}N_{\mathbf{u}}(v^*)$,
2. $PN_{\mathbf{u}}(v) = \textit{anti-}PN_{\mathbf{u}}(v^*)$,
3. $\mathbf{u}\textit{-core}(v) = \mathbf{u}\textit{-anti-core}(v^*)$,
4. $\mathbf{u}\textit{-least-core}(v) = \mathbf{u}\textit{-least-anti-core}(v^*)$.

Proof For every $S \in \mathcal{A}$, $x \in PI(v)$ it holds that

$$\begin{aligned} u_S \circ f_{v^*}(N \setminus S, x) &= u_S \circ (x(N \setminus S) - v^*(N \setminus S)) \\ &= u_S \circ (x(N) - x(S) - (v(N) - v(S))) \\ &= u_S \circ (v(S) - x(S)) = u_S \circ e_v(S, x). \end{aligned}$$

3: Then it follows

$$\begin{aligned} \mathbf{u}\text{-core}(v) &= \{x \in PI(v) \colon u_S \circ e_v(S, x) \le 0,\ \forall S \in \hat{\mathcal{A}}\} \\ &= \{x \in PI(v^*) \colon u_{N\setminus S} \circ f_{v^*}(S, x) \le 0,\ \forall S \in \hat{\mathcal{A}}^{\complement}\} \\ &= \mathbf{u}\text{-anti-core}(v^*)\,. \end{aligned}$$

4: Similarly, for any $\alpha \in \mathbb{R}$

$$\begin{aligned} \mathbf{u}\text{-core}_{\varepsilon}(v) &= \{x \in PI(v) \colon \max_{S \in \hat{\mathcal{A}}} u_S \circ e_v(S, x) \le \varepsilon\} \\ &= \{x \in PI(v) \colon \max_{S \in \hat{\mathcal{A}}} u_S \circ f_{v^*}(N \setminus S, x) \le \varepsilon\} \\ &= \{x \in PI(v) \colon \max_{S \in \hat{\mathcal{A}}^{\complement}} u_{N\setminus S} \circ f_{v^*}(S, x) \le \varepsilon\} = \mathbf{u}\text{-anti-core}_{\varepsilon}(v^*)\,. \end{aligned}$$

Therefore, $\mathbf{u}$-least-core$(v) = \mathbf{u}$-least-anti-core(v^*).

1–2: $v^*(N) = v(N) - v(\emptyset) = v(N)$, therefore $PI(v) = PI(v^*)$ and $\mathbf{u}\text{-}I(v) = \mathbf{u}\text{-anti-}I(v^*)$.

Since $u_S \circ f_{v^*}(N \setminus S, x) = u_S \circ e_v(S, x)$ for all $S \in \mathcal{A}$, $x \in PI(v)$ we have that $E_v^{\mathbf{u}}(x) = F_{v^*}^{\mathbf{u}}(x)$ for all $x \in PI(v)$. Therefore, $N_{\mathbf{u}}(v) = \text{anti-N}_{\mathbf{u}}(v^*)$ and $PN_{\mathbf{u}}(v) = \text{anti-}PN_{\mathbf{u}}(v^*)$. □

Example 9 Consider the game in Example 3. Then $\mathbf{u}\text{-core}(v) = \{x \in \mathbb{R}^3 \colon x_1 + x_2 + x_3 = 6,\ \frac{1-x_1}{1} \le 0,\ \frac{1-x_2}{1} \le 0,\ \frac{2-x_1-x_3}{2} \le 0,\ \frac{2-x_1-x_2}{2} \le 0\} = \{x \in \mathbb{R}^3 \colon x_1 + x_2 + x_3 = 6, x_1 \ge 1, x_2 \ge 1, x_1 + x_3 \ge 2, x_1 + x_2 \ge 2\}$, which equals to $\mathbf{u}$-anti-core(v^*) calculated in Example 7.

4 The lexicographic center method for games with utility functions

Later, we will use the lexicographic center method (Kopelowitz, 1967; Maschler et al., 1979), hence here we shortly discuss it. More precisely, we discuss its $\mathbf{u}$-modified version (Dornai & Pintér, 2024).

Let $v \in \mathcal{G}^{N,\mathcal{A}}$ be a game with a utility function $\mathbf{u}$. The modified lexicographic center algorithm for calculating the $\mathbf{u}$-prenucleolus of game v solves the following optimization problems iteratively:

$$\begin{aligned} & t \to \min \\ \text{s.t. } & u_S \circ e_v(S,x) \leq t, \;\; S \in \hat{\mathcal{A}} \setminus \left(\cup_{r=0}^{k-1} W_r\right) \\ & x \in X_{k-1} \\ & t \in R_{\mathbf{u}}, \end{aligned} \tag{1}$$

where $X_0 := PI(v)$ and $W_0 := \emptyset$, and for $k \geq 1$ t_k denotes the optimum of (1) if it exists, and

$$\begin{aligned} X_k &:= \left\{x \in X_{k-1} \colon u_S \circ e_v(S,x) \leq t_k, \; \forall S \in \hat{\mathcal{A}} \setminus \left(\cup_{r=0}^{k-1} W_r\right)\right\}, \\ W_k &:= \{S \in \hat{\mathcal{A}} \colon \exists c_S \in \mathbb{R}, \text{ such that } u_S \circ e_v(S,x) = c_S, \; \forall x \in X_k\}. \end{aligned}$$

It is easy to see that $t_k \geq t_{k+1}$ and $X_k \supseteq X_{k+1}$ for all $k \in \mathbb{N}_+$, and there exists k^* such that for all $l \geq k^*$ it holds that $X_{k^*} = X_l$. Then according to Maschler et al. (1992) $X_{k^*} = PN_{\mathbf{u}}(v)$, as discussed in Dornai and Pintér (2024).

Similarly, the modified lexicographic center algorithm for calculating the $\mathbf{u}$-anti-prenucleolus of the dual game v^* solves the following optimization problems iteratively:

$$\begin{aligned} & t \to \min \\ \text{s.t. } & u_{N \setminus S} \circ f_{v^*}(S,x) \leq t, \;\; S \in \hat{\mathcal{A}}^{\complement} \setminus \left(\cup_{r=0}^{k-1} W_r^d\right) \\ & x \in X_{k-1}^d \\ & t \in R_{\mathbf{u}}, \end{aligned} \tag{2}$$

where $X_0^d := PI(v^*)$ and $W_0^d := \emptyset$ and for $k \geq 1$ t_k^d denotes the optimum of (1) if it exists, and

$$\begin{aligned} X_k^d &:= \left\{x \in X_{k-1}^d \colon u_{N \setminus S} \circ f_{v^*}(S,x) \leq t_k^d, \; \forall S \in \hat{\mathcal{A}}^{\complement} \setminus \left(\cup_{r=0}^{k-1} W_r^d\right)\right\}, \\ W_k^d &:= \left\{S \in \hat{\mathcal{A}}^{\complement} \colon \exists c_S \in \mathbb{R}, \text{ such that } f_{N \setminus S} \circ f_{v^*}(S,x) = c_S, \; \forall x \in X_k^d\right\}. \end{aligned}$$

Again, it is easy to see that $t_k^d \geq t_{k+1}^d$ and $X_k^d \supseteq X_{k+1}^d$ for all $k \in \mathbb{N}_+$, and there exists k^* such that for all $l \geq k^*$ it holds that $X_{k^*} = X_l$. Since for every $x \in PI(v), S \in \hat{\mathcal{A}}$ $u_S \circ e_v(S,x) = u_S \circ f_{v^*}(N \setminus S, x)$, we have that $t_i^d = t_i$ and $X_i^d = X_i$ for all $i \in \mathbb{N}_+$. In addition, we learned in Theorem 8, that the $\mathbf{u}$-prenucleolus of the primal game equals the $\mathbf{u}$-anti-prenucleolus of the dual game. Therefore, we can apply the result of Maschler et al. (1992), saying, that $X_{k^*}^d = \text{anti-}PN_{\mathbf{u}}(v^*)$.

5 u-essential and dually-u-essential coalitions

In this section, we shortly discuss the notion of **u**-essential coalitions (Dornai & Pintér, 2024), and a related result which generalizes Huberman (1980). In most of this section, we discuss the notion of **u**-anti-essential coalitions and show that those form a characterization set of the **u**-anti-prenucleolus for games having non-empty **u**-anti-core. Finally, using the results for **u**-anti-essential coalitions, we show that dually-**u**-essential coalitions form a characterization set for the **u**-prenucleolus of **u**-balanced games.

First, we consider the **u**-essential coalitions:

Definition 10 Consider a game $v \in \mathcal{G}^{N,\mathcal{A}}$ with a utility function $\mathbf{u}$. A coalition $S \in \hat{\mathcal{A}}$ is $\mathbf{u}$-*essential* if either $\mathcal{D}_S^{\hat{\mathcal{A}}} = \emptyset$ or if there exists $x \in \mathbf{u}$-least-core(v) such that

$$u_S \circ e_v(S,x) > \max_{\mathcal{B} \in \mathcal{D}_S^{\hat{\mathcal{A}}}} \sum_{T \in \mathcal{B}} u_T \circ e_v(T,x) \,.$$

Let $\mathcal{E}_v^{\mathbf{u}}$ denote the class of $\mathbf{u}$-essential coalitions of the game v.

Comparing the definition above to the one in Dornai and Pintér (2024) (Definition 6), we can see that the two definitions are not the same. While in the version of Dornai and Pintér (2024) the considered x is form the $\mathbf{u}$-core of the game, in the definition above it is from the $\mathbf{u}$-least-core of the game. This approach yields a subset of $\mathbf{u}$-essential coalitions as defined in Dornai and Pintér (2024), resulting in a smaller set $\mathcal{E}_v^{\mathbf{u}}$. Consequently, Theorem 11 is a stronger result than Theorem 8 in Dornai and Pintér (2024). However, as noted in Remark 1 of Dornai and Pintér (2024), this distinction does not affect the results discussed subsequently. Therefore, we can rely on the proofs provided in Dornai and Pintér (2024).

The following theorem generalizes the result by Huberman (1980) and it is a reformulation of Theorem 8 in Dornai and Pintér (2024). Remark 1 in Dornai and Pintér (2024) applies here, and in the proof of Theorem 15 we use the same idea as the one that can be applied to prove this theorem, hence we omit its proof.

Theorem 11 *Consider a* $\mathbf{u}$*-balanced game* $v \in \mathcal{G}^{N,\mathcal{A}}$*, and let*

$$Y_1 := \{x \in PI(v) \colon u_S \circ e_v(S,x) \le t_1,\ \forall S \in \mathcal{E}_v^{\mathbf{u}}\},$$

and for all $k \ge 2$ *let*

$$Y_k := \left\{ x \in X_{k-1} \colon u_S \circ e_v(S,x) \le t_k,\ \forall S \in \mathcal{E}_v^{\mathbf{u}} \setminus (\cup_{r=1}^{k-1} W_r) \right\}.$$

Then $X_k = Y_k$*, for all* $k \ge 1$.

In other words, Theorem 11 claims that the $\mathbf{u}$-essential coalitions give a characterization set for the $\mathbf{u}$-prenucleolus of $\mathbf{u}$-balanced games.

Now, we turn our attention to dual essentiality.

Definition 12 Consider a game $v \in \mathcal{G}^{N,\mathcal{A}}$ with a utility function $\mathbf{u}$. Then a coalition $S \in \hat{\mathcal{A}}^{\complement}$ is $\mathbf{u}$-anti-essential if $\mathcal{D}_S^{\hat{\mathcal{A}}^{\complement}} = \emptyset$ or if there exists $x \in \mathbf{u}$-least-anti-core(v^*) such that

$$u_{N \setminus S} \circ f_{v^*}(S,x) > \max_{\mathcal{B} \in \mathcal{D}_S^{\hat{\mathcal{A}}^{\complement}}} \sum_{T \in \mathcal{B}} u_{N \setminus T} \circ f_{v^*}(T,x) \,.$$

The complementer set of the class of $\mathbf{u}$-anti-essential coalitions (of game v^*) is called the class of dually-$\mathbf{u}$-essential coalitions (of game v). Let $\mathcal{E}_{v^*}^{a-\mathbf{u}}$ denote the class of $\mathbf{u}$-anti-essential coalitions (of v^*) and $\mathcal{E}_v^{d-\mathbf{u}}$ denote the class of dually-$\mathbf{u}$-essential coalitions (of v).

Example 13 Consider an additive game with the pecapita utility function $\mathbf{u}$, where $N = \{1,2,3\}$, $v(\{i\}) = 1$ for all $i \in N$ and $\hat{\mathcal{A}} = \{\{1\},\{2\},\{1,2\},\{1,3\}\}$.

Then $\mathbf{u}$-least-core(v) $=$ $\mathbf{u}$-least-anti-core(v^*) $=$ $\{(1,1,1)\}$ and $\hat{\mathcal{A}}^{\complement} = \{\{2\},\{3\},\{2,3\},\{1,3\}\}$.

The $\mathbf{u}$-essential coalitions of the game are $\mathcal{E}_v^{\mathbf{u}} = \{\{1\},\{2\},\{1,3\}\}$, since $\mathcal{D}_S^{\hat{\mathcal{A}}} = \emptyset$ for all $S \in \mathcal{E}_v^{\mathbf{u}}$ and for all $x \in \mathbf{u}$-least-core(v) $\frac{v(\{1,2\}-x_1-x_2)}{2} = \frac{2-1-1}{2} = v(\{1\})-x_1+v(\{2\})-x_2 =$

$1 - 1 + 1 - 1$. Note that although $\{1, 3\}$ is not singleton, it is **u**-essential by definition, since it does not have a decomposition of feasible coalitions.

The **u**-anti-essential coalitions are $\mathcal{E}_{v^*}^{a-\mathbf{u}} = \{\{2\}, \{3\}, \{1, 3\}\}$, since for all three of these coalitions $\mathcal{D}_S^{\hat{\mathcal{A}}^{\complement}} = \emptyset$. $\{2, 3\}$ is not **u**-anti-essential, because for all $x \in$ **u**-least-anti-core(v^*) $\frac{x_2+x_3-v^*(\{2,3\})}{1} = \frac{x_2+x_3-2}{1} = 0$ and $\mathcal{D}_{\{2,3\}}^{\hat{\mathcal{A}}^{\complement}} = \{\{\{2\}, \{3\}\}\}$. Therefore $\max_{\mathcal{B} \in \mathcal{D}_{\{2,3\}}^{\hat{\mathcal{A}}^{\complement}}} \sum_{T \in \mathcal{B}} u_{N \setminus T} \circ f_{v^*}(T, x) = \frac{x_2-1}{2} + \frac{x_3-1}{2} = 0$.

The dually-**u**-essential coalitions of v are the complementer set of the anti-**u**-essential coalitions of v^*, hence $\mathcal{E}_v^{d-\mathbf{u}} = \{\{2\}, \{1, 3\}, \{1, 2\}\}$.

Notice that the dually-**u**-essential coalitions (Definition 12) do not coincide with the **u**-essential coalitions (Definition 10).

Consider the following notation: for a class of coalitions $\mathcal{S} \subseteq \hat{\mathcal{A}}$ and $t \in \mathbb{R}$ let $X^d(\mathcal{S}, t) := \{x \in PI(v) \colon u_{N \setminus S} \circ f_{v^*}(S, x) \leq t, \forall S \in \mathcal{S}\}$.

Before proving the main theorem of this section, we need to prove that the **u**-anti-essential coalitions characterize the **u**-least-anti-core. For this result we need Lemmata 29, 30, 31 and Lemma 32 (see Appendix 5), which provides the main step in proving that the **u**-anti-essential coalitions characterize the **u**-least-anti-core. When proving this lemma, it is important to acknowledge that, unlike in the general case of TU-games, the utility functions prevent x from being removed from the inequality in Definition 12. Consequently, we must examine whether $X^d(\hat{\mathcal{A}}^{\complement}, t_1^d)$ and $X^d(\mathcal{E}_{v^*}^{a-\mathbf{u}}, t_1^d)$ truly represent the same sets. (This result would have been easier to prove if we had used $PI(v)$ instead of **u**-least-anti-core(v) in Definition 12; however, doing so would significantly increase the size of the set of dually-u-essential coalitions.)

