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RESEARCH ARTICLE

Projects with uncertain requirements and deadlines

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Abstract
I analyze a dynamic moral hazard problem in teams with imperfect monitoring in continuous time. In the model, players work together to achieve a breakthrough in a project while facing a deadline. The target effort needed to achieve a breakthrough is unknown, but the players have a common prior about its distribution. I characterize the equilibrium and the effort path that maximizes the team's welfare for general distributions of this target effort and show that three effects are at work: free-riding (i.e., working less), last-minute rush (i.e., working later), and a past-effort effect (i.e., working more if others worked more in the past). This past-effort effect increases or decreases the amount of work players put into the project, depending on the type of project faced.

Keywords Moral hazard in teams · Public good provision · Procrastination · Projects · Strategic experimentation

JEL Classification C72 · C73 · D81 · D82 · H41

1 Introduction

In this paper, I analyze a dynamic moral hazard problem in teams that work on a project with imperfect monitoring in continuous time. In addition to a deadline, the team faces uncertain objectives, meaning that the team members do not know the target effort level, which is the level of effort required to complete the project. The novelty of the work presented in this paper is related to the range of possible forms of uncertainty the team members face regarding this required target effort level. The main results reported in the paper demonstrate that different forms of this uncertainty lead qualitatively to very different forms of team behavior. The crucial determinant of equilibrium behavior is whether or not the implied hazard rate of the target effort level distribution increases or decreases in terms of effort. An increasing hazard rate is a

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probability of reaching the target effort that increases with cumulative past investment of effort. This rate implies the existence of a past-effort effect that at least partially counteracts the free-riding effect. In contrast, a decreasing hazard rate is a probability of a breakthrough that decreases with the cumulative past investment of effort. This rate exacerbates the free-riding effect. An increasing (or decreasing) hazard rate of the target effort distribution has similar behavioral implications as if the team members' effort choices were strategic complements (substitutes). Regardless of this hazard rate, the equilibrium effort always exhibits a last-minute rush: The effort increases just before the deadline is reached.

I derive these results in a model with the following features:

- **public benefits**, which are realized upon completion of a project in the form of a lump-sum payment,
- **private costs**, which are assumed to be quadratic in the instantaneous effort,
- an **unknown threshold for success** or uncertain requirements with a commonly known distribution, which I will call the *target effort distribution*,
- **unobservable efforts**, so that only the player knows their effort,
- and a **deadline**, after which the team can no longer complete the project.

In the model, the players exert effort over time either until the deadline is reached or the project is successful (before the deadline is reached). While doing so, they only know if they have been successful yet. A project is considered to be successful when an unknown target effort threshold is reached, which is perceived as random by the players. Projects in this model are described by the assumed distribution of the target effort.

This *target effort distribution* can apply to many different types of projects. For example, these could be projects in which players continuously learn something about the project's target effort while they are trying to complete it or projects in which only the current investment of effort influences the probability of success. The latter would be modeled by using an exponential target effort distribution.[1]

In the model, players learn about the target effort level not only from the effort they exert but also from the effort of others. Therefore, this model is a model of strategic experimentation (see, e.g., Bolton and Harris (1999), or Keller and Rady (2010)). More specifically, it is a good news model following Keller et al. (2005) (see Hoelzemann and Klein (2021) for an experimental test of the predictions of the good news model).

However, it differs from most models of strategic experimentation in three aspects: First, in most of the literature, news-arrival processes are memoryless. Notable exceptions to this are Boyarchenko (2018) or Boyarchenko (2020). This paper presents a very general news-arrival process (or, as referred to in this paper, a more general target effort distribution). The general target effort distribution allows the chance of success to depend on the past investment of effort, both negatively as well as positively. Second, players have not only information externalities but also very strong payoff externalities: If any player achieves a breakthrough, the game ends, and each player receives the same payoff right away. Finally, players do not observe other players'

[1] For more explanations and examples of modeling projects with the target effort distribution, see Appendix C.

actions, whereas experimentation outcomes are perfectly observable, as in Bonatti and Hörner (2011).

Due to the similar information structure, Bonatti and Hörner (2011) is closely related to the model described in this article. The two main differences are that they assume linear costs (instead of quadratic costs, as in this article) and that their news-arrival process is a Poisson process with a chance of failure. Their news-arrival process is equivalent to a target effort that is incompletely exponentially distributed, namely exponentially distributed with a probability mass point at ∞. Therefore, both the free-riding and the past-effort effect (which they call procrastination) are present in both models. In Bonatti and Hörner's (2011) model, the past-effort effect is negative, as the incomplete exponential distribution has a decreasing hazard rate. However, although they are very similar, the equilibrium behavior in Bonatti and Hörner (2011) does not include a last-minute rush, as convex costs drive this effect.

Another closely related paper is Georgiadis (2017). In this author's model, there is uncertainty about how effort affects the provision of the public good but no uncertainty about the target effort. The author assumes that effort affects the drift of a standard Brownian motion towards this commonly known target effort, which, in this paper, could be replicated with a target effort distribution with an increasing hazard rate. They also show that behavior under infrequent monitoring conditions can lead to first-best behavior.[2] However, even in the case without monitoring, we do not observe a last-minute rush in Georgiadis (2017). This is because, unlike in my model, rewards are not paid out as soon as the target effort is reached but only once the deadline has been reached.

The past-effort effect described in this paper operates by influencing other players' beliefs about the project's likelihood of success. A related mechanism appears in Dong (2023), where a better-informed player exerts more effort, which boosts the optimism of a less-informed teammate. Similarly, Cetemen (2024) examines a model where players, each privately informed about a common productivity parameter, experience a mutual encouragement effect in equilibrium, which counteracts free-riding incentives. However, unlike the models in Dong (2023) and Cetemen (2024), where the positive effect arises due to asymmetric information and its transmission, this paper assumes symmetric information, with all players sharing the same prior and updating their beliefs at the same pace in equilibrium.

Cetemen et al. (2020) identify a mutual encouragement effect driven by uncertainty-induced optimism, where interim performance feedback enables explicit belief manipulation among team members. This effect motivates higher effort, counteracting free-riding similarly to the increasing hazard rate case in this paper. The mechanism behind their encouragement via belief manipulation closely resembles the past-effort effect discussed in this paper. However, an important distinction is that in their model, belief manipulation occurs explicitly through observable interim feedback. In this paper, optimism arises implicitly through accumulated effort alone and is, therefore, an inherent property of projects with an increasing hazard rate of the target effort dis-

[2] The benchmark case without monitoring (Georgiadis 2017) is very similar to this model with a known effort threshold, as can be seen in Section 3.3.

tribution. Moreover, unlike Cetemen et al. (2020), this paper also considers scenarios where additional effort decreases optimism, generating a discouragement effect.

Finally, Chen (2020) analyzes a revision game where players choose effort levels while receiving a steady stream of payoffs. Players minimize effort while trying to avoid falling below an unknown threshold, incurring a lump-sum cost, and forfeiting future benefits if they do. While both models involve experimentation to uncover an unknown threshold, in Chen's (2020) framework, no past-effort effect occurs. This difference arises from the nature of the experimentation and the underlying distribution of the unknown threshold: their model features conclusive bad news and an increasing hazard of failure, making experimentation more and more risky, and players always stop experimenting at some point in time. In contrast, this paper presents a good-news model where, due to the assumption of quadratic costs, experimentation always occurs, and past effort can influence beliefs positively.

The paper is organized as follows: In Section 2, I describe the model. In Section 3, I derive the optimal effort path for the non-cooperative and the welfare-maximizing case and show that three effects are at work: free-riding, last-minute rush, and the past-effort effect. Furthermore, I identify a series of qualitative properties of the Nash equilibrium effort function and their behavioral interpretations. Then, I examine three special cases of the model in more detail: projects with a constant hazard rate of the target effort distribution, behavior of patient players, and behavior under certain objectives. Finally, in Section 5 I conclude the paper.

2 The model

There are n symmetric players, the team members. Time t is continuous and runs from time 0 to an exogenously given deadline T with $0 < T < \infty$. Each player $i \in \{1, ..., n\}$ chooses a level of instantaneous effort as a function of time, denoted by $a_i : [0, T] \to \mathbb{R}_+$. The set of all such measurable functions constitutes the strategy space for each player. Therefore, it is implicitly assumed that players do not observe each other's effort choices. The total aggregate cumulative effort up to time $t \in [0, T]$ is given by $y(t) = \int_0^t \sum_{i=1}^{n} a_i(s)\mathrm{d}s$. Note that $y(t)$ is a weakly increasing function over t.

There is an uncertain and exogenously given amount of target effort X that is required for the project to succeed. Ex ante, at time 0, X is drawn from a distribution over $\mathbb{R}_+$ with density function f, with $f(x)$ positive and continuously differentiable and bounded above for all $x \in \mathbb{R}_+$.

At the moment $t \in [0, T]$, when the total aggregate effort $y(t)$ achieves the realized target effort x, each team member receives a payoff of 1. If the aggregate effort by the deadline $y(T)$ falls short of the realized target effort x, then all players receive a benchmark payoff of zero. In addition, players have quadratic instantaneous costs of effort $ca_i(t)^2$, with $c > 0$. Finally, players evaluate streams of payoffs with an exponential discounted utility and with a given discount rate $r \geq 0$ (discount factor e^{-r}).

For each realization x of the target effort and for a given total aggregate effort function y, let $t^*(x) = \min\{t \in [0, T] \mid y(t) \geq x\}$ denotes the time when this target effort is achieved.

