Mathematical Programming
https://doi.org/10.1007/s10107-025-02260-x

FULL LENGTH PAPER

Series A

Parabolic target-space interior-point algorithm for monotone weighted linear complementarity problem

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Received: 12 September 2024 / Accepted: 11 July 2025

Abstract

We revisit the main principles for constructing polynomial-time primal-dual interior-point algorithms (IPAs). We investigate the *weighted* Linear Complementarity Problem (WLCP), by extending the framework of Parabolic Target Space (PTS), proposed by Nesterov (2008) for primal-dual Linear Programming (LP) Problems. This approach has several advantages. The proposed method based on the PTS approach starts from an arbitrary strictly feasible primal-dual pair and follows a single central path toward the solution. It has the best known worst-case complexity bound. Finally, it works in a large neighborhood of the deviated central path, allowing very large steps. The latter ability results in a significant acceleration at the end of the process, confirmed by our preliminary computational experiments.

Keywords Interior-point algorithm · Parabolic target-space · Monotone weighted linear complementarity problems · Bisymmetric matrices · Polynomial complexity

Mathematics Subject Classification 90C51 · 90C33

1 Introduction

In this paper, we deal with the *weighted Linear Complementarity Problem (WLCP)*

$$-M\mathbf{u} + \mathbf{v} = \mathbf{q}, \quad \mathbf{u}, \mathbf{v} \geq \mathbf{0}, \quad \mathbf{u}\,\mathbf{v} = \mathbf{p}, \tag{$WLCP$}$$

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Published online: 12 August 2025

where $M \in \mathbb{R}^{n\times n}$ is a given matrix, $\mathbf{q}, \mathbf{p} \in \mathbb{R}^n$ and $\mathbf{p} \geq \mathbf{0}$ are given vectors, and n is a natural number.

If we consider $(WLCP)$ with $\mathbf{p} = \mathbf{0}$, then we get the class of *Linear Complementarity Problems (LCPs)*. The most important classical results about the theory, methods, and applications to solve LCPs are summarized in the monographs written by Cottle et al. [3] and Kojima et al. [19]. The largest class of matrices, called *sufficient matrices* that guarantees important properties of LCPs (e.g., the solution set of the LCP is convex) has been defined by Cottle et al. [4]. Kojima et al. [19] introduced a new class of matrices, called $P_*(\kappa)$-matrices, where κ is a nonnegative parameter, and a related class of P_*-matrices, $P_* = \cup_{\kappa\geq 0} P_*(\kappa)$. They showed that the interior-point algorithm (IPA) which solves $P_*(\kappa)$-LCP has polynomial iteration complexity that depends on the size of the problem, the starting point's duality gap, the accuracy parameter, and the parameter κ. Later, Väliaho [34] showed that the class of sufficient matrices and the class of P_*-matrices coincide. It is clear that if $\kappa = 0$, then we get back the monotone WLCPs (monotone LCPs). It is worth mentioning that in addition to IPAs, finite pivot algorithms for solving sufficient LCPs have also been introduced, for details see [5, 6] and references therein.

In the last few decades many IPAs have been introduced to solve $P_*(\kappa)$-LCPs, for details see [7, 8, 17]. Illés et al. [16] introduced an IPA to solve (general) LCPs that in polynomial time either gives a solution of the original problem or detects the lack of $P_*(\tilde{\kappa})$-property, with arbitrarily large, but apriori fixed $\tilde{\kappa}$. In the latter case, the IPA gives a polynomial size certificate depending on parameter $\tilde{\kappa}$, the initial interior point and the input size of the LCP.

We revisit the main principles for constructing polynomial-time primal-dual IPAs. Starting from the break-through paper by Gonzaga [12], their development was related to the *barrier methods*, where the barrier function for the feasible set is added to the objective function. With this construction, using the theory of self-concordant functions proposed by Nesterov and Nemirovski [27], it was possible to develop variants of IPAs for different convex problems, as well. However, in order to solve the original problem, the most efficient *primal-dual methods*, [27, 35, 36] need to follow several *central paths* (up to three), which correspond to different *stages* of the solution process. This multistage structure of the methods significantly reduces their efficiency.

For the weighted analytic center problem with nonnegative weights, Ye [37] derived the first order optimality conditions that led to a special WLCP (see [37], problem (3) on page 319). In the same paper, Ye introduced a modified primal-dual path-following algorithm to solve this WLCP and derived the polynomial complexity for it. During the same year, in 2008, starting from a different idea, Nesterov [25] derived exactly the same WLCP (see [25] problem (2.1) on page 2081) and used it to introduce a new IPA for solving primal-dual pair of Linear Programming (LP) problems.

Anstreicher [1] introduced a joint generalization of the LP and the weighted analytic center problem, called LPWC. The dual problem (DPWC) of LPWC has been derived and both weak and strong duality results for problem pairs have been proved, as well. Anstreicher studied complexity results for several different IPAs to solve LPWC and DPWC. As an application, Anstreicher studied Fischer equilibrium problem, with linear utility functions, in the form of the Eisenberg-Gale formulation (for details, see

[37]) and using volumetric and logarithmic barriers obtained an improved complexity result.

Potra [28] observed that both problems of Ye [37] and Anstreicher [1] related to the Eisenberg-Gale formulation of the Fischer equilibrium problem, with linear utility functions led to WLCP [1]. He pointed out that the WLCPs of Anstreicher [1] and Ye [37] are monotone since the matrices in the linear constraints are skew symmetric. This observation inspired him to introduce a more general class of convex optimization problems that generalize the LPWC, called *quadratic programming and weighted centering (QPWC)* problem (see subsection 2.3 in [28]). Potra [28] defined the dual problem of QPWC and derived the duality theory for this special convex programming problem class (Theorem 2.1, page 1640 in [28]). The optimality conditions of QPWC give rise to a monotone WLCP. Potra [28] proposed two IPAs with polynomial iteration complexity for solving these monotone WLCPs.

Potra [29] introduced the sufficient WLCPs, studied the properties of this problem class. Most of these results (Theorem 1-4 in [29]) generalize the similar statements known in the literature of sufficient LCPs. Potra proposed a predictor-corrector IPA for solving sufficient WLCPs. The complexity result depends on the initial, strictly feasible solution $\left(\mathbf{u}^{(0)}, \mathbf{v}^{(0)}\right)$, the violation in the equation $\mathbf{u}^{(0)}\mathbf{v}^{(0)} = \mathbf{p}$, the accuracy parameter, and the parameter κ.

In this paper we consider WLCPs with positive semidefinite matrices (special cases: skew symmetric and bisymmetric matrices). Such WLCPs are called *monotone WLCPs* and they can be derived from LP problems, linearly constrained convex quadratic programming problems and some other optimization problems, see [1, 28, 37]. We generalize the result of Nesterov [25], the primal–dual IPA to solve LP problems, which is based on the concept of *parabolic target space* (PTS) to solve monotone WLCPs. We come back to the Renegar's idea [31] of using the *methods of centers* proposed by Huard [13], where the objective function is treated by a logarithmic term.

The concept of *weighted central path* (WCP) in the literature of sufficient LCPs, first occurs in the paper of Illés et al. [14]. Following the idea of Nesterov [25], first, we introduce a relaxation of WCP, and show that the solution set of the relaxed problem is convex. Later, an additional (convex) constraint on the duality gap of the monotone WLCP is added. Finally, we arrive at a *convex feasibility problem* (CFP) that has original variables of the monotone WLCP, and those related to the relaxation, and the additional constraint (Section 3). The new variables naturally satisfy an extra condition leading to the observation of a PTS. The use of the PTS allows us to discuss the new IPA for both the weighted and classical monotone LCP at the same time.

The solution of a classical monotone WLCP reduces to the solution of a sequence of CFPs. The driving force of our new IPA lies in the structure of CFPs and the *self-concordant barrier function F* assigned to the CFP, and its properties (Section 4).

Nevertheless, the new, adaptive *parabolic target-space interior-point algorithm* (PTS IPA) for monotone WLCPs possesses the best known complexity result (Section 5). The computational efficiency of the new method has been illustrated on a set of test problems (Section 6).

[1] Potra was the first, who talked about *weighted complementarity problems*.

Throughout the paper, we use the following notations. We use $\mathbb{R}^n_+$ and $\mathbb{R}^n_{++}$ for the positive orthant and its interior. We denote by $\mathbf{e}_i$, $i = 1, \ldots, n$, the coordinate vectors in $\mathbb{R}^n$, and $\mathbf{e}$ is the vector of all ones. In general, with boldface small letters we denote finite dimensional vectors, while real numbers, coordinates of vectors are denoted by small letters. All arithmetic operations and relations involving vectors, like $\mathbf{xs}$, $\mathbf{x}/\mathbf{s}$ for $\mathbf{x}, \mathbf{s} \in \mathbb{R}^n$, are understood in the component-wise sense. The scalar products and the norms are defined in the standard way: $\mathbf{x}^T\mathbf{s} = \sum_{i=1}^n x_i\, s_i$, $\quad \|\mathbf{x}\|^2 = \mathbf{x}^T\mathbf{x}$, $\quad \mathbf{x}, \mathbf{s} \in \mathbb{R}^n$.

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a function, denote by $\nabla f(\mathbf{x}) \in \mathbb{R}^n$ its gradient, where $\mathbf{x} \in \mathbb{R}^n$. For the function $F : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$, notations $\nabla_1 F(\mathbf{u}, \mathbf{t}) \in \mathbb{R}^n$ and $\nabla_2 F(\mathbf{u}, \mathbf{t}) \in \mathbb{R}^m$ are used for its partial gradients related to variables $\mathbf{u} \in \mathbb{R}^n$ and $\mathbf{t} \in \mathbb{R}^m$, respectively. A similar notation is applied to the partial Hessians, too

$$\nabla^2_{11} F(\mathbf{u}, \mathbf{t}) \in \mathbb{R}^{n\times n}, \quad \nabla^2_{12} F(\mathbf{u}, \mathbf{t}) = \nabla^2_{21} F(\mathbf{u}, \mathbf{t})^T \in \mathbb{R}^{n\times m}, \quad \nabla^2_{22} F(\mathbf{u}, \mathbf{t}) \in \mathbb{R}^{m\times m}.$$

In this paper, we often use different facts from the general theory of self-concordant functions. For the reader's convenience, we summarize most of the important and useful notations and results related to this theory in Appendix II.

2 From weighted central path problem to a sequence of convex feasibility problems

2.1 Central path and weighted central path

In this paper, we assume that the matrix M is positive semidefinite. It is easy to show that this matrix class coincides with the class of $P_*(0)$-matrices, that is a subclass of $P_*(\kappa)$-matrices introduced by Kojima et al. [19], where $\kappa \geq 0$.

We denote by $\mathcal{F} = \{(\mathbf{u}, \mathbf{v}) \in \mathbb{R}^n_+ \times \mathbb{R}^n_+ : -M\mathbf{u} + \mathbf{v} = \mathbf{q}\}$ and $\mathcal{F}^+ = \{(\mathbf{u}, \mathbf{v}) \in \mathbb{R}^n_{++} \times \mathbb{R}^n_{++} : -M\mathbf{u} + \mathbf{v} = \mathbf{q}\}$ the set of feasible and strictly feasible solutions of the WLCP. Note that the definition of these sets is independent of $\mathbf{p}$, namely the definitions are the same in the case of LCP ($\mathbf{p} = \mathbf{0}$). In contrast, the solution set of the WLCP depends on $\mathbf{p}$, let us denote it by $\mathcal{F}^*_{\mathbf{p}} = \{(\mathbf{u}, \mathbf{v}) \in \mathcal{F} : \mathbf{uv} = \mathbf{p}\}$.

For any $(\mathbf{u}, \mathbf{v}) \in \mathcal{F}^+$ the complementarity condition of LCP could not be satisfied, since $\mathbf{u}\,\mathbf{v} > \mathbf{0}$. Thus, the complementarity condition needs to be relaxed. Now, we are ready to introduce the corresponding *central path problem (CPP)*

$$-M\mathbf{u} + \mathbf{v} = \mathbf{q}, \quad \mathbf{u}, \mathbf{v} > \mathbf{0}, \quad \mathbf{u}\,\mathbf{v} = \mu\,\mathbf{e}, \tag{1}$$

for a given $\mu > 0$ and to define the *central path*

$$\mathcal{C} = \{(\mathbf{u}, \mathbf{v}) \in \mathcal{F}^+ \,:\, \mathbf{u}\,\mathbf{v} = \mu\,\mathbf{e} \text{ for some } \mu > 0\},$$

which contains all those strictly feasible solutions of the LCP that solve CPP for some $\mu > 0$. The central path for linear optimization problem has been introduced by Sonnevend [32] and Megiddo [21] independently. Illés et al. [14] gave an elementary proof for the following theorem:

Theorem 2.1 *Let the matrix M of the LCP be a given $\mathcal{P}_*(\kappa)$-matrix. Then, the following statements are equivalent:*

i) $\mathcal{F}^+ \neq \emptyset$,
ii) $\forall\, \mathbf{w} \in \mathbb{R}^n_{++},\ \exists!(\mathbf{u}, \mathbf{v}) \in \mathcal{F}^+ :\ \mathbf{u}\,\mathbf{v} = \mathbf{w}$,
iii) $\exists!(\mathbf{u}, \mathbf{v}) \in \mathcal{F}^+ :\ \mathbf{u}\,\mathbf{v} = \mu\,\mathbf{e}$.