We need Lemma 33 to show that not only $X^d(\hat{\mathcal{A}}^{\complement}, t_1^d) = X^d(\mathcal{E}_{v^*}^{a-\mathbf{u}}, t_1^d)$ (see Lemma 32), but also $t_1^d = \min\{t \colon X^d(\mathcal{E}_{v^*}^{a-\mathbf{u}}, t) \neq \emptyset\}$.

The following proposition is a corollary of Lemmata 32 and 33.

Proposition 14 *Consider a utility function* **u**, *and a* **u**-*balanced game* $v \in \mathcal{G}^{N,\mathcal{A}}$. *Let* $t_1^{d'} := \min\{t \colon X^d(\mathcal{E}_{v^*}^{a-\mathbf{u}}, t) \neq \emptyset\}$, *and* $X_1^{d'} := X^d(\hat{\mathcal{A}}^{\complement}, t_1^{d'})$. *Then the following hold:* $t_1^d = t_1^{d'}$ *and* $X_1^d = X_1^{d'}$.

Proof We know that $t_1^{d'} \leq t_1^d$. Therefore, by Lemmata 32 and 33 we have that $X_1^{d'} = X^d(\mathcal{E}_{v^*}^{a-\mathbf{u}}, t_1^{d'}) = X^d(\hat{\mathcal{A}}^{\complement}, t_1^{d'})$. However, $X^d(\hat{\mathcal{A}}^{\complement}, t_1^{d'}) \neq \emptyset$ if and only if $t_1^{d'} \geq t_1^d$, hence $t_1^d = t_1^{d'}$ and $X_1^d = X_1^{d'}$. □

The following theorem states that the **u**-anti-essential coalitions form a characterization set for the **u**-anti-prenucleolus of v^*, if v is **u**-balanced.

Theorem 15 *Consider a utility function* **u**, *and a* **u**-*balanced game* $v \in \mathcal{G}^{N,\mathcal{A}}$, *and for* $k \geq 1$ *let*

$$Y_k^d := \left\{x \in X_{k-1}^d \colon u_{N \setminus S} \circ f_{v^*}(S, x) \leq t_k^d \ \forall S \in \mathcal{E}_{v^*}^{a-\mathbf{u}}\right\}.$$

Then $X_k^d = Y_k^d$ *for all* $k \geq 1$. *In other words, the* **u**-*anti-essential coalitions (of* v^**) form a characterization set for the* **u**-*anti-prenucleolus of* v^*.

Proof $X_k^d \subseteq Y_k^d$ for all $k \geq 1$ by definition.

By Proposition 14, we have that $X_1^d = Y_1^d$. Therefore, $Y_1^d = X_1^d =$ **u**-least-anti-core(v^*).

Indirectly assume that there exists $k \geq 2$ such that $X_k^d \not\supseteq Y_k^d$, that is, there exists $y^* \in Y_k^d$ and $S \in (\hat{\mathcal{A}}^{\complement}) \setminus (\mathcal{E}_{v^*}^{a-\mathbf{u}} \cup (\cup_{r=0}^{k-1} W_r^d))$ such that $u_{N \setminus S} \circ f_{v^*}(S, y^*) > t_k^d$.

It follows from Lemma 28 that for all $x \in \mathbf{u}$-least-anti-core(v^*) there exists $\mathcal{B}_x \in \mathcal{D}_S^{\hat{\mathcal{A}}^\complement}$, $\mathcal{B}_x \subseteq \mathcal{E}_{v^*}^{a-\mathbf{u}}$, such that $u_{N\setminus S} \circ f_{v^*}(S, x) \leq \sum_{T \in \mathcal{B}_x} u_{N\setminus T} \circ f_{v^*}(T, x)$. In particular, since $Y_k^d \subseteq \mathbf{u}$-least-anti-core$(v^*)$

$$u_{N\setminus S} \circ f_{v^*}(S, y^*) \leq \sum_{T \in \mathcal{B}_{y^*}} u_{N\setminus T} \circ f_{v^*}(T, y^*)\,.$$

Since $\mathbf{u}$-anti-core$(v^*) \neq \emptyset$, we have that $u_{N\setminus S} \circ f_{v^*}(S, x) \leq 0$, for all $S \in \hat{\mathcal{A}}^\complement$, $x \in X_{k-1}^d \subseteq \mathbf{u}$-least-anti-core$(v^*)$. Therefore, for any $T \in \mathcal{B}_{y^*}$ it holds that $u_{N\setminus S} \circ f_{v^*}(S, y^*) \leq u_{N\setminus T} \circ f_{v^*}(T, y^*)$. By Lemma 34, there exists $T^* \in \mathcal{B}_{y^*}$ such that $T^* \notin \cup_{r=0}^{k-1} W_r^d$. Therefore, $u_{N\setminus S} \circ f_{v^*}(S, y^*) \leq u_{N\setminus T^*} \circ f_{v^*}(T^*, y^*) \leq t_k^d$, which is a contradiction. □

The following theorem establishes the connection between characterization sets for the $\mathbf{u}$-prenucleolus of the primal game and characterization sets for the $\mathbf{u}$-anti-prenucleolus of the dual game.

Theorem 16 *Consider a game $v \in \mathcal{G}^{N,\mathcal{A}}$ with a utility function $\mathbf{u}$. Then a set system is a characterization set for the $\mathbf{u}$-prenucleolus of v if and only if its complementer set system is a characterisation set for the $\mathbf{u}$-anti-prenucleolus of v^*.*

Proof Let $\mathcal{B} \subseteq \mathcal{A}$ be an arbitrary set system. In the first step of the generalized lexicographic center algorithm (see Sect. 4) for calculating the $\mathbf{u}$-prenucleolus of v with respect to $\mathcal{B}$, we have to solve the following optimization problem:

$$\begin{aligned} &t \to \min \\ \text{s.t. } &u_S \circ e_v(S, x) \leq t, \quad S \in \mathcal{B} \\ &x(N) = v(N) \\ &t \in R_{\mathbf{u}} \end{aligned} \tag{3}$$

Since $e_v(S, x) = v(S) - (x(N) - x(N\setminus S)) = x(N\setminus S) - (v(N) - v(S)) = x(N\setminus S) - v^*(N\setminus S))$ Problem (3) is equivalent with the following problem

$$\begin{aligned} &t \to \min \\ \text{s.t. } &u_S \circ (x(N \setminus S) - v^*(N \setminus S)) \leq t, \quad S \in \mathcal{B} \\ &v(N) = x(N) \\ &t \in R_{\mathbf{u}} \end{aligned} \tag{4}$$

From the dual perspective Problem (4) can be reformulated as follows

$$\begin{aligned} &t \to \min \\ \text{s.t. } &u_{N\setminus S} \circ f_{v^*}(S, x) \leq t, \quad S \in \mathcal{B}^\complement \\ &v^*(N) = x(N) \\ &t \in R_{\mathbf{u}} \end{aligned} \tag{5}$$

Problem (5) is the first step of the generalized lexicographic center algorithm (see Sect. 4) for calculating the $\mathbf{u}$-anti-prenucleolus of v^* with respect to $\mathcal{B}^\complement$. Therefore, if its optimum exists then $t_1 = t_1^d$, $X_1 = X_1^d$ and $W_1 = W_1^d$.

Applying the steps in the lexicographic center algorithm for each $k \geq 2$ we have $t_{k-1} = t_{k-1}^d$ (if the optimum exists), $X_{k-1} = X_{k-1}^d$ and $W_{k-1} = W_{k-1}^d$, and we have to solve the following problem as the kth step of the algorithm

$$\begin{aligned} &t \to \min \\ \text{s.t. } &u_S \circ e_v(S, x) \leq t, \quad S \in \mathcal{B} \setminus \cup_{r=0}^{k-1} W_r \\ &x \in X_{k-1} \\ &t \in R_{\mathbf{u}} \end{aligned}$$

or equivalently

$$
\begin{aligned}
& t \to \min \\
\text{s.t. } & u_{N\setminus S} \circ f_{v^*}(S, x) \leq t, \quad S \in \mathcal{B}^{\complement} \setminus \cup_{r=0}^{k-1} W_r^d \\
& x \in X_{k-1}^d \\
& t \in R_{\mathbf{u}}.
\end{aligned}
$$

Therefore, if the optimum exists $t_k = t_k^d$, $X_k = X_k^d$ and $W_k = W_k^d$.

Summing up, for all $k \geq 1$ $X_k = X_k^d$, hence the **u**-prenucleolus of v with respect to $\mathcal{B}$ coincides with the **u**-anti-prenucleolus of v^* with respect to $\mathcal{B}^{\complement}$. Moreover, $\mathcal{B}$ is a characterization set of the **u**-prenucelolus of v if and only if $\mathcal{B}^{\complement}$ is a characterizaton set of the **u**-anti-prenculeolus of v^*. □

The following theorem is a corollary of Theorems 15 and 16.

Theorem 17 *Consider a game $v \in \mathcal{G}^{N,\mathcal{A}}$ with a utility function* **u**, *such that v is* **u**-*balanced. Then the dually-***u**-*essential coalitions form a characterization set for the* **u**-*prenucleolus of v.*

6 The intersection of two characterization sets

Solymosi and Sziklai (2016) showed that the intersection of the set of essential coalitions and the set of dually-essential coalitions is a characterization set for the nucleolus for games without restricted cooperation, where the least-core is a proper subset of the core (Theorem 11 on p. 523 in Solymosi and Sziklai (2016)).

In this section, we show that the intersection of the set of **u**-essential coalitions and the set of dually-**u**-essential coalitions is a characterization set for the **u**-prenucleolus for games where the **u**-least-core is a proper subset of the **u**-core. Therefore we generalize (Solymosi & Sziklai, 2016) in two directions: we consider games with restricted cooperation and we consider games with utility functions.

First, we prove a generalization of Theorem 2.3 on p. 362 in Granot et al. (1998). In case of games with restricted cooperation, the original conditions do not imply the equality of the considered characterization sets, but only less:

Theorem 18 *Consider a* **u**-*balanced game $v \in \mathcal{G}^{N,\mathcal{A}}$ with the utility function* **u**, *and a set of coalitions $\mathcal{F} \subseteq \hat{\mathcal{A}}$, $\mathcal{F} \neq \emptyset$. Let $v' \in \mathcal{G}^{N,\mathcal{F}\cup\{N,\emptyset\}}$ be $v' = v|_{\mathcal{F}\cup\{N,\emptyset\}}$. If $x \in PN_{\mathbf{u}}(v')$, and for every $S \in \hat{\mathcal{A}} \setminus \mathcal{F}$ there exists $\mathcal{F}_{S,x} \subseteq \mathcal{F}$, $\mathcal{F}_{S,x} \neq \emptyset$ such that*

1. *$u_S \circ e_v(S, x) \leq u_T \circ e_v(T, x)$ for all $T \in \mathcal{F}_{S,x}$,*
2. *$\chi_S \in \mathrm{Lin}\{\chi_T : T \in \mathcal{F}_{S,x} \cup \{N\}\}$,*

then $x \in PN_{\mathbf{u}}(v)$.

To prove this theorem we need Lemma 35 and a generalization of Kohlberg's theorem (Kohlberg, 1971).

Maschler et al. (1992) proposed a generalization of Kohlberg's theorem for the general prencucleolus (Theorem 7.2 on pp. 102–105 in Maschler et al. (1992)). The **u**-prenucleolus is a special case of the general prenucleolus, therefore the following generalization of Kohlberg's theorem is a corollary of the result by Maschler et al. (1992):

Theorem 19 *Consider a game $v \in \mathcal{G}^{N,\mathcal{A}}$ with a utility function $\mathbf{u}$, and $x \in PI(v)$. Then x is an element of the $\mathbf{u}$-prenucleolus, that is, $x \in PN_{\mathbf{u}}(v)$ if and only if $\mathcal{D}_{\mathbf{u}}^{\hat{\mathcal{A}}}(\alpha, x)$ is a balanced set of coalitions for every α such that $\mathcal{D}_{\mathbf{u}}^{\hat{\mathcal{A}}}(\alpha, x) \neq \emptyset$, where $\mathcal{D}_{\mathbf{u}}^{\hat{\mathcal{A}}}(\alpha, x) := \{S \in \hat{\mathcal{A}} \colon u_S \circ e_v(S, x) \geq \alpha\}$.*

Now we are ready to prove Theorem 18.

The proof of Theorem 18 Since $\mathcal{F} \subseteq \hat{\mathcal{A}}$ we have that $\mathcal{D}_{\mathbf{u}}^{\mathcal{F}}(\alpha, x) \subseteq \mathcal{D}_{\mathbf{u}}^{\hat{\mathcal{A}}}(\alpha, x)$. Therefore, if $\mathcal{D}_{\mathbf{u}}^{\mathcal{F}}(\alpha, x) \neq \emptyset$ then $\mathcal{D}_{\mathbf{u}}^{\hat{\mathcal{A}}}(\alpha, x) \neq \emptyset$. By Point *1.* if $\mathcal{D}_{\mathbf{u}}^{\hat{\mathcal{A}}}(\alpha, x) \neq \emptyset$ then $\mathcal{D}_{\mathbf{u}}^{\mathcal{F}}(\alpha, x) \neq \emptyset$.

By Theorem 19, if $x \in PN_{\mathbf{u}}(v')$ then for every $\alpha \in \mathbb{R}$ it holds that $\mathcal{D}_{\mathbf{u}}^{\mathcal{F}}(\alpha, x)$ is either empty or balanced. If $\mathcal{D}_{\mathbf{u}}^{\mathcal{F}}(\alpha, x) = \emptyset$ then by Point *1.* $\mathcal{D}_{\mathbf{u}}^{\hat{\mathcal{A}}}(\alpha, x) = \emptyset$. If $\mathcal{D}_{\mathbf{u}}^{\mathcal{F}}(\alpha, x)$ is balanced then by Lemma 35 and Point *2.* $\mathcal{D}_{\mathbf{u}}^{\hat{\mathcal{A}}}(\alpha, x)$ is balanced as well. Therefore, by Theorem 19 $x \in PN_{\mathbf{u}}(v)$.

Notice that by Theorem 18 we have that $PN_{\mathbf{u}}(v') \subseteq PN_{\mathbf{u}}(v)$. In case of games with restricted cooperation, we need an extra condition to get $PN_{\mathbf{u}}(v') = PN_{\mathbf{u}}(v)$.

Theorem 20 *Consider a game $v \in \mathcal{G}^{N,\mathcal{A}}$ with a utility function $\mathbf{u}$ such that v is $\mathbf{u}$-balanced, and a set sytem $\mathcal{F} \subseteq \hat{\mathcal{A}}$, $\mathcal{F} \neq \emptyset$. Let $v' \in \mathcal{G}^{N,\mathcal{F}\cup\{N,\emptyset\}}$ be such that $v' = v|_{\mathcal{F}\cup\{N,\emptyset\}}$ and $X \subseteq PI(v)$ be such that $PN_{\mathbf{u}}(v), PN_{\mathbf{u}}(v') \subseteq X$. If for every $x \in X$, $S \in \hat{\mathcal{A}} \backslash \mathcal{F}$ there exists $\mathcal{F}_{S,x} \subseteq \mathcal{F}$, $\mathcal{F}_{S,x} \neq \emptyset$ such that*

1. *$u_S \circ e_v(S, x) \leq u_T \circ e_v(T, x)$ for all $T \in \mathcal{F}_{S,x}$,*
2. *$\chi_S \in \mathrm{Lin}\{\chi_T \colon T \in \mathcal{F}_{S,x} \cup \{N\}\}$,*
3. *For all $\alpha \in \mathbb{R}$ if $\mathcal{D}_{\mathbf{u}}^{\hat{\mathcal{A}}}(x, \alpha)$ is balanced then $\mathcal{D}_{\mathbf{u}}^{\mathcal{F}}(x, \alpha)$ is balanced,*

Then $\mathcal{F}$ is a characterization set for $PN_{\mathbf{u}}(v)$.