Formally, a player i's utility, given their strategy a_i, aggregate effort function y (which depends on own and the team members' strategies), and realized target effort x are then given by

$$V(a_i, y, x) = \begin{cases} re^{-rt^*(x)} - r \int\limits_0^{t^*(x)} e^{-rs} ca_i(s)^2 \mathrm{d}s & \text{if } t^*(x) \leq T \\ -r \int\limits_0^T e^{-rs} ca_i(s)^2 \mathrm{d}s & \text{if } t^*(x) > T. \end{cases} \tag{1}$$

The ex ante expected utility to player i given the strategy profile $a = (a_1, ..., a_n)$ is then given by

$$\mathbb{E}_X V\left(a_i, \sum_{j=1}^n a_j, X\right) = \int_{x=0}^{\infty} V\left(a_i, \sum_{j=1}^n a_j, x\right) f(x)\mathrm{d}x. \tag{2}$$

Taking this expectation and denoting the hazard rate of the distribution F by $h(y(t)) = \frac{f(y(t))}{1-F(y(t))}$, we obtain the following useful representation of the expected utility of a player as a function of the instantaneous effort function profile a and the cumulative aggregate effort function y. Its proof, using a change of variable in the integral, is given in Appendix B.1.

Lemma 1 (Expected utility) *The expected utility of a player i is*

$$\mathbb{E}_X V(a_i, y, X) = r \int\limits_0^T e^{-rt}(1 - F(y(t))) \left(h(y(t)) \sum_{j=1}^n a_j(t) - ca_i(t)^2\right) \mathrm{d}t.$$

By design, if all other factors are fixed, an increase in instantaneous effort $a_i(t)$ at some point in time t increases the chance to succeed over t and increases the instantaneous costs $ca_i(t)^2$. Furthermore, again if all other factors are fixed, an increase in the instantaneous effort of any of the other players $a_j(t)$, $j \neq i$, at time t increases the chance to succeed at time t without affecting the given player's costs.

However, the major novelty of the work presented in this paper, and which is reflected in the expected utility given in Lemma 1, is the effect of $h(y)$, the hazard rate of the target effort distribution F, on the player's payoffs. The current investment of effort not only influences the current likelihood of a breakthrough but also the effectiveness of the future effort (invested by all players) with respect to achieving this breakthrough. When the hazard rate is increasing, the current investment of effort makes future effort more effective, whereas a decreasing hazard rate has the opposite effect.

3 Results

The following theorem is the main technical result of this work. It shows that the game always has a unique symmetric Nash equilibrium and characterizes the symmetric Nash equilibrium effort function in terms of a differential equation.

Theorem 1 (Equilibrium Effort) *A unique symmetric Nash equilibrium in pure strategies exists in which, along the equilibrium path, the individual effort of a player i evolves according to*

$$\dot{a}_i(t) = \frac{2n-1}{2} h(y(t)) a_i(t)^2 + r a_i(t) - \frac{r}{2c} h(y(t)) \tag{3}$$

and reaches $a_i(T) = \frac{1}{2c} h\left(y(T)\right)$ by the deadline T.

The detailed proof is given in Appendix B.2, but I provide the main arguments here.

I use the Pontryagin maximum principle to identify a candidate solution for the optimal control problem, which is the problem of finding any player i's best response as a function of any aggregate opponent effort function $\sum_{j \neq i} a_j(t)$. This candidate solution is described in terms of a second-order boundary value problem. Using the symmetry (where all players use the same effort function), I then derive the induced boundary value problem as stated in Theorem 1 and show that this problem only allows exactly one solution.

What remains to be shown is that this solution is indeed an equilibrium and that no other symmetric equilibria exist.

To show that this candidate solution is indeed an equilibrium, "standard" conditions (i.e., global concavity or Arrow/Mangasarian sufficiency conditions) cannot be used, because they are not satisfied for all general target effort distributions F.

Using a result from optimal control theory, however, it can be shown that every maximizer of the best response optimal control problem has to be a Pontryagin extremal, meaning that it satisfies the Pontryagin maximum principle. The solution I identified above is, therefore, the only candidate for a symmetric equilibrium effort function.

Finally, applying a Tonelli-type existence theorem allows me to show that the general best response effort problem always has a global maximizer. The underlying reason for this is that boundary arcs (i.e., solutions at the boundary of the space of all effort functions) are precluded by the assumed boundedness and continuity of the hazard rate $h(y)$, which implies that the value of additional effort is always bounded, while additional costs are not.

Therefore, the solution to the boundary value problem given above is indeed the unique symmetric Nash equilibrium effort function.

Remark (Equilibrium concept) The game is, in principle, a dynamic one: Players make decisions as a function of time and based on their own past decisions. But because no player ever learns anything about any of the other players' decisions as long as the game continues, Nash equilibria in this game are trivially and sequentially rational.

Remark (Asymmetric equilibria) While there are many asymmetric equilibria, I only consider symmetric equilibria in this paper. Given the symmetric setting, restricting our attention to symmetric equilibria seems natural. Furthermore, every welfare-maximizing solution to the problem is also symmetric due to the assumption of quadratic costs (see Appendix A).

In the section below, I use Theorem 1, which is hard to interpret on its own, to identify a series of qualitative properties of the unique symmetric Nash equilibrium effort function and their behavioral interpretations and economic consequences. The results are presented in a series of propositions, with full proofs provided in the appendix.

Proposition 1 (Non-decreasing Effort) *The equilibrium effort $a_i(t)$ of a player i is a non-decreasing function of time t*

(a) if the hazard rate $h(y(t))$ is a non-decreasing function of $y(t)$ or
(b) if the discount rate r is equal to 0.

Thus, increasing effort over time is, in this model, the expected equilibrium behavior. It is only possible to have decreasing effort over some period of time if the hazard rate $h(y)$ decreases at some point and players discount future payoffs .

To understand the intuition that underlies this result, note that a positive discount rate r (i.e., some degree of impatience) creates an incentive for front-loading effort to achieve the breakthrough (and thus the reward) earlier. Similarly, a decreasing hazard rate means that the chances of a breakthrough decrease as the past aggregate effort decreases. Thus, front-loading the effort makes completing the project less likely; this, in turn, can lead to a decrease in effort over at least some interval of time.

The next proposition provides a comparison between the equilibrium effort function and the welfare-maximizing solution. The *welfare-maximizing effort function* is the function $a : [0, T] \to \mathbb{R}_+$ (in the space of all such functions denoted by $\mathcal{A}$) that solves the problem $\max_{a\in\mathcal{A}} \mathbb{E}_X V(a, na, X)$. The welfare-maximizing effort function is derived in ways that are similar to the derivation of the equilibrium effort function, as a boundary value problem, which does also not allow for direct interpretation. Its characterization and the proof thereof are given in Appendix A.

Proposition 2 (Equilibrium vs. welfare-maximizing effort function) *If there are two or more players, then for every positive, continuously differentiable, and bounded target effort distribution density $f(y(t))$,*

(a) the cumulative welfare maximizing effort function is at least as high as the cumulative equilibrium effort function at every point over time t,
(b) the welfare-maximizing effort function is strictly higher than the equilibrium effort function at $t = 0$ and the deadline $t = T$ for non-decreasing hazard rates $h(y(t))$.

In particular, the players never invest an excessive amount of effort compared to the welfare-maximizing solution; therefore, the equilibrium effort is always inefficiently low.

Definition 1 (Last-minute rush) A player i's effort function $a_i(t)$ exhibits a *last-minute rush* if point in time $\hat{T} < T$ exist such that for all t and t' such that $\hat{T} \le t < t' \le T$, we have $a_i(t) < a_i(t')$.

Proposition 3 (Last-minute rush) *For every positive, continuously differentiable, and bounded target-effort distribution density f, both the unique symmetric equilibrium effort function and the welfare-maximizing effort function exhibit a last-minute rush.*

A last-minute rush is present for any target-effort distribution, even when the hazard rate decreases sharply near the deadline. This is due to the following interaction between convex costs and uncertain objectives: Close to the deadline, the effect of discounting is negligible, and the effectiveness of effort is essentially constant because $f(x)$ is continuous and bounded. Due to the convex costs of effort, effort is optimally distributed evenly over the short remaining time.

As time passes without the target being reached, the time remaining to achieve a breakthrough is shorter. Now two effects occur: First, players would still like to put in the same total amount of effort that gives them a reasonable expectation that a breakthrough can be achieved. These incentives have hardly changed. Second, as the same total amount of effort now has to be distributed over a shorter amount of time, the marginal cost of instantaneous effort increases (due to convex costs). The latter effect is a second-order effect induced by the first effect and is, therefore, always clearly outweighed by the former. For this reason, close to the deadline, the players (in equilibrium as well as in the welfare-maximizing solution) put in a higher amount of instantaneous effort as time passes.

Apart from the free-riding effect, which is observed in all such models, there is another strategic effect: the past-effort effect. This effect stems from the fact that the effort a player invests in the project right now might affect the effort of every other player in the future.

Definition 2 (Past-effort effect) A player i's effort function $a_i(t)$ exhibits a *positive past-effort effect* at time t if the best response of this player involves an increasing cumulative past investment of effort $y(t)$. It exhibits a *negative past-effort effect* if the best response involves a decreasing $y(t)$.