Statement $i)$ is called the *interior point condition (IPC)* of the LCP.
Statement $ii)$ of the previous theorem defines a very important problem[2]:

$$-M\mathbf{u} + \mathbf{v} = \mathbf{q}, \quad \mathbf{u}, \mathbf{v} > \mathbf{0}, \quad \mathbf{u}\,\mathbf{v} = \mathbf{w}, \tag{$WCPP$}$$

that we call the *weighted central path problem (WCPP)* for a given $\mathbf{w} \in \mathbb{R}^n_{++}$. The unique solution of the (WCPP) can be denoted by $(\mathbf{u}(\mathbf{w}), \mathbf{v}(\mathbf{w})) \in \mathcal{F}^+$.

Statement $iii)$ of the previous theorem says that the central path of an LCP with $\mathcal{P}_*(\kappa)$-matrix is unique.[3]
In our case (M is $P_*(0)$-matrix), assuming that LCP satisfies the IPC, namely $\mathcal{F}^+ \neq \emptyset$, it can be proved that $\mathcal{F}^* \neq \emptyset$ is a compact (see Corollary 3.4. in [22]) and convex set (see Theorem 5 in [4] or Corollary 3.3. in [15]).

In the classical IPAs we consider WCPPs with $\mathbf{w} = \mu\,\mathbf{e},\ \mu \to 0+$. By solving these WCPPs approximately, we converge to a solution of the LCP, namely we change the entries of $\mathbf{w}$ proportionally. This, in some ways, restricts the possible optimization strategies. Moreover, as we will see later, the limiting value $\mathbf{w} = \mathbf{0}$ is not the only interesting target. In this paper, we show that the general theory of self-concordant functions gives the opportunity to justify different strategies for updating approximations to $(\mathbf{u}(\mathbf{w}), \mathbf{v}(\mathbf{w}))$ with unbalanced weights. However, for that, we need to introduce the trajectory $(\mathbf{u}(\mathbf{w}), \mathbf{v}(\mathbf{w}))$ not by the weighted barriers, but by a version of *method of centers* [10, 30] (that was utilized in [25] for LP problems). This is the subject of the next section.

2.2 Building the model for parabolic target space IPA

Since we assume that the matrix M is positive semidefinite, the (WCPP) has a unique solution based on the previous subsection. Now, following the idea presented in [25], we are ready to define a slight modification of (WCPP), that we call the *relaxed weighted central path problem (RWCPP)*, as follows

$$-M\mathbf{u} + \mathbf{v} = \mathbf{q}, \quad \mathbf{u}, \mathbf{v} \geq \mathbf{0}, \quad \mathbf{u}\,\mathbf{v} \geq \mathbf{w}^2, \tag{$RWCPP$}$$

where $\mathbf{w} \in \mathbb{R}^n$. Thus, $\mathbf{w}^2$ serves as a replacement of an old nonnegative vector $\mathbf{w}$ in ($WCPP$). Clearly, the solution set of ($RWCPP$) contains the solution of the corresponding ($WCPP$), and it is non-empty if the IPC holds. Furthermore, it is easy

[2] This problem is a ($WLCP$) with $\mathbf{w} > \mathbf{0}$.

[3] Since Illés et al. [14] did not publish their approach, the details of the proof can be found in the thesis of his former PhD student [22].

to show that the nonlinear inequalities

$$u_i\, v_i - w_i^2 \geq 0,$$

define a convex cone in $\mathbb{R}^3$ [25], which is a second-order cone. Thus, the solution set of (RWCPP) is a convex set. Since M is a positive semidefinite matrix, $\mathbf{u}^T(M\mathbf{u} + \mathbf{q})$ is a convex function of $\mathbf{u}$, thus, the level set

$$\mathcal{L}_{w_0} = \{(\mathbf{u}, \mathbf{v}) \in \mathcal{F} : \mathbf{u}^T\mathbf{v} \leq w_0\},$$

for all $w_0 \in \mathbb{R}$ is a convex set. Furthermore, if $w_0 \geq 0$, then it is nonempty and compact. Taking into consideration the definition of (RWCPP) and the level set $\mathcal{L}_{w_0}$, we can define the following *convex feasibility problem (CFP)*

$$-M\mathbf{u} + \mathbf{v} = \mathbf{q}, \quad \mathbf{u}, \mathbf{v} \geq \mathbf{0}, \quad \mathbf{u}\,\mathbf{v} \geq \mathbf{w}^2, \quad \text{and} \quad w_0 \geq \mathbf{u}^T\mathbf{v}. \tag{CFP}$$

The next statement follows from our construction and the unique solvability of (WCPP).

Proposition 2.1 *Let us assume that $\mathcal{F}_+ \neq \emptyset$. For a given pair of $(w_0, \mathbf{w})$ (CFP) has feasible solution if and only if $w_0 \geq \|\mathbf{w}\|^2$.*

From the linear constraint of the monotone WLCP, we can express the variable $\mathbf{v}$ as $\mathbf{v} = \mathbf{q} + M\,\mathbf{u}$ and we can reformulate (CFP) in the following way

$$\mathbf{q} + M\mathbf{u} \geq \mathbf{0}, \quad \mathbf{u} \geq \mathbf{0}, \quad \mathbf{u}\,(\mathbf{q} + M\mathbf{u}) \geq \mathbf{w}^2, \quad \text{and} \quad w_0 \geq \mathbf{q}^T\mathbf{u} + \mathbf{u}^T M\mathbf{u}. \tag{2}$$

When $\mathbf{u}$ is a solution of the (RWCPP), then $w_0 \geq \|\mathbf{w}\|^2$ follows. The $w_0 \geq \|\mathbf{w}\|^2$ is an important inequality defined during our relaxation process for the new variables $\mathbf{t} = (w_0, \mathbf{w})$, that we call the variables in the *parabolic target space*

$$\mathcal{T} = \{\mathbf{t} = (w_0, \mathbf{w}) \in \mathbb{R}^{1+n} : w_0 \geq \|\mathbf{w}\|^2\},$$

to distinguish from the original variables $(\mathbf{u}, \mathbf{v})$ of the LCP. Clearly, $\mathcal{T}$ is a convex set.

Now, we shall define all those vectors that satisfy the system of convex inequalities (2) as

$$\mathcal{F}_z = \{\mathbf{z} = (\mathbf{u}, \mathbf{t}) \in \mathbb{R}^n \times \mathbb{R}^{1+n} :\ \mathbf{q} + M\mathbf{u} \geq \mathbf{0},\ \mathbf{u} \geq \mathbf{0},\ \mathbf{u}\,(\mathbf{q} + M\mathbf{u}) \geq \mathbf{w}^2,\ w_0 \geq \mathbf{q}^T\mathbf{u} + \mathbf{u}^T M\mathbf{u}\}.$$

Clearly, from $\mathcal{F}^+ \neq \emptyset$ follows that $\mathcal{F}_z$, has an interior point solution, too. The convex set $\mathcal{F}_z$ admits a standard self-concordant barrier ($M_F = 1$)

$$F(\mathbf{z}) \;=\; F(\mathbf{u}, \mathbf{t}) = -\ln\left(w_0 - \mathbf{u}^T(\mathbf{q} + M\mathbf{u})\right) - \sum_{i=1}^{n} \ln\left(u_i\,(\mathbf{q} + M\mathbf{u})_i - w_i^2\right),$$

with barrier parameter $\nu_F = 2n + 1$.

We will use the restriction of the function F on the $\mathbf{u}$-space defined as $F_{\mathbf{t}} = F(.,\mathbf{t}) : \mathbb{R}^n \to \mathbb{R}$ for a fixed vector $\mathbf{t} \in \mathbb{R}^{1+n}$. Since F is a self-concordant function with $M_F = 1$, then, Theorem 7.1 in Appendix II implies that the function $F_{\mathbf{t}}$ is a self-concordant function, with $M_{F_{\mathbf{t}}} = 1$, as well.

Now, we can define the control barrier function, $\phi : \mathbb{R}^{1+n} \to \mathbb{R}$ as follows

$$\phi(\mathbf{t}) = \min_{\mathbf{u}:(\mathbf{u},\mathbf{t})\in\mathcal{F}_z} F(\mathbf{u},\mathbf{t}). \tag{3}$$

In what follows, we use the notation $\mathbf{z}(\mathbf{t}) = (\mathbf{u}(\mathbf{t}), \mathbf{t})$ for the optimal solution of (3), and $\mathbf{v}(\mathbf{t}) = M\mathbf{u}(\mathbf{t}) + \mathbf{q}$. Note that

$$\nabla\phi(\mathbf{t}) = \nabla_2 F\left(\mathbf{u}(\mathbf{t}),\mathbf{t}\right), \qquad \mathbf{t} \in \operatorname{dom}\phi, \tag{4}$$

$$\nabla^2\phi(\mathbf{t}) = \nabla^2_{22} F\left(\mathbf{u}(\mathbf{t}),\mathbf{t}\right) - \nabla^2_{21} F\left(\mathbf{u}(\mathbf{t}),\mathbf{t}\right)\left[\nabla^2_{11} F\left(\mathbf{u}(\mathbf{t}),\mathbf{t}\right)\right]^{-1}\nabla^2_{12} F\left(\mathbf{u}(\mathbf{t}),\mathbf{t}\right). \tag{5}$$

As in [25], function $\phi(\cdot)$ has a closed form representation.

Theorem 2.2 *Let us assume that $\mathcal{F}_+ \neq \emptyset$, then $\operatorname{dom}\phi = \operatorname{int}(\mathcal{T}) \neq \emptyset$ and the optimization problem (3) has a unique solution, and the corresponding optimal vector $\mathbf{u}(\mathbf{t})$ satisfies the following equation*

$$\mathbf{u}(\mathbf{t})\left(\mathbf{q} + M\mathbf{u}(\mathbf{t})\right) = \mathbf{u}(\mathbf{t})\,\mathbf{v}(\mathbf{t}) = \mathbf{w}^2 + \frac{w_0 - \|\mathbf{w}\|^2}{n+1}\mathbf{e}. \tag{6}$$

Moreover, for all $\mathbf{t} \in \operatorname{dom}\phi$, we have

$$\phi(w_0,\mathbf{w}) = -(n+1)\ln\frac{w_0 - \|\mathbf{w}\|^2}{n+1}. \tag{7}$$

Proof We follow the proof of Lemma 1 in [25]. Let $\mathbf{t} = (w_0, \mathbf{w}) \in \operatorname{int}\mathcal{T}$, $\varepsilon = \frac{1}{2n}(w_0 - \|\mathbf{w}\|^2)$ and $\bar{\mathbf{w}} = \mathbf{w}^2 + \varepsilon\mathbf{e}$. Using Theorem 2.1 there exists $\bar{\mathbf{u}} > \mathbf{0}$ and $\bar{\mathbf{v}} = M\bar{\mathbf{u}} + \mathbf{q} > \mathbf{0}$ such that $\bar{\mathbf{u}}\,\bar{\mathbf{v}} = \bar{\mathbf{w}}$. Note that for $(\bar{\mathbf{u}}, \mathbf{t})$ the function F is well defined. Moreover, from the unique correspondence between $\bar{\mathbf{u}}$ and $\bar{\mathbf{w}}$, the optimization problem (3) becomes

$$\phi(\mathbf{t}) = \min_{\bar{\mathbf{w}}>\mathbf{0}}\left[-\ln\left(w_0 - \sum_{i=1}^n \bar{w}_i\right) - \sum_{i=1}^n \ln\left(\bar{w}_i - w_i^2\right)\right]. \tag{8}$$

The first order optimality conditions of (8) are

$$\frac{1}{w_0 - \sum_{i=1}^n \bar{w}_i} - \frac{1}{\bar{w}_i - w_i^2} = 0, \qquad i = 1, 2, \ldots, n$$

and the optimal vector $\bar{\mathbf{w}}^*$ can be found from the following equations

$$w_0 - \sum_{i=1}^{n} \bar{w}_i^* = \bar{w}_i^* - w_i^2, \qquad i = 1, 2, \dots, n.$$

That is, $\bar{\mathbf{w}}^* = \mathbf{w}^2 + \frac{w_0 - \|\mathbf{w}\|^2}{n+1}\,\mathbf{e}$, proving (6). Furthermore,

$$\bar{w}_i^* - w_i^2 = \frac{w_0 - \|\mathbf{w}\|^2}{n+1}, \qquad i = 1, 2, \dots, n,$$

and then using (8), we obtain (7). □

It can be observed that in (6) and (7) the difference of w_0 and $\|\mathbf{w}\|^2$ appears. For this reason, let us introduce the function $\rho(\mathbf{t}) = \rho(w_0, \mathbf{w}) = w_0 - \|\mathbf{w}\|^2$, which in some sense could serve as a measure of the distance from the boundary of the parabolic target space $\mathcal{T}$.

Note that the relation (6) works in two ways. Indeed, for $\mathbf{t} \in \operatorname{dom} \phi$, we can easily compute the right-hand side of this equality, which gives us the exact value of the product of unknown vectors $\mathbf{u}(\mathbf{t})$ and $\mathbf{v}(\mathbf{t}) = \mathbf{q} + M\mathbf{u}(\mathbf{t})$.

On the other hand, if we have $\mathbf{u} > \mathbf{0}$ with $\mathbf{v} = M\,\mathbf{u} + \mathbf{q} > \mathbf{0}$, that are (strictly) feasible solutions of the WLCP, then it is always possible to find a vector $\mathbf{t}(\mathbf{u}) \in \operatorname{dom} \phi$, such that $\mathbf{u} = \mathbf{u}(\mathbf{t}(\mathbf{u}))$ and $\mathbf{v} = \mathbf{v}(\mathbf{t}(\mathbf{u}))$. Indeed, define

$$\xi(\mathbf{u}) = \min_{1 \le i \le n} \; u_i\,(\mathbf{q} + M\mathbf{u})_i \; > \; 0.$$

Then, we can define

$$\mathbf{w}(\mathbf{u}) = \Big[\mathbf{u}\,(\mathbf{q} + M\mathbf{u}) - \xi(\mathbf{u})\,\mathbf{e}\Big]^{1/2}, \qquad w_0(\mathbf{u}) = \mathbf{u}^T(\mathbf{q} + M\mathbf{u}) + \xi(\mathbf{u}). \tag{9}$$

It is easy to see that

$$\xi(\mathbf{u}) = \frac{w_0(\mathbf{u}) - \|\mathbf{w}(\mathbf{u})\|^2}{n+1}. \tag{10}$$

Therefore, by denoting $\mathbf{t}(\mathbf{u}) = (w_0(\mathbf{u}), \mathbf{w}(\mathbf{u}))$, we have $\mathbf{u} = \mathbf{u}(\mathbf{t}(\mathbf{u}))$ and $\mathbf{v} = \mathbf{v}(\mathbf{t}(\mathbf{u}))$.