Proof Take $x \in X \subseteq PI(v)$. Since $\mathcal{F} \subseteq \hat{\mathcal{A}}$ we have that $\mathcal{D}_{\mathbf{u}}^{\mathcal{F}}(\alpha, x) \subseteq \mathcal{D}_{\mathbf{u}}^{\hat{\mathcal{A}}}(\alpha, x)$. Therefore, if $\mathcal{D}_{\mathbf{u}}^{\mathcal{F}}(\alpha, x) \neq \emptyset$ then $\mathcal{D}_{\mathbf{u}}^{\hat{\mathcal{A}}}(\alpha, x) \neq \emptyset$. By Point *1.* if $\mathcal{D}_{\mathbf{u}}^{\hat{\mathcal{A}}}(\alpha, x) \neq \emptyset$ then $\mathcal{D}_{\mathbf{u}}^{\mathcal{F}}(\alpha, x) \neq \emptyset$.

By Lemma 35 if $\mathcal{D}_{\mathbf{u}}^{\mathcal{F}}(\alpha, x)$ is balanced then $\mathcal{D}_{\mathbf{u}}^{\hat{\mathcal{A}}}(\alpha, x)$ is balanced. By Point *3.* if $\mathcal{D}_{\mathbf{u}}^{\hat{\mathcal{A}}}(\alpha, x)$ is balanced then $\mathcal{D}_{\mathbf{u}}^{\mathcal{F}}(\alpha, x)$ is balanced.

By Theorem 19 $x \in PN_{\mathbf{u}}(v')$ if and only if for every $\alpha \in \mathbb{R}$ it holds that $\mathcal{D}_{\mathbf{u}}^{\mathcal{F}}(\alpha, x)$ is either empty or balanced. In the first paragraph we concluded that $\mathcal{D}_{\mathbf{u}}^{\mathcal{F}}(\alpha, x)$ is empty if and only if $\mathcal{D}_{\mathbf{u}}^{\hat{\mathcal{A}}}(\alpha, x)$ is empty. In the second paragraph we concluded that $\mathcal{D}_{\mathbf{u}}^{\mathcal{F}}(\alpha, x)$ is balanced if and only if $\mathcal{D}_{\mathbf{u}}^{\hat{\mathcal{A}}}(\alpha, x)$ is balanced. Therefore, by Theorem 19 $x \in PN_{\mathbf{u}}(v)$ if and only if $x \in PN_{\mathbf{u}}(v')$.

Since, for every $x \in X$ it holds that $x \in PN_{\mathbf{u}}(v)$ if and only if $x \in PN_{\mathbf{u}}(v')$ and $PN_{\mathbf{u}}(v), PN_{\mathbf{u}}(v') \subseteq X$, we conclude that $\mathcal{F}$ is a characterisation set for $PN_{\mathbf{u}}(v)$. $\square$

Consider a game $v \in \mathcal{G}^{N,\mathcal{A}}$ with a utility function $\mathbf{u}$ such that v is $\mathbf{u}$-balanced. Then for every $S \in \hat{\mathcal{A}} \backslash (\mathcal{E}_v^{d-\mathbf{u}} \cap \mathcal{E}_v^{\mathbf{u}})$ it holds that either S is not $\mathbf{u}$-essential or it is not dually-$\mathbf{u}$-essential. Indeed:

Case 1: If S is not $\mathbf{u}$-essential then by Lemma 36 for every $x \in \mathbf{u}$-least-core(v) there exists $\mathcal{B}_S \in \mathcal{D}_S^{\hat{\mathcal{A}}}$, $\mathcal{B}_S \subseteq \mathcal{E}_v^{\mathbf{u}}$ such that

$$u_S \circ e_v(S, x) \leq \sum_{T \in \mathcal{B}_S} u_T \circ e_v(T, x)\,.$$

Case 2: If S is $\mathbf{u}$-essential, that is, it is not dually-$\mathbf{u}$-essential then for every $x \in$ $\mathbf{u}$-least-core(v)

$$\begin{aligned} u_S \circ e_v(S, x) &= u_S \circ (v(S) - x(S)) \\ &= u_S \circ (v(N) - v^*(N \setminus S) - (x(N) - x(N \setminus S))) \\ &= u_S \circ (x(N \setminus S) - v^*(N \setminus S)) = u_S \circ f_{v^*}(N \setminus S, x)\,, \end{aligned} \tag{6}$$

and by Lemma 28 there exists $\mathcal{B} \in \mathcal{D}^{\hat{\mathcal{A}}^{\complement}}_{N\setminus S}$, $\mathcal{B} \subseteq \mathcal{E}^{a-\mathbf{u}}_v$ such that $u_S \circ f_{v^*}(N\setminus S, x) \leq \sum_{T\in\mathcal{B}} u_{N\setminus T} \circ f_{v^*}(T, x)$. Then

$$\begin{aligned} u_S \circ f_{v^*}(N \setminus S, x) &\leq \sum_{T\in\mathcal{B}} u_{N\setminus T} \circ f_{v^*}(T, x) \\ &= \sum_{T\in\mathcal{B}} u_{N\setminus T} \circ e_v(N \setminus T, x) = \sum_{T\in\mathcal{B}^{\complement}} u_T \circ e_v(T, x). \end{aligned} \tag{7}$$

Let $\mathcal{B}_S := \mathcal{B}^{\complement} \subseteq \mathcal{E}^{d-\mathbf{u}}_v$. Then by Eq. (6) and (7) we have that

$$u_S \circ e_v(S, x) \leq \sum_{T\in\mathcal{B}_S} u_T \circ e_v(T, x)\,.$$

Let a directed graph $\Gamma_{v,\mathbf{u}}(x)$ be defined as follows: the nodes are the coalitions of $\hat{\mathcal{A}}\setminus(\mathcal{E}^{\mathbf{u}}_v \cap \mathcal{E}^{d-\mathbf{u}}_v)$, and there is a directed edge from S to S', $S, S' \in \hat{\mathcal{A}}\setminus(\mathcal{E}^{\mathbf{u}}_v \cap \mathcal{E}^{d-\mathbf{u}}_v)$, if S' appears in the partition or in the anti-partition of S, that is, if $S' \in \mathcal{B}_S\setminus(\mathcal{E}^{\mathbf{u}}_v \cap \mathcal{E}^{d-\mathbf{u}}_v)$.

Remark 21 Note that since the considered partitions and anti-partitions can be arbitrarily chosen, the graph $\Gamma_{v,\mathbf{u}}(x)$ can be defined multiple ways. However, this does not make any problem in our analysis, since we need only an instance of such graphs, we do not need to consider all of them.

The following example illustrates the idea of the directed graph above.

Example 22 Consider the following game: $N = \{1, 2, 3, 4\}$, $\hat{\mathcal{A}} = \{\{1\}, \{2\}, \{3\}, \{4\}, \{1, 2\}, \{1, 3\}, \{2, 4\}, \{1, 2, 3\}, \{1, 2, 4\}, \{1, 3, 4\}, \{2, 3, 4\}\}$, $v(\{1\}) = v(\{2\}) = 1.5, v(\{3\}) = v(\{4\}) = 1, v(\{1, 2\}) = 3.5, v(\{1, 3\}) = v(\{2, 4\}) = 2, v(\{1, 2, 3\}) = v(\{1, 2, 4\}) = v(\{1, 3, 4\}) = v(\{2, 3, 4\}) = 4$, $v(N) = 6$. Let $\mathbf{u}$ be the identity function.

The essential coalitions are: $\mathcal{E}_v = \{\{1\}, \{2\}, \{3\}, \{4\}, \{1, 2\}, \{1, 3, 4\}, \{2, 3, 4\}\}$.

The dual game is the following: $v^*(\{1\}) = v^*(\{2\}) = v^*(\{3\}) = v^*(\{4\}) = 2, v^*(\{1, 3\}) = v^*(\{2, 4\}) = 4, v^*(\{3, 4\}) = 2.5, v^*(\{1, 2, 3\}) = v^*(\{1, 2, 4\}) = 5, v^*(\{1, 3, 4\}) = v^*(\{2, 3, 4\}) = 4.5$, $v^*(N) = 6$.

The anti-essential coalitions of the dual game are: $\mathcal{E}^a_{v^*} = \{\{1\}, \{2\}, \{3\}, \{4\}, \{3, 4\}, \{1, 2, 3\}, \{1, 2, 4\}\}$. Therefore, the dually-essential coalitions of the original game are: $\mathcal{E}^d_v = \{\{1, 2, 3\}, \{1, 2, 4\}, \{1, 3, 4\}, \{2, 3, 4\}, \{1, 2\}, \{3\}, \{4\}\}$.

Therefore, $\mathcal{E}^{\mathbf{u}}_v \cap \mathcal{E}^{d-\mathbf{u}}_v = \{\{3\}, \{4\}, \{1, 2\}, \{1, 3, 4\}, \{2, 3, 4\}\}$ and the nodes of the graph are $\hat{\mathcal{A}}\setminus(\mathcal{E}^{\mathbf{u}}_v \cap \mathcal{E}^{d-\mathbf{u}}_v) = \{\{1\}, \{2\}, \{1, 3\}, \{2, 4\}, \{1, 2, 3\}, \{1, 2, 4\}\}$.

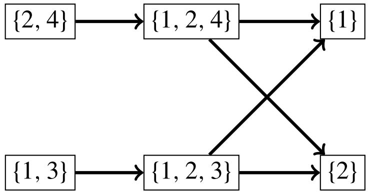
For all $S \in \hat{\mathcal{A}}\setminus(\mathcal{E}^{\mathbf{u}}_v \cap \mathcal{E}^{d-\mathbf{u}}_v)$ there exists a $\mathcal{B}_S \in \mathcal{E}^{\mathbf{u}}_v$ partition (Case 1) or a $\mathcal{B}^{d-\mathbf{u}}_v$ anti-partition (Case 2) as described above. To define the graph, fix one of these coalition sets for each coalition.

$\mathcal{B}_{\{1,2,4\}} = \{\{1\}, \{2\}, \{4\}\}$, because $v(\{1, 2, 4\}) \leq v(\{1\}) + v(\{2\}) + v(\{4\})$ and $\{\{1\}, \{2\}, \{4\}\} \subseteq \mathcal{E}^{\mathbf{u}}_v$.

$\mathcal{B}_{\{2,4\}} = \{\{2, 3, 4\}, \{1, 2, 4\}\}$, because $-v^*(\{1, 3\}) \leq -v^*(\{1\}) - v^*(\{3\})$ and $\{\{2, 3, 4\}, \{1, 2, 4\}\} \subseteq \mathcal{E}^{d-\mathbf{u}}_v$.

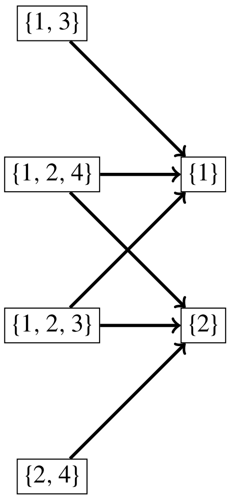
Similarly, let $\mathcal{B}_{\{1,2,3\}} = \{\{1\}, \{2\}, \{3\}\}$, $\mathcal{B}_{\{1,3\}} = \{\{1, 2, 3\}, \{1, 3, 4\}\}$, $\mathcal{B}_{\{1\}} = \{\{1, 2\}, \{1, 3, 4\}\}$, $\mathcal{B}_{\{2\}} = \{\{1, 2\}, \{2, 3, 4\}\}$

For an arbitrary $x \in \mathbf{u}\text{-core}(v)$, $\Gamma_{v,\mathbf{u}}(x)$ is the following



There is an edge from $\{2, 4\}$ to $\{1, 2, 4\}$, because $\{1, 2, 4\} \in \mathcal{B}_{\{2,4\}}$ and there is an edge from $\{1, 2, 4\}$ to $\{1\}$, because $\{1\} \in \mathcal{B}_{\{1,2,4\}}$, etc.

As mentioned in Remark 21, $\Gamma_{v,\mathbf{u}}(x)$ can be defined in multiple ways depending on the choice of the $\mathcal{B}_S$ partitions and anti-partitions. For example, $\mathcal{B}_{\{2,4\}}$ can be chosen as $\{\{2\}, \{4\}\}$, because $v(\{2, 4\}) \leq v(\{2\})+v(\{4\})$ and $\{\{2\}, \{4\}\} \subseteq \mathcal{E}_v^{\mathbf{u}}$ and $\mathcal{B}_{\{1,3\}}$ can be chosen as $\{\{1\}, \{3\}\}$. Then the graph would be



Lemma 38 shows that the above defined graph is acyclic, if $\mathbf{u}\text{-least-core}(v) \neq \mathbf{u}\text{-core}(v)$. In Example 24, we show that if $\mathbf{u}\text{-least-core}(v) = \mathbf{u}\text{-core}(v)$ then $\Gamma_{v,\mathbf{u}}(x)$ can be cyclic.

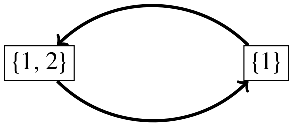
Remark 23 From a geometric perspective, the condition $\mathbf{u}\text{-core}(v) \neq \emptyset$ and $\mathbf{u}\text{-least-core}(v) \neq \mathbf{u}\text{-core}(v)$ is equivalent to the $\mathbf{u}$-core having maximal dimension $(|N| - 1)$ or, equivalently, to the $\mathbf{u}$-core having a nonempty relative interior. This equivalence holds in the absence of utility functions, and since utility functions homeomorphically transform the excesses, the equivalence still applies.

Example 24 Consider the game from Example 13. We have already seen that $\mathcal{E}_v^{\mathbf{u}} = \{\{1\}, \{2\}, \{1, 3\}\}$, $\mathcal{E}_v^{d-\mathbf{u}} = \{\{2\}, \{1, 3\}, \{1, 2\}\}$ and $\mathbf{u}\text{-least-core}(v) = \{(1, 1, 1)\}$. Since $\mathbf{u}\text{-core}(v) = \{(1, 1, 1)\}$, it equals the $\mathbf{u}$-least-core.

$\mathcal{E}_v^{\mathbf{u}} \cap \mathcal{E}_v^{d-\mathbf{u}} = \{\{2\}, \{1, 3\}\}$; therefore, $\hat{A} \backslash (\mathcal{E}_v^{\mathbf{u}} \cap \mathcal{E}_v^{d-\mathbf{u}}) = \{\{1\}, \{1, 2\}\}$. Let $x = (1, 1, 1) \in \mathbf{u}\text{-least-core}(v)$.

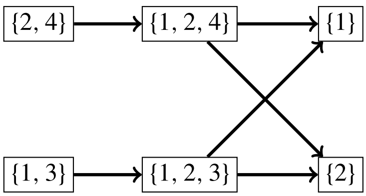
Then the nodes of $\Gamma_{v,\mathbf{u}}(x)$ are $\{1\}, \{1,2\}$. There is an edge from $\{1,2\}$ to $\{1\}$, because $\frac{v(\{1,2\})-x_1-x_2}{2} \leq \frac{v(\{1\})-x_1}{1} + \frac{v(\{2\})-x_2}{1}$, hence $\{1\}$ appears in the partition of $\{1,2\}$. In addition, there is an edge from $\{1\}$ to $\{1,2\}$, because $\frac{x_2+x_3-v^*(\{2,3\})}{1} \leq \frac{x_2-v^*(\{2\})}{2} + \frac{x_3-v^*(\{3\})}{2}$, hence $\{1,2\}$ appears in the anti-partition of $\{1\}$.