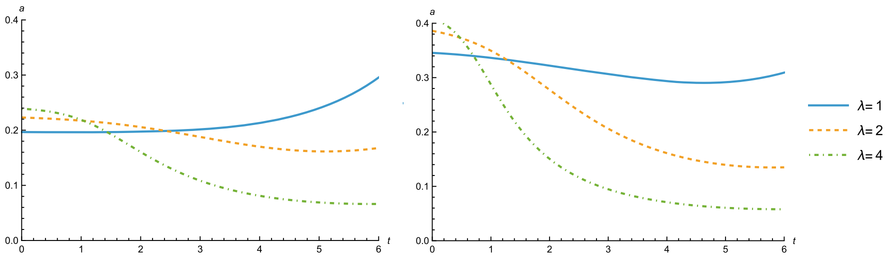
Proposition 4 (Past-effort effect in equilibrium) *If the hazard rate of the target effort distribution $h(y(t))$ increases (is constant, or decreasing) at time t, the players' equilibrium effort functions exhibit a positive (no, or a negative) past-effort effect at t.*

This result is proved in Appendix B.8.

The past-effort effect in this model is directly linked to different types of projects as characterized by the hazard rate. If a positive past-effort effect occurs, any amount of effort a player invests now increases the effectiveness and, therefore, everyone's effort in the future. Therefore, players are incentivized to work earlier rather than later. However, in the event of a decreasing hazard rate, this effect is a negative past-effort effect: If a player invests more effort now and the players do not succeed, they have a lower belief in the chance of succeeding in the future. They will, therefore, invest less effort, and especially in the earlier periods.

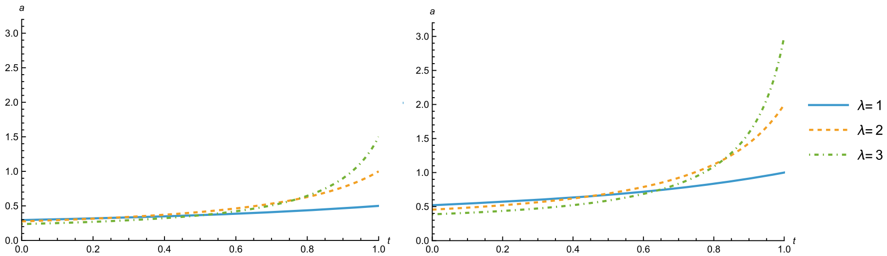
In Figure 1, you can find the equilibrium and welfare-maximizing effort functions for three examples with decreasing hazard rates, and in Figure 2, three examples with a constant hazard rate.[3]

[3] All examples assume 2 players, $c = 1$, and $r = 0.1$. The target effort distribution in Figure 1 is an incomplete exponential distribution with different rate parameters λ and a probability mass point of 0.05



(a) Equilibrium effort function (b) Welfare-maximizing effort function

Fig. 1 Examples: Decreasing hazard rate



(a) Equilibrium effort function (b) Welfare-maximizing effort function

Fig. 2 Examples: Constant hazard rate

These examples illustrate several of the properties discussed above: While all non-decreasing hazard rate projects exhibit increasing effort over time, the decreasing hazard rate examples demonstrate the negative past-effort effect — players reduce their effort the more effort has been invested. This effect is eventually outweighed by the last-minute rush, which drives effort up as the deadline approaches. The decreasing hazard rate examples with rate parameters $\lambda = 2$ and $\lambda = 3$ further show that the welfare-maximizing effort function can lie below the equilibrium effort function.

Remark (The past-effort effect without a deadline) Unlike the last-minute rush, the past-effort effect is not a consequence of deadlines. When $T \to \infty$, the last-minute rush disappears, and the hazard rate remains one of the main drivers of equilibrium behavior, as long as players have a positive discount rate $r > 0$.

From Section 3.1, we know that for a constant hazard rate, each player exerts constant instantaneous effort at every point in time t. Moreover, the proof of Proposition 4 in Appendix B.8 shows that the proposition still holds when $T \to \infty$. Therefore, if the hazard rate of the target effort distribution $h(y(t))$ is increasing (decreasing) everywhere, the instantaneous effort is increasing (decreasing) everywhere.

Remark (The past-effort effect and the encouragement effect) In an earlier version of this paper, the past-effort effect was called the encouragement effect. However,

Footnote 3 continued
at ∞, i.e., a 5% chance that the project cannot be completed. The three examples in Figure 2 use an exponential distribution with different rate parameters λ.

the literature on strategic experimentation (see Bolton and Harris (1999), Keller et al. (2005), and Hörner et al. (2022)) defines the encouragement effect differently, and it is, as Hörner et al. (2022) have shown, a direct consequence of Markovian strategies. Therefore, the encouragement effect, as described in the strategic experimentation literature, does not occur in this work.

The next three sections examine special cases of the model in more detail: projects where success depends only on current effort — i.e., with a constant hazard rate of the target effort distribution; behavior of patient players, corresponding to the model without discounting; and behavior under certain objectives, where the threshold is common knowledge.

3.1 Projects with a constant hazard rate

We now examine the case of projects with a constant hazard rates of the target effort distribution, i.e., the exponential distribution $F(y) = 1 - e^{-\lambda y}$ with rate parameter $\lambda > 0$. This distribution captures the idea that the chance of success depends only on the current investment of effort, with past effort having no effect.

Under the exponential distribution, the solution from Theorem 1 simplifies to:

$$\begin{aligned} \dot{a}_i &= \frac{2n-1}{2}\lambda a_i^2 + r a_i - \frac{r}{2c}\lambda \\ a_i(T) &= \frac{\lambda}{2c}, \end{aligned} \tag{4}$$

which has the following explicit solution:

$$a_i(t) = \frac{\lambda\left(e^{\frac{(t-T)\sqrt{cr\left(cr+\lambda^2(2n-1)\right)}}{c}}+1\right)\left(\lambda^2(2n-1)+cr-\sqrt{cr\left(cr+\lambda^2(2n-1)\right)}\right)+2\sqrt{cr\left(cr+\lambda^2(2n-1)\right)}}{2c\left(cr+\lambda^2(2n-1)\right)\left(\left(1-\frac{2cr+\lambda^2(2n-1)}{2\sqrt{cr\left(cr+\lambda^2(2n-1)\right)}}\right)e^{\frac{(t-T)\sqrt{cr\left(cr+\lambda^2(2n-1)\right)}}{c}}+\frac{2cr+\lambda^2(2n-1)}{2\sqrt{cr\left(cr+\lambda^2(2n-1)\right)}}+1\right)}.$$

The welfare-maximizing effort a has the following explicit solution:

$$a(t) = \frac{\lambda n e^{-T\sqrt{r\left(\frac{\lambda^2 n}{c}+r\right)}}\left(\left(r-\sqrt{r\left(\frac{\lambda^2 n}{c}+r\right)}\right)e^{t\sqrt{r\left(\frac{\lambda^2 n}{c}+r\right)}}-\left(\sqrt{r\left(\frac{\lambda^2 n}{c}+r\right)}+r\right)e^{T\sqrt{r\left(\frac{\lambda^2 n}{c}+r\right)}}\right)}{\lambda^2 n\left(e^{(t-T)\sqrt{r\left(\frac{\lambda^2 n}{c}+r\right)}}-1\right)+2cr\left(e^{(t-T)\sqrt{r\left(\frac{\lambda^2 n}{c}+r\right)}}-1\right)-2\sqrt{cr\left(cr+\lambda^2 n\right)}\left(e^{(t-T)\sqrt{r\left(\frac{\lambda^2 n}{c}+r\right)}}+1\right)}.$$

Given these results, some observations about behavior in projects with a constant hazard rate of the target effort distribution can be made: Individual (both instantaneous and cumulative) effort in the equilibrium and the welfare-maximizing solution is decreasing in the number of players n, whereas total (instantaneous and cumulative) effort is increasing in n. As established in Proposition 1, the equilibrium effort $a_i(t)$ of a player i is an increasing function of time t. The same holds for the welfare-maximizing effort function.

The instantaneous effort exerted right before the deadline depends only on the marginal costs c and the rate parameter λ; it is independent of both the number of players n and the discount rate r. At the deadline, players no longer need to consider future consequences of their actions. Whether they succeed or fail, the game ends. Hence, players' optimization problems reduce to a static problem. Since the expected marginal benefit of effort under a constant hazard rate do not depends on others' effort, each player exerts effort such that their marginal cost of effort ($\frac{1}{2c}$) equals the marginal expected benefit (λ).

Finally, in the absence of a deadline (i.e., $T \to \infty$), effort in both the equilibrium and the welfare-maximizing solution remains constant over time. To illustrate this, consider comparing the problem at $t_0 = 0$ and any other time $t > 0$: the only differences are the elapsed time t and accumulated effort $y(t)$. Since past time does not influence current decisions and $y(t)$ does not affect beliefs under a constant hazard rate, the problem faced at t_0 and at t is the same. Given a unique best response, continuation strategies must therefore be identical for all t, implying constant effort over time.

3.2 Patient players

So far, we have focused on the problem in which players are discounting. However, when players are patient ($r = 0$), the solution from Theorem 1 simplifies to

$$\begin{aligned} \dot{a}_i &= \frac{2n-1}{2} h(y) a_i^2, \\ a(T) &= \frac{1}{2c} h(y(T)). \end{aligned} \tag{5}$$

Proposition 5 *The equilibrium effort function $a_i(t)$ of patient players (i.e., $r = 0$) is increasing in t. It is concave for decreasing hazard rates, and convex for increasing hazard rates.*

Proposition 5 follows directly from Equation (5), as $h(y)$ and a are always positive.

For $r = 0$, players have no incentive to complete the project early, and thus no incentive to exert effort earlier. Due to quadratic costs, the optimal way to distribute a given cumulative effort over time is to spread it evenly across the entire period $[0, T]$. At the deadline, players face a static problem and choose their effort level a such that the marginal cost of effort, $\frac{1}{2c}$, equals the marginal expected benefit, $h(y(T))$. Before the deadline ($t < T$), players also consider contingent payoffs: If they succeed now, they forgo potential future payoffs that could be achieved at lower cost–both because other players will continue to exert effort and because spreading effort more evenly reduces total cost. As a result, effort for patient players increases in t.