3 Parabolic Target-Space Interior Point Algorithm

In this section, we are interested in tracing the surface $\mathbf{u}(\mathbf{t})$ for $\mathbf{t} \in \mathcal{T}$. In fact, we cannot compute the point $\mathbf{u}(\mathbf{t})$ exactly. However, for our goals, it is sufficient to give the update strategies only for an approximate $\mathbf{u}$ to this surface. The closeness of $\mathbf{u}$ to $\mathbf{u}(\mathbf{t})$ can be measured by the (dual) local norm of the partial gradient

$$\lambda_{\mathbf{t}}(\mathbf{u}) := \|\nabla_1 F(\mathbf{u}, \mathbf{t})\|_{[\nabla_{11}^2 F(\mathbf{u},\mathbf{t})]^{-1}} = \sqrt{\nabla_1 F(\mathbf{u}, \mathbf{t})^T [\nabla_{11}^2 F(\mathbf{u}, \mathbf{t})]^{-1} \nabla_1 F(\mathbf{u}, \mathbf{t})}$$

based on the self-concordant property of function F. The (dual) local norm $\lambda_{\mathbf{t}}(\mathbf{u})$ is called the partial Newton decrement of the function F. Observe that $\lambda_{\mathbf{t}}(\mathbf{u})$ is the Newton decrement of the function $F_{\mathbf{t}} = F(., \mathbf{t})$, namely $\lambda_{\mathbf{t}}(\mathbf{u}) = \|\nabla F_{\mathbf{t}}(\mathbf{u})\|_{[\nabla^2 F_{\mathbf{t}}(\mathbf{u})]^{-1}}$. For more details, see Appendix II.

Using $\lambda_{\mathbf{t}}(\mathbf{u})$ for a given solution $\mathbf{z} = (\mathbf{u}, \mathbf{t}) \in \mathcal{F}_z$, we can define a neighbourhood of $\mathbf{u}(\mathbf{t})$:

$$\mathcal{N}_\lambda(\beta) = \{\mathbf{z} = (\mathbf{u}, \mathbf{t}) \in \mathcal{F}_z : \; \lambda_{\mathbf{t}}(\mathbf{u}) \leq \beta\}$$

which is called the λ-*neighbourhood* of the point $\mathbf{u}(\mathbf{t})$, where $0 < \beta < 1$.

However, since the exact value $F(\mathbf{u}(\mathbf{t}), \mathbf{t}) = \phi(\mathbf{t})$ is known, it is possible to use a *functional proximity measure*

$$\Psi(\mathbf{z}) = F(\mathbf{u}, \mathbf{t}) - \phi(\mathbf{t}) \geq 0, \quad \mathbf{z} = (\mathbf{u}, \mathbf{t}) \in \mathcal{F}_z.$$

Based on the functional proximity measure $\Psi(\mathbf{z})$ we define the following set:

$$\mathcal{W}(\gamma_1, \gamma_2) = \{\mathbf{z} = (\mathbf{u}, \mathbf{t}) \in \mathcal{F}_z : \; \gamma_1 \leq \Psi(\mathbf{z}) \leq \gamma_2\},$$

where $0 \leq \gamma_1 \leq \gamma_2$.

Lemma 3.1 *Let* $0 < \beta < 1$ *and* $\mathbf{z} = (\mathbf{u}, \mathbf{t}) \in \mathcal{N}_\lambda(\beta)$. *Then,*

$$\omega\big(\lambda_{\mathbf{t}}(\mathbf{u})\big) \leq \Psi(\mathbf{z}) \; \leq \; \omega_*\big(\lambda_{\mathbf{t}}(\mathbf{u})\big). \tag{11}$$

Proof The functions ω and ω_* are defined in the Appendix II. Feasible solutions $\mathbf{z}(\mathbf{t}) = (\mathbf{u}(\mathbf{t}), \mathbf{t})$ and $\mathbf{z} = (\mathbf{u}, \mathbf{t})$ are given. Since $\mathbf{z}(\mathbf{t}) = (\mathbf{u}(\mathbf{t}), \mathbf{t})$ is optimal solution of the problem (3), we have $\phi(\mathbf{t}) = F(\mathbf{z}(\mathbf{t})) = F(\mathbf{u}(\mathbf{t}), \mathbf{t})$ and $\nabla_1 F(\mathbf{u}(\mathbf{t}), \mathbf{t}) = \mathbf{0}$.

Now, let us apply Theorem 7.9 to the self-concordant function $F_{\mathbf{t}}$ at points $\mathbf{u}, \mathbf{u}(\mathbf{t}) \in$ dom $F_{\mathbf{t}}$. From the inequality (50), with $M_{F_{\mathbf{t}}} = 1$, follows that

$$F_{\mathbf{t}}(\mathbf{u}) \geq F_{\mathbf{t}}(\mathbf{u}(\mathbf{t})) + (\mathbf{u} - \mathbf{u}(\mathbf{t}))^T \nabla F_{\mathbf{t}}(\mathbf{u}(\mathbf{t})) + \omega\left(\|\nabla F_{\mathbf{t}}(\mathbf{u}) - \nabla F_{\mathbf{t}}(\mathbf{u}(\mathbf{t}))\|_{[\nabla^2 F_{\mathbf{t}}(\mathbf{u})]^{-1}}\right).$$

Clearly, $\nabla F_{\mathbf{t}}(\mathbf{u}(\mathbf{t})) = \nabla_1 F(\mathbf{u}(\mathbf{t}), \mathbf{t}) = \mathbf{0}$, thus, the second term on the right side of the inequality is 0. Rearranging the inequality, we get

$$F(\mathbf{u}, \mathbf{t}) - F(\mathbf{u}(\mathbf{t}), \mathbf{t}) = F_{\mathbf{t}}(\mathbf{u}) - F_{\mathbf{t}}(\mathbf{u}(\mathbf{t})) \geq \omega\left(\|\nabla F_{\mathbf{t}}(\mathbf{u})\|_{[\nabla^2 F_{\mathbf{t}}(\mathbf{u})]^{-1}}\right) = \omega\big(\lambda_{\mathbf{t}}(\mathbf{u})\big).$$

Therefore,

$$\Psi(\mathbf{z}) = F(\mathbf{u}, \mathbf{t}) - \phi(\mathbf{t}) \geq \omega\big(\lambda_{\mathbf{t}}(\mathbf{u})\big),$$

proving the lower bound in (11).

Taking into consideration that $\mathbf{z} = (\mathbf{u}, \mathbf{t}) \in \mathcal{N}_\lambda(\beta)$, we have $\lambda_{\mathbf{t}}(\mathbf{u}) \leq \beta < 1$, so the second inequality of Theorem 7.9 can be used with similar computations as before, getting the

$$\Psi(\mathbf{z}) \leq \omega_*\big(\lambda_{\mathbf{t}}(\mathbf{u})\big)$$

upper bound in (11). ◻

Now, we are ready to present our, new *Adaptive Parabolic Target-Space Interior Point Algorithm (PTS IPA)* to solve monotone WLCPs.

3.1 PTS IPA for monotone WLCPs

We define a vector $\mathbf{w}^*$ to the weight vector $\mathbf{p}$ of the WLCP such that $\mathbf{p} = (\mathbf{w}^*)^2$. The vector $(\mathbf{w}^*)^2$ is called the target vector. The algorithm starts with a given point $\mathbf{z}^{(0)} = (\mathbf{u}^{(0)}, \mathbf{t}^{(0)}) \in \operatorname{int} \mathcal{F}_z$, so it satisfies all inequalities in (2) as strict inequalities. Starting from $\mathbf{u}^{(0)} > \mathbf{0}$ with $\mathbf{v}^{(0)} = M\,\mathbf{u}^{(0)} + \mathbf{q} > \mathbf{0}$, we can compute a $\mathbf{t}^{(0)}$ by using (9). The algorithm consists of two different types of iterations. In the outer loop, the goal is to use such a direction in the parabolic target space that ensures a large enough decrease in the stopping criteria, namely the new iterates come closer to the target vector $(\mathbf{w}^*)^2 \in \mathbb{R}^n_+$ with respect to some proximity measure. Although, the result of such computation achieves its goal of better approximating the target value, it may end up in a vector that is not well centered in the sense of the Newton decrement, $\lambda_{\mathbf{t}}(\mathbf{u})$. The goal of the inner iteration is to restore this "well centered" property of the computed new solution, namely to ensure that the computed solution belongs to the λ-neighbourhood $\mathcal{N}_\lambda(\beta)$ of the current point $\mathbf{u}(\mathbf{t})$. During the inner loop, the parameter vector $\mathbf{t}$ of the parabolic target space does not change.

Algorithm 1: Adaptive Parabolic Target-Space Interior Point Algorithm for Monotone WLCPs

Input: the initial point $\mathbf{z}^{(0)} = (\mathbf{u}^{(0)}, \mathbf{t}^{(0)}) \in \operatorname{int} \mathcal{F}_z$,
the target vector $(\mathbf{w}^*)^2 \in \mathbb{R}^n_+$ such that $(\mathbf{w}^*)^2 = \mathbf{p}$,
the accuracy parameter $\varepsilon > 0$,
the margins $\delta_u \geq \delta_l$,
the proximity level $\beta \in \left(0, \frac{1}{2}\right)$: $0 < \frac{2\beta^2}{1-2\beta} \leq \delta_l$.

Iteration:
$\mathbf{z} := \mathbf{z}^{(0)}$.
while $\|\mathbf{u}\,\mathbf{v} - (\mathbf{w}^*)^2\| > \varepsilon$ **do**
  Choose a target direction $\Delta\mathbf{t} \in \mathbb{R}^{n+1} \setminus \{\mathbf{0}\}$.
  Compute the direction $\mathbf{d} = (\Delta\mathbf{u}, \Delta\mathbf{t}) \in \mathbb{R}^{2n+1}$ as
  $$\mathbf{d} = \begin{pmatrix} \Delta\mathbf{u} \\ \Delta\mathbf{t} \end{pmatrix} = \begin{pmatrix} -[\nabla^2_{11} F(\mathbf{z})]^{-1}\, \nabla^2_{12} F(\mathbf{z})\, \Delta\mathbf{t} \\ \Delta\mathbf{t} \end{pmatrix}.$$
  Find a step length $\alpha > 0$ such that $\hat{\mathbf{z}} := \mathbf{z} + \alpha\,\mathbf{d} \in \mathcal{W}(\delta_l, \delta_u)$.
  Make an update $\hat{\mathbf{t}} := \mathbf{t} + \alpha\,\Delta\mathbf{t}$ and $\hat{\mathbf{u}} := \mathbf{u} + \alpha\,\Delta\mathbf{u}$.
  while $\lambda_{\hat{\mathbf{t}}}(\hat{\mathbf{u}}) > \beta$ **do**
    Compute the direction $\Delta\hat{\mathbf{u}} = -\left[\nabla^2_{11} F(\hat{\mathbf{u}}, \hat{\mathbf{t}})\right]^{-1} \nabla_1 F(\hat{\mathbf{u}}, \hat{\mathbf{t}})$.
    Compute the step size as $\hat{\alpha} = \frac{1}{1+\lambda_{\hat{\mathbf{t}}}(\hat{\mathbf{u}})}$.
    Make an update $\hat{\mathbf{u}} := \hat{\mathbf{u}} + \hat{\alpha}\,\Delta\hat{\mathbf{u}}$.
  end while
  Set $\mathbf{z} := (\hat{\mathbf{u}}, \hat{\mathbf{t}})$.
end while

In the next section, we will analyse the validity and convergence of Algorithm 1.

4 Analysis of the Algorithm 1

In the analysis of our predictor-corrector strategy, we use both measurements $\lambda_{\mathbf{t}}(\mathbf{u})$ and $\Psi(\mathbf{z})$. Let us analyse the performance of Algorithm 1 to solve monotone WLCPs.

4.1 Analysis of the corrector step

At the start of the analysis, let us first observe that the process at the inner while loop (corrector step) is a standard Damped Newton Method (DNM) (see Appendix II or pages 348-349 in [26]).

Lemma 4.1 *Let us assume that $\hat{\mathbf{z}} \in \mathcal{W}(\delta_l, \delta_u) \setminus \mathcal{N}_\lambda(\beta)$, where $\beta \in \left(0, \frac{1}{2}\right)$ such that $0 < \frac{2\beta^2}{1-2\beta} \leq \delta_l$. Let k denote the number of steps in an inner loop. Then, we need*

$$k \leq \left\lfloor \frac{\delta_u}{\omega(\beta)} \right\rfloor + 1 \tag{12}$$

corrector steps in order to have $\mathbf{z} \in \mathcal{N}_\lambda(\beta)$.

Proof In the inner while loop, we are minimizing the self-concordant function $F_{\hat{\mathbf{t}}}$, where $\hat{\mathbf{z}} = (\hat{\mathbf{u}}, \hat{\mathbf{t}}) \in \mathcal{F}_z$.