$\Gamma_{v,\mathbf{u}}(x)$ is cyclic:

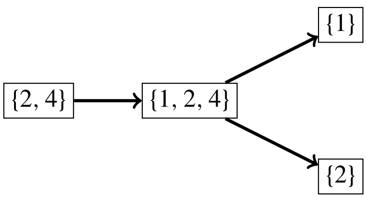


In the following example, we illustrate the idea behind the proof of Lemma 40. In particular, we want to illustrate the idea behind the proof of the following: for a $\mathbf{u}$-balanced game v, a coalition $S \in \hat{\mathcal{A}} \backslash (\mathcal{E}_v^{\mathbf{u}} \cap \mathcal{E}_v^{d-\mathbf{u}})$, if $\mathbf{u}$-least-core$(v) \neq \mathbf{u}$-core(v), then for all $x \in \mathbf{u}$-least-core(v), there exists $\mathcal{F}_{S,x} \subseteq \mathcal{E}_v^{\mathbf{u}} \cap \mathcal{E}_v^{d-\mathbf{u}}$ such that $u_S \circ e_v(S,x) \leq \sum_{T \in \mathcal{F}_{S,x}} u_T \circ e_v(T,x)$.

Example 25 Consider the game of Example 22 with the identity utility function, $x \in \mathbf{u}$-least-core(v), $\mathcal{B}_{\{1,2,4\}} = \{\{1\},\{2\},\{4\}\}$, $\mathcal{B}_{\{2,4\}} = \{\{2,3,4\},\{1,2,4\}\}$, $\mathcal{B}_{\{1,2,3\}} = \{\{1\},\{2\},\{3\}\}$, $\mathcal{B}_{\{1,3\}} = \{\{1,2,3\},\{1,3,4\}\}$, $\mathcal{B}_{\{1\}} = \{\{1,2\},\{1,3,4\}\}$, $\mathcal{B}_{\{2\}} = \{\{1,2\},\{2,3,4\}\}$ and $\Gamma_{v,\mathbf{u}}(x)$:



The game is $\mathbf{u}$-balanced and $\mathbf{u}$-least-core$(v) \neq \mathbf{u}$-core(v). Consider the coalition $S = \{2,4\}$. Let $\Gamma'_{v,\mathbf{u}}(x)$ be the induced subgraph of $\Gamma_{v,\mathbf{u}}(x)$ consisting only of the nodes $\{S\} \cup \{T \in \hat{\mathcal{A}} \backslash (\mathcal{E}_v^{\mathbf{u}} \cap \mathcal{E}_v^{d-\mathbf{u}}) :$ there is a directed path in $\Gamma_{v,\mathbf{u}}(x)$ from S to $T\}$, that is:



Since $\Gamma_{v,\mathbf{u}}(x)$ is acyclic, according to Lemma 38 $\Gamma'_{v,\mathbf{u}}(x)$ is acyclic as well. Therefore, there is a topological ordering on the nodes of $\Gamma'_{v,\mathbf{u}}$: $T_1 = \{2,4\}$, $T_2 = \{1,2,4\}$, $T_3 = \{1\}$, $T_4 = \{2\}$.

$$v(S) = v(\{2,4\}) \leq \sum_{T \in \mathcal{B}_{\{1,2\}}} v(T) = v(\{2,3,4\}) + v(\{1,2,4\}), \tag{8}$$

where $\{2,3,4\} \in \mathcal{E}_v^{\mathbf{u}} \cap \mathcal{E}_v^{d-\mathbf{u}}$, but $\{1,2,4\} \notin \mathcal{E}_v^{\mathbf{u}} \cap \mathcal{E}_v^{d-\mathbf{u}}$. Let us remove $T_2 = \{1,2,3\}$ from the sum by using $v(T_2) \leq \sum_{T \in \mathcal{B}_{T_2}} v(T)$. Then $\mathcal{T}_2 = \{\{2,3,4\},\{1\},\{2\},\{4\}\}$ gives the coalitions appearing in the sum:

$$v(\{2,4\}) \leq v(\{2,3,4\}) + v(\{1\}) + v(\{2\}) + v(\{4\}). \tag{9}$$

here $\{2,3,4\}, \{4\} \in \mathcal{E}_v^{\mathbf{u}} \cap \mathcal{E}_v^{d-\mathbf{u}}$, but $\{1\}, \{2\} \notin \mathcal{E}_v^{\mathbf{u}} \cap \mathcal{E}_v^{d-\mathbf{u}}$. Now we can remove $T_3 = \{1\}$ from the sum as above by using $\mathcal{B}_{T_3}$. Then $\mathcal{T}_3 = \{\{2,3,4\}, \{1,2\}, \{1,3,4\}, \{2\}, \{4\}\}$ gives the coalitions appearing in the sum:

$$v(\{2,4\}) \leq v(\{2,3,4\}) + v(\{1,2\}) + v(\{1,3,4\}) + v(\{2\}) + v(\{4\}). \tag{10}$$

here $\{2,3,4\}, \{1,2\}, \{1,3,4\}, \{4\} \in \mathcal{E}_v^{\mathbf{u}} \cap \mathcal{E}_v^{d-\mathbf{u}}$, but $\{2\} \notin \mathcal{E}_v^{\mathbf{u}} \cap \mathcal{E}_v^{d-\mathbf{u}}$. Remove $T_4 = \{2\}$ from the sum by using $\mathcal{B}_{T_4}$. Then $\mathcal{T}_4 = \{\{2,3,4\}, \{1,2\}, \{1,3,4\}, \{4\}\}$ gives the coalitions appearing in the sum:

$$v(\{2,4\}) \leq 2v(\{2,3,4\}) + 2v(\{1,2\}) + v(\{1,3,4\}) + v(\{4\}), \tag{11}$$

where $\mathcal{T}_4 \subseteq \mathcal{E}_v^{\mathbf{u}} \cap \mathcal{E}_v^{d-\mathbf{u}}$.

Finally, we have a set of coalitions ($\mathcal{T}_4$) such that $v(S) \leq \sum_{T \in \mathcal{T}_4} v(T)$ and $\mathcal{T}_4 \subseteq \mathcal{E}_v^{\mathbf{u}} \cap \mathcal{E}_v^{d-\mathbf{u}}$. We can follow this idea, whenever $\Gamma_{v,\mathbf{u}}(x)$ is acyclic. The method for finding such a set of coalitions is explained in more detail in the proof of the Lemma 40.

In order to apply Theorem 20 to prove that $\mathcal{E}_v^{\mathbf{u}} \cap \mathcal{E}_v^{d-\mathbf{u}}$ is a characterization set for the $\mathbf{u}$-prenucleolus we must find a set $X \subseteq PI(v)$ such that $PN_{\mathbf{u}}(v), PN_{\mathbf{u}}(v') \subseteq X$, where $v' \in \mathcal{G}^{N, \mathcal{E}_v^{\mathbf{u}} \cap \mathcal{E}_v^{d-\mathbf{u}}}$ is such that $v' = v|_{\mathcal{E}_v^{\mathbf{u}} \cap \mathcal{E}_v^{d-\mathbf{u}}}$.

We know that $PN_{\mathbf{u}}(v) \subseteq \mathbf{u}$-least-core$(v)$ and $PN_{\mathbf{u}}(v') \subseteq \mathbf{u}$-least-core$(v')$. Lemmata 42 and 43 state that $\mathbf{u}$-least-core$(v) = \mathbf{u}$-least-core(v'), hence we can apply Theorem 20 with $X = \mathbf{u}$-least-core$(v) = \mathbf{u}$-least-core(v').

Similarly to the notion in Sect. 5, we introduce the following: for a class of coalitions $\mathcal{S} \subseteq \hat{\mathcal{A}}$ and $t \in \mathbb{R}$ let $X(\mathcal{S}, t) := \{x \in PI(v) \colon u_S \circ e_v(S, x) \leq t, \forall S \in \mathcal{S}\}$.

The following proposition is a direct corollary of Lemmata 42 and 43, hence we omit its proof.

Proposition 26 *Consider a* $\mathbf{u}$*-balanced game* $v \in \mathcal{G}^{N,\mathcal{A}}$ *with the utility function* $\mathbf{u}$*, and* $v' = v|_{\mathcal{E}_v^{\mathbf{u}} \cap \mathcal{E}_v^{d-\mathbf{u}}}$. *If* $\mathbf{u}$*-least-core*$(v) \neq \mathbf{u}$*-core*(v) *then* $\mathbf{u}$*-least-core*$(v) = \mathbf{u}$*-least-core*(v').

The following theorem is the main result of this section.

Theorem 27 *Consider a game* $v \in \mathcal{G}^{N,\mathcal{A}}$ *with a utility function* $\mathbf{u}$ *such that* v *is* $\mathbf{u}$*-balanced. If* $\mathbf{u}$*-least-core*$(v) \neq \mathbf{u}$*-core*(v) *then* $\mathcal{E}_v^{d-\mathbf{u}} \cap \mathcal{E}_v^{\mathbf{u}}$ *is a characterization set for the* $\mathbf{u}$*-prenucleolus.*

Proof If $\mathbf{u}$-least-core$(v) \neq \mathbf{u}$-core(v) then by Proposition 26 we have that $\mathbf{u}$-least-core$(v) = \mathbf{u}$-least-core(v'), where $v' = v|_{\mathcal{E}_v^{\mathbf{u}} \cap \mathcal{E}_v^{d-\mathbf{u}}}$. Therefore, $PN_{\mathbf{u}}(v), PN_{\mathbf{u}}(v') \subseteq \mathbf{u}$-least-core$(v)$.

By Lemma 40 for all $S \in \hat{\mathcal{A}} \backslash (\mathcal{E}_v^{d-\mathbf{u}} \cap \mathcal{E}_v^{\mathbf{u}})$, $x \in \mathbf{u}$-least-core(v), there exists $\mathcal{F}_{S,x} \subseteq \mathcal{E}_v^{d-\mathbf{u}} \cap \mathcal{E}_v^{\mathbf{u}}$ such that $u_S \circ e_v(S,x) \leq \sum_{T \in \mathcal{F}_{S,x}} u_T \circ e_v(T,x)$ and $\chi_S \in \text{Lin}\{\chi_T : T \in \mathcal{F}_{S,x} \cup \{N\}\}$. Since for all $T \in \mathcal{F}_{S,x}$ it holds that $u_T \circ e_v(T,x) < 0$ we have that $u_S \circ e_v(S,x) \leq u_T \circ e_v(T,x)$, for all $T \in \mathcal{F}_{S,x}$.

Moreover, by Lemma 39 for all $\alpha \in \mathbb{R}$ it holds that if $\mathcal{D}_{\mathbf{u}}^{\hat{\mathcal{A}}}(x,\alpha)$ is balanced then $\mathcal{D}_{\mathbf{u}}^{\mathcal{E}_v^{d-\mathbf{u}} \cap \mathcal{E}_v^{\mathbf{u}}}(x,\alpha)$ is balanced.

Therefore, we can apply Theorem 20 on $X = \mathbf{u}$-least-core(v), $\mathcal{F} = \mathcal{E}_v^{d-\mathbf{u}} \cap \mathcal{E}_v^{\mathbf{u}}$ and conclude that $\mathcal{E}_v^{d-\mathbf{u}} \cap \mathcal{E}_v^{\mathbf{u}}$ is a characterization set for $PN_{\mathbf{u}}(v)$. ◻

7 Conclusion

In this paper, we define a variant of essential coalitions, namely the class of dually-**u**-essential coalitions, which generalizes the concept of dually-essential coalitions by Solymosi and Sziklai (2016) to TU-games with utilities (Dornai & Pintér, 2024). Using this notion, we prove a variant of Huberman (1980)'s theorem, which also generalizes a theorem by Solymosi and Sziklai (2016), stating that the dually-**u**-essential coalitions form a characterization set for the **u**-prenucleolus of **u**-balanced games.

Furthermore, we explore the dual of TU-games with utility functions and define the **u**-anti-nucleolus, **u**-anti-prenucleolus, **u**-anti-core, **u**-least-anti-core, and dually-**u**-essential coalitions.

In addition, we generalize a result by Granot et al. (1998) about characterization sets of the prenucleolus to TU-games with utility functions. Leveraging this theorem, we generalize a result by Solymosi and Sziklai (2016), demonstrating that the intersection of the set of **u**-essential coalitions and the set of dually-**u**-essential coalitions forms a characterization set for the **u**-prenucleolus, provided the **u**-least-core is a proper subset of the **u**-core. Furthermore, we show that this intersection also characterizes the **u**-least-core under these conditions.

8 Lemmata and proofs for Sect. 5

Lemma 28 *Consider a game $v \in \mathcal{G}^{N,\mathcal{A}}$ with a utility function $\mathbf{u}$. Let $S \in \hat{\mathcal{A}}^{\complement}$ be a non $\mathbf{u}$-anti-essential coalition (of v^*). Then for every $x \in \mathbf{u}$-least-anti-core(v^*) there exists $\mathcal{B}^* \in \mathcal{D}_S^{\hat{\mathcal{A}}^{\complement}}$ such that $u_{N\setminus S} \circ f_{v^*}(S,x) \leq \sum_{T \in \mathcal{B}^*} u_{N\setminus T} \circ f_{v^*}(T,x)$ and $\mathcal{B}^* \subseteq \mathcal{E}_{v^*}^{a-\mathbf{u}}$.*

Proof Since S is not **u**-anti-essential, by definition for each $x \in \mathbf{u}$-least-anti-core(v^*) there exists $\mathcal{B} \in \mathcal{D}_S^{\hat{\mathcal{A}}^{\complement}}$ such that $u_{N\setminus S} \circ f_{v^*}(S,x) \leq \sum_{T \in \mathcal{B}} u_{N\setminus T} \circ f_{v^*}(T,x)$. For each $x \in \mathbf{u}$-least-anti-core(v^*) let $\mathbf{B}(x) := \{\mathcal{B} \in \mathcal{D}_S^{\hat{\mathcal{A}}^{\complement}} : u_{N\setminus S} \circ f_{v^*}(S,x) \leq \sum_{T \in \mathcal{B}} u_{N\setminus T} \circ f_{v^*}(T,x)\}$, and let $\mathcal{B}^* \in \mathbf{B}(x)$ be such that for every partition $\mathcal{B} \in \mathbf{B}(x)$ it holds that $|\mathcal{B}^*| \geq |\mathcal{B}|$.