3.3 Projects with certain objectives

We now examine the model without uncertain objectives, i.e., when the realized target effort x is common knowledge. As this case closely resembles the model of Georgiadis (2017) without monitoring, we set $c = \frac{1}{2}$ as in his paper to facilitate comparison.

The key difference between the two models is the timing of rewards: in our model, rewards are paid out as soon as the project is completed, whereas in Georgiadis (2017), players are paid only after a monitoring event — in this case at the deadline T. Thus, there is no incentive to complete the project early in his model, while in ours, players are incentivized to do so.

Unfortunately, the assumptions on f no longer hold, as there is a probability mass point of 1 at the known threshold, and $h(y) = 0$ otherwise. Consequently, most of the earlier results, including those on existence and uniqueness, no longer apply. As shown by Georgiadis (2017), multiple equilibria exist, including those in which no one exerts effort. We therefore focus on symmetric project-completing equilibria (i.e., $y(t) = x$ for some $t \leq T$) in the following proposition.

Proposition 6 (Known target effort)
For a known target effort x and profitable projects (i.e., the expected value for each player is positive, i.e., $rx^2 < 2n^2$), a symmetric project-completing equilibrium

1. *for $r = 0$ is given by constant effort $a_i = \frac{1}{n}\frac{x}{T}$ exerted between $t = 0$ and $t^* = T$, and*
2. *for $r > 0$ is given by an increasing amount of effort a_i between $t = 0$ and $t^* \leq T$, with*

$$t^* = \min\left\{\frac{1}{r}\log\left(n\left(\frac{\sqrt{2}x\sqrt{r}}{2n^2 - x^2r} + \frac{2n}{2n^2 - x^2r}\right)\right), T\right\},$$

and

$$a_i(t) = \frac{rx}{n}\frac{e^{rt}}{e^{rt^*} - 1} \quad \forall t \in [0, t^*],$$

and $a_i(t) = 0$ for any $t > t^*$.

The proof splits the problem into two parts: the optimal effort allocation, already solved by Georgiadis (2017, Section 5.1), and the determination of the optimal completion time t^*.

The equilibrium depicted is only one of many possible equilibria, but it shows two important things: First, unlike in Georgiadis (2017), discounting in our model can induce early project completion, even when the threshold is known. Second, with a known threshold and a positive discount rate r, the equilibrium effort function is increasing in t. Finally, very profitable projects or those with sufficiently late deadlines will be completed before the deadline.

4 Applications to project management

The established link between team members' effort and their beliefs has practical implications for team management. Specifically, our results suggest concrete managerial practices for teams facing uncertain objectives and deadlines. In this section, you can find four different suggestions for managerial practices.

Re-evaluating procrastination: This paper shows that delaying effort and increasing activity closer to the deadline is not necessarily indicative of inefficient procrastination. Instead, this behavior naturally emerges from uncertainty regarding project requirements and is even present in the welfare-maximizing solution. Consequently, sanctioning delayed effort may be unwarranted when requirements are uncertain.

Optimal team size based on the type of uncertain objective: Projects characterized by a decreasing hazard rate of the target effort distribution strongly suffer from negative past-effort effects and free-riding, making smaller teams preferable. Conversely, projects with increasing hazard rates foster optimism through positive past-effort effects, counteracting free-riding and thus benefiting from larger teams.

Shaping project objectives to reduce free-riding: If managers can influence the (real or perceived) target effort distribution, they can prevent negative past-effort effects and invoke positive past-effort effects to mitigate free-riding. This might be done, for example, by redefining project success criteria, effectively shifting the target effort distribution from decreasing to increasing hazard rates. A project initially defined as unsuccessful could instead be considered successful if it clarifies methodological limitations or reveals the necessity for alternative approaches. Similarly, a project initially intended to solve a specific problem might be considered successful if all reasonable solution paths have been explored. Breaking projects into smaller, clearly achievable subprojects could also enhance optimism and team effort.

Introducing uncertain objectives: As suggested by Cetemen et al. (2020), appropriately introducing uncertainty into projects may trigger positive past-effort effects and therefore help to mitigate free-riding. For assessing this, comparing equilibrium paths for uncertain objectives to those outlined in Section 3.3 might be helpful. An illustrative example of welfare improvement through the introduction of uncertainty into objectives is provided in Appendix D.

These instruments illustrate how understanding the relationship between team beliefs, effort dynamics, and project uncertainty can meaningfully inform and improve team management practices.

5 Conclusion

In this paper, I have analyzed the behavior of a team working together on a project, where the individual team members cannot observe how much effort other team members invest in the project. More importantly, they do not know how much total effort is needed to complete the project. I modeled this behavior by assuming that the team members collectively need to invest an unknown amount of total effort, denoted the target effort, in the project for this project to be successful. I also assumed that the distribution of this target effort over possible levels was commonly known. This target effort distribution is the major novel aspect of this work. By varying this distribution, one can describe a wide range of different types of projects. The key characteristic of this distribution is the hazard rate, which describes the effect the past investment of effort has on the chance of success when investing effort at a given moment in time.

I have shown that, for any distribution of target effort levels, a unique symmetric equilibrium always exists. I was able to characterize the gradient of equilibrium effort (as well as of the welfare-maximizing effort) over time by means of a differential equation. This technical result, while hard to interpret by itself, is key to providing behavioral predictions in this model. In particular, I have shown that at least two additional behavioral effects operate beyond the usual free-riding effect. The extent of these effects depends on the hazard rate: In equilibrium, players are subject to a past-effort effect, and their behavior in both the equilibrium and the welfare-maximizing solution exhibit a last-minute rush. A positive past-effort effect, which somewhat counteracts the typically detrimental free-riding effect, is a strategic effect. When this effect occurs, the players' best responses increase with the team's cumulative past investment of effort. This effect occurs if and only if the target effort's hazard rate increases at that level of the past investment of effort. If the hazard rate decreases at that level of the past investment of effort, players may experience a negative past-effort effect, in the sense that higher investments of effort by the team in the past decreases any player's best response effort at that time. A last-minute rush describes behavior seen when players increase their efforts near the deadline, i.e., they work harder and harder until the deadline is reached. This is not a strategic effect; it is present in the equilibrium and the welfare-maximizing solution and with any target effort distribution. Behavior that exhibits a last-minute rush could be interpreted as a result of procrastination, but here it is derived (in equilibrium and the welfare maximizing solution) from completely standard preferences. Even more so, if the team members are very patient or the project is very short, the effort invested along the equilibrium path increases everywhere.

The model displays the feature that, in equilibrium, a link is established between the players' effort choices over time and their underlying beliefs about the target effort distribution of the players. This link has both positive and normative implications: On the positive side, if team members' beliefs about the target effort are known or can be influenced, the model yields testable theoretical predictions. On the normative side, the results offer guidance, e.g., for how a manager might effectively steer a team, as discussed in Section 4.

Appendix

A Welfare-Maximizing Solution

The welfare-maximizing solution maximizes the (equally weighted) sum of expected utilities from Lemma 1 for each player i.

Thus, the social planner's optimization problem is given by

$$\max_{a_1(t),\ldots,a_n(t)} \int_0^T e^{-rt}\,(1-F(y(t)))\left(n\frac{f(y(t))\left(\sum_{i=1}^{n} a_i(t)\right)}{1-F(y(t))} - \sum_{i=1}^{n} ca_i(t)^2\right)\mathrm{d}t, \quad (6)$$

with $y(t) = \int_0^t \sum_{i=1}^{n} a_i(t)\mathrm{d}t$ and $y(0) = 0$.

Lemma 2 shows that we can focus on the symmetric problem in which every player exerts $asoc(t)$ without loss of generality.

Lemma 2 *Along every welfare-maximizing effort path, all players exert the same effort at every point in time, i.e., $a_i(t) = a(t)$ for every player i and time t.*

The intuition behind Lemma 2 is as follows: Due to the assumptions associated with symmetric and additive-separable effects of efforts and convex costs, an equal distribution of the efforts invested at every point in time results in the lowest cumulative costs at a given effort level. For the proof, see Appendix B.4.

Therefore, the social planner's optimization problem is simplified to

$$\max_a r \int_0^T e^{-rt}(1 - F(y(t))) \left(\frac{f(y(t))(n^2 a(t))}{1 - F(y(t))} - nc\bar{a}(t) \right) \mathrm{d}t. \tag{7}$$

Proposition 7 (Welfare-Maximizing Effort) *Under socially optimal conditions, every player i exerts an effort a which develops according to*

$$\dot{a}(t) = \frac{1}{2} h(y(t)) a(t)^2 + ra(t) - \frac{nr}{2c} h(y(t))$$

and reaches $a(T) = \frac{n}{2c} h(y(T))$ at time T.

Proof The proof is analogous to the proof for the equilibrium effort function in Appendix B.2. Following the same steps, we obtain the welfare-maximizing cumulative effort:

$$\begin{aligned} \ddot{y} &= -\frac{rn^2}{2c} h(y) + \frac{1}{2} h(y) \dot{y}^2 + r\dot{y} \\ y(0) &= 0, \\ \dot{y}_T &= \frac{n^2}{2c} h(y(T)). \end{aligned}$$

Or in terms of instantaneous effort a:

$$\dot{a} = \frac{1}{2} h(y) a^2 + ra - \frac{nr}{2c} h(y)$$

which reaches $a(T) = \frac{n}{2c} h(y(T))$ at time T.