Based on Theorem 7.10, the full Newton step is feasible. Since dom $F_{\hat{\mathbf{t}}}$ is convex, a damped Newton step also gives a feasible solution. From Theorem 7.11 and using the monotone increasing property of the function ω, for two consecutive solutions of the inner loop $\bar{\mathbf{z}} = (\bar{\mathbf{u}}, \hat{\mathbf{t}})$ and $\mathbf{z}^+ = (\mathbf{u}^+, \hat{\mathbf{t}})$ we have

$$F_{\hat{\mathbf{t}}}(\bar{\mathbf{u}}) - F_{\hat{\mathbf{t}}}(\mathbf{u}^+) = F(\bar{\mathbf{u}}, \hat{\mathbf{t}}) - F(\mathbf{u}^+, \hat{\mathbf{t}}) \geq \omega\big(\lambda_{\hat{\mathbf{t}}}(\bar{\mathbf{u}})\big) \geq \omega(\beta). \tag{13}$$

Hence, the decrease of the self-concordant function F after each step in the inner loop is at least $\omega(\beta) > 0$, until we reach the neighborhood $\mathcal{N}_\lambda(\beta)$, thus, k is finite. Since k is the first index for which $\hat{\mathbf{z}}^{(k)} \in \mathcal{N}_\lambda(\beta)$, the iterates $\hat{\mathbf{z}} = \hat{\mathbf{z}}^{(0)} = (\hat{\mathbf{u}}^{(0)}, \hat{\mathbf{t}})$, $\hat{\mathbf{z}}^{(1)} = (\hat{\mathbf{u}}^{(1)}, \hat{\mathbf{t}}), \ldots, \hat{\mathbf{z}}^{(k-1)} = (\hat{\mathbf{u}}^{(k-1)}, \hat{\mathbf{t}}) \notin \mathcal{N}_\lambda(\beta)$ and $\hat{\mathbf{z}}^{(k)} = (\hat{\mathbf{u}}^{(k)}, \hat{\mathbf{t}}) \in \mathcal{N}_\lambda(\beta)$. On the other hand, we know that the starting point of the inner loop $\hat{\mathbf{z}}^{(0)} = (\hat{\mathbf{u}}^{(0)}, \hat{\mathbf{t}}) \in \mathcal{W}(\delta_l, \delta_u)$, thus

$$\delta_u \geq \Psi(\hat{\mathbf{z}}^{(0)}) = F(\hat{\mathbf{u}}^{(0)}, \hat{\mathbf{t}}) - \phi(\hat{\mathbf{t}}) \geq F(\hat{\mathbf{u}}^{(0)}, \hat{\mathbf{t}}) - F(\hat{\mathbf{u}}^{(i)}, \hat{\mathbf{t}}), \qquad i = 1, \ldots, k-1, \tag{14}$$

since $\phi(\hat{\mathbf{t}}) \leq F(\hat{\mathbf{u}}^{(i)}, \hat{\mathbf{t}})$ for any $\hat{\mathbf{z}}^{(i)} = (\hat{\mathbf{u}}^{(i)}, \hat{\mathbf{t}}) \in \mathcal{F}_z$ $(i = 1, 2, \ldots, k-1)$. Using (13) and (14), we have

$$\delta_u \geq F(\hat{\mathbf{u}}^{(0)}, \hat{\mathbf{t}}) - F(\hat{\mathbf{u}}^{(k-1)}, \hat{\mathbf{t}}) \geq (k-1)\,\omega(\beta). \tag{15}$$

Now, the iteration bound of the inner loop follows. □

Inequality (12) shows that we need no more than a constant number of steps in the corrector stage.

4.2 Analysis of the predictor step

It is clear that after the inner loop of Algorithm 1 the solution $\mathbf{z} = (\mathbf{u}, \mathbf{t}) \in \mathcal{N}_\lambda(\beta)$.

Analysing the step of the outer loop of Algorithm 1, we find that the solution $\hat{\mathbf{z}} = (\hat{\mathbf{u}}, \hat{\mathbf{t}}) \in \mathcal{W}(\delta_l, \delta_u)$, due to the fact that the function F is a self-concordant barrier, thus the step length $\alpha > 0$ can be chosen to satisfy the inequality $\delta_l \leq \Psi(\hat{\mathbf{z}}) \leq \delta_u$.

Let us show now that the step length $\alpha_k > 0$ computed at the k^{th} predictor iteration is sufficiently big. For that, we need to compare the size of the steps in $\mathbf{z}$-space $\left(\mathbb{R}^{2n+1}\right)$ with that in $\mathbf{t}$-space $\left(\mathbb{R}^{n+1}\right)$. The local norm in $\mathbf{z}$-space is defined using the self-concordant function F and the set int $\mathcal{F}_z$, while in $\mathbf{t}$-space using ϕ and the set int $\mathcal{T}$.

First, we show that if $\mathbf{z}$ is in the neighborhood $\mathcal{N}_\lambda(\beta)$, then $\mathbf{z}(\mathbf{t})$ is close to $\mathbf{z}$ and that is also true for the gradient vectors.

Lemma 4.2 *Let* $\mathbf{z} = (\mathbf{u}, \mathbf{t}) \in \mathcal{N}_\lambda(\beta)$*, where* $\beta \in \left(0, \frac{1}{2}\right)$*. Then,*

$$\|\mathbf{z} - \mathbf{z}(\mathbf{t})\|_{\nabla^2 F(\mathbf{z})} \leq \frac{\lambda_{\mathbf{t}}(\mathbf{u})}{1 - \lambda_{\mathbf{t}}(\mathbf{u})} \leq \frac{\beta}{1-\beta} < 1, \tag{16}$$

$$\|\nabla F(\mathbf{z}) - \nabla F(\mathbf{z}(\mathbf{t}))\|_{[\nabla^2 F(\mathbf{z})]^{-1}} \leq \frac{\lambda_{\mathbf{t}}(\mathbf{u})}{1 - 2\lambda_{\mathbf{t}}(\mathbf{u})} \leq \frac{\beta}{1-2\beta}. \tag{17}$$

Proof By definition, $\|\mathbf{z} - \mathbf{z}(\mathbf{t})\|_{\nabla^2 F(\mathbf{z})} = \|\mathbf{u} - \mathbf{u}(\mathbf{t})\|_{\nabla^2 F_t(\mathbf{u})}$. Using Corollary 7.6 with the self-concordant function $F_{\mathbf{t}}$, we get

$$\frac{\|\mathbf{u} - \mathbf{u}(\mathbf{t})\|_{\nabla^2 F_t(\mathbf{u})}}{1 + \|\mathbf{u} - \mathbf{u}(\mathbf{t})\|_{\nabla^2 F_t(\mathbf{u})}} \leq \lambda_{\mathbf{t}}(\mathbf{u}).$$

Since $\mathbf{z} \in \mathcal{N}_\lambda(\beta)$, $\lambda_{\mathbf{t}}(\mathbf{u}) \leq \beta < 1$, the inequalities in (16) are proved by rearranging the previous inequality and then substituting the upper bound β on $\lambda_{\mathbf{t}}(\mathbf{u})$.

In a similar way, we can give an estimation for the gradient. Applying Corollary 7.7, we get

$$\frac{\|\nabla F(\mathbf{z}) - \nabla F(\mathbf{z}(\mathbf{t}))\|_{[\nabla^2 F(\mathbf{z})]^{-1}}}{1 + \|\nabla F(\mathbf{z}) - \nabla F(\mathbf{z}(\mathbf{t}))\|_{[\nabla^2 F(\mathbf{z})]^{-1}}} \leq \|\mathbf{z} - \mathbf{z}(\mathbf{t})\|_{\nabla^2 F(\mathbf{z})}.$$

Since $\|\mathbf{z} - \mathbf{z}(\mathbf{t})\|_{\nabla^2 F(\mathbf{z})} < 1$, we can reformulate it as (17) and again substitute the upper bound β. □

To give an estimation on the length of the predictor direction $\mathbf{d}$ by the length of $\Delta\mathbf{t}$ (so considering only its $\mathbf{t}$-space part), we need one simple fact on the monotonicity of Schur complement.

Lemma 4.3 (Schur monotonicity lemma) *Consider two symmetric matrices*

$$Q_i = \begin{bmatrix} A_i & B_i \\ B_i^T & C_i \end{bmatrix} \in \mathbb{R}^{(n+m)\times(n+m)}, \qquad i = 1, 2,$$

such that $0 \prec Q_1 \preceq Q_2$*. Then,* $\left[C_1 - B_1^T A_1^{-1} B_1\right] \preceq \left[C_2 - B_2^T A_2^{-1} B_2\right]$.

Proof Let $X_i = A_i^{-1} B_i$ and $S_i = C_i - B_i^T A_i^{-1} B_i$ $(i = 1, 2)$, then the well-known identity says

$$Q_i = \begin{bmatrix} I & O \\ X_i^T & I \end{bmatrix} \begin{bmatrix} A_i & O \\ O & S_i \end{bmatrix} \begin{bmatrix} I & X_i \\ O & I \end{bmatrix}, \qquad i = 1, 2,$$

where O is the zero matrix and I is the identity matrix with appropriate size. This means by $A_i \succ 0$ that

$$\begin{pmatrix} \mathbf{u} \\ \mathbf{v} \end{pmatrix}^T Q_i \begin{pmatrix} \mathbf{u} \\ \mathbf{v} \end{pmatrix} = (\mathbf{u} + X_i \mathbf{v})^T A_i (\mathbf{u} + X_i \mathbf{v}) + \mathbf{v}^T S_i \mathbf{v} \geq \mathbf{v}^T S_i \mathbf{v}, \qquad i = 1, 2,$$

and the quadratic form is minimal in $\mathbf{u} = -X_i \mathbf{v}$. Therefore, for any $\mathbf{v} \in \mathbb{R}^m$,

$$\begin{aligned} \mathbf{v}^T [S_2 - S_1] \mathbf{v} &= \begin{bmatrix} -X_2 \mathbf{v} \\ \mathbf{v} \end{bmatrix}^T Q_2 \begin{bmatrix} -X_2 \mathbf{v} \\ \mathbf{v} \end{bmatrix} - \begin{bmatrix} -X_1 \mathbf{v} \\ \mathbf{v} \end{bmatrix}^T Q_1 \begin{bmatrix} -X_1 \mathbf{v} \\ \mathbf{v} \end{bmatrix} \\ &\geq \begin{bmatrix} -X_2 \mathbf{v} \\ \mathbf{v} \end{bmatrix}^T (Q_2 - Q_1) \begin{bmatrix} -X_2 \mathbf{v} \\ \mathbf{v} \end{bmatrix} \geq 0, \end{aligned}$$

since $Q_2 - Q_1 \succeq 0$. □

Lemma 4.4 *Let* $\mathbf{z} = (\mathbf{u}, \mathbf{t}) \in \mathcal{N}_\lambda(\beta)$, *where* $\beta \in \left(0, \frac{1}{2}\right)$ *and* $\mathbf{d}$ *be the predictor direction of Algorithm 1. Then,*

$$\|\mathbf{d}\|_{\nabla^2 F(\mathbf{z})} \leq \frac{1 - \beta}{1 - 2\beta} \|\Delta \mathbf{t}\|_{\nabla^2 \phi(\mathbf{t})}. \tag{18}$$

Proof We know that the Hessian $\nabla^2 F(\mathbf{z})$ is nondegenerate for any $\mathbf{z} \in \operatorname{int} \mathcal{F}_z$ (see Theorem 7.2) and based on the result of Theorem 7.4 and (16), we have

$$\nabla^2 F(\mathbf{z}) \preceq \frac{1}{(1 - \|\mathbf{z} - \mathbf{z}(\mathbf{t})\|_{\nabla^2 F(\mathbf{z})})^2} \nabla^2 F(\mathbf{z}(\mathbf{t})) \preceq \left(\frac{1 - \beta}{1 - 2\beta} \right)^2 \nabla^2 F(\mathbf{z}(\mathbf{t})).$$

Therefore, using the monotonicity of Schur complement (Lemma 4.3), we complete the proof:

$$\begin{aligned} \|\mathbf{d}\|^2_{\nabla^2 F(\mathbf{z})} &= \mathbf{d}^T \nabla^2 F(\mathbf{z}) \mathbf{d} = \begin{pmatrix} \Delta \mathbf{u} \\ \Delta \mathbf{t} \end{pmatrix}^T \begin{pmatrix} \nabla^2_{11} F(\mathbf{z}) & \nabla^2_{12} F(\mathbf{z}) \\ \nabla^2_{21} F(\mathbf{z}) & \nabla^2_{22} F(\mathbf{z}) \end{pmatrix} \begin{pmatrix} \Delta \mathbf{u} \\ \Delta \mathbf{t} \end{pmatrix} \\ &= \Delta \mathbf{t}^T \left(\nabla^2_{22} F(\mathbf{z}) - \nabla^2_{21} F(\mathbf{z}) [\nabla^2_{11} F(\mathbf{z})]^{-1} \nabla^2_{12} F(\mathbf{z}) \right) \Delta \mathbf{t} \\ &\leq \left(\frac{1 - \beta}{1 - 2\beta} \right)^2 \Delta \mathbf{t}^T \left(\nabla^2_{22} F(\mathbf{z}(\mathbf{t})) - \nabla^2_{21} F(\mathbf{z}(\mathbf{t})) [\nabla^2_{11} F(\mathbf{z}(\mathbf{t}))]^{-1} \nabla^2_{12} F(\mathbf{z}(\mathbf{t})) \right) \Delta \mathbf{t} \\ &= \left(\frac{1 - \beta}{1 - 2\beta} \right)^2 \Delta \mathbf{t}^T \nabla^2 \phi(\mathbf{t}) \Delta \mathbf{t} = \left(\frac{1 - \beta}{1 - 2\beta} \right)^2 \|\Delta \mathbf{t}\|^2_{\nabla^2 \phi(\mathbf{t})}. \end{aligned}$$

□

Now, we are ready to give a lower bound on the displacement in the $\mathbf{t}$-space.