Indirectly assume that there exists a coalition $T^* \in \mathcal{B}^*$ such that it is not **u**-anti-essential. Since T^* is not **u**-anti-essential, by Definition 12 $\exists \mathcal{B}' \in \mathcal{D}_{T^*}^{\hat{\mathcal{A}}^{\complement}}$ such that $u_{N\setminus T^*} \circ f_{v^*}(T^*,x) \leq \sum_{T' \in \mathcal{B}^*} u_{N\setminus T'} \circ f_{v^*}(T',x)$. Therefore,

$$u_{N\setminus S} \circ f_{v^*}(S,x) \leq \sum_{T' \in (\mathcal{B}^* \setminus \{T^*\}) \cup \mathcal{B}'} u_{N\setminus T'} \circ f_{v^*}(T',x),$$

and $|(\mathcal{B}^* \setminus \{T\}) \cup \mathcal{B}'| > |\mathcal{B}^*|$, which is a contradiction. □

Lemma 29 *Consider a game $v \in \mathcal{G}^{N,\mathcal{A}}$ with a utility function $\mathbf{u}$, and $t', t'' \in \mathbb{R}$ such that $t' \leq t''$. Then $X^d(\mathcal{S},t') \subseteq X^d(\mathcal{S},t'')$.*

Proof For every $x \in X^d(\mathcal{S},t')$ it holds that $x \in PI(v^*)$, and for all $S \in \mathcal{S}$:

$$u_{N\setminus S} \circ f_{v^*}(S,x) \leq t' \leq t''.$$

Therefore, $x \in X^d(\mathcal{S},t'')$. □

Lemma 30 *Consider a utility function $\mathbf{u}$ and $\mathbf{u}$-balanced game $v \in \mathcal{G}^{N,\mathcal{A}}$. Then for every $t_1^d \leq t$ it holds that both $X^d(\hat{\mathcal{A}}^{\complement}, t)$ and $X^d(\mathcal{E}_{v^*}^{a-\mathbf{u}}, t)$ are nonempty, convex and closed, where t_1^d is from Problem 2.*

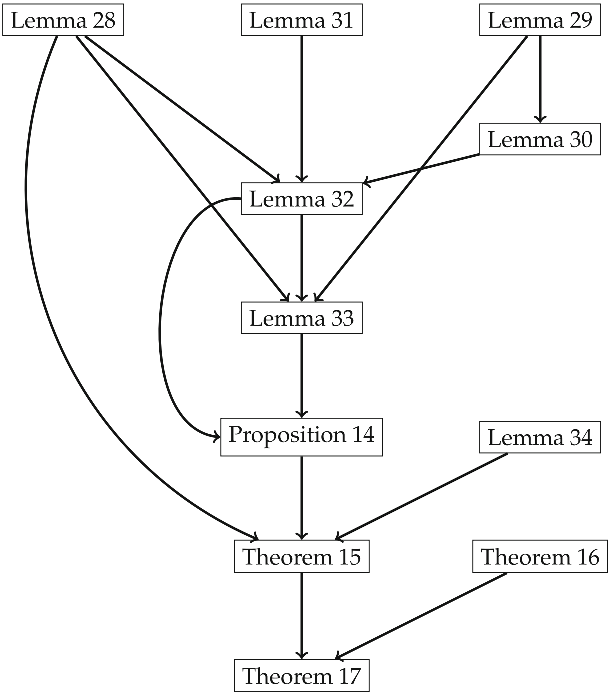


Fig. 1 The relationships of the results of Sect. 5

Proof By Lemma 29 $X^d(\hat{\mathcal{A}}^{\complement}, t) \supseteq X^d(\hat{\mathcal{A}}^{\complement}, t_1^d) = \mathbf{u}$-least-anti-core$(v^*)$. We know that $\mathbf{u}$-least-anti-core$(v^*) \neq \emptyset$, hence $X^d(\hat{\mathcal{A}}^{\complement}, t) \neq \emptyset$, and $X^d(\hat{\mathcal{A}}^{\complement}, t) \subseteq X^d(\mathcal{E}_{v^*}^{a-\mathbf{u}}, t)$, because $\mathcal{E}_{v^*}^{a-\mathbf{u}} \subseteq \hat{\mathcal{A}}^{\complement}$. Therefore, $X^d(\mathcal{E}_{v^*}^{a-\mathbf{u}}, t) \neq \emptyset$ as well.

Let $H_S := \{x \in \mathbb{R}^N : u_{N\setminus S} \circ f_{v*}(S, x) \leq t\}$, for all $S \in \hat{\mathcal{A}}^{\complement}$. Then $H_S = \{x \in \mathbb{R}^N : f_{v^*}(S, x) \leq u_{N\setminus S}^{-1}(t)\}$, hence H_S is a closed half-space, therefore, it is convex and closed. $X_0^d = PI(v) = \{x \in \mathbb{R}^N : x(N) = v^*(N)\}$ is a hyperplane, therefore it is convex and closed. Finally,

$$\begin{aligned} X^d(\hat{\mathcal{A}}^{\complement}, t) &= \{x \in PI(v^*) : u_{N\setminus S} \circ f_{v^*}(S, x) \leq t \ \forall S \in \hat{\mathcal{A}}^{\complement}\} \\ &= \left(\cap_{S \in \hat{\mathcal{A}}^{\complement}} H_S\right) \cap X_0^d \end{aligned}$$

hence $X^d(\hat{\mathcal{A}}^{\complement}, t)$ is the intersection of convex closed sets, therefore, it is convex and closed.

Similarly, $X^d(\mathcal{E}_{v^*}^{a-\mathbf{u}}, t) = \left(\cap_{S \in \mathcal{E}_v^{\mathbf{u}}} H_S\right) \cap X_0^d$ is the intersection of convex closed sets, hence it is convex and closed. □

Lemma 31 *Consider a game* $v \in \mathcal{G}^{N,\mathcal{A}}$ *with a utility function* $\mathbf{u}$*, and take* $x_1, x_2 \in \mathbb{R}^N$, $S \in \mathcal{A}^{\complement}$. *If* $u_{N\setminus S} \circ f_{v^*}(S, x_1) < u_{N\setminus S} \circ f_{v^*}(S, x_2)$ *then for every* $\lambda \in (0, 1)$ *it holds that*

$$u_{N\setminus S} \circ f_{v^*}(S, x_1) < u_{N\setminus S} \circ f_{v^*}(S, \lambda x_1 + (1-\lambda)x_2) < u_{N\setminus S} \circ f_{v^*}(S, x_2)\,.$$

Proof Let $\lambda \in (0, 1)$, then

$$(\lambda x_1 + (1-\lambda)x_2)(S) = \lambda x_1(S) + (1-\lambda)x_2(S).$$

Therefore,

$$\begin{aligned} f_{v^*}(S, \lambda x_1 + (1-\lambda)x_2) &= (\lambda x_1 + (1-\lambda)x_2)(S) - v^*(S) \\ &= \lambda x_1(S) + (1-\lambda)x_2(S) - \lambda v^*(S) - (1-\lambda)v^*(S) \\ &= \lambda f_{v^*}(S, x_1) + (1-\lambda) f_{v^*}(S, x_2)\,. \end{aligned}$$

Since $u_{N\setminus S}$ is strictly monotone increasing we have that

$$u_{N\setminus S} \circ f_{v^*}(S, x_1) < u_{N\setminus S} \circ f_{v^*}(S, \lambda x_1 + (1-\lambda)x_2) < u_{N\setminus S} \circ f_{v^*}(S, x_2)\,.$$

□

Lemma 32 *Consider a game $v \in \mathcal{G}^{N,\mathcal{A}}$ with a utility function $\mathbf{u}$. If v is $\mathbf{u}$-balanced, then $X^d(\hat{\mathcal{A}}^{\complement}, t_1^d) = X^d(\mathcal{E}_{v^*}^{a-\mathbf{u}}, t_1^d)$.*

Proof Since $\mathcal{E}_{v^*}^{a-\mathbf{u}} \subseteq \hat{\mathcal{A}}^{\complement}$, it holds that $X^d(\hat{\mathcal{A}}^{\complement}, t_1^d) \subseteq X^d(\mathcal{E}_{v^*}^{a-\mathbf{u}}, t_1^d)$.

Indirectly assume that $\exists x^1 \in X^d(\mathcal{E}_{v^*}^{a-\mathbf{u}}, t_1^d) \setminus X^d(\hat{\mathcal{A}}^{\complement}, t_1^d)$. This means that $\exists S \in \hat{\mathcal{A}}^{\complement}$ such that

$$u_{N\setminus S} \circ f_{v^*}(S, x^1) > t_1^d\,. \tag{12}$$

Let $\mathcal{S}_{x^1} := \{S \in \hat{\mathcal{A}}^{\complement} : u_{N\setminus S} \circ f_{v^*}(S, x^1) > t_1^d\}$. Then for all $S \in \mathcal{S}_{x^1}$ it holds that $S \notin \mathcal{E}_{v^*}^{a-\mathbf{u}}$.

Let $x^* \in X^d(\hat{\mathcal{A}}^{\complement}, t_1^d)$ be the closest point of set $X^d(\hat{\mathcal{A}}^{\complement}, t_1^d)$ to the point x^1. It is clear that such x^* exists, since $X^d(N \setminus \hat{\mathcal{A}}, t_1^d)$ is nonempty and closed by Lemma 30.

Since $X^d(\mathcal{E}_{v^*}^{a-\mathbf{u}}, t_1^d)$ is a convex set, for every $\lambda \in [0,1]$ it holds that $\lambda x^* + (1-\lambda)x^1 \in X^d(\mathcal{E}_{v^*}^{a-\mathbf{u}}, t_1^d)$.

By Lemma 28 we have that for each $S \in \mathcal{S}_{x^1}$ there exists $\mathcal{B}_S^* \subseteq \mathcal{E}_{v^*}^{a-\mathbf{u}}$, $\mathcal{B}_S^* \in \mathcal{D}_S^{\hat{\mathcal{A}}^{\complement}}$ such that

$$u_{N\setminus S} \circ f_{v^*}(S, x^*) \le \sum_{T \in \mathcal{B}_S^*} u_{N\setminus T} \circ f_{v^*}(T, x^*)\,. \tag{13}$$

Since $t_1^d \le 0$ and for all $x \in X^d(\mathcal{E}_{v^*}^{a-\mathbf{u}}, t_1^d)$ and $T \in \mathcal{E}_{v^*}^{a-\mathbf{u}}$ it holds that $u_{N\setminus T} \circ f_{v^*}(T, x) \le t_1^d$, therefore for all $S \in \mathcal{S}_{x^1}$

$$u_{N\setminus S} \circ f_{v^*}(S, x^*) \le \sum_{T \in \mathcal{B}_S^*} u_{N\setminus T} \circ f_{v^*}(T, x^*) \le t_1^d\,. \tag{14}$$

By (12), (14) and the continuity of $u_{N\setminus S}$, for every $S \in \mathcal{S}_{x^1}$ there exists $\lambda_S \in [0,1]$ such that

$$u_{N\setminus S} \circ f_{v^*}\left(S, \lambda_S x^* + (1-\lambda_S)x^1\right) = t_1^d\,.$$

Let $S^1 \in \operatorname{argmin}_{S \in \mathcal{S}_{x^1}} ||x^* - (\lambda_S x^* + (1-\lambda_S)x^1))||$, and let $x^2 = \lambda_{S^1} x^* + (1-\lambda_{S^1})x^1$. Then

$$u_{N\setminus S^1} \circ f_{v^*}\left(S^1, x^2\right) \ge \sum_{T \in \mathcal{B}_{S^1}^*} u_{N\setminus T} \circ f_{v^*}\left(T, x^2\right),$$

because $t_1^d \le 0$ and $u_{N\setminus T} \circ f_{v^*}(T, x^2) \le t_1^d$, for all $T \in \mathcal{B}_{S^1}^*$.

Then there are two cases:

Case 1:

$$u_{N\setminus S^1}\circ f_{v^*}\big(S^1,x^2\big)=\sum_{T\in\mathcal{B}^*_{S^1}} u_{N\setminus T}\circ f_{v^*}\big(T,x^2\big)\,.$$

In this case, $t_1^d=u_{N\setminus S^1}\circ f_{v^*}(S^1,x^2)=\sum_{T\in\mathcal{B}^*_{S^1}} u_{N\setminus T}\circ f_{v^*}(T,x^2)\le|\mathcal{B}^*_{S^1}|t_1^d\le t_1^d$; therefore $t_1^d=0$.

Then $\sum_{T\in\mathcal{B}^*_{S^1}} u_{N\setminus T}\circ f_{v^*}(T,x^2)=0$, hence for all $T\in\mathcal{B}^*_{S^1}$ it holds that $u_{N\setminus T}\circ f_{v^*}(T,x^2)=0$.

We know that $u_{N\setminus S^1}\circ f_{v^*}(S^1,x^1)>t_1^d\ge\sum_{T\in\mathcal{B}^*_{S^1}} u_{N\setminus T}\circ f_{v^*}(T,x^1)$ and that $x^2(S^1)=\sum_{T\in\mathcal{B}^*_{S^1}} x^2(T)$ and $x^1(S^1)=\sum_{T\in\mathcal{B}^*_{S^1}} x^1(T)$. Then $x^1(S^1)>x^2(S^1)$ and $\exists T'\in\mathcal{B}^*_{S^1}$ such that $x^1(T')>x^2(T')$. However, $u_{N\setminus T'}$ is a strictly monotone increasing function, hence we have that $u_{N\setminus T'}\circ f_{v^*}(T',x^1)>t_1^d$, which is a contradiction, because $x^1\in X^d(\mathcal{E}^{a-\mathbf{u}}_{v^*},t_1^d)$.

Case 2:

$$u_{N\setminus S^1}\circ f_{v^*}\big(S^1,x^2\big)>\sum_{T\in\mathcal{B}^*_{S^1}} u_{N\setminus T}\circ f_{v^*}\big(T,x^2\big)\,. \tag{15}$$

Since for all $S\in\hat{\mathcal{A}}^{\complement}\setminus\mathcal{S}_{x_1}$ it holds that $u_{N\setminus S}\circ f_{v^*}(S,x^1)\le t_1^d$, by Lemma 31

$$u_{N\setminus S}\circ f_{v^*}\big(S,x^2\big)\le t_1^d\,.$$

This means that for all $S\in\hat{\mathcal{A}}^{\complement}$ it holds that $u_{N\setminus S}\circ f_{v^*}(S,x^2)\le t_1^d$, hence $x^2\in X^d(\hat{\mathcal{A}}^{\complement},t_1^d)$.

Since $x^*\in X^d(\hat{\mathcal{A}}^{\complement},t_1^d)$ is the closest point of set $X^d(\hat{\mathcal{A}}^{\complement},t_1^d)$ to the point x^1, we have that $x^2=x^*$. However, then (15) contradicts (13). □

Lemma 33 *Consider a utility function* $\mathbf{u}$*, and a* $\mathbf{u}$*-balanced game* $v\in\mathcal{G}^{N,\mathcal{A}}$ *and* $t\in\mathbb{R}$*. If* $t<t_1^d$ *then* $X^d(\hat{\mathcal{A}}^{\complement},t)=X^d(\mathcal{E}^{a-\mathbf{u}}_{v^*},t)=\emptyset$.

Proof By the definition of t_1^d (see Problem 2), we have that $X^d(\hat{\mathcal{A}}^{\complement},t)=\emptyset$ and we know that $X^d(\hat{\mathcal{A}}^{\complement},t)\subseteq X^d(\mathcal{E}^{a-\mathbf{u}}_{v^*},t)$.

By Lemma 29, since $t<t_1^d$, we have that $X^d(\mathcal{E}^{a-\mathbf{u}}_{v^*},t)\subseteq X^d(\mathcal{E}^{a-\mathbf{u}}_{v^*},t_1^d)$. Furthermore, by Lemma 32 it holds that $X^d(\mathcal{E}^{a-\mathbf{u}}_{v^*},t_1^d)=X^d(\hat{\mathcal{A}}^{\complement},t_1^d)=\mathbf{u}$-least-anti-core$(v^*)$. Therefore,

$$X^d(\mathcal{E}^{a-\mathbf{u}}_{v^*},t)\subseteq X^d(\mathcal{E}^{a-\mathbf{u}}_{v^*},t_1^d)=X^d(\hat{\mathcal{A}}^{\complement},t_1^d)=\mathbf{u}\text{-least-anti-core}(v^*)\,.$$

Then for every $x\in X^d(\mathcal{E}^{a-\mathbf{u}}_{v^*},t)$ and for every $S\in(N\setminus\hat{\mathcal{A}})\setminus\mathcal{E}^{a-\mathbf{u}}_{v^*}$ by Lemma 28 and by the non-positivity of t, it holds that $\exists\mathcal{B}\in\mathcal{D}_S^{\hat{\mathcal{A}}^{\complement}}$, $\mathcal{B}\subseteq\mathcal{E}^{a-\mathbf{u}}_{v^*}$ such that

$$u_{N\setminus S}\circ f_{v^*}(S,x)\le\sum_{T\in\mathcal{B}} u_{N\setminus T}\circ f_{v^*}(T,x)\le t\,.$$

Then $x\in X^d(\hat{\mathcal{A}}^{\complement},t)$, that is, $X^d(\mathcal{E}^{a-\mathbf{u}}_{v^*},t)\subseteq X^d(\hat{\mathcal{A}}^{\complement},t)$. Summing up, we can conclude that $X^d(\mathcal{E}^{a-\mathbf{u}}_{v^*},t)=X^d(\hat{\mathcal{A}}^{\complement},t)=\emptyset$. □

Lemma 34 *Consider a utility function* $\mathbf{u}$*, and a* $\mathbf{u}$*-balanced game* $v\in\mathcal{G}^{N,\mathcal{A}}$*. Consider Problem (2), and let k be an arbitrary positive integer. Then for every* $S\in\hat{\mathcal{A}}^{\complement}\setminus(\cup_{r=0}^{k-1}W_r^d)$ *such that* $\mathcal{D}_S^{\hat{\mathcal{A}}^{\complement}}\neq\emptyset$*, and for every* $\mathcal{B}^*\in\mathcal{D}_S^{\hat{\mathcal{A}}^{\complement}}$ *there exists* $T^*\in\mathcal{B}^*$ *such that* $T^*\notin\cup_{r=0}^{k-1}W_r^d$.