The properties derived in Lemma 4 and Lemma 5 also apply to the solution for the social planner's problem (Proposition 7). ◻

B Proofs

B.1 Expected utility

We know that the target effort X is based on a distribution with the probability distribution function f and the cumulative distribution function F, and that the total aggregate cumulative effort up to time t is given by $y(t) = \int_0^t a_i(s) + a_{-i}(s)\mathrm{d}s$ with $a_{-i}(s) = \sum_{j \neq i} a_j(s)$ representing the cumulative effort of every player except i at time s.

Given some breakthrough time t^*, the utility is then:

$$\tilde{V}_i(a_i, t^*) = re^{-rt^*} - r \int_0^{t^*} e^{-rt} ca_i(t)^2 \mathrm{d}t.$$

Therefore, we know that the payoff part (re^{-rt^*}) of the expected utility is equal to the distribution of t^*: $f^*(t)$. Let us now rewrite $f^*(t)$ and $F^*(t)$ as a distribution over the cumulative effort y:

$$\begin{aligned} F^*(t) &= P[t \geq t^*] = P[y(t) \geq x] = F(y(t)) \\ \Rightarrow f^*(t) &= f(y(t))\,(a_i(t) + a_{-i}(t))\,. \end{aligned}$$

Therefore, the expected payoff of player i is:

$$\mathbb{E}_{t^*}\left[re^{-rt^*}\right] = r \int_0^T e^{-rt} f(y(t))\,(a_i(t) + a_{-i}(t))\,\mathrm{d}s. \tag{8}$$

For the expected costs $r \int_0^{t^*} e^{rt} ca_i(t)^2 \mathrm{d}t$, we have to distinguish between two cases: One in which the project is successful (i.e., $\bar{t} < T$) and we pay until $\bar{t}$, and one in which it is unsuccessful ($t^* \geq T$), and we only pay until T. As we know that:$1 - Ft^*(\infty) = P[t^* = \infty] = P[x > y(\infty)] = 1 - F(y(\infty))$ we get:

$$\begin{aligned} &\int_0^\infty \int_0^{\min\{t,T\}} e^{-rs} ca_i(s)^2 \mathrm{d}s\mathrm{d}F(y(t)) + (1 - F(y(\infty))) \int_0^T e^{-rs} ca_i(s)^2 \mathrm{d}s \\ &= \int_0^\infty \int_0^\infty \mathbb{1}_{s<t}\mathbb{1}_{s<T} e^{-rs} ca_i(s)^2 \mathrm{d}s\mathrm{d}F(y(t)) + (1 - F(y(\infty))) \int_0^T e^{-rs} ca_i(s)^2 \mathrm{d}s \\ &\overset{\text{Fubini's Theorem}}{=} \int_0^\infty \int_0^\infty \mathbb{1}_{s<t}\mathbb{1}_{s<T} e^{-rs} ca_i(s)^2 \mathrm{d}F(y(t))\mathrm{d}s + (1 - F(y(\infty))) \int_0^T e^{-rs} ca_i(s)^2 \mathrm{d}s \\ &= \int_0^\infty \mathbb{1}_{s<T} e^{-rs} ca_i(s)^2 \int_0^\infty \mathbb{1}_{s<t} \mathrm{d}F(y(t))\mathrm{d}s + (1 - F(y(\infty))) \int_0^T e^{-rs} ca_i(s)^2 \mathrm{d}s \end{aligned}$$

$$
\begin{aligned}
&= \int_0^\infty \mathbb{1}_{s<T} e^{-rs} c a_i(s)^2 (1 - F(y(s)) \mathrm{d}s \\
&= \int_0^T e^{-rs} c a_i(s)^2 (1 - F(y(s)) \mathrm{d}s. \qquad (9)
\end{aligned}
$$

If we add the expected payoff (Equation (8)) and the expected costs (Equation (9)), we get the expected utility, as stated in Lemma 1:

$$
\mathbb{E}_X V(a_i, y, X) = r \int_0^T e^{-rt} (1 - F(y(t))) \left(\frac{f(y(t))}{1 - F(y(t)} \sum_{j=1}^n a_j(t)) - c a_i(t)^2 \right) \mathrm{d}t.
$$

B.2 Theorem 1 (Optimal Effort)

Candidate solution

From Lemma 1, we know that a player's maximization problem (omitting the time index t from $y(t)$ and $a_i(t)$) for all i,

$$
\max_{a_i} V(a_i, a_{-i}) = r \int_0^T e^{-rt} (1 - F(y)) \left(\frac{f(y)(a_i + a_{-i})}{1 - F(y)} - c a_i^2 \right) \mathrm{d}t \qquad (10)
$$

with boundary condition $y_0 = 0$ for the cumulative effort at time 0.

Finding the best response a_i of some player i to the strategies of the other players, a_{-i} in the problem stated in Equation (10) is a discounted optimal control problem of the following form:

$$
\begin{aligned}
H(y, \lambda, t) &= f(y)(a_i + \sum_{j \neq i} a_j) - c a_i^2 (1 - F(y)) + \lambda (a_i + a_{-i}) \\
\dot{y} &= a_i + \sum_{j \neq i} a_j \\
y(0) &= 0 \qquad (11) \\
\lambda(T) &= 0 \\
a_i, y &\in \mathbb{R}_+ \quad \forall i.
\end{aligned}
$$

Applying the Pontryagin maximum principle, we know that the necessary conditions for a maximum are:

$$
\frac{\partial H}{\partial a} = 0 \qquad (12)
$$

$$
\frac{\partial H}{\partial y} = r\lambda - \dot{\lambda} \qquad (13)
$$

$$
\frac{\partial H}{\partial \lambda} = \dot{y} \qquad (14)
$$

$$\lambda(T)y(T) = 0 \quad \Rightarrow \quad \lambda(T) = 0. \tag{15}$$

with the Hamiltonian (Equation (12)), the equation of motion for the state variable (Equation (13)), the equation of motion for the costate variable (Equation (14)), and the transversality condition (Equation (15)) for $y(T)$ being free. In addition, we can see that the optimal control does not depend on the a_j's of the other players but only on the sum $a_{-i} := \sum_{j \neq i} a_j$. Therefore, we get:

$$\begin{aligned}
a_i &= \frac{f(y) + \lambda}{2c\,(1 - F(y))} \\
\dot{\lambda} &= r\lambda - f'(y)(a_i + a_{-i}) - c a_i^2 f(y) \\
\dot{y} &= a_i + a_{-i} \\
y(0) &= 0, \quad \lambda(T) = 0.
\end{aligned}$$

From here on, we only consider symmetric equilibria. Therefore, we can replace a_{-i} by $(n-1)a_i$. Hence, the necessary conditions for a best response are:

$$\begin{aligned}
a_i &= \frac{f(y) + \lambda}{2c\,(1 - F(y))} \\
\dot{\lambda} &= r\lambda - f'(y) n a_i - c a_i^2 f(y) \\
\dot{y} &= n a_i \\
y(0) &= 0, \quad \lambda(T) = 0.
\end{aligned}$$

Using a_i we get:

$$\begin{aligned}
\dot{\lambda} &= r\lambda - f'(y) n \left(\frac{f(y) + \lambda}{2c\,(1 - F(y))} \right) - c \left(\frac{f(y) + \lambda}{2c\,(1 - F(y))} \right)^2 f(y) \\
\dot{y} &= n \left(\frac{f(y) + \lambda}{2c\,(1 - F(y))} \right) \\
y(0) &= 0, \quad \lambda(T) = 0.
\end{aligned}$$

So, the equation of motion for the costate and its time derivative are:

$$\begin{aligned}
\lambda &= \frac{2c}{n}\,(1 - F(y))\,\dot{y} - f(y) \\
\dot{\lambda} &= \frac{2c}{n}\,(1 - F(y))\,\ddot{y} - \frac{2c}{n} f(y)\dot{y}^2 - f'(y)\dot{y}.
\end{aligned}$$

Using this, the necessary conditions can be reduced to the following boundary value problem:

$$(1 - F(y))\,\ddot{y} = -\frac{nr}{2c} f(y) - \frac{1}{2n} f(y)\dot{y}^2 + f(y)\dot{y}^2 + r\,(1 - F(y))\,\dot{y}$$

$$y(0) = 0$$
$$\lambda(T) = \frac{2c}{n}\,(1 - F(y_T))\,\dot{y}_T - f(y_T) = 0.$$

Introducing the hazard rate $h(y) := \frac{f(y)}{1-F(y)}$, we have necessary conditions for Equation (10)

$$\begin{aligned} \ddot{y} &= -\frac{rn}{2c}h(y) + \frac{2n-1}{2n}h(y)\dot{y}^2 + r\dot{y} \\ y(0) &= 0 \\ \dot{y}_T &= \frac{n}{2c}h(y_T). \end{aligned} \tag{16}$$

Rewritten in terms of the individual effort, i.e., using that $\dot{y} = na_i$ and $\ddot{y} = n\dot{a}_i$, we get:

$$\begin{aligned} \dot{a} &= -\frac{r}{2c}h(y) + \frac{2n-1}{2}h(y)a_i^2 + ra_i \\ a_i(T) &= \frac{1}{2c}h(y(T)), \end{aligned} \tag{17}$$

which is a non-linear boundary value problem of the second order.

However, a solution to this problem might not exist, and even if it exists, it might not be unique. The following lemma shows that it does exist uniquely.

