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Apportionment methods in resource allocation

László Á. Kóczy [a,b,*], Tamás Koltai [b], Balázs R. Sziklai [a,c]

[a] ELTE Centre for Economic and Regional Studies, Hungary
[b] Budapest University of Technology and Economics, Hungary
[c] Corvinus University Budapest, Hungary

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ABSTRACT

The efficient use of resources is of paramount importance in any organization, and this is especially true of the use of human resources. We are considering an organization with a fixed number of workers over a network of subunits. The relative efficiency of these units can be uncovered using Data Envelopment Analysis (DEA) methods. How do we overcome efficiency differences when resources are constrained? Motivated by concerns for quality assurance, we use a method that lexicographically minimizes the tasks per worker. We present a fast algorithm to calculate the optimal allocation. Connections to the apportionment literature — mostly focusing on the fair allocation of voting districts among geographical or administrative regions, — are discussed. The allocation can be validated using DEA: efficiency indicators are now levelled.

The method is illustrated using data from salary administrators at the Hungarian State Treasury's 19 county-level subsidiaries. Knowing the number of standard administrative tasks for each subsidiary and each month, we allocate workers proportionally to the busiest months. After the reallocation, the relative efficiency of the worst-performing counties moves from about 60% to over 90%.

1. Introduction

The efficient use of resources is of paramount importance in any organization, and this is especially true of the use of human resources. We are considering an organization consisting of a network of subunits operating with workers as inputs and a handful of fairly standardized tasks. Such a standardized environment allows us to compare the efficiency of different subunits and make recommendations for an optimal choice of inputs subject to constraints in overall worker headcount.

Comparing and benchmarking the performance of the different units or branches, such as shops, restaurants, departments or governmental organizations, is a standard management task. Often, there is no single indicator that captures all elements of their performance, including profit, turnover, consumer satisfaction, service time, quality, etc., and these data are often not measured in the same units. Much of these inputs and outputs can be expressed in monetary terms. For non-profit organizations, an approach to attach monetary values to outputs and sometimes to inputs is not only ill-suited but may be downright impossible.

Data Envelopment Analysis (DEA) can compare the relative performance of units with different combinations of in- and outputs: Here, *output* is a broad term encompassing these results, while *inputs* refer to the resources utilized to achieve them, as deemed relevant by management for the organization's efficient operation. In the case of a shop, for example, outputs do not only include sales but also consumer satisfaction and service speed, while the size of the staff and the shop, as well as the logistics tools and personnel, may all be included among the inputs. As the examined units decide independently on the amount of inputs and outputs, these are referred to as Decision-Making Units (DMUs). DMUs are compared based on the *ratio of the weighted sums of outputs and inputs*, where the weights are endogenously determined by mathematical tools using the characteristics of the studied organizational units [1].

It is important to stress that the efficiency (or inefficiency) of the DMUs in question is *relative* to each other. We do not claim that our most efficient DMU cannot be more efficient. Throughout the whole paper, efficiency is always used in a relative context. DEA is extensively used to evaluate the performance of for-profit, non-profit, and public administration organizations. Its application grows exponentially [2–5], and new areas of application are the results of the popularity of the method and its constant theoretical development [6–9]. In this paper, DEA is used to evaluate the performance of a public administration organization and verify resource allocation results.

As we have said, DMUs decide independently on the inputs and outputs, while in our case, the input decisions are subject to overall budgetary conditions. In other words, what we are considering is the

* Corresponding author at: ELTE Centre for Economic and Regional Studies, Hungary.
E-mail address: koczy.laszlo@krtk.hu (L.Á. Kóczy).

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problem of allocating indivisible resources based on certain *claims* to these. This can be written as an integer programming problem, but this is rather difficult to solve in general. However, the problem is analogous to the allocation of parliamentary seats based on representation claims in elections, such as distributing seats to counties or states according to voter or population numbers, or to political parties based on the votes they receive on their party lists. The so-called apportionment problem has a history going as far back as the 18th century [10–12]. Most of the apportionment methods have undergone the scrutiny of academics — unfortunately, Balinski and Young [11] have shown that no method is immune to certain "paradoxes" or the corresponding criticisms. Lately, the Venice Commission [13] introduced a new aspect of equality, and the existing methods perform rather poorly with respect to this idea [14]. The approach we take here is built on the idea of the Leximin method [15] but is based on a notion of quality assurance rather than equality.

The structure of the paper is as follows. First, we present our use case: workforce allocation within the Hungarian State Treasury. Next, we describe the methods employed, including both Data Envelopment Analysis (DEA) and apportionment techniques. The following section provides details on the data used. We then present the results of our calculations. The paper concludes with a summary and a discussion of alternative approaches.

2. Problem statement

The Hungarian State Treasury (HST) is a central budgetary institution with an independent legal personality, national authority, independent operation, and management, and it employs nearly four thousand people. Since 1 January 1996, the organization has been carrying out its tasks as defined in the Act on Public Finances. The Budapest headquarters of the HST is responsible for ensuring that the funds available for expenditure in the Budget Act are used in accordance with the law. The offices of the 19 county directorates at the headquarters are responsible for specific, decentralized, high-turnover client transactions.