Proof $k = 1$: Then $\cup_{r=0}^{k-1} W_r^d = \emptyset$, hence for every $\mathcal{B}^* \in \mathcal{D}_S^{\hat{\mathcal{A}}^\complement}$ it holds that $T^* \notin \cup_{r=0}^{k-1} W_r^d$, for all $T^* \in \mathcal{B}^*$.

$k \geq 2$: Indirectly assume that for every $\mathcal{B}^* \in \mathcal{D}_S^{\hat{\mathcal{A}}^\complement}$ it holds that $\mathcal{B}^* \subseteq \cup_{r=0}^{k-1} W_r^d$. Then for every $\mathcal{B}^* \in \mathcal{D}_S^{\hat{\mathcal{A}}^\complement}$ and $x \in X_{k-1}^d$

$$u_{N\setminus S} \circ f_{v^*}(S, x) = u_{N\setminus S} \circ (x(S) - v^*(S)) = u_{N\setminus S} \circ \left(\sum_{T \in \mathcal{B}^*} x(T) - v^*(S) \right)$$
$$= u_{N\setminus S} \circ \left(\sum_{T \in \mathcal{B}^*} (u_{N\setminus T}^{-1}(c_T) + v^*(T)) - v^*(S) \right),$$

where c_T is a constant such that $u_{N\setminus T} \circ f_{v^*}(T, x) = c_T$, for all $x \in X_l^d$, where $T \in W_l^d$.

Therefore, for each $x, x' \in X_{k-1}^d$ it holds that $u_{N\setminus S} \circ f_{v^*}(S, x) = u_{N\setminus S} \circ f_{v^*}(S, x')$ meaning that $S \in \cup_{r=0}^{k-1} W_r^d$, which is a contradiction. □

9 Lemmata and proofs for Sect. 6

Lemma 35 *Let $\mathcal{F} \subseteq 2^N$ be a balanced set of coalitions and $S \notin \mathcal{F}$ be such that $\chi_S \in$ $\mathrm{Lin}\{\chi_T : T \in \mathcal{F} \cup \{N\}\}$. Then $\mathcal{F} \cup \{S\}$ is a balanced set of coalitions.*

Proof $\mathcal{F}$ is balanced, hence there exist $\lambda_T > 0$, $T \in \mathcal{F}$ weight system such that

$$\sum_{T \in \mathcal{F}} \lambda_T \chi_T = \chi_N .$$

$\chi_S \in \mathrm{Lin}\{\chi_T : T \in \mathcal{F} \cup \{N\}\}$, hence there exist $k_T \in \mathbb{R}$, $T \in \mathcal{F}$ and $k_N \in \mathbb{R}$ such that

$$k_N \chi_N + \sum_{T \in \mathcal{F}} k_T \chi_T = \chi_S .$$

Then there exists $\varepsilon > 0$ such that $\lambda_T - \varepsilon k_T > 0$ for all $T \in \mathcal{F}$ and $1 + \varepsilon k_N > 0$.

$$\chi_N = \sum_{T \in \mathcal{F}} \lambda_T \chi_T + \varepsilon \chi_S - \varepsilon \chi_S$$
$$= \sum_{T \in \mathcal{F}} (\lambda_T - \varepsilon k_T) \chi_T - \varepsilon k_N \chi_N + \varepsilon \chi_S .$$

Therefore,

$$(1 + \varepsilon k_N) \chi_N = \textstyle\sum_{T \in \mathcal{F}} (\lambda_T - \varepsilon k_T) \chi_T + \varepsilon \chi_S$$
$$\chi_N = \textstyle\sum_{i \in \mathcal{F}} \frac{\lambda_T - \varepsilon k_T}{1 + \varepsilon k_N} \chi_T + \frac{\varepsilon}{1 + \varepsilon k_N} \chi_S,$$

where $\frac{\lambda_T - \varepsilon k_T}{1 + \varepsilon k_N} > 0$ and $\frac{\varepsilon}{1 + \varepsilon k_N} > 0$. Therefore, the set $\mathcal{F} \cup \{S\}$ is balanced. □

Lemma 36 is a technical result. It is a variant of Lemma 3 in Dornai and Pintér (2024). Remark 1 in Dornai and Pintér (2024) applies to this lemma, hence we omit its proof.

Lemma 36 *Consider a game $v \in \mathcal{G}^{N,\mathcal{A}}$ with a utility function $\mathbf{u}$. Let $S \in \hat{\mathcal{A}} \setminus \mathcal{E}_v^{\mathbf{u}}$. Then for every $x \in \mathbf{u}$-least-core(v) there exists $\mathcal{B}^* \in \mathcal{D}_S^{\hat{\mathcal{A}}}$ such that $u_S \circ e_v(S, x) \leq \sum_{T \in \mathcal{B}^*} u_T \circ e_v(T, x)$ and $\mathcal{B}^* \subseteq \mathcal{E}_v^{\mathbf{u}}$.*

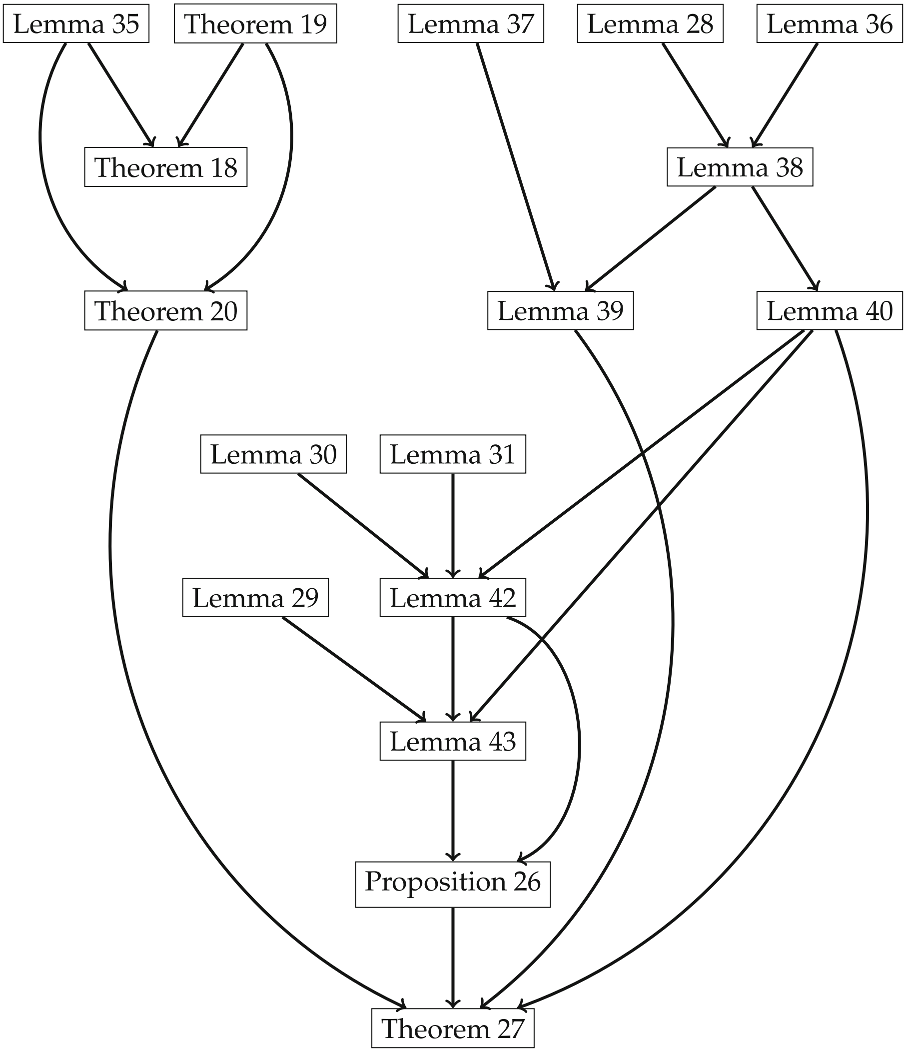


Fig. 2 The relationships of the results of Sect. 6

We need Lemma 37 to show the balancedness of certain set systems.

Lemma 37 *Consider the set systems $\mathcal{S}, \mathcal{T} \subseteq \mathcal{P}(N)$ such that $\mathcal{T} \subseteq \mathcal{S}$, and a sequence of set systems $\mathcal{S} = \mathcal{F}_0, \mathcal{F}_1, \mathcal{F}_2, \ldots$ such that*

1. *if $\mathcal{F}_n$ is balanced then $\mathcal{F}_{n+1}$ is balanced, for every $n \in \mathbb{N}$,*
2. *$|\mathcal{T} \cap \mathcal{F}_n| \leq |\mathcal{T} \cap \mathcal{F}_{n+1}|$, for every $n \in \mathbb{N}$,*
3. *if $\mathcal{T} \subsetneq \mathcal{F}_n$ then $|\mathcal{F}_n \cap (\mathcal{S} \setminus \mathcal{T})| > |\mathcal{F}_{n+1} \cap (\mathcal{S} \setminus \mathcal{T})|$.*

Then there exists $n^* \in \mathbb{N}$ such that $\mathcal{F}_{n^*} = \mathcal{T}$, and if $\mathcal{S}$ is balanced then $\mathcal{T}$ is balanced.

Proof By Point 3. there exist $n^* \leq |\mathcal{S} \setminus \mathcal{T}|$ such that $|\mathcal{F}_{n^*} \cap (\mathcal{S} \setminus \mathcal{T})| = 0$, that is, $\mathcal{F}_{n^*} \subseteq \mathcal{T}$.

By $\mathcal{T} \subseteq \mathcal{S} = \mathcal{F}_0$ and Point 2. it holds that $\mathcal{T} \subseteq \mathcal{F}_n$, for all $n \in \mathbb{N}$, in particular $\mathcal{T} \subseteq \mathcal{F}_{n^*}$.

Therefore, $\mathcal{T} = \mathcal{F}_{n^*}$, and by Point *1.* if $\mathcal{S}$ is balanced then for all $n \in \mathbb{N}$ it holds that $\mathcal{F}_n$ is balanced. In particular $\mathcal{F}_{n^*} = \mathcal{T}$ is balanced. □

Lemma 38 *Consider a* $\mathbf{u}$*-balanced game* $v \in \mathcal{G}^{N,\mathcal{A}}$ *with the utility function* $\mathbf{u}$*, and* $x \in \mathbf{u}$*-least-core*(v)*. If* $\mathbf{u}$*-least-core*$(v) \neq \mathbf{u}$*-core*(v) *then* $\Gamma_{v,\mathbf{u}}(x)$ *is acyclic.*

Proof Indirectly assume that the graph is not acyclic. Then there is a cycle $T_1, T_2, \ldots, T_r \subseteq \hat{\mathcal{A}} \backslash (\mathcal{E}_v^{\mathbf{u}} \cap \mathcal{E}_v^{d-\mathbf{u}})$ such that

$$\begin{aligned}
u_{T_1} \circ e_v(T_1, x) &\leq u_{T_2} \circ e_v(T_2, x) + \sum_{j=1}^{k_1} u_{S_j^1} \circ e_v\left(S_j^1, x\right)\\
u_{T_2} \circ e_v(T_2, x) &\leq u_{T_3} \circ e_v(T_3, x) + \sum_{j=1}^{k_2} u_{S_j^2} \circ e_v\left(S_j^2, x\right)\\
&\vdots\\
u_{T_r} \circ e_v(T_r, x) &\leq u_{T_1} \circ e_v(T_1, x) + \sum_{j=1}^{k_r} u_{S_j^r} \circ e_v\left(S_j^r, x\right),
\end{aligned}$$

where $\mathcal{B}_{T_i} = \{T_{i+1}\} \cup \{S_j^i\}_{j=1}^{k_i}$, $i = 1, \ldots, r-1$ and $\mathcal{B}_{T_r} = \{T_1\} \cup \{S_j^r\}_{j=1}^{k_r}$ are the corresponding partitions or anti-partitions.

Summing up the inequalities above, we have that

$$0 \leq \sum_{i=1}^{r} \sum_{j=1}^{k_r} u_{S_j^i} \circ e_v\left(S_j^i, x\right). \tag{16}$$

By Eq. (16), we have that $u_{S_j^i} \circ e_v(S_j^i, x) = 0$, for all $i = 1, 2, \ldots, r$, $j = 1, 2, \ldots, k_r$. However, since $x \in \mathbf{u}$-least-core$(v) \neq \mathbf{u}$-core(v), we have that $u_{S_j^i} \circ e_v(S_j^i, x) < 0$, for all $i = 1, 2, \ldots, r$, $j = 1, 2, \ldots, k_r$, which is a contradiction. □

We need Lemma 39 to show that Point *3.* of Theorem 20 holds for the intersection of the set of $\mathbf{u}$-essential and the set of dually-$\mathbf{u}$-essential coalitions, if $\mathbf{u}$-least-core(v) is a proper subset of $\mathbf{u}$-core(v).

Lemma 39 *Consider a* $\mathbf{u}$*-balanced game* $v \in \mathcal{G}^{N,\mathcal{A}}$ *with a utility function* $\mathbf{u}$*, and* $x \in \mathbf{u}$*-least-core*(v)*. If* $\mathbf{u}$*-least-core*$(v) \neq \mathbf{u}$*-core*(v) *then the following holds: if* $\hat{\mathcal{A}}$ *is balanced then* $\mathcal{E}_v^{\mathbf{u}} \cap \mathcal{E}_v^{d-\mathbf{u}}$ *is balanced. In addition, for all* $\alpha \in \mathbb{R}$ *it holds that if* $\mathcal{D}_{\mathbf{u}}^{\hat{\mathcal{A}}}(\alpha, x)$ *is balanced then* $\mathcal{D}_{\mathbf{u}}^{\mathcal{E}_v^{\mathbf{u}} \cap \mathcal{E}_v^{d-\mathbf{u}}}(\alpha, x)$ *is balanced.*

Proof By Lemma 38, we have that $\Gamma_{v,\mathbf{u}}(x)$ is acyclic, hence there is a topological ordering on its nodes $T_1, T_2, \ldots, T_{|\hat{\mathcal{A}} \backslash (\mathcal{E}_v^{\mathbf{u}} \cap \mathcal{E}_v^{d-\mathbf{u}})|}$. Then we can define the following sequence of coalitions:

$$\begin{aligned}
&\mathcal{F}_0 = \hat{\mathcal{A}},\\
&\mathcal{F}_n = \mathcal{F}_{n-1} \setminus \{T_n\} \cup \mathcal{B}_{T_n}, \quad n = 1, 2, \ldots, |\hat{\mathcal{A}} \setminus (\mathcal{E}_v^{\mathbf{u}} \cap \mathcal{E}_v^{d-\mathbf{u}})|,\\
&\mathcal{F}_n = \mathcal{F}_{n-1}, n \in \mathbb{N}, \qquad n > |\hat{\mathcal{A}} \setminus (\mathcal{E}_v^{\mathbf{u}} \cap \mathcal{E}_v^{d-\mathbf{u}})|.
\end{aligned}$$

Let $\mathcal{B}_{T_n}$ denote the partition or anti-partition of T_n used in defining $\Gamma_{v,\mathbf{u}}(x)$. Next we check the three conditions of Lemma 37:

Point *1*.: If $\mathcal{F}_n$ is balanced then $\mathcal{F}_{n+1}$ is balanced, for every $n \in \mathbb{N}$.

If $n \geq |\hat{\mathcal{A}} \backslash (\mathcal{E}_v^{\mathbf{u}} \cap \mathcal{E}_v^{d-\mathbf{u}})|$ then $\mathcal{F}_n = \mathcal{F}_{n+1}$, hence Point *1*. holds.