Lemma 3 (Existence and Uniqueness of the Solution to the Initial Value Problem) *A solution to the initial value problem from Equation (16) (and, therefore, also for Equation (17)) exists and is unique.*

Proof As $a_i : [0, T] \to \mathbb{R}_+$ is continuous and maps from a compact space to a metric space, we know that it is bounded. Therefore, $\frac{r}{2c}h(y) + \frac{2n-1}{2n}h(y)a^2 - ra$ is Lipschitz-continuous in (a_i, y) as a_i is bounded (and, therefore, also a_i^2), $y(t) = \int\limits_0^t na_i(s)\mathrm{d}s$ and $y_0 = 0$ and $h(y)$ is bounded due to the assumptions on f and F. Thus, by Picard-Lindelöf, we know that a unique solution to the initial value problem for a_i exists.

As $y(t) = \int\limits_0^t na_i(s)\mathrm{d}s$ and $y(0) = 0$ and a_i exists and is unique, $y(t)$ also exists uniquely. □

This result does not mean that a Nash equilibrium exists. The candidate solution might not be a global maximizer for the problem (i.e., the conditions in Equation (17) might not be sufficient). Furthermore, even if Equation (17) represents a Nash equilibrium, other Nash equilibria might exist. There could be, for example, boundary arcs or other solutions that might not be found by applying the Pontryagin maximum principle.

Therefore, we have to show evidence for the sufficiency and uniqueness of the solution.

Sufficiency and uniqueness of the solution

To do that, we first show that our optimal control problem has at least one global maximizer via a Tonelli-type existence proof. Then, we show that every maximizer has to be a Pontryagin extremal (i.e., it will be found by applying the Pontryagin maximum principle). Thus, we know that the candidate solution is the only maximum, as the candidate solution is unique.

To demonstrate the existence, we use a Tonelli-type existence theorem. For convenience, we use the version from Torres (2004), Theorem 10, as we are using results reported in this paper later on.

Lemma 4 (Existence of a Global Maximizer) *The maximization problem, as stated in Equation (10), has an absolute maximum within the space of the control.*

Proof Based on the Torres (2004), we have to show that the problem has to satisfy conditions of **Coercivity** and **Convexity** to have a global minimum.

Following the notation of the paper, $-L(t, y, a_i) = -\left(f(y)(a_i + \sum_{j \neq i} a_j) - ca_i^2 (1 - F(y)) \right)$ is the Lagrangian function of our problem from Equation (11) multiplied by -1 to transform the maximization problem into a minimization problem, and $\varphi(t, x, a) = a_i + \sum_{j \neq i} a_j$ is the evolution of the state.

Then we know that a global maximizer always exists if the following conditions exist for all (t, y, a_i):

Coercivity: A function $\theta : \mathbb{R}_+ \to \mathbb{R}$, bounded below, exists such that:

$$\lim_{r->+\infty} \frac{\theta(r)}{r} = +\infty,$$
$$-L(t, y, a_i) \geq \theta(\varphi(t, y, a_i)),$$
$$\lim_{a_i->+\infty} \varphi(t, y, a_i)) = +\infty.$$

As $\varphi(t, x, a)) = a_i + \sum_{j \neq i} a_j$ is linear in a_i, the last condition is fulfilled. Now, we have to show that a function θ exists that fulfills the other two conditions.

$-L(t, y, a_i) = -\left(f(y)(a_i + \sum_{j \neq i} a_j) - ca_i^2 (1 - F(y)) \right) = ca_i^2 (1 - F(y)) - f(y)(a_i + \sum_{j \neq i} a_j)$is, by assumption, as $f(y)$ is bounded above and $1 - F(y)$ is bounded below, bounded and we can always find a $\theta()$, s.t. $\theta(a_i) \leq -L(t, y, a_i)$ and $\lim_{r->+\infty} \frac{\theta(r)}{r} = +\infty$.[4]

[4] For example $\theta(r) = \alpha r^2 - \beta r - \gamma$ with $\alpha \leq c(1 - F^-)$, $\beta \geq f^+$ and $\gamma > f^+ a_{-i}$ with f^+ being the upper bound of f and F^- the lower bound of F fulfills the conditions.

Convexity: $-L(t, y, a_i) = -\left(f(y)(a_i + \sum_{j \neq i} a_j) - c a_i^2 (1 - F(y)) \right)$ and the evolution of the state $\varphi(t, y, a_i) = a_i + a_{-i}$ both have to be convex in a_i.[5]

We have assumed that $c > 0$ and $f(y) > 0$, $F(y) < 1$, $L(t, y, a_i)$ is concave in a_i, and $-L(t, y, a_i)$ is convex in a_i. Furthermore, $\varphi(t, y, a_i)$ is linear in a_i and, therefore, also convex.

Thus, as Convexity and Coercivity are given, we know that our optimal control problem has a global maximizer. □

However, this does not ensure that we can find this maximizer by applying the Pontryagin maximum principle. For example, we might have singular arcs or upper or lower boundary arcs.[6]

We establish that the maximizer we have found is, in fact, the global maximizer by using the main result of Torres (2004) (Theorem 13 and Corollary 4):

Lemma 5 (Maximizers are Pontryagin extremals) *All maximizers of the problem, as stated in Equation (11), are Pontryagin extremals.*

Proof Using Torres (2004), Corollary 4, we know that Coercivity (which we have shown in Lemma 4) and the following conditions imply that all maximizers are found by applying the Pontryagin maximum principle if constants $k_1 > 0$ and k_2 exist such that:

$$\left| \frac{\partial L}{\partial t} \right| \leq k_1 |L| + k_2, \quad \left| \frac{\partial L}{\partial x} \right| \leq k_1 |L| + k_2,$$
$$\left| \frac{\partial \varphi}{\partial t} \right| \leq k_1 |\varphi| + k_2, \quad \left| \frac{\partial \varphi_{a_i}}{\partial x} \right| \leq k_1 |\varphi_{a_i}| + k_2.$$

The first inequality is trivially fulfilled, as L does not depend on t. The second inequality is fulfilled, as $f(y)$ is assumed to be continuously differentiable. The third inequality is fulfilled, as φ does not depend on t and, as φ (and thus φ_{a_i}) is independent of x, the last inequality is also fulfilled. □

Thus, we know that every maximizer has to be found by applying the Pontryagin maximum principle and that at least one global maximizer exists. Furthermore, we know that Equation (17) is the solution found by applying the Pontryagin maximum principle and that it is the unique solution to the application of the Pontryagin maximum principle. Therefore, the following corollary follows.

Corollary (Sufficiency and uniqueness) *The solution to Equation (17) gives us the unique global maximizer to the optimal control problem in Equation (11) and, therefore, the unique symmetric Nash equilibrium.*

Thus, we know that the initial value problem describes the only symmetric Nash equilibrium in pure strategies in Equation (16).

[5] For existence, we do not need global convexity or convexity in the state.

[6] We would have, for example, a lower boundary arc if a player would stop working at some point in time.

B.3 Proposition 1 (Increasing Effort)

(a) Non-decreasing hazard rates

Proof Assume your effort is decreasing at some point t_1, i.e., $\dot{a}_i(t_1) < 0$. Based on Proposition 3, we know that there is a t_3 close to T, such that $\dot{a}_i(t_3) > 0$. Therefore, due to the continuity of a, a t_2 exists such that $t_1 \leq t_2 \leq t_3$ with $\dot{a}_2 \geq 0$ and $a_2 \leq a_1$. As we have a non-decreasing hazard rate, we know that $h(y(t_2)) \geq h(y(t_1))$. Thus, we have for $r \neq 0$:

$$
\begin{aligned}
& \dot{a}_i(t_1) && < 0 \\
\Leftrightarrow\ & \frac{2n-1}{2} h(y(t_1)) a_i(t_1)^2 + r a_i(t_1) - \frac{r}{2c} h(y(t_1)) && < 0 \\
\xRightarrow{r \geq 0 \text{ and } a_i(t_2) \leq a_i(t_1)}\ & \frac{2n-1}{2} h(y_1) a_i(t_1)^2 + r a_i(t_2) - \frac{r}{2c} h(y(t_1)) && < 0 \\
\xRightarrow{a_i(t_2) \leq a_i(t_1)}\ & \left(\frac{2n-1}{2} a_i(t_2)^2 - \frac{r}{2c}\right) h(y_1) + r a_i(t_2) && < 0 \\
\xRightarrow[r>0 \text{ and } a_i(t_2)>0]{h(y(t_2)) \geq h(y(t_1)), h(y(t_1)) \geq 0,}\ & \left(\frac{2n-1}{2} a_i(t_2)^2 - \frac{r}{2c}\right) h(y(t_2)) + r a_i(t_2) && < 0 \\
\Rightarrow\ & \frac{2n-1}{2} h(y(t_2)) a_i(t_2)^2 + r a_i(t_2) - \frac{r}{2c} h(y(t_2)) && < 0 \\
\Leftrightarrow\ & \dot{a}_i(t_2) && < 0 \quad \textbf{Contradiction}
\end{aligned}
$$

□

(b) $r = 0$

Proof For $r = 0$ $\dot{a}_i > 0$ results trivially from Theorem 1, as $\frac{2n-1}{2} h(y) a_i^2 \geq 0$. □

B.4 Lemma 2 (asymmetric equilibria)

Proof Assume there is an asymmetric equilibrium that is welfare-maximizing. Then $\exists i, t, j : a_i(t) > a_j(t)$. However, it would be possible to improve the welfare by changing the effort paths of i and j to $a_i^*(t)$ and $a_j^*(t)$ as follows: $a_i^*(t) = a_j^*(t) = \frac{a_i(t)+a_i(t)}{2}$ as this does not change the overall effort (and, therefore, the chance of success) but reduces the combined expected costs of the project due to the quadratic costs. Therefore, an asymmetric equilibrium can never be welfare-maximizing. □

B.5 Proposition 2 (equilibrium vs welfare-maximizing effort function)

Proof (a):
It follows directly from the definition of the welfare-maximizing solution. Assume that there is an equilibrium path on which the cumulative effort is higher than on the welfare-maximizing path for the first time at t_1. A player can then unilaterally invest some ϵ effort less at time t_1 and, if the cumulative amount of effort invested when on the equilibrium path ever falls below the amount invested when on the welfare-maximizing path, they invest ϵ more at that point in time (or never, if the amount of effort invested when on the equilibrium path is always higher).