Theorem 4.5 *In the predictor step of Algorithm 1, if* $\mathbf{z} = (\mathbf{u}, \mathbf{t}) \in \mathcal{N}_\lambda(\beta)$*, with* $\beta \in \left(0, \frac{1}{2}\right)$*, and* $\alpha > 0$ *is such that* $\mathbf{z} + \alpha\, \mathbf{d} \in \mathcal{W}(\delta_l, \delta_u)$ *holds, then*

$$\alpha\, \|\Delta\mathbf{t}\|_{\nabla^2\phi(\mathbf{t})} \;\geq\; \frac{1-2\beta}{1-\beta}\, \omega_*^{-1}\left(\frac{1}{2}\left[\delta_l - \frac{2\beta^2}{1-2\beta}\right]\right) \;>\; 0, \tag{19}$$

where $\Delta\mathbf{t}$ *is the chosen target direction.*

Proof On the contrary, assume that

$$\alpha\, \|\Delta\mathbf{t}\|_{\nabla^2\phi(\mathbf{t})} \;<\; \frac{1-2\beta}{1-\beta}\, \omega_*^{-1}\left(\frac{1}{2}\left[\delta_l - \frac{2\beta^2}{1-2\beta}\right]\right).$$

Using that $\omega_*^{-1}(x) < 1$ for $x > 0$, it means that $\alpha\, \|\Delta\mathbf{t}\|_{\nabla^2\phi(\mathbf{t})} < \frac{1-2\beta}{1-\beta}$. Hence, based on (18), this assumption means that $\alpha\, \|\mathbf{d}\|_{\nabla^2 F(\mathbf{z})} < 1$.

Now, using this, the fact that F is a self-concordant function, and inequality (49) in Theorem 7.8, we have the following inequality for the solutions $\mathbf{z}$ and $\mathbf{z} + \alpha\, \mathbf{d}$.

$$F(\mathbf{z} + \alpha\, \mathbf{d}) \leq F(\mathbf{z}) + \alpha\, \mathbf{d}^T \nabla F(\mathbf{z}) + \omega_*\big(\alpha\, \|\mathbf{d}\|_{\nabla^2 F(\mathbf{z})}\big). \tag{20}$$

Due to the fact that ϕ defined by (3) is a convex function,

$$\phi(\mathbf{t} + \alpha\, \Delta\mathbf{t}) \geq \phi(\mathbf{t}) + \alpha\, \Delta\mathbf{t}^T \nabla\phi(\mathbf{t}) = \phi(\mathbf{t}) + \alpha\, \mathbf{d}^T \nabla F(\mathbf{z}(\mathbf{t})). \tag{21}$$

After subtracting (21) from (20), we get

$$\begin{aligned} \Psi(\mathbf{z} + \alpha\, \mathbf{d}) &= F(\mathbf{z} + \alpha\, \mathbf{d}) - \phi(\mathbf{t} + \alpha\, \Delta\mathbf{t}) \\ &\leq \Psi(\mathbf{z}) + \alpha\, \mathbf{d}^T \big(\nabla F(\mathbf{z}) - \nabla F(\mathbf{z}(\mathbf{t}))\big) + \omega_*\big(\alpha\, \|\mathbf{d}\|_{\nabla^2 F(\mathbf{z})}\big). \end{aligned}$$

Recall that in the predictor step of Algorithm 1 the step length α has been chosen to satisfy the following condition $\mathbf{z} + \alpha\, \mathbf{d} \in \mathcal{W}(\delta_l, \delta_u)$, namely $\delta_l \leq \Psi(\mathbf{z} + \alpha\, \mathbf{d})$. On the other hand, $\mathbf{z} \in \mathcal{N}_\lambda(\beta)$. From Lemma 3.1, we have $\Psi(\mathbf{z}) \leq \omega_*(\lambda_\mathbf{t}) \leq \omega_*(\beta)$ using the monotone increasing property of ω_*. Combining these estimations, we have

$$\delta_l \leq \omega_*(\beta) + \alpha\, \|\mathbf{d}\|_{\nabla^2 F(\mathbf{z})}\, \|\nabla F(\mathbf{z}) - \nabla F(\mathbf{z}(\mathbf{t}))\|_{[\nabla^2 F(\mathbf{z})]^{-1}} + \omega_*\left(\alpha\, \|\mathbf{d}\|_{\nabla^2 F(\mathbf{z})}\right).$$

Now, we can use inequality (40) of Lemma 7.3, since $\alpha\, \|\mathbf{d}\|_{\nabla^2 F(\mathbf{z})} < 1$, and we get

$$\delta_l \leq \omega_*(\beta) + \omega\left(\|\nabla F(\mathbf{z}) - \nabla F(\mathbf{z}(\mathbf{t}))\|_{[\nabla^2 F(\mathbf{z})]^{-1}}\right) + 2\, \omega_*\left(\alpha\, \|\mathbf{d}\|_{\nabla^2 F(\mathbf{z})}\right).$$

Functions ω and ω_* are monotone increasing, so we can use the upper bounds given in (17) and (18), which yields

$$\delta_l \leq \omega_*(\beta) + \omega\left(\frac{\beta}{1-2\beta}\right) + 2\omega_*\left(\frac{1-\beta}{1-2\beta}\alpha\, \|\Delta\mathbf{t}\|_{\nabla^2\phi(\mathbf{t})}\right). \tag{22}$$

We provide an upper bound on the first two terms in (22) using the nonnegativity of the logarithm,

$$\omega_*(\beta) + \omega\left(\frac{\beta}{1-2\beta}\right) = -\beta - \ln(1-\beta) + \frac{\beta}{1-2\beta} - \ln\left(1 + \frac{\beta}{1-2\beta}\right) \leq \frac{2\beta^2}{1-2\beta}.$$

We conclude from (22), that

$$0 < \delta_l - \frac{2\beta^2}{1-2\beta} \leq 2\,\omega_*\left(\frac{1-\beta}{1-2\beta}\alpha\,\|\Delta\mathbf{t}\|_{\nabla^2\phi(\mathbf{t})}\right), \tag{23}$$

which contradicts our initial assumption. □

Corollary 4.6 *In the predictor step of Algorithm 1, if* $\mathbf{z} = (\mathbf{u}, \mathbf{t}) \in \mathcal{N}_\lambda(\beta)$, *with* $\beta \in (0, \frac{1}{2})$, *and* $\alpha > 0$ *is such that* $\mathbf{z} + \alpha\,\mathbf{d} \in \mathcal{W}(\delta_l, \delta_u)$, *where* $\delta_l = (2+\kappa^2)\frac{\beta^2}{1-2\beta}$ *with some* $\kappa > 0$, *then*

$$\alpha\,\|\Delta\mathbf{t}\|_{\nabla^2\phi(\mathbf{t})} \;\geq\; \frac{\beta(1-2\beta)\kappa}{(1-\beta)(1-\beta+\beta\kappa)}, \tag{24}$$

where $\Delta\mathbf{t}$ *is the chosen target direction.*

Proof Based on (19), with the given special value of δ and properties of the function ω_*, we have

$$\begin{aligned}\frac{1-\beta}{1-2\beta}\alpha\,\|\Delta\mathbf{t}\|_{\nabla^2\phi(\mathbf{t})} &\geq \omega_*^{-1}\left(\frac{\kappa^2\beta^2}{2\,(1-2\beta)}\right) \geq \omega_*^{-1}\left(\frac{\kappa^2\beta^2}{2\,(1-\beta)^2}\right)\\ &\geq \omega_*^{-1}\left(\frac{\kappa^2\beta^2}{2\,(1-\beta)(1-\beta+\kappa\beta)}\right).\end{aligned} \tag{25}$$

By inequality (41) of Lemma 7.3, we get

$$\frac{\kappa^2\beta^2}{2\,(1-\beta)(1-\beta+\kappa\beta)} = \frac{\frac{\kappa^2\beta^2}{(1-\beta+\kappa\beta)^2}}{2\left(1-\frac{\kappa\beta}{1-\beta+\kappa\beta}\right)} \geq \omega_*\left(\frac{\kappa\beta}{1-\beta+\kappa\beta}\right).$$

Combining the above inequality with (25), we get (24). □

Remark 4.1 If $\beta = \frac{1}{4}$, $\delta_l = \frac{2+\kappa^2}{8}$, where $\kappa \geq 0$, then in each predictor step of Algorithm 1 to solve monotone WLCPs, we have

$$\alpha\|\Delta\mathbf{t}\|_{\nabla^2\phi(\mathbf{t})} \geq \frac{2\,\kappa}{3\,(3+\kappa)}. \tag{26}$$

5 Derivation of iteration bounds for Algorithm 1

The goal of this section is to give an upper bound on the number of outer iterations. As we have seen in Lemma 4.1, after a finite number of corrector steps, the solution satisfies $\mathbf{z} \in \mathcal{N}_\lambda(\beta)$. Thus, the sequence of points $\mathbf{z}^{(0)}, \mathbf{z}^{(1)}, \mathbf{z}^{(2)}, \ldots, \mathbf{z}^{(k)}$ computed by the algorithm starting from the initial feasible solution $\mathbf{z}^{(0)}$, belongs to $\mathcal{N}_\lambda(\beta) \subset \mathcal{F}_z$. In particular, this means that for $k \geq 0$

$$\mathbf{u}^{(k)} \geq \mathbf{0}, \quad M\mathbf{u}^{(k)}+\mathbf{q} \geq \mathbf{0}, \quad \mathbf{u}^{(k)}(M\mathbf{u}^{(k)}+\mathbf{q}) \geq \left(\mathbf{w}^{(k)}\right)^2, \quad \left(\mathbf{u}^{(k)}\right)^T(M\mathbf{u}^{(k)}+\mathbf{q}) \leq w_0^{(k)}.$$

Thus, if we have a target point $\mathbf{t}^* = (w_0^*, \mathbf{w}^*) \in \operatorname{dom}\phi$, it is enough to study the rate of convergence of $\mathbf{w}^{(k)} \to \mathbf{w}^*$. For obtaining the complexity result, we focus on a special type of updating strategy in the target space, $\mathcal{T}$, namely the *Greedy Step*:

$$\mathbf{t}(\alpha) = \mathbf{t} + \alpha\,(\mathbf{t}^* - \mathbf{t}) = \alpha\,\mathbf{t}^* + (1-\alpha)\,\mathbf{t}, \tag{27}$$

where $\alpha \in (0, 1]$ and $\mathbf{t} = (w_0, \mathbf{w})$ is the current iterate, while $\Delta\mathbf{t} := \mathbf{t}^* - \mathbf{t}$.

Let $w_0^* := \|\mathbf{w}^*\|^2$, then $\rho(\mathbf{t}^*) = 0$, which means that the target point $\mathbf{t}^*$ is on the boundary of $\mathcal{T}$. We introduce the function $\ell^*(\mathbf{t}) := w_0 - w_0^* - 2\,\mathbf{w}^{*T}(\mathbf{w}-\mathbf{w}^*)$. Simple computations show that

$$\ell^*(\mathbf{t}) = w_0 - \|\mathbf{w}\|^2 + \|\mathbf{w}-\mathbf{w}^*\|^2 = \rho(\mathbf{t}) + \|\mathbf{w}-\mathbf{w}^*\|^2.$$

Thus, we have

$$\ell^*(\mathbf{t}) \geq \max\{w_0 - \|\mathbf{w}\|^2, \|\mathbf{w}-\mathbf{w}^*\|^2\} = \max\{\rho(\mathbf{t}), \|\mathbf{w}-\mathbf{w}^*\|^2\}, \qquad \mathbf{t} \in \operatorname{dom}\phi,$$

and clearly $\ell^*(\mathbf{t}^*) = 0$. Hence, we can use $\ell^*(\mathbf{t}^{(k)})$ as a natural measure for the quality of our approximate solutions, since it gives an upper bound for the distance of the point from the boundary and the target point, as well.

Now, we can measure the distance of vector $\mathbf{t} \in \operatorname{dom}\phi$ from the boundary of $\operatorname{dom}\phi$ in the opposite direction (to the greedy one) by defining

$$\underline{\alpha}(\mathbf{t}) = \max\{\alpha \geq 0 : \mathbf{t}(1-\alpha) \in \operatorname{dom}\phi\}.$$

Note that the maximum is achieved when $\mathbf{t}(1-\alpha)$ is on the boundary of $\operatorname{dom}\phi$, so $w_0(1-\alpha) = \|\mathbf{w}(1-\alpha)\|^2$, therefore

$$\underline{\alpha}(\mathbf{t}) = \frac{\ell^*(\mathbf{t})}{\|\mathbf{w}-\mathbf{w}^*\|^2} = \frac{\rho(\mathbf{t})}{\|\mathbf{w}-\mathbf{w}^*\|^2} + 1 > 1. \tag{28}$$

Using the function $\underline{\alpha}$, we can estimate the local norm of the search direction $\Delta\mathbf{t}$.