The HST's payroll accounting department accounts for nearly nine hundred thousand employees in the public sector (salaries, other allowances), health insurance benefits (sickness benefits, maternity benefits, maternity allowance for accidents) and employers' public charges. Its main activities include tax return, accounting and financing (e.g. monthly tax return, net pay transfer), salary accounting (net salary calculations, deductions and blockings, the determination of health insurance cash benefits) and the operation of the institutional subsystem (e.g. provision of operating conditions, electronic data transmission). The salary accounting offices of the county directorates are responsible for, inter alia:

1. carrying out social security duties,
2. reporting the insured,
3. reporting personal income taxes,
4. reporting to the state tax authority,
5. the provision of data on wages, labour and statistics by budgetary bodies,
6. operating the headcount and payroll information system. (Website of the HST, 2019)

At the time of our study, there were apparent differences among the different county directorates. Both anecdotal evidence and available data (such as overtime requests) indicated discrepancies in the load of the salary accountants and the time it takes to handle ad-hoc requests. Due to budgetary constraints, simply hiring new staff to ease the load of overloaded directorates is not a possibility, while positions may be shifted around among directorates.

This leads to the following optimization problem: determine the number of the salary accounting staff for each directorate while keeping the necessary overtime minimal, or in other words, maximize the attention devoted to each request.

3. Methods

The allocation of constrained resources is an everyday task in management. When facing such a problem, one has to identify the main objective of the allocation. Our focus is on quality management: we define *service quality* q as the *resource per task ratio* of a decision-making unit. What is the highest service quality q we can have, such that each unit has at least the required amount of resources allocated? The problem can be written as follows:

$$\max_{q\in\mathbb{R}} q \quad \text{such that} \tag{MILP1}$$
$$p_i q - a_i \le 0 \quad \text{for all } i$$
$$\sum_i a_i = H$$
$$a_i, q \ge 0 \quad \text{for all } i$$

where $p_i \ge 0$ is the amount of tasks to process in unit $i \in \mathbb{N} = \{1, \dots, n\}$, a_i is the allocated resource, H is the total resource available. While in general, this linear programming problem is easy to solve, it becomes *seemingly* computationally hard if the allocated resources are discrete, if

$$a_i \in \mathbb{N} \quad \text{for all } i.$$

3.1. Data envelopment analysis

The common approach to help decision-making units (DMUs) operate efficiently is by Data Envelopment Analysis (DEA). Here, the objective is to find an efficient combination of m inputs (resources) x to produce an optimal combination of s outputs y. For each of the n decision-making units, we have the following problem:

$$\max_{u,v} \frac{\sum_r u_r y_{r0}}{\sum_i v_i x_{i0}} \quad \text{such that} \tag{1}$$
$$\frac{\sum_r u_r y_{rj}}{\sum_i v_i x_{ij}} \le 1 \quad \forall j \in N, \text{and}$$
$$u, v > 0,$$

where $u, v > 0$ denotes weak inequalities with $u = 0, v = 0$ naturally ruled out. Note that Charnes et al. [1] successfully converted this optimization problem into linear form.

Two cases require special attention: In the *input-oriented* model, we set $\sum_i v_i x_{i0} = 1$ and the objective is to maximize the attainable output, while in the *output-oriented* model we set $\sum_r u_r y_{r0} = 1$ and the objective is to produce the same output using an optimal combination of inputs. We study an output-oriented problem with a given number of tasks to complete, but using resources optimally.

While DEA is a natural approach, there are a number of reasons why we need to reach for other techniques. Firstly, DEA uses continuous optimization, while in the problem we have discrete resources. Another problem is the pooled resource our DMUs must use: In a standard DEA problem, DMUs optimize independently, while in the problem at hand, if one DMU plans to use more resources, others must use less. In other words, we face an allocation problem and in the following, we propose an alternative approach to solving this allocation problem.

Allocation problems arise in various aspects of our lives, from university admissions to disputes among heirs of an estate. Such conflicts are neither rare nor new: searching for optimal allocations has been the focus of academic interest for hundreds of years. Today, it is mostly game theory and its related fields that study these problems. For an outsider, the literature is surprisingly varied with very different approaches taken for the various problems. For us, the first task is also to identify the main characteristics of the problem at hand.

Firstly, observe that we look for a *proportional* allocation. If the allocation is not proportional, one of the units gets too many resources, taking away some from the less endowed ones. The proportional allocation would not be too interesting with divisible goods, such as monetary resources. When the resources are, for example, human resources, we must consider integer allocations.

Example 1. Consider the following example with only two units, with 30 tasks each. If we want to allocate two workers, the obvious allocation is 1 worker to each unit. Allocating three workers, the only sensible allocation is asymmetric: $a = (1, 2)$. On average, the workers handle 20 tasks each; in one of the units, the average workload is 50% above, while in the other, 25% below the average. Increasing the resource made only one of the units better off.

Example 2. Let us modify Example 1: now unit 1 has only $p_1 = 20$ tasks, while $p_2 = 30$ remains the same. Let us try to allocate $H = 4$ workers. The second unit has more tasks: an allocation of $a = (1, 3)$ yields average loads of 20 and 10, while $a = (2, 2)$ yields 10 and 15 tasks on average. In this case, the quality of service consideration gives preference to the second allocation.

How do we allocate staff to the different units in general?

The academic literature studies two related but slightly different problems. The first is the apportionment problem linked to the design of voting systems. Any change to the voting rules leads to heated debates, often centring around the problem of drawing up constituencies. The manipulative forming of voting districts may drastically affect the election outcomes. When politicians change district borders to increase their chances of getting reelected, we talk about Gerrymandering. See [16] for a remarkable paper that uses DEA techniques to select the best districting plans objectively.

Few realize that a more fundamental way to influence election outcomes is by designing constituencies of drastically different sizes. A good electoral law designs districts in a way that minimizes inequality. Now, it is also clear how this is related to our problem at hand: In a representative democracy, the representation of voters of a state or county is the "task", and representatives are the resource allocated.

Over the centuries, various principles and algorithms have been introduced to allocate parliamentary seats fairly. They can be compared by looking at their properties. Unfortunately even the most elementary properties are conflicting, they cannot be true for the same algorithm [11] and the ideas of good apportionment