In this case, due to the definition of the welfare-maximizing solution, the difference in welfare is positive. As the player has shifted some investment of effort from an earlier point in time to a later point in time, she is better off than all other players.

Thus, her decision to shift the effort is profitable for her, and therefore, this path can not be an equilibrium path.

(b):

Part 1 ($t = 0$) is a direct result of part (a). I denote the welfare-maximizing effort function by a^* (and the corresponding cumulative welfare-maximizing effort by y^*) and the equilibrium effort function by a (and the corresponding cumulative equilibrium effort by y). Proof by contradiction: If there is a t': $\forall t \in [0, t']$: $a(t) > a^*(t)$, then $y(t) > y^*(t) \forall t \in [0, t']$, which contradicts part (a).

Part 2 ($t = T$) is easy to see from Theorem 1 and Proposition 7 that

$$a(T) = \frac{1}{2c} h(y(T)) < a^*(T) = \frac{n}{2c} h(y^*(T)),$$

as $y^*(T) \geq y(T)$, $n \geq 1$ and $h(\cdot)$ is non-decreasing. □

B.6 Proposition 3 (last-minute rush)

To prove Proposition 3, we use the continuity of y to show that the negative part of $\dot{a}_i$ vanishes near the deadline and is therefore strictly positive.

Proof As $h(y(t))$ is continuous in y, it is also continuous in t. We also know from Theorem 1 that $a_i(t)$ is continuous in t and, from Equation (17), that it satisfies

$$\begin{aligned} \dot{a} &= -\frac{r}{2c} h(y) + \frac{2n-1}{2} h(y) a_i^2 + r a_i, & a_i(T) &= \frac{1}{2c} h(y(T)) \\ \Leftrightarrow \dot{a} &= \frac{2n-1}{2} h(y) a_i^2 - r \left(\frac{1}{2c} h(y) - a_i \right), & a_i(T) &= \frac{1}{2c} h(y(T)). \end{aligned}$$

As $a_i(t)$ and $h(y(t))$ are continuous in t, we know:

$$\lim_{t \to T} \left(\frac{1}{2c} h(y(t)) - a_i(t)\right) \to \left(\frac{1}{2c} h(y(T)) - a_i(T)\right) = 0.$$

Furthermore, as $n > \frac{1}{2}$, $h(y(t)) > 0$, $a_i(t) > 0$ we know that:

$$\lim_{t \to T} \frac{2n-1}{2} h(y(t)) a_i(t)^2 \to \epsilon > 0.$$

Therefore $\lim_{t \to T} \dot{a}_i > 0$, and thus an δ exists in which $\dot{a}_i(t) > 0$ for all $[T - \delta, T]$. □

B.7 Free-riding

We use the best responses from Appendix B.2 to show that

$$\frac{\partial a_i^*(\cdot)}{\partial a_{-i}(t)} \leq 0$$

everywhere.

Proof

$$\begin{aligned}
(12) \Leftrightarrow a_i(t) &= \frac{f(y(t)) + \lambda}{2c\,(1 - F(y(t)))} \\
&\Leftrightarrow \lambda = 2ca_i(t)\,(1 - F(y(t))) - f(y(t)) \\
&\Rightarrow \dot{\lambda} = 2c\,(1 - F(y(t)))\,\dot{a}_i(t) - \dot{y}(t)\left(2ca_i(t) f(y(t)) + f'(y(t))\right) \\
(13) &\Rightarrow 2c\,(1 - F(y(t)))\,\dot{a}_i(t) - 2cf(y(t))a_i(t)^2 \\
&- 2cf(y(t))a_i(t)a_{-i}(t) - f(y(t))a_i(t) - f(y(t))a_{-i}(t) \\
(14) &\Rightarrow \dot{\lambda} = 2rca_i(t)\,(1 - F(y(t)))\,a_i(t) \\
&- f(y(t))r - f'(y(t))a_i(t) - f'(y(t))a_{-i}(t) - ca_i(t)^2 f(y(t)) \\
&\Rightarrow (1 - r)\,2c\,(1 - F(y(t)))\,a_i(t) - cfa_i(t)^2 = -f(y(t))r + 2cf(y(t))a_i(t)a_{-i}(t) \\
&\Leftrightarrow 2h(y(t))a_i(t) - a_i^2 - 2a_i(t)a_{-i}(t) = -\frac{r}{c}
\end{aligned}$$

Solving for $a_i(\cdot)$ and taking the partial derivative gives:

$$\frac{\partial a_i^*(\cdot)}{\partial a_{-i}(t)} = \frac{(a_{-i}(t) - h(y(t)))}{\sqrt{\frac{c}{r} + (a_{-i}(t) - h(y(t)))^2}} - 1.$$

This is always negative if $h(y(t)) \neq a_{-i}(t)$ and 0 otherwise. Therefore, we always see a free-riding effect. □

B.8 Proposition 4 (past-effort effect)

Proof Using the best response $a_i^*(\cdot)$ from Appendix B.2 and Appendix B.7 and taking the partial derivative with respect to $y(t)$ yields:

$$\begin{aligned}
\frac{\partial a_i^*(\cdot)}{\partial y(t)} &= \frac{(h(y(t)) - a_{-i}(t))\,h'(y(t))}{\sqrt{\frac{r}{c} - 2a_{-i}(t)h(y(t)) + h(y(t))^2 + a_{-i}(t)^2}} + h'(y(t)) \\
&= h'(y(t))\left(\frac{h(y(t)) - a_{-i}(t)}{\sqrt{\frac{r}{c} + (h(y(t)) - a_{-i}(t))^2}} + 1\right).
\end{aligned}$$

This is always 0 for $h'(y(t)) = 0$ (i.e., a constant hazard rate), always positive for $h'(y(t)) > 0$, and always negative for $h'(y(t)) < 0$. $\square$

B.9 Proposition 6 (known target effort)

The proof splits the problem into two parts: the optimal effort allocation, which has already been solved by Georgiadis (2017, Section 5.1) and an allocation for finding the optimal time to complete the problem t^*.

Proof: Proposition 6 The solution to the case of $r = 0$ is trivial: $\bar{x}$ is distributed evenly between $t = 0$ and T.

In the case of $r > 0$, the problem can be split into two problems: one for the optimal effort allocation, which has been solved by Georgiadis (2017) and one for finding the optimal t^* to finish the project.

The solution to the first part, based on Georgiadis (2017), is that the players will choose an amount of effort to invest so that their discounted marginal costs are constant, i.e., $a_i = \frac{r\bar{x}}{n}\frac{e^{rt}}{e^{rt^*}-1}$.

The solution to the second part is an optimization problem, in which we maximize the expected discounted playoff of the players, given that the project is profitable, i.e., $r\bar{x}^2 < 2n^2$:

$$\max_{t'} e^{-rt'} - \frac{r\bar{x}^2}{2n^2}\frac{1}{e^{rt'}-1}$$
$$\Rightarrow t' = \frac{1}{r}\log\left(n\left(\frac{\sqrt{2}\bar{x}\sqrt{r}}{2n^2-\bar{x}^2 r} + \frac{2n}{2n^2-\bar{x}^2 r}\right)\right).$$

Thus, the stopping time is $t^* = \min\{t', T\}$.

Intuitively, it is clear that deviations from this t^* can never be profitable in the symmetric equilibrium, as one would either invest effort earlier without moving the project's completion date or moving t^* back, as it would lower the payoff. $\square$

C Interpretations of the target effort distribution

The most important characteristic of a project in this model is the target effort distribution or, in other words, how much effort one has to invest to have a certain chance of success, given the effort that the team has invested in the past. Therefore, this distribution describes how likely every possible effort threshold is at the project's present stage. If the player thinks finding a cure for a disease costs around 100 billion staff hours of research, she could assume some normal distribution of around 100 billion. If I am certain that I have lost my keys in my apartment (again) but have no idea where they could be, assuming a uniform distribution in every room in my apartment seems reasonable.

In this section, we are looking at three classes of distribution and how these can be interpreted in the context of the model. The distributions will be denoted by their hazard rates $h(y(t)) := \frac{f(y(t))}{1-F(y(t))}$, which describes the effect that past investments of effort have had on the effectiveness of the current investments of effort, or, in other words, the probability that the project will be successful now, given that it has not been successful so far.

While the current interpretation of a fixed effort threshold and the simple deterministic effort-project progress relation is fairly restrictive, the results of the model hold for more flexible interpretations of projects.

Let me give two simple examples:

- Let us say that we want to model a project in which prior effort does not influence the chances of a breakthrough right now (e.g., when rolling a die). While this is not a project with a fixed target effort, we can rewrite $f(y(t))$ as $f(y(t)) = \lambda y(t)$, meaning that the chance of a breakthrough only depends on the current amount of effort invested. Rewritten in terms of this model, f is then the exponential distribution.
- Let us say that we want a structure similar to the one developed by Bonatti and Hörner (2011): This features independent draws (as described above), but there is additionally a chance p that the project is bad and will never be successful. This type of project can be replicated by $f(y(t))$, which is an incomplete exponential distribution with a mass point p at ∞.