Lemma 5.1 *Let* $\mathbf{t} \in \operatorname{dom} \Phi$ *and* $\Delta\mathbf{t}$ *be the target direction. Then, we have*

$$\sqrt{\frac{n+1}{2}} \leq \frac{\underline{\alpha}(\mathbf{t})-1}{\underline{\alpha}(\mathbf{t})} \|\Delta\mathbf{t}\|_{\nabla^2\phi(\mathbf{t})} \leq \sqrt{n+1}.$$

Proof Consider the function $\psi(\mathbf{t}) := -\ln \rho(\mathbf{t})$. Let us compute its derivatives at the point $\mathbf{t} \in \operatorname{int}\operatorname{dom}\phi$ along the direction

$$\Delta\mathbf{t} = (w_0^* - w_0, \mathbf{w}^* - \mathbf{w}) = (\|\mathbf{w}^*\|^2 - w_0, \mathbf{w}^* - \mathbf{w})$$

namely, compute the directional derivatives of

$$\psi(\mathbf{t} + \alpha\,\Delta\mathbf{t}) = -\ln(w_0 + \alpha\,\Delta w_0 - \|\mathbf{w} + \alpha\,\Delta\mathbf{w}\|^2).$$

Using (28) we have

$$\begin{aligned}
D\psi(\mathbf{t})[\Delta\mathbf{t}] &= -\frac{\Delta w_0 - 2\,\mathbf{w}^T\Delta\mathbf{w}}{\rho(\mathbf{t})} = -\frac{\|\mathbf{w}-\mathbf{w}^*\|^2 - (w_0 - \|\mathbf{w}\|^2)}{\rho(\mathbf{t})}\\
&= 1 - \frac{\|\mathbf{w}-\mathbf{w}^*\|^2}{\rho(\mathbf{t})},\\
D^2\psi(\mathbf{t})[\Delta\mathbf{t}]^2 &= \frac{\left(\|\mathbf{w}-\mathbf{w}^*\|^2 - \rho(\mathbf{t})\right)^2}{\rho^2(\mathbf{t})} + \frac{2\|\mathbf{w}-\mathbf{w}^*\|^2}{\rho(\mathbf{t})} = \frac{\|\mathbf{w}-\mathbf{w}^*\|^4}{\rho^2(\mathbf{t})} + 1\\
&= \frac{1}{(\underline{\alpha}(\mathbf{t})-1)^2} + 1.
\end{aligned} \tag{29}$$

Using that $D^2\psi(\mathbf{t})[\Delta\mathbf{t}]^2 = \|\Delta\mathbf{t}\|^2_{\nabla^2\psi(\mathbf{t})}$ and the previous equation, we get

$$\left(\frac{\underline{\alpha}(\mathbf{t})-1}{\underline{\alpha}(\mathbf{t})}\right)^2 \|\Delta\mathbf{t}\|^2_{\nabla^2\psi(\mathbf{t})} = \frac{1}{\underline{\alpha}^2(\mathbf{t})} + \left(1 - \frac{1}{\underline{\alpha}(\mathbf{t})}\right)^2.$$

From the previous expression, we obtained the following bounds

$$1 \geq \frac{\underline{\alpha}(\mathbf{t})-1}{\underline{\alpha}(\mathbf{t})} \|\Delta\mathbf{t}\|_{\nabla^2\psi(\mathbf{t})} \geq \frac{1}{\sqrt{2}}. \tag{30}$$

In Theorem 4.5 the bound is given on $\|\Delta\mathbf{t}\|_{\nabla^2\phi(\mathbf{t})} = D^2\phi(\mathbf{t})[\Delta\mathbf{t}]^2$, where the closed form of the function $\phi(\mathbf{t}) = -(n+1)\ln\frac{\rho(\mathbf{t})}{n+1}$ is stated in (7). Similarities between the functions ψ and ϕ are clear, thus we need to understand the connections between $\|\Delta\mathbf{t}\|_{\nabla^2\psi(\mathbf{t})}$ and $\|\Delta\mathbf{t}\|_{\nabla^2\phi(\mathbf{t})}$. It can be shown that

$$D^2\phi(\mathbf{t})[\Delta\mathbf{t}]^2 = (n+1)\,D^2\psi(\mathbf{t})[\Delta\mathbf{t}]^2, \qquad \text{so} \qquad \|\Delta\mathbf{t}\|_{\nabla^2\phi(\mathbf{t})} = \sqrt{n+1}\,\|\Delta\mathbf{t}\|_{\nabla^2\psi(\mathbf{t})},$$

which combining with (30), completes the proof. □

The expression of the distance function $\underline{\alpha}(\mathbf{t})$ appearing in Lemma 5.1 gives the idea to define the following *merit function*

$$\mu^*(\mathbf{t}) := \frac{\underline{\alpha}(\mathbf{t})}{\underline{\alpha}(\mathbf{t}) - 1}\, \ell^*(\mathbf{t}) = \frac{\ell^*(\mathbf{t})^2}{\ell^*(\mathbf{t}) - \|\mathbf{w} - \mathbf{w}^*\|^2} \geq \ell^*(\mathbf{t}). \tag{31}$$

The following lemma plays an important role in the analysis. It shows that if the step length is big enough, then the decrease of the merit function value $\mu^*(\mathbf{t}(\alpha))$ is large enough.

Lemma 5.2 *Let* $\alpha \in (0, 1)$ *be a feasible step length in the predictor step of Algorithm 1. If* $\alpha \geq \gamma\, \frac{\underline{\alpha}(\mathbf{t})-1}{\underline{\alpha}(\mathbf{t})}$*, then* $\mu^*(\mathbf{t}(\alpha)) \leq \frac{1}{1+\gamma}\, \mu^*(\mathbf{t})$ *with some* $\gamma \in (0, 1)$.

Proof Note that $\ell^*(\mathbf{t}(\alpha)) = (1-\alpha)\ell^*(\mathbf{t})$ and $\|\mathbf{w}(\alpha) - \mathbf{w}^*\| = (1-\alpha)\|\mathbf{w} - \mathbf{w}^*\|$, thus $\underline{\alpha}(\mathbf{t}(\alpha)) = \frac{\underline{\alpha}(\mathbf{t})}{1-\alpha} > \underline{\alpha}(\mathbf{t})$. Using the definition of the merit function μ^*, we have

$$\mu^*(\mathbf{t}(\alpha)) = \frac{\underline{\alpha}(\mathbf{t}(\alpha))}{\underline{\alpha}(\mathbf{t}(\alpha)) - 1}\, \ell^*(\mathbf{t}(\alpha)). \tag{32}$$

We estimate the two factors from (32) separately. Using the lower bound on the step length α, we get

$$\begin{aligned} \frac{\underline{\alpha}(\mathbf{t}(\alpha)) - 1}{\underline{\alpha}(\mathbf{t}(\alpha))} &= 1 - \frac{1-\alpha}{\underline{\alpha}(\mathbf{t})} \\ &\geq 1 - \frac{1}{\underline{\alpha}(\mathbf{t})} + \gamma\, \frac{\underline{\alpha}(\mathbf{t}) - 1}{\underline{\alpha}^2(\mathbf{t})} = \left(1 - \frac{1}{\underline{\alpha}(\mathbf{t})}\right)\left(1 + \frac{\gamma}{\underline{\alpha}(\mathbf{t})}\right), \end{aligned} \tag{33}$$

and

$$\ell^*(\mathbf{t}(\alpha)) = (1-\alpha)\ell^*(\mathbf{t}) \leq \left(1 - \gamma\, \frac{\underline{\alpha}(\mathbf{t}) - 1}{\underline{\alpha}(\mathbf{t})}\right) \ell^*(\mathbf{t}). \tag{34}$$

Substituting the bounds obtained in (33) and (34) into (32), we get

$$\begin{aligned} \mu^*(\mathbf{t}(\alpha)) &\leq \frac{1 - \gamma\, \frac{\underline{\alpha}(\mathbf{t})-1}{\underline{\alpha}(\mathbf{t})}}{\left(1 - \frac{1}{\underline{\alpha}(\mathbf{t})}\right)\left(1 + \frac{\gamma}{\underline{\alpha}(\mathbf{t})}\right)}\, \ell^*(\mathbf{t}) = \frac{\underline{\alpha}(\mathbf{t}) - \gamma(\underline{\alpha}(\mathbf{t}) - 1)}{\underline{\alpha}(\mathbf{t}) + \gamma}\, \mu^*(\mathbf{t}) \\ &= \left(1 - \frac{\underline{\alpha}(\mathbf{t})\gamma}{\underline{\alpha}(\mathbf{t}) + \gamma}\right) \mu^*(\mathbf{t}) \leq \frac{1}{1+\gamma}\, \mu^*(\mathbf{t}), \end{aligned} \tag{35}$$

since $1 - x\gamma/(x + \gamma)$ is a monotone decreasing function for $x \geq 1$ and $\underline{\alpha}(t) > 1$ by (28). $\square$

Let us show now that the condition of Lemma 5.2 can be derived from inequality (26). Indeed, we have

$$\alpha_k \geq \frac{1}{\|\Delta \mathbf{t}_k\|_{\nabla^2 \phi(\mathbf{t}_k)}}\, \frac{2\kappa}{3\,(3+\kappa)} \geq \frac{\underline{\alpha}(\mathbf{t}) - 1}{\underline{\alpha}(\mathbf{t})}\, \gamma, \tag{36}$$

with $\gamma = \frac{2\,\kappa}{3\,(3+\kappa)\,\sqrt{n+1}}$.

In the following theorem, we derive an upper bound on the total number of iterations.

Theorem 5.3 *Let $\beta = \frac{1}{4}$, $\delta_u = \eta\,\delta_l$, where $\delta_l = \frac{2+\kappa^2}{8}$ and $\eta \geq 1$, $\kappa > 0$, $\gamma = \frac{2\kappa}{3(3+\kappa)\sqrt{n+1}}$ and let $\mathbf{z}^{(0)} = (\mathbf{u}^{(0)}, \mathbf{t}^{(0)}) \in \mathcal{F}_z$ be the starting point. Then, Algorithm 1 performs at most*

$$\frac{1}{\gamma}\log_2 \frac{\mu^*(\mathbf{t}^{(0)})}{\varepsilon} = O\left(\sqrt{n}\ln \frac{\mu^*(\mathbf{t}^{(0)})}{\varepsilon}\right)$$

iterations.

Proof Using (36) and Lemma 5.2, we get that after each iteration $\mu^*(\mathbf{t}(\alpha)) \leq \frac{1}{1+\gamma}\,\mu^*(\mathbf{t})$. Hence, $\mu^*(\mathbf{t}) \leq \varepsilon$ if

$$\left(\frac{1}{1+\gamma}\right)^k \mu^*(\mathbf{t}^{(0)}) \leq \varepsilon.$$

Rearranging the inequality, we obtain that $\mu^*(\mathbf{t}) \leq \varepsilon$ holds if

$$k \geq \frac{1}{\ln(1+\gamma)} \ln\left(\frac{\mu^*(\mathbf{t}^{(0)})}{\varepsilon}\right).$$

Using $\ln(1+\gamma) \geq \gamma\,\ln 2$ for $\gamma \in (0, 1)$, we obtain the result. □

6 Numerical results

We tested Algorithm 1 on two sets of instances: in the first case, we used matrices given on the website [33], while in the second case, we randomly generated the coefficient matrix of (WLCP). We considered $\beta = \frac{1}{4}$, $\delta_l = 0.9$, $\delta_u = 1$ and $\varepsilon = 10^{-8}$.

The website [33] gives a detailed description of how the collected matrices were generated. All the matrices are either bisymmetric or symmetric PSD. For these instances, we generated the vector $\mathbf{q}$ as $-M\mathbf{e} + \mathbf{e}$ and used the vector $\mathbf{e}$ as the initial solution.

A total of 66 problems (21 symmetric PSD and 45 bisymmetric) were considered, 47 of which were solved by the algorithm with an accuracy of at least 10^{-6} (see Tables 1 and 2), while in case of 7 problems we got a solution with accuracy worse than 10^{-3} (see Table 3). The required accuracy in all these runs was 10^{-8}. The earlier terminations are related with difficulties in linear algebra, which can be eliminated by utilizing more robust linear algebra software and better implementation of other details which can suffer from numerical inaccuracies caused by finite precision arithmetic. As we can see, the total number of solutions of Newton systems is always quite moderate.

Let us present the computational results for randomly generated problems. Recall that we considered the following problem: Find $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n_+ \;:\; \mathbf{u}\mathbf{v} = \mathbf{p}$ with $\mathbf{v} = M\mathbf{u} + \mathbf{q}$, where $\mathbf{p} \in \mathbb{R}^n_+$ and $M + M^T \succeq 0$. A necessary and sufficient condition for the solvability of this problem is the existence of the strictly feasible primal-dual pair $(\hat{\mathbf{u}}, \hat{\mathbf{v}}) \in \mathbb{R}^{2n}_{++}$, such that $\hat{\mathbf{v}} = M\hat{\mathbf{u}} + \mathbf{q}$.