Otherwise, if $\mathcal{F}_n$ is balanced then there exists $(\lambda_S)_{S \in \mathcal{F}_n} \geq 0$ such that

$$\sum_{S \in \mathcal{F}_n} \lambda_S \chi_S = \chi_N \,. \tag{17}$$

If $\mathcal{B}_{T_{n+1}}$ is the corresponding partition of T_{n+1}, then T_{n+1} is not $\mathbf{u}$-essential and

$$\chi_{T_{n+1}} = \sum_{T \in \mathcal{B}_{T_{n+1}}} \chi_T \,.$$

Therefore, Eq. (17) can be rewritten as

$$\sum_{S \in \mathcal{F}_n \backslash \{T_{n+1}\}} \lambda_S \chi_S + \sum_{T \in \mathcal{B}_{T_{n+1}}} \lambda_{T_{n+1}} \chi_T = \chi_N,$$

hence $\mathcal{F}_{n+1} = (\mathcal{F}_i \backslash \{T_{n+1}\}) \cup \mathcal{B}_{T_{n+1}}$ is balanced.

If $\mathcal{B}_{T_{n+1}}$ is the corresponding anti-partition of T_{n+1}, then T_{n+1} is not dually-$\mathbf{u}$-essential and

$$\chi_{T_{n+1}} = \sum_{T \in \mathcal{B}_{T_{n+1}}} \chi_T - (|\mathcal{B}_{T_{n+1}}| - 1)\chi_N \,.$$

Therefore, Eq. (17) can be rewritten as

$$\begin{aligned}
&\sum_{S \in \mathcal{F}_n \backslash \{T_{n+1}\}} \lambda_S \chi_S + \sum_{T \in \mathcal{B}_{T_{n+1}}} \lambda_{T_{n+1}} \chi_T - \lambda_{T_{n+1}}(|\mathcal{B}_{T_{n+1}}| - 1)\chi_N = \chi_N \\
&\sum_{S \in \mathcal{F}_n \backslash \{T_{n+1}\}} \lambda_S \chi_S + \sum_{T \in \mathcal{B}_{T_{n+1}}} \lambda_{T_{n+1}} \chi_T = (1 + \lambda_{T_{n+1}}(|\mathcal{B}_{T_{n+1}}| - 1))\chi_N \\
&\sum_{S \in \mathcal{F}_n \backslash \{T_{n+1}\}} \frac{\lambda_S}{1 + \lambda_{T_{n+1}}(|\mathcal{B}_{T_{n+1}}| - 1)} \chi_S + \sum_{T \in \mathcal{B}_{T_{n+1}}} \frac{\lambda_{T_{n+1}}}{1 + \lambda_{T_{n+1}}(|\mathcal{B}_{T_{n+1}}| - 1)} \chi_T = \chi_N,
\end{aligned}$$

hence, $\mathcal{F}_{n+1} = (\mathcal{F}_n \backslash \{T_{n+1}\}) \cup \mathcal{B}_{T_{n+1}}$ is balanced.

Point *2*.: $|(\mathcal{E}_v^{\mathbf{u}} \cap \mathcal{E}_v^{d-\mathbf{u}}) \cap \mathcal{F}_n| \leq |(\mathcal{E}_v^{\mathbf{u}} \cap \mathcal{E}_v^{d-\mathbf{u}}) \cap \mathcal{F}_{n+1}|$, for every $n \in \mathbb{N}$.

If $n \geq |\hat{\mathcal{A}} \backslash (\mathcal{E}_v^{\mathbf{u}} \cap \mathcal{E}_v^{d-\mathbf{u}})|$ then $\mathcal{F}_n = \mathcal{F}_{n+1}$, hence $|(\mathcal{E}_v^{\mathbf{u}} \cap \mathcal{E}_v^{d-\mathbf{u}}) \cap \mathcal{F}_n| = |(\mathcal{E}_v^{\mathbf{u}} \cap \mathcal{E}_v^{d-\mathbf{u}}) \cap \mathcal{F}_{n+1}|$.

Otherwise, since $\mathcal{F}_{n+1} = (\mathcal{F}_n \backslash \{T_{n+1}\}) \cup \mathcal{B}_{T_{n+1}}$, where $T_{n+1} \notin (\mathcal{E}_v^{\mathbf{u}} \cap \mathcal{E}_v^{d-\mathbf{u}})$, the number of coalitions in the intersection never decreases, that is, $|(\mathcal{E}_v^{\mathbf{u}} \cap \mathcal{E}_v^{d-\mathbf{u}}) \cap \mathcal{F}_n| \leq |(\mathcal{E}_v^{\mathbf{u}} \cap \mathcal{E}_v^{d-\mathbf{u}}) \cap \mathcal{F}_{n+1}|$.

Point *3*.: If $(\mathcal{E}_v^{\mathbf{u}} \cap \mathcal{E}_v^{d-\mathbf{u}}) \subsetneq \mathcal{F}_n$, then $|\mathcal{F}_n \cap (\hat{\mathcal{A}} \backslash (\mathcal{E}_v^{\mathbf{u}} \cap \mathcal{E}_v^{d-\mathbf{u}}))| > |\mathcal{F}_{n+1} \cap (\hat{\mathcal{A}} \backslash (\mathcal{E}_v^{\mathbf{u}} \cap \mathcal{E}_v^{d-\mathbf{u}}))|$.

By the topological ordering of the nodes (coalitions), for all $i, j \in \mathbb{N}_+$, $i < j \leq |\hat{\mathcal{A}} \backslash (\mathcal{E}_v^{\mathbf{u}} \cap \mathcal{E}_v^{d-\mathbf{u}})|$ it holds that $T_i \notin \mathcal{B}_{T_j}$. Therefore, $T_i \notin \mathcal{F}_j$. Since $T_i \in \mathcal{F}_{i-1}$ and if a coalition – not from the intersection – is removed then it never comes back, we have that $|\mathcal{F}_i \cap (\hat{\mathcal{A}} \backslash (\mathcal{E}_v^{\mathbf{u}} \cap \mathcal{E}_v^{d-\mathbf{u}}))| > |\mathcal{F}_{i+1} \cap (\hat{\mathcal{A}} \backslash (\mathcal{E}_v^{\mathbf{u}} \cap \mathcal{E}_v^{d-\mathbf{u}}))|$.

In conclusion, by Lemma 37 we have that $\mathcal{F}_{|\hat{\mathcal{A}} \backslash (\mathcal{E}_v^{\mathbf{u}} \cap \mathcal{E}_v^{d-\mathbf{u}})|} = (\mathcal{E}_v^{\mathbf{u}} \cap \mathcal{E}_v^{d-\mathbf{u}})$, and if $\hat{\mathcal{A}}$ is balanced, then $(\mathcal{E}_v^{\mathbf{u}} \cap \mathcal{E}_v^{d-\mathbf{u}})$ is balanced.

To prove the second part of the theorem, for each $\alpha \in \mathbb{R}$ consider the induced subgraph of $\Gamma_{v,\mathbf{u}}(x)$ with nodes from $(\hat{\mathcal{A}} \backslash (\mathcal{E}_v^{\mathbf{u}} \cap \mathcal{E}_v^{d-\mathbf{u}})) \cap \mathcal{D}_{\mathbf{u}}^{\hat{\mathcal{A}}}(x, \alpha)$. Let $\Gamma_{v,\mathbf{u}}^{\alpha}(x)$ denote this induced subgraph. It is easy to check that $\Gamma_{v,\mathbf{u}}^{\alpha}(x)$ is a directed acyclic graph, hence there is a topological ordering on its nodes: $T'_1, T'_2, \ldots T'_{|(\hat{\mathcal{A}} \backslash (\mathcal{E}_v^{\mathbf{u}} \cap \mathcal{E}_v^{d-\mathbf{u}})) \cap \mathcal{D}_{\mathbf{u}}^{\hat{\mathcal{A}}}(x,\alpha)|}$.

Consider the following sequence of sets:

$$\begin{aligned}
&\mathcal{F}'_0 = \mathcal{D}_{\mathbf{u}}^{\hat{\mathcal{A}}}(x,\alpha) \\
&\mathcal{F}'_n = \left(\mathcal{F}'_{n-1} \setminus \{T'_n\}\right) \cup \mathcal{B}_{T'_n}, \quad && n = 1,2,\dots, \left|\left(\mathcal{D}_{\mathbf{u}}^{\hat{\mathcal{A}}}(x,\alpha) \setminus \left(\mathcal{E}_v^{\mathbf{u}} \cap \mathcal{E}_v^{d-\mathbf{u}}\right)\right)\right|, \\
&\mathcal{F}'_n = \mathcal{F}'_{n-1}, && n \in \mathbb{N},\ n > \left|\left(\mathcal{D}_{\mathbf{u}}^{\hat{\mathcal{A}}}(x,\alpha) \setminus \left(\mathcal{E}_v^{\mathbf{u}} \cap \mathcal{E}_v^{d-\mathbf{u}}\right)\right)\right|.
\end{aligned}$$

The proof of that Points *1.* and *2.* of Lemma 37 hold for the sequence of sets $(\mathcal{F}'_n)$ goes as it goes for the sequence of sets $(\mathcal{F}_n)$, hence we omit its proof. For Point *3.* of Lemma 37 consider the following:

For every $n = 1,2,\dots,|(\mathcal{D}_{\mathbf{u}}^{\hat{\mathcal{A}}}(x,\alpha)\setminus(\mathcal{E}_v^{\mathbf{u}} \cap \mathcal{E}_v^{d-\mathbf{u}}))|$ and $S \in \mathcal{B}_{T'_n}\setminus(\mathcal{E}_v^{\mathbf{u}} \cap \mathcal{E}_v^{d-\mathbf{u}})$ it holds that S is a node in $\Gamma_{v,\mathbf{u}}^{\alpha}(x)$. Indeed, since $x \in \mathbf{u}$-least-core(v), $\mathbf{u}$-least-core$(v) \neq \mathbf{u}$-core(v) and $u_{T'_n} \circ e_v(T'_n, x) \leq \sum_{S\in\mathcal{B}_{T'_n}} u_S \circ e_v(S,x)$, we have that for every $S \in \mathcal{B}_{T'_n}$ it holds that $u_S \circ e_v(S,x) \geq u_{T'_n} \circ e_v(T'_n, x) \geq \alpha$.

Therefore, we can apply Lemma 37 and get that $\mathcal{F}'_{|(\mathcal{D}_{\mathbf{u}}^{\hat{\mathcal{A}}}(x,\alpha)\setminus(\mathcal{E}_v^{\mathbf{u}}\cap\mathcal{E}_v^{d-\mathbf{u}}))|} = \mathcal{D}_{\mathbf{u}}^{(\mathcal{E}_v^{\mathbf{u}}(x,\alpha)\cap\mathcal{E}_v^{d-\mathbf{u}}))}(x,\alpha)$ and that, if $\mathcal{D}_{\mathbf{u}}^{\hat{\mathcal{A}}}(x,\alpha)$ is balanced then $\mathcal{D}_{\mathbf{u}}^{(\mathcal{E}_v^{\mathbf{u}}(x,\alpha)\cap\mathcal{E}_v^{d-\mathbf{u}}))}(x,\alpha)$ is balanced. □

Lemma 40 shows that Points *1.* and *2.* of Theorem 20 hold for the intersection of the set of $\mathbf{u}$-essential and the set of dually-$\mathbf{u}$-essential coalitions, if $\mathbf{u}$-least-core(v) is a proper subset of $\mathbf{u}$-core(v).

Lemma 40 *Consider a game $v \in \mathcal{G}^{N,\mathcal{A}}$ with a utility function $\mathbf{u}$ such that the game is $\mathbf{u}$-balanced, and a coalition $S \in \hat{\mathcal{A}}\setminus(\mathcal{E}_v^{\mathbf{u}} \cap \mathcal{E}_v^{d-\mathbf{u}})$. If $\mathbf{u}$-least-core$(v) \neq \mathbf{u}$-core(v) then for every $x \in \mathbf{u}$-least-core(v) there exists $\mathcal{B}^* \subseteq \mathcal{E}_v^{\mathbf{u}} \cap \mathcal{E}_v^{d-\mathbf{u}}$ such that $u_S \circ e_v(S,x) \leq \sum_{T\in\mathcal{B}^*} u_T \circ e_v(T,x)$ and $\chi_S \in \mathrm{Lin}\{\chi_T : \mathcal{B}^* \cup \{N\}\}$.*

Proof If $\mathbf{u}$-least-core$(v) \neq \mathbf{u}$-core(v) then by Lemma 38 $\Gamma_{v,\mathbf{u}}(x)$ is acyclic. Take the induced subgraph of $\Gamma_{v,\mathbf{u}}(x)$ consisting only of the nodes $\{S\} \cup \{T \in \hat{\mathcal{A}}\setminus(\mathcal{E}_v^{\mathbf{u}} \cap \mathcal{E}_v^{d-\mathbf{u}})$: there is a directed path in $\Gamma_{v,\mathbf{u}}(x)$ from S to $T\}$. Let $\Gamma'_{v,\mathbf{u}}(x)$ denote this induced subgraph. Then $\Gamma'_{v,\mathbf{u}}(x)$ is also acyclic, hence there is a topological ordering on its nodes: $T_1, T_2, \dots T_r$, where $T_1 = S$.

Consider the following sequence of sets:

$$\begin{aligned}
&\mathcal{T}_0 = \{S\}\,, \\
&\mathcal{T}_n = (\mathcal{T}_{n-1} \setminus \{T_n\}) \cup \mathcal{B}_{T_n}\,, \qquad n = 1,\dots,r\,.
\end{aligned}$$

By the definition of $\mathcal{B}_{T_n}$, $n = 1,\dots,r$

$$u_S \circ e_v(S,x) \leq \sum_{T\in\mathcal{T}_n} u_T \circ e_v(T,x)\,,$$

in particular

$$u_S \circ e_v(S,x) \leq \sum_{T\in\mathcal{T}_r} u_T \circ e_v(T,x)\,.$$

Moreover, since for all $1 \leq i \leq j \leq r$ it holds that $T_i \notin \mathcal{B}_{T_j}$, and for all $T \in \mathcal{B}_{T_i}\setminus(\mathcal{E}_v^{\mathbf{u}} \cap \mathcal{E}_v^{d-\mathbf{u}})$ we have that T is a node of the graph $\Gamma'_{v,\mathbf{u}}(x)$, we have that $\mathcal{T}_r \subseteq \mathcal{E}_v^{\mathbf{u}} \cap \mathcal{E}_v^{d-\mathbf{u}}$.

Since for each $n = 1, \ldots, r$ it holds that $\mathcal{B}_{T_n}$ is either a partition or an anti-partition of T_n, we have that $\chi_{T_n} \in \mathrm{Lin}\{\chi_T : T \in \mathcal{B}_{T_n} \cup \{N\}\}$. If $\mathcal{B}_{T_n}$ is a partition of T_n then

$$\chi_{T_n} = \sum_{T \in \mathcal{B}_{T_n}} \chi_T \,.$$

If $\mathcal{B}_{T_n}$ is an anti-partition of T_n then

$$\chi_{T_n} = \sum_{T \in \mathcal{B}_{T_n}} \chi_T - (|\mathcal{B}_{T_n}| - 1)\chi_N \,.$$

Therefore, for every $n = 1, \ldots, r$ it holds that $\chi_S \in \mathrm{Lin}\{\chi_T : T \in \mathcal{T}_n \cup \{N\}\}$. In particular $\chi_S \in \mathrm{Lin}\{\chi_T : T \in \mathcal{T}_r \cup \{N\}\}$.

In conclusion, let $\mathcal{B}^* := \mathcal{T}_r$. Then $\mathcal{B}^* \subseteq \mathcal{E}_v^{\mathbf{u}} \cap \mathcal{E}_v^{d-\mathbf{u}}$, $u_S \circ e_v(S, x) \leq \sum_{T \in \mathcal{B}^*} u_T \circ e_v(T, x)$, and $\chi_S \in \mathrm{Lin}\{\chi_T : T \in \mathcal{B}^* \cup \{N\}\}$. □

Remark 41 Notice that Lemmata 29, 30 and 31 hold if one changes $X^d(\mathcal{S}, t)$ and $u_{N \setminus S} \circ f_v(S, x)$ for $X(\mathcal{S}, t)$ and $u_S \circ e_v(S, x)$ respectively. The proofs are practically the same as the proofs of Lemmata 4, 5 and 6 in Dornai and Pintér (2024) respectively, hence we omit them.