However, throughout this paper, I stay with the interpretation of a fixed effort threshold and the simple deterministic effort-project progress relation.

Example 1 (Constant hazard rate) The first type of distribution has a constant hazard rate, i.e., the exponential distribution ($F(y) = 1 - e^{-\lambda x}$ with a rate parameter $\lambda > 0$). This distribution conveys the idea that the chance of success only depends on the current investment of effort, and the past investment of effort does not matter at all. For example, if trying to roll a six on a die. The chance of success is independent of prior rolls of the dice.

Example 2 (Decreasing hazard rate) A variation in the exponential distribution is an incomplete exponential distribution (i.e., an exponential distribution with a probability mass at infinity).[7] Although the intuition behind this result is technically a distribution with a decreasing hazard rate, it is similar to the memoryless distribution example: The probability distribution itself is memoryless. However, there is a chance of failure: As time goes on, the expected probability of failure is updated; therefore, the amount of effort already invested increases. A popular example of a decreasing hazard rate is a search model, similar to the one described in Keller et al. (2005), where you search in the most likely places first or invest in R&D: The more you invest without success, the greater your belief is that no solution to the problem exists.

Example 3 (Increasing hazard rate) The last example is increasing hazard rates (e.g., when the target effort is distributed uniformly over some interval). Possible applications are projects with a strong learning-by-doing effect and projects where success in

[7] Using this distribution in this model yields us a model very similar to the so-called good news bandit models. One example is Bonatti and Hörner's (2011) benchmark model; the only difference is that we use quadratic instead of linear costs.

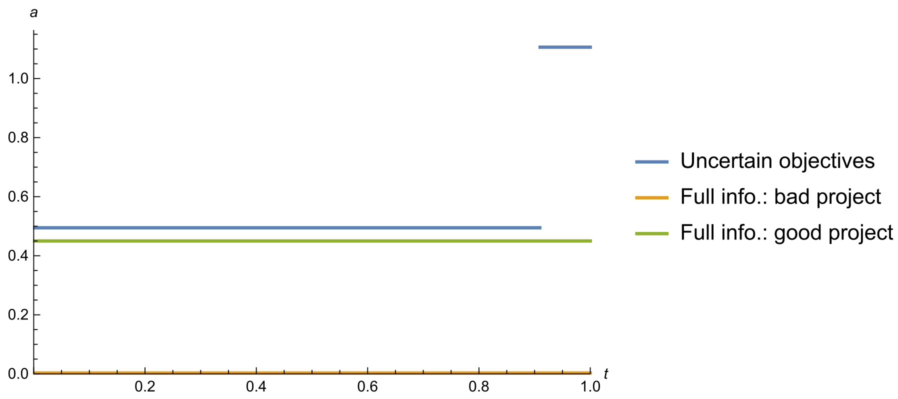


Fig. 3 Full information vs uncertain objectives

a certain period depends on the cumulative amount of effort, not on the current amount of effort.[8] For example, the effort a legal team invests while trying to find a particular file in a room full of documents.

For more examples, see Section 2 in Khan and Stinchcombe (2015), which provides an overview of the meaning of success probability distributions, their hazard rates, and their relations to different projects.

D Welfare improvement due to uncertain objectives

While it is generally beneficial to know the objectives, there are scenarios in which uncertainty about objectives can improve welfare. An example is illustrated in Figure 3, which considers two patient players with a target effort X that is 0.9 with an 80% probability and 1.1 with a 20% probability, and a deadline $T = 1$. Consequently, regardless of the realized value of X, it is always beneficial for the team to complete the project.

Under uncertain objectives, there is only one symmetric Nash equilibrium (depicted by the blue line in Figure 3), in which the project is completed regardless of the drawn target effort. In contrast, under complete information, when the target effort is high ($X = 1.1$), two symmetric Nash equilibria emerge: a "good" equilibrium where both players work efficiently and complete the project, and a "bad" equilibrium where neither player works. Therefore, the resulting outcome depends on the equilibrium selected. However, in the "good" equilibrium, where both players choose to work, private costs exceed private benefits, potentially making this equilibrium less likely to be chosen (see, for example, Myatt and Wallace (2008)).

Thus, in this example, uncertainty ensures that the project is always completed, while known target efforts may lead to project non-completion.

[8] One example is Georgiadis (2015). In his model, the uncertainty is about the effect of effort and not the threshold, but this is just a different way to model uncertainty about the relationship between effort spent and success. One can, therefore, generate a very similar model in the framework presented by choosing the appropriate target effort distribution, which would have an increasing hazard rate.

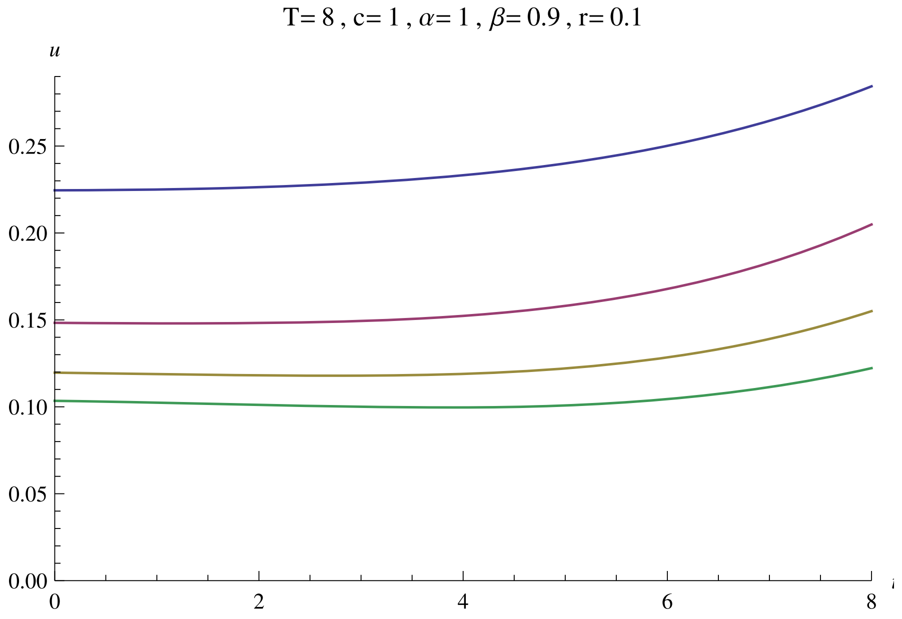


Fig. 4 Individual effort for different team sizes

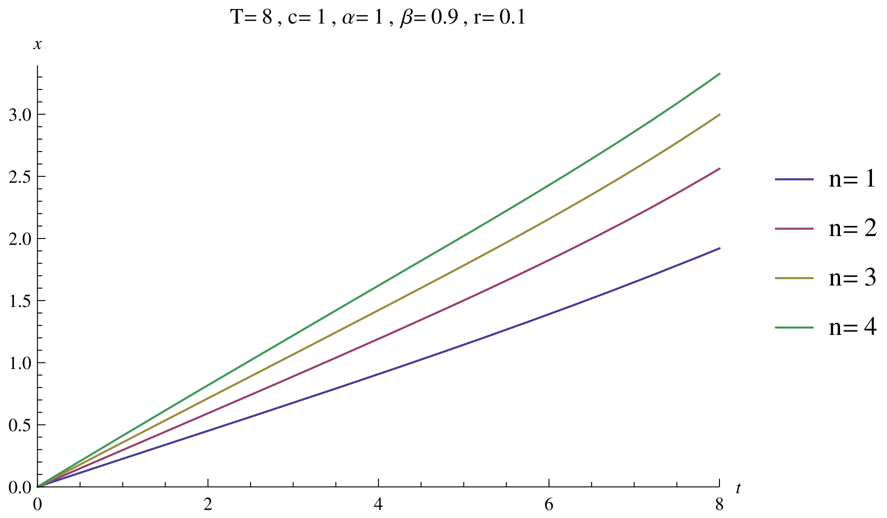


Fig. 5 Cumulative effort for different team sizes

E The effect of team size

So far, we have assumed a fixed number of players. What happens if the number of players changes?

Given that the reward from the successful completion of the project is modeled as non-rivalrous, it is not surprising that the welfare increases as the number of players increases.

To show this, remember that players never over-exert themselves in the equilibrium in this model. Let's start by stating $n = 1 \rightarrow n = 2$: Assume a 2-player equilibrium in which the players are worse off than in the $n = 1$ case. In this case, each player profits from deviating their strategy to play the $n = 1$ player strategy, since they have the same costs as in the $n = 1$ player case and a strictly higher cumulative effort $y(t)$ at every point in time and, therefore, at least marginally higher chances of winning before time t.[9] As the same argument also works for $N \rightarrow N + 1$, we know that welfare strictly increases as the number of players increases.

But what about individual efforts? We know that there are two strategic effects at work, the free-riding and the past-effort effect.

When the hazard rate is constant, we only have a free-riding and no past-effort effect. Thus, every individual will work less. As the past-effort effect only depends on the cumulative amount of effort, we can also expect to see the same effect when a decreasing hazard rate is present, which we can observe in the example of an incomplete exponential distribution in Figure 4 and Figure 5. Here, we can see the lower individual investments of effort and higher cumulative amounts of effort for larger team sizes. When the increasing hazard rate is present, however, individual investments of effort can be higher as long as the past-effort effect is larger than the free-riding effect.

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[9] Strictly higher cumulative effort since the other player will never exert 0 effort as shown before. Strictly higher chances of winning before time t is due to our assumptions on f.

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