Table 1 Numerical results for symmetric PSD matrices with reached accuracy at least 10^{-6}

Problem	Dim	Predictor	Corrector	Total
TR_PSD_1	27	18	35	53
TR_PSD_2	32	17	32	49
TR_PSD_5	48	20	39	59
TR_PSD_6	47	20	40	60
TR_PSD_7	46	20	40	60
TR_PSD_8	50	34	59	93
TR_PSD_9	50	29	58	87
TR_PSD_10	49	29	58	87
TR_PSD_11	48	34	59	93
TR_PSD_12	74	18	36	54
TR_PSD_13	77	28	48	76
TR_PSD_14	83	21	42	63
TR_PSD_15	91	47	63	110
TR_PSD_16	96	16	33	49
TR_PSD_17	97	19	38	57
TR_PSD_20	50	29	58	87

Table 2 Numerical results for bisymmetric matrices with reached accuracy at least 10^{-6}

Problem	Dim	Predictor	Corrector	Total
TR_BS_1,2	59	19	38	57
TR_BS_4	68	19	39	58
TR_BS_5,6	73	17	34	51
TR_BS_7,8	74	24	48	72
TR_BS_9	75	20	84	104
TR_BS_10	75	20	40	60
TR_BS_11,12	80	25	49	74
TR_BS_13,14	81	26	51	77
TR_BS_15	84	16	34	50
TR_BS_16	84	16	33	49
TR_BS_17,18	88	24	46	70
TR_BS_19	89	24	70	94
TR_BS_20	89	19	39	58
TR_BS_21	89	19	39	58
TR_BS_22	95	23	45	68
TR_BS_23	96	24	47	71
TR_BS_24,25	95	23	46	69
TR_BS_28,29	102	29	57	86
TR_BS_30,31	103	24	47	71
TR_BS_42	90	20	42	62
TR_BS_43	90	19	38	57
TR_BS_45	90	24	49	73

Table 3 Numerical results for difficult problems (reached accuracy is worse than 10^{-3})

Problem	Dim	Predictor	Corrector	Total
TR_PSD_3	41	11	23	34
TR_PSD_4	43	7	14	21
TR_PSD_22	79	6	12	18
TR_BS_3	68	14	28	42
TR_BS_26	97	15	30	45
TR_BS_27	97	17	35	52
TR_BS_34	140	23	46	69

Table 4 Average iteration numbers for random problems

Dimension	16	32	64	128	256	512
Predictions	11.3	12.5	14.7	18.5	22.9	29.0
Corrections	22.8	24.9	30.7	50.1	58.0	74.9

Hence, this condition serves as a starting element for our random generator. It performs the following four steps.

1. Choose randomly $\hat{\mathbf{u}}, \hat{\mathbf{v}} \in \mathbb{R}^n_{++}$, whose components are uniformly distributed in $(0, 1)$.

2. Form a random matrix $A \in \mathbb{R}^{n\times n}$ and a lower triangular random matrix $L \in \mathbb{R}^{n\times n}$ with elements uniformly distributed in $(-1, 1)$.

3. Set $M = AA^T + \xi(L - L^T)$, where $\xi \geq 0$ is a parameter. Define $\mathbf{q} = \hat{\mathbf{v}} - M\hat{\mathbf{u}}$.

4. With probability $\pi \in [0, 1]$, decide if the element p_i is positive. Positive p_i are chosen uniformly distributed in $(0, 1]$.

Thus, our generator has three parameters: n, $\xi \geq 0$, and $\pi \in [0, 1]$. For our experiments, they are chosen as follows: $\pi = \frac{1}{2}$, $\xi = 10$.

In this way, we generated 100 problems of each size indicated in Table 4. The columns of Table 4 contain the average numbers of predictor and corrector steps for different dimensions in the series of one hundred random test problems.

As we see, for one predictor step, we need approximately 2–3 corrector steps. With the growth of the dimension, the Newton system inside the algorithm becomes more and more degenerate. Hence, for dimension $n = 512$ our algorithm reached only the accuracy $\varepsilon = 10^{-7}$.

The most interesting observation from our preliminary experiments is the significant acceleration of convergence in the end of the process. We can see it from Table 5, which relates the progress in the proximity measure $\mu^*(\cdot)$, achieved at a random problem of dimension $n = 512$, with the number of predictor and corrector steps. In the last column of Table 5, we can see the relative size of the predictor step as compared with the maximal step preserving feasibility.

In the end of the process, the algorithm often demonstrates a good linear rate. However, the possibility to achieve a local super-linear rate of convergence for methods of this type remains an interesting open question.

Table 5 Acceleration of convergence for a random problem

$\mu^*(\mathbf{t})$	N_{pred}	N_{cor}	$\alpha_{\max}$
$5 \cdot 10^2$	0	0	0.84
$4 \cdot 10^1$	15	32	0.87
$3 \cdot 10^0$	21	45	0.92
$3 \cdot 10^{-1}$	24	51	0.95
$2 \cdot 10^{-2}$	26	55	0.96
$2 \cdot 10^{-3}$	27	57	0.97
$3 \cdot 10^{-4}$	28	59	0.97
$2 \cdot 10^{-5}$	29	61	0.97
$2 \cdot 10^{-6}$	30	63	0.96
$1 \cdot 10^{-7}$	31	65	0.96

7 Conclusion

In this paper, we introduced an IPA for monotone WLCPs (see Algorithm 1) by extending the PTS framework proposed by Nesterov (2008) in [25] for primal-dual LP problems. The method performs two types of steps. In the predictor stage, the goal is to use a search direction in the PTS which ensures a large enough decrease in the stopping criteria. The goal of the corrector stage is to ensure that the computed solution belongs to the neighbourhood $\mathcal{N}_\lambda(\beta)$ of the current point $\mathbf{u}(\mathbf{t})$. The algorithm works in a wide neighborhood, allowing very large steps. This results in a significant acceleration in the end of the process, which was also illustrated in our preliminary numerical experiments. Furthermore, we showed that the proposed PTS IPA has the best known worst-case complexity bound. One direction for future research is to generalize the algorithm to more general classes of problems, such as $P_*(\kappa)$-WLCPs (sufficient-WLCPs), or WLCPs over symmetric cones. Considering the local superlinear rate of convergence for the method is another open question. Furthermore, analysing other strategies instead of the greedy step in the PTS is also an interesting future research topic.

Appendix I: Some optimization problems leading to monotone LCPs and WLCPs

In this section, we introduce two different forms of the linearly constrained convex quadratic programming problems. The first classical model contains only sign restricted variables, while the second one has free variables, as well. Due to the fact that the objective function is quadratic, the elimination of free variables can not be done in the similar way as in LP.

Let us consider the following linearly constrained, primal convex quadratic programming problem

$$\left.\begin{array}{c}\min \frac{1}{2}\,\mathbf{x}^T Q\mathbf{x} + \mathbf{c}^T\mathbf{x} \\ A\mathbf{x} \le \mathbf{b}, \quad \mathbf{x} \ge \mathbf{0}\end{array}\right\} \quad (P-QP)$$

where $Q \in \mathbb{R}^{\bar{n}\times\bar{n}}$ is a given positive semidefinite matrix, and $A \in \mathbb{R}^{\bar{m}\times\bar{n}}$ is a given matrix. Furthermore, $\mathbf{c} \in \mathbb{R}^{\bar{n}}$ and $\mathbf{b} \in \mathbb{R}^{\bar{m}}$ are given vectors. Vector $\mathbf{x} \in \mathbb{R}^{\bar{n}}$ is the vector of the (primal) decision variables.

Let us consider the Lagrange function, $L : \mathbb{R}_+^{\bar{n}+\bar{m}+\bar{n}} \to \mathbb{R}$ assigned to the $(P-QP)$ problem as

$$L(\mathbf{x},\mathbf{y},\mathbf{s}) = \frac{1}{2}\,\mathbf{x}^T Q\mathbf{x} + \mathbf{c}^T\mathbf{x} + \mathbf{y}^T(A\mathbf{x}-\mathbf{b}) - \mathbf{s}^T\mathbf{x}.$$

The first order optimality conditions, the Karush-Kuhn-Tucker constraints [2, 20] can be derived as

$$\begin{aligned} A\,\mathbf{x}\ + \mathbf{z} = \mathbf{b}, &\quad \mathbf{x}\ge \mathbf{0},\ \ \mathbf{z}\ge\mathbf{0},\\ -A^T\mathbf{y}\ - Q\,\mathbf{x}\ + \mathbf{s} = \mathbf{c}, &\quad \mathbf{y}\ge\mathbf{0},\ \ \mathbf{s}\ge\mathbf{0},\\ \mathbf{x}\mathbf{s}=\mathbf{0},\quad \mathbf{y}\mathbf{z}=\mathbf{0}. & \end{aligned} \tag{37}$$

where $\mathbf{z} = A\mathbf{x} - \mathbf{b} \in \mathbb{R}_+^{\bar{m}}$. By introducing the following notations

$$M = \begin{bmatrix} Q & A^T \\ -A & O \end{bmatrix} \in \mathbb{R}^{(\bar{n}+\bar{m})\times(\bar{n}+\bar{m})},\quad \mathbf{q} = \begin{pmatrix}\mathbf{c}\\ \mathbf{b}\end{pmatrix},\ \mathbf{u} = \begin{pmatrix}\mathbf{x}\\ \mathbf{y}\end{pmatrix},\ \mathbf{v} = \begin{pmatrix}\mathbf{s}\\ \mathbf{z}\end{pmatrix} \in \mathbb{R}^{\bar{n}+\bar{m}},$$

and $n = \bar{n}+\bar{m}$. We can define a special LCP, as the *first order optimality criteria* of the $(P-QP)$ problem.

The second linearly constrained convex quadratic programming model was introduced by Klafszky and Terlaky [18] as

$$\left.\begin{array}{llll} \min\ \mathbf{c}^T\mathbf{x} + \frac{1}{2}\,\mathbf{x}^T C^T C\,\mathbf{x} + \frac{1}{2}\,\mathbf{z}^T\mathbf{z} \\ \qquad A\mathbf{x}\ + \qquad B\mathbf{z} \qquad \ge \mathbf{b} \\ \qquad\qquad \mathbf{x} \qquad\qquad \ge \mathbf{0} \end{array}\right\}\quad (P-QP_{KT}),$$

$$\left.\begin{array}{llll} \max\ \mathbf{y}^T\mathbf{b} - \frac{1}{2}\,\mathbf{y}^T BB^T\mathbf{y} - \frac{1}{2}\,\mathbf{w}^T\mathbf{w} \\ \qquad \mathbf{y}^T A\ - \qquad \mathbf{w}^T C \qquad \le \mathbf{c} \\ \qquad\qquad \mathbf{y} \qquad\qquad \ge \mathbf{0} \end{array}\right\}\quad (D-QP_{KT}),$$

where $A \in \mathbb{R}^{\bar{m}\times\bar{n}}$, $B \in \mathbb{R}^{\bar{m}\times\bar{k}}$, $C \in \mathbb{R}^{\bar{l}\times\bar{n}}$ are given matrices, and $\mathbf{c},\mathbf{x} \in \mathbb{R}^{\bar{n}}$, $\mathbf{b},\mathbf{y} \in \mathbb{R}^{\bar{m}}$, $\mathbf{z} \in \mathbb{R}^{\bar{k}}$, $\mathbf{w} \in \mathbb{R}^{\bar{l}}$ are vectors used in the problem description. It is easy to derive the *weak duality theorem* in the following form.

Proposition 7.1 *Let* $(\mathbf{x},\mathbf{z})$ *and* $(\mathbf{y},\mathbf{w})$ *be arbitrary primal and dual feasible solutions. Then,*

$$\mathbf{c}^T\mathbf{x} + \frac{1}{2}\,\mathbf{x}^T C^T C\,\mathbf{x} + \frac{1}{2}\,\mathbf{z}^T\mathbf{z}\ \ge\ \mathbf{y}^T\mathbf{b} - \frac{1}{2}\,\mathbf{y}^T BB^T\mathbf{y} - \frac{1}{2}\,\mathbf{w}^T\mathbf{w}$$

holds. The previous inequality is satisfied with equality, if and only if

$$\mathbf{w} = C\mathbf{x},\quad \mathbf{z} = B^T\mathbf{y},\qquad \textit{and}\qquad \mathbf{r}^T\mathbf{y} = 0,\quad \mathbf{s}^T\mathbf{x} = 0,$$

are fulfilled, where $\mathbf{r} = A\mathbf{x} + B\mathbf{z} - \mathbf{b}$ *and* $\mathbf{s} = \mathbf{c} - \mathbf{y}^T A + \mathbf{w}^T C$*, are primal and dual slack variables.*

Those primal- and dual feasible solutions that satisfy the weak duality inequality with equality are called *optimal primal and dual solutions*. From those constraints that ensure the previous inequality is satisfied with equality, the following LCP can be derived.

$$\left.\begin{array}{rcl} -P\mathbf{y} - A\mathbf{x} + \bar{\mathbf{y}} & = & -\mathbf{b} \\ A^T\mathbf{y} - Q\mathbf{x} \qquad + \bar{\mathbf{x}} & = & \mathbf{c} \\ \mathbf{x},\ \mathbf{y},\ \bar{\mathbf{x}},\ \bar{\mathbf{y}} & \geq & \mathbf{0} \\ \mathbf{x}\,\bar{\mathbf{x}} = \mathbf{0}, \quad \mathbf{y}\,\bar{\mathbf{y}} & = & \mathbf{0}, \end{array}\right\} \quad (BLCP)$$

where $P = BB^T$ and $Q = C^TC$ are positive semidefinite matrices. The $(BLCP)$ is the corresponding Karush–Kuhn–Tucker system to $(P - QP_{KT})$ and $(D - QP_{KT})$ problems [2, 18]. If we denote by M the matrix of the linear system of $(BLCP)$, then the matrix M has the following structure

$$M = \begin{bmatrix} P & A \\ -A^T & Q \end{bmatrix}.$$

Furthermore, we introduce the following notations $n := \bar{m} + \bar{n}$, $\mathbf{q} := \binom{-\mathbf{b}}{\mathbf{c}}$, $\mathbf{u} := \binom{\mathbf{x}}{\mathbf{y}}$ and $\mathbf{v} := \binom{\bar{\mathbf{y}}}{\bar{\mathbf{x}}}$. The $(BLCP)$ problem with the given bisymmetric matrix M[4] is an LCP with a special structure.

Let us note here that we can define the corresponding (WLCP) problem by changing the right-hand side of the last equations in (37) or in (BLCP) from $\mathbf{0}$ to a nonnegative vector $\mathbf{p}$.

Appendix II: Self-concordant functions and some useful results

This part of the paper is based on the theory of self-concordant functions presented in [9, 11, 23, 26].