Lemma 42 *Consider a game $v \in \mathcal{G}^{N,\mathcal{A}}$ with a utility function $\mathbf{u}$ such that v is $\mathbf{u}$-balanced. If $\mathbf{u}$-least-core$(v) \neq \mathbf{u}$-core(v) then $X(\hat{\mathcal{A}}, t_1) = X(\mathcal{E}_v^{\mathbf{u}} \cap \mathcal{E}_v^{d-\mathbf{u}}, t_1)$.*

Proof Since $\mathcal{E}_v^{\mathbf{u}} \cap \mathcal{E}_v^{d-\mathbf{u}} \subseteq \hat{\mathcal{A}}$ it holds that $X(\hat{\mathcal{A}}, t_1) \subseteq X(\mathcal{E}_v^{\mathbf{u}} \cap \mathcal{E}_v^{d-\mathbf{u}}, t_1)$, where t_1 is from Sect. 4.

Indirectly assume that $\exists x^1 \in X(\mathcal{E}_v^{\mathbf{u}} \cap \mathcal{E}_v^{d-\mathbf{u}}, t_1) \setminus X(\hat{\mathcal{A}}, t_1)$. This means that $\exists S \in \hat{\mathcal{A}}$ such that

$$u_S \circ e_v(S, x^1) > t_1 \,. \tag{18}$$

Let $\mathcal{S}_{x^1} := \{S \in \hat{\mathcal{A}} \colon u_S \circ e_v(S, x^1) > t_1\}$. Then for all $S \in \mathcal{S}_{x^1}$ it holds that $S \notin \mathcal{E}_v^{\mathbf{u}} \cap \mathcal{E}_v^{d-\mathbf{u}}$.

Let $x^* \in X(\hat{\mathcal{A}}, t_1)$ be the closest point of set $X(\hat{\mathcal{A}}, t_1)$ to the point x^1. It is clear that such x^* exists, since $X(\hat{\mathcal{A}}, t_1)$ is nonempty and closed by Lemma 30 and Remark 41.

Since $X(\mathcal{E}_v^{\mathbf{u}} \cap \mathcal{E}_v^{d-\mathbf{u}}, t_1)$ is a convex set, for every $\lambda \in [0, 1]$ it holds that $\lambda x^* + (1-\lambda)x^1 \in X(\mathcal{E}_v^{\mathbf{u}} \cap \mathcal{E}_v^{d-\mathbf{u}}, t_1)$.

By Lemma 40 we know that for each $S \in \mathcal{S}_{x^1}$ there exists $\mathcal{B}_S^* \subseteq \mathcal{E}_v^{\mathbf{u}} \cap \mathcal{E}_v^{d-\mathbf{u}}$ such that

$$u_S \circ e_v(S, x^*) \leq \sum_{T \in \mathcal{B}_S^*} u_T \circ e_v(T, x^*) \,. \tag{19}$$

Since $t_1 < 0$, for every $x \in X(\mathcal{E}_v^{\mathbf{u}} \cap \mathcal{E}_v^{d-\mathbf{u}}, t_1)$ and $T \in \mathcal{E}_v^{\mathbf{u}} \cap \mathcal{E}_v^{d-\mathbf{u}}$ it holds that $u_T \circ e_v(T, x) \leq t_1$. Therefore, for all $S \in \mathcal{S}_{x^1}$

$$u_S \circ e_v(S, x^*) \leq \sum_{T \in \mathcal{B}_S^*} u_T \circ e_v(T, x^*) \leq t_1 \,. \tag{20}$$

By Eqs. (18), (20) and the continuity of u_S, we have that for every $S \in \mathcal{S}_{x^1}$ there exists $\lambda_S \in [0, 1]$ such that

$$u_S \circ e_v\left(S, \lambda_S x^* + (1 - \lambda_S)x^1\right) = t_1 \,.$$

Let $S^1 \in \mathrm{argmin}_{S \in \mathcal{S}_{x^1}} ||x^* - (\lambda_S x^* + (1 - \lambda_S)x^1))||$, and let $x^2 := \lambda_{S^1} x^* + (1 - \lambda_{S^1})x^1$.

Since $t_1 < 0$ and $u_T \circ e_v(T, x^2) \leq t_1$, for all $T \in \mathcal{B}^*_{S^1}$ we have that

$$u_{S^1} \circ e_v\big(S^1, x^2\big) > \sum_{T \in \mathcal{B}^*_{S^1}} u_T \circ e_v\big(T, x^2\big) . \tag{21}$$

By the choice of x^2, Lemma 31 and Remark 41, for all $S \in \mathcal{S}_{x^1}$

$$u_S \circ e_v(S, x^2) \leq t_1 .$$

Since for all $S \in \hat{\mathcal{A}} \backslash \mathcal{S}_{x_1}$ it holds that $u_S \circ e_v(S, x^1) \leq t_1$. By Lemma 31 and Remark 41

$$u_S \circ e_v\big(S, x^2\big) \leq t_1 .$$

This means that for each $S \in \hat{\mathcal{A}}$ it holds that $u_S \circ e_v(S, x^2) \leq t_1$, hence $x^2 \in X(\hat{\mathcal{A}}, t_1)$.

Since $x^* \in X(\hat{\mathcal{A}}, t_1)$ is the closest point of set $X(\hat{\mathcal{A}}, t_1)$ to the point x^1, we have that $x^2 = x^*$. However, then (21) contradicts (19). □

Lemma 43 *Consider a utility function* $\mathbf{u}$*, a* $\mathbf{u}$*-balanced game* $v \in \mathcal{G}^{N,\mathcal{A}}$ *and* $t \in \mathbb{R}$*. If* $t < t_1$ *then* $X(\hat{\mathcal{A}}, t) = X(\mathcal{E}_v^{\mathbf{u}} \cap \mathcal{E}_v^{d-\mathbf{u}}, t) = \emptyset$*.*

Proof By the definition of t_1 (see Sect. 4), we have that $X(\hat{\mathcal{A}}, t) = \emptyset$. Moreover, we know that $X(\hat{\mathcal{A}}, t_1) \subseteq X(\mathcal{E}_v^{\mathbf{u}} \cap \mathcal{E}_v^{d-\mathbf{u}}, t_1)$.

By Lemma 29, Remark 41 and that $t < t_1$ we have that $X(\mathcal{E}_v^{\mathbf{u}} \cap \mathcal{E}_v^{d-\mathbf{u}}, t) \subseteq X(\mathcal{E}_v^{\mathbf{u}} \cap \mathcal{E}_v^{d-\mathbf{u}}, t_1)$. Furthermore, by Lemma 42 it holds that $X(\mathcal{E}_v^{\mathbf{u}} \cap \mathcal{E}_v^{d-\mathbf{u}}, t_1) = X(\hat{\mathcal{A}}, t_1) =$ $\mathbf{u}$-least-core(v). Therefore,

$$X(\mathcal{E}_v^{\mathbf{u}} \cap \mathcal{E}_v^{d-\mathbf{u}}, t) \subseteq X(\mathcal{E}_v^{\mathbf{u}} \cap \mathcal{E}_v^{d-\mathbf{u}}, t_1) = X(\hat{\mathcal{A}}, t_1) = \mathbf{u}\text{-least-core}(v) .$$

Then by Lemma 40 and by the non-positivity of t for every $x \in X(\mathcal{E}_v^{\mathbf{u}} \cap \mathcal{E}_v^{d-\mathbf{u}}, t)$ and $S \in \hat{\mathcal{A}} \backslash \mathcal{E}_v^{\mathbf{u}} \cap \mathcal{E}_v^{d-\mathbf{u}}$ it holds that $\exists \mathcal{B} \subseteq \mathcal{E}_v^{\mathbf{u}} \cap \mathcal{E}_v^{d-\mathbf{u}}$, such that

$$u_S \circ e_v(S, x) \leq \sum_{T \in \mathcal{B}} u_T \circ e_v(T, x) \leq t .$$

Then $x \in X(\hat{\mathcal{A}}, t)$, that is, $X(\mathcal{E}_v^{\mathbf{u}} \cap \mathcal{E}_v^{d-\mathbf{u}}, t) \subseteq X(\hat{\mathcal{A}}, t)$. Summing up, we can can conclude that $X(\mathcal{E}_v^{\mathbf{u}} \cap \mathcal{E}_v^{d-\mathbf{u}}, t) = X(\hat{\mathcal{A}}, t) = \emptyset$. □

10 Summary

See Table 1.

Table 1 Summary

Primal game	Dual game	Coincide
Game: $v(S)$	Dual game: $v^*(S) = v(N) - v(N \setminus S)$	✗
Feasible coalitions	Complementer set of feasible coalitions	✗
$\mathcal{A} \subseteq \mathcal{P}(N)$, where $\emptyset, N \in \mathcal{A}$	$\mathcal{A}^{\complement} := \{N \setminus S \colon S \in \mathcal{A}\}$	
excess: $e_v(S, x) = v(S) - x(S)$	satisfaction: $f_{v^*}(S, x) = x(S) - v^*(S)$	✗
$E_v(x) = (\ldots \geq e_v(S, x) \geq \ldots \colon S \in \hat{\mathcal{A}})$	$F_{v^*}(x) = (\ldots \geq f_{v^*}(S, x) \geq \ldots \colon S \in \hat{\mathcal{A}}^{\complement})$	✗
$\text{core}(v) = \{x \in PI(v) \colon x(S) \geq v(S),\ \forall S \in \hat{\mathcal{A}}\}$	$\text{anti-core}(v^*) = \{x \in PI(v^*) \colon x(S) \leq v^*(S),\ \forall S \in \hat{\mathcal{A}}^{\complement}\}$	✓
ε-core: $\{x \in PI(v) \colon \max_{S \in \hat{\mathcal{A}}} e_v(S, x) \leq \varepsilon\}$	ε-anti-core: $\{x \in PI(v^*) \colon \max_{S \in \hat{\mathcal{A}}^{\complement}} f_{v^*}(S, x) \leq \varepsilon\}$	✓
least-core: ε^*-core, where $\varepsilon^* = \min_{\text{core}_\varepsilon(v) \neq \emptyset} \varepsilon$	least-anti-core: ε^*-anti-core, where $\varepsilon^* = \min_{\text{anti-core}_\varepsilon(v) \neq \emptyset} \varepsilon$	✓
nucleolus	anti-nucleolus: anti-$N(v^*) =$	✓
$N(v) = \{x \in I(v) \colon E_v(x) \leq_L E_v(y),\ \forall y \in I(v)\}$	$\{x \in \text{anti-}I(v^*) \colon F_{v^*}(x) \leq_L F_{v^*}(y),\ \forall y \in \text{anti-}I(v^*)\}$	
prenucleolus	anti-prenucleolus	✓
$PN(v) = \{x \in PI(v) \colon E_v(x) \leq_L E_v(y),\ \forall y \in PI(v)\}$	anti-$PN(v^*) = \{x \in PI(v^*) \colon F_{v^*}(x) \leq_L F_{v^*}(y),\ \forall y \in PI(v^*)\}$	
essential coalitions ($\mathcal{A} = \mathcal{P}(N)$): $S \in \hat{\mathcal{P}}(N)$, if	anti-essential coalitions ($\mathcal{A} = \mathcal{P}(N)$): $S \in \hat{\mathcal{P}}(N)$, if	✗
$\lvert S \rvert = 1$ or $v(S) > \max_{\mathcal{B} \in \mathcal{D}_S} \sum_{T \in \mathcal{B}} v(T)$	$\lvert S \rvert = 1$ or $v^*(S) < \min_{\mathcal{B} \in \mathcal{D}_S} \sum_{T \in \mathcal{B}} v^*(T)$	
essential coalitions: $S \in \hat{\mathcal{A}}$, if	anti-essential coalitions: $S \in \hat{\mathcal{A}}^{\complement}$, if	✗
$\mathcal{D}_S^{\hat{\mathcal{A}}} = \emptyset$ or $v(S) > \max_{\mathcal{B} \in \mathcal{D}_S} \sum_{T \in \mathcal{B}} v(T)$	$\mathcal{D}_S^{\hat{\mathcal{A}}^{\complement}} = \emptyset$ or $v^*(S) < \min_{\mathcal{B} \in \mathcal{D}_S} \sum_{T \in \mathcal{B}} v^*(T)$	
$\mathbf{u}$-excess: $u_S \circ e_v(S, x)$	$\mathbf{u}$-satisfaction: $u_{N \setminus S} \circ f_{v^*}(S, x)$	✗
$E_v^{\mathbf{u}}(x) = (\cdots \geq u_S \circ e_v(S, x) \geq \ldots \colon S \in \hat{\mathcal{A}})$	$F_{v^*}^{\mathbf{u}}(x) = (\cdots \geq u_{N \setminus S} \circ f_{v^*}(S, x) \geq \ldots \colon S \in \hat{\mathcal{A}}^{\complement})$	✗
$\mathbf{u}$-core$(v) =$	$\mathbf{u}$-anti-core$(v^*) =$	✓
$\{x \in PI(v) \colon u_S \circ e_v(S, x) \leq 0,\ \forall S \in \hat{\mathcal{A}}\}$	$\{x \in PI(v^*) \colon u_{N \setminus S} \circ f_{v^*}(S, x) \leq 0,\ \forall S \in \hat{\mathcal{A}}^{\complement}\}$	
$\mathbf{u}$-nucleolus: $N_{\mathbf{u}}(v) =$	$\mathbf{u}$-anti-nucleolus: anti-$N_{\mathbf{u}}(v^*) =$	✓
$\{x \in \mathbf{u}\text{-}I(v) \colon E_v^{\mathbf{u}}(x) \leq_L E_v^{\mathbf{u}}(y),\ \forall y \in \mathbf{u}\text{-}I(v)\}$	$\{x \in \mathbf{u}\text{-anti-}I(v^*) \colon F_{v^*}^{\mathbf{u}}(x) \leq_L F_{v^*}^{\mathbf{u}}(y),\ \forall y \in \mathbf{u}\text{-anti-}I(v^*)\}$	
$\mathbf{u}$-prenucleolus	$\mathbf{u}$-anti-prenucleolus	✓
$PN_{\mathbf{u}}(v) = \{x \in PI(v) \colon E_v^{\mathbf{u}}(x) \leq_L E_v^{\mathbf{u}}(y),\ \forall y \in PI(v)\}$	anti-$PN_{\mathbf{u}}(v^*) = \{x \in PI(v^*) \colon F_{v^*}^{\mathbf{u}}(x) \leq_L F_{v^*}^{\mathbf{u}}(y),\ \forall y \in PI(v^*)\}$	

Table 1 continued

Primal game	Dual game	Coincide
u-essential coalitions: $S \in \hat{\mathcal{A}}$ if $\mathcal{D}_S^{\hat{\mathcal{A}}} = \emptyset$ or if there exists $x \in$ **u**-least-core(v), such that $u_S \circ e_v(S, x) > \max_{\mathcal{B} \in \mathcal{D}_S^{\hat{\mathcal{A}}}} \sum_{T \in \mathcal{B}} u_T \circ e_v(T, x)$	**u**-anti-essential coalitions: $S \in \hat{\mathcal{A}}^{\complement}$ if $\mathcal{D}_S^{\hat{\mathcal{A}}^{\complement}} = \emptyset$ or if there exists $x \in$ **u**-least-anti-core(v^*), such that $u_{N \setminus S} \circ f_{v^*}(S, x) > \max_{\mathcal{B} \in \mathcal{D}_S^{\hat{\mathcal{A}}^{\complement}}} \sum_{T \in \mathcal{B}} u_{N \setminus T} \circ f_{v^*}(T, x)$	✗

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Declarations

Conflict of interest Both authors declare that they have no Conflict of interest.

Ethical Approval This article does not contain any studies with human participants or animals performed by any of the authors.

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