Definition 7.1 A function $f : \mathbb{R}^n \to \mathbb{R}$ is called *self-concordant* if it is a closed convex function with open domain and there exists a constant $M_f \geq 0$ such that the inequality

$$|D^3 f(\mathbf{x})[\mathbf{h}, \mathbf{h}, \mathbf{h}]| \leq 2M_f \left(D^2 f(\mathbf{x})[\mathbf{h}, \mathbf{h}]\right)^{\frac{3}{2}} \tag{38}$$

holds for all $\mathbf{x} \in \operatorname{dom} f$ and $\mathbf{h} \in \mathbb{R}^n$, where $D^2 f(\mathbf{x})[\mathbf{h}, \mathbf{h}]$ and $D^3 f(\mathbf{x})[\mathbf{h}, \mathbf{h}, \mathbf{h}]$ are the second and third order differentials of function f taken at point $\mathbf{x}$ in the direction $\mathbf{h}$. If $M_f = 1$, the function is called *standard self-concordant*.

A specific subfamily of self-concordant functions is the family of *self-concordant barriers*.

[4] The

$$M = \begin{bmatrix} P & A \\ -A^T & Q \end{bmatrix} = \begin{bmatrix} P & 0 \\ 0 & Q \end{bmatrix} + \begin{bmatrix} 0 & A \\ -A^T & 0 \end{bmatrix} \in \mathbb{R}^{n \times n}$$

is *bisymmetric matrix*, if P and Q are symmetric positive semidefinite matrices.

Definition 7.2 Let f be a standard self-concordant function. We call it a *ν-self-concordant barrier* for the set of dom f, if

$$(Df(\mathbf{x})[\mathbf{h}])^2 \leq \nu\, D^2 f(\mathbf{x})[\mathbf{h}, \mathbf{h}] \tag{39}$$

for all $\mathbf{x} \in \operatorname{dom} f$, where $Df(\mathbf{x})[\mathbf{h}]$ is the first order differential of function f taken at point $\mathbf{x}$ in the direction of $\mathbf{h}$. The value ν is called the parameter of the barrier.

The next theorem states that self-concordance is an affine-invariant property.

Theorem 7.1 ([26], Theorem 5.1.2) *Let a function f be self-concordant with constant M_f, and $\mathcal{A} : \mathbb{R}^n \to \mathbb{R}^m$ be a linear operator of the form $\mathcal{A}(\mathbf{x}) := A\mathbf{x} + \mathbf{b}$, where $A \in \mathbb{R}^{m\times n}$ and $\mathbf{b} \in \mathbb{R}^m$. Then, the function*

$$\phi(\mathbf{x}) = f(\mathcal{A}(\mathbf{x}))$$

is also self-concordant and $M_\phi = M_f$.

Theorem 7.2 ([26], Theorem 5.1.6) *Let a function f be self-concordant and* dom f *contains no straight lines. Then, the Hessian $\nabla^2 f(\mathbf{x})$ is nondegenerate at all points $\mathbf{x} \in \operatorname{dom} f$.*

We will use the following notation:

$$\|\mathbf{h}\|_{\mathbf{x}}^2 = \|\mathbf{h}\|_{\nabla^2 f(\mathbf{x})}^2 = D^2 f(\mathbf{x})[\mathbf{h}, \mathbf{h}] = \mathbf{h}^T \nabla^2 f(\mathbf{x})\mathbf{h},$$

by referring to the point where the Hessian matrix of the function f has been computed. This is called the primal local norm of direction $\mathbf{h}$ with respect to $\mathbf{x}$.

Under the assumption that dom f contains no straight lines, we can define *dual local norm* for any $\mathbf{x} \in \operatorname{dom} f \subset \mathbb{R}^n$ and $\mathbf{g} \in \mathbb{R}^n$ as follows

$$\|\mathbf{g}\|_{\mathbf{x}}^* = \|\mathbf{g}\|_{[\nabla^2 f(\mathbf{x})]^{-1}} = \sqrt{\mathbf{g}^T [\nabla^2 f(\mathbf{x})]^{-1}\mathbf{g}}.$$

It can be shown that for any $\mathbf{g}, \mathbf{h} \in \mathbb{R}^n$: $|\mathbf{h}^T\mathbf{g}| \leq \|\mathbf{h}\|_{\mathbf{x}}\, \|\mathbf{g}\|_{\mathbf{x}}^*$ holds.

Using the dual local norm we can introduce the *Newton decrement* of the function f at $\mathbf{x} \in \operatorname{dom} f$ in the following way

$$\lambda_f(\mathbf{x}) = \|\nabla f(\mathbf{x})\|_{\mathbf{x}}^* = \|\nabla f(\mathbf{x})\|_{[\nabla^2 f(\mathbf{x})]^{-1}} = \sqrt{\nabla f(\mathbf{x})^T [\nabla^2 f(\mathbf{x})]^{-1} \nabla f(\mathbf{x})}.$$

Let us define the following functions:

$$\omega(t) = t - \ln(1+t), \quad t \geq 0,$$

and

$$\omega_*(\tau) = -\tau - \ln(1-\tau), \quad \tau \in [0, 1).$$

Note that ω and ω_* are monotone increasing convex functions.

We summarize from Lemmas 5.1.4 and 5.1.5 of [26] some useful results that we use in our analysis.

Lemma 7.3 *For any* $t \geq 0$ *and* $\tau \in [0, 1)$*, we have*

$$\omega(t) + \omega_*(\tau) \geq t\,\tau, \tag{40}$$

$$\frac{\tau^2}{2-\tau} \leq \omega_*(\tau) \leq \frac{\tau^2}{2\,(1-\tau)}. \tag{41}$$

Theorem 7.4 ([26], Theorem 5.1.7) *Let* $\mathbf{x} \in \operatorname{dom} f$*. Then, for any* $\mathbf{y} \in W^0\left(\mathbf{x}, \frac{1}{M_f}\right)$ *we have*

$$(1 - M_f\, r)^2\, \nabla^2 f(\mathbf{x}) \preceq \nabla^2 f(\mathbf{y}) \preceq \frac{1}{(1 - M_f\, r)^2}\, \nabla^2 f(\mathbf{x}), \tag{42}$$

where $r = \|\mathbf{y} - \mathbf{x}\|_{\nabla^2 f(\mathbf{x})}$.

We need additional inequalities that characterize self-concordant functions.

Theorem 7.5 ([26], Theorem 5.1.8) *Let a function* f *be self-concordant. For any* $\mathbf{x}, \mathbf{y} \in \operatorname{dom} f$*, we have*

$$(\mathbf{y} - \mathbf{x})^T\, (\nabla f(\mathbf{y}) - \nabla f(\mathbf{x})) \geq \frac{\|\mathbf{y} - \mathbf{x}\|^2_{\nabla^2 f(\mathbf{x})}}{1 + M_f\, \|\mathbf{y} - \mathbf{x}\|_{\nabla^2 f(\mathbf{x})}}. \tag{43}$$

$$f(\mathbf{y}) \geq f(\mathbf{x}) + (\mathbf{y} - \mathbf{x})^T \nabla f(\mathbf{x}) + \frac{1}{M_f^2}\, \omega(M_f\, \|\mathbf{y} - \mathbf{x}\|_{\nabla^2 f(\mathbf{x})}), \tag{44}$$

where $\omega(t) = t - \ln(1 + t)$.

An easy consequence of the inequality (43) is the following statement.

Corollary 7.6 *Let a function* f *be self-concordant. For any* $\mathbf{x}, \mathbf{y} \in \operatorname{dom} f$*, we have*

$$\frac{\|\mathbf{y} - \mathbf{x}\|_{\nabla^2 f(\mathbf{x})}}{1 + M_f\, \|\mathbf{y} - \mathbf{x}\|_{\nabla^2 f(\mathbf{x})}} \leq \|\nabla f(\mathbf{y}) - \nabla f(\mathbf{x})\|_{[\nabla^2 f(\mathbf{x})]^{-1}}. \tag{45}$$

Define the *Fenchel conjugate function* (or *Fenchel dual function*) f_* of the self-concordant function f for $\mathbf{s} \in \mathbb{R}^n$, the value of this function is defined as follows:

$$f_*(\mathbf{s}) = \sup_{\mathbf{x} \in \operatorname{dom} f} \left(\mathbf{s}^T \mathbf{x} - f(\mathbf{x})\right). \tag{46}$$

Clearly, $\operatorname{dom} f_* = \{\mathbf{s} \in \mathbb{R}^n : f(\mathbf{x}) - \mathbf{s}^T \mathbf{x} \text{ is bounded below on } \operatorname{dom} f\}$.

Corollary 7.7 *Let a self-concordant function* f *be given and denote by* f_* *its Fenchel conjugate function. For any* $\mathbf{x}, \mathbf{y} \in \operatorname{dom} f$ *we have*

$$\frac{\|\nabla f(\mathbf{x}) - \nabla f(\mathbf{y})\|_{[\nabla^2 f(\mathbf{x})]^{-1}}}{1 + M_f\, \|\nabla f(\mathbf{x}) - \nabla f(\mathbf{y})\|_{[\nabla^2 f(\mathbf{x})]^{-1}}} \leq \|\mathbf{y} - \mathbf{x}\|_{\nabla^2 f(\mathbf{x})}. \tag{47}$$

Similarly to the lower bounds (43) and (44), under mild, natural assumptions upper bounds can be provided for the same expressions. Interestingly enough the function ω has been replaced by its Fenchel conjugate ω_*.

Theorem 7.8 ([26], Theorem 5.1.9) *Let a function f be self-concordant. For any $\mathbf{x} \in \operatorname{dom} f$ with $\|\mathbf{y}-\mathbf{x}\|_{\nabla^2 f(\mathbf{x})} < \frac{1}{M_f}$, we have*

$$(\mathbf{y}-\mathbf{x})^T\left(\nabla f(\mathbf{y})-\nabla f(\mathbf{x})\right) \leq \frac{\|\mathbf{y}-\mathbf{x}\|^2_{\nabla^2 f(\mathbf{x})}}{1-M_f\,\|\mathbf{y}-\mathbf{x}\|_{\nabla^2 f(\mathbf{x})}}, \tag{48}$$

$$f(\mathbf{y}) \leq f(\mathbf{x}) + (\mathbf{y}-\mathbf{x})^T\nabla f(\mathbf{x}) + \frac{1}{M_f^2}\,\omega_*(M_f\,\|\mathbf{y}-\mathbf{x}\|_{\nabla^2 f(\mathbf{x})}), \tag{49}$$

where $\omega_*(t) = -t - \ln(1-t), \quad t \in [0, 1)$.

Furthermore, the following important inequalities hold, as well.

Theorem 7.9 ([26], Theorem 5.1.12) *Let a function f be self-concordant. For any $\mathbf{x}, \mathbf{y} \in \operatorname{dom} f$, we have*

$$f(\mathbf{y}) \geq f(\mathbf{x}) + (\mathbf{y}-\mathbf{x})^T\nabla f(\mathbf{x}) + \frac{1}{M_f^2}\,\omega\left(M_f\,\|\nabla f(\mathbf{y})-\nabla f(\mathbf{x})\|_{[\nabla^2 f(\mathbf{y})]^{-1}}\right). \tag{50}$$

If in addition $\|\nabla f(\mathbf{y})-\nabla f(\mathbf{x})\|_{[\nabla^2 f(\mathbf{y})]^{-1}} < \frac{1}{M_f}$, then

$$f(\mathbf{y}) \leq f(\mathbf{x}) + (\mathbf{y}-\mathbf{x})^T\nabla f(\mathbf{x}) + \frac{1}{M_f^2}\,\omega_*\left(M_f\,\|\nabla f(\mathbf{y})-\nabla f(\mathbf{x})\|_{[\nabla^2 f(\mathbf{y})]^{-1}}\right). \tag{51}$$

The following theorem estimates the local convergence of the standard Newton Method. (For details, see pages 190-191 in [24].)

Theorem 7.10 ([24], Theorem 4.1.14) *Let a function f be self-concordant. Assume that $\operatorname{dom} f$ contains no straight line, and $\mathbf{x} \in \operatorname{dom} f$ with $\lambda_f(\mathbf{x}) < 1$. Then, the point*

$$\mathbf{x}^+ = \mathbf{x} - [\nabla^2 f(\mathbf{x})]^{-1}\nabla f(\mathbf{x})$$

belongs to $\operatorname{dom} f$ and we have $\lambda_f(\mathbf{x}^+) \leq \left(\frac{\lambda_f(\mathbf{x})}{1-\lambda_f(\mathbf{x})}\right)^2$.

Let us consider now the scheme of the Damped Newton Method, namely the new iterate $\mathbf{x}^+$ is computed as follows

$$\mathbf{x}^+ = \mathbf{x} - \frac{1}{1+M_f\,\lambda_f(\mathbf{x})}[\nabla^2 f(\mathbf{x})]^{-1}\nabla f(\mathbf{x}).$$

Theorem 7.11 ([26], Theorem 5.1.15) *Let a function f be self-concordant and apply the Damped Newton's method for minimizing the function f. Then, we have*

$$f(\mathbf{x}^{+}) \leq f(\mathbf{x}) - \frac{1}{M_f^2}\,\omega(M_f\,\lambda_f(\mathbf{x})).$$

Acknowledgements The authors would like to thank the anonymous reviewers and the associated editor for the valuable suggestions that helped the improvement of the paper.

Funding Open access funding provided by Corvinus University of Budapest. This research was supported by the National Research, Development and Innovation Office (NKFIH) under grant number 2024-1.2.3-HU-RIZONT-2024-00030. Petra Renáta Rigó is thankful for the support provided by the Ministry of Culture and Innovation of Hungary from the National Research, Development and Innovation Fund, financed under the PD_22 funding scheme, project no. 142154. The research of Marianna E.-Nagy was supported by the János Bolyai Research Scholarship of the Hungarian Academy of Sciences.

Declarations

Conflicts of Interest The authors declare that they have no conflict of interest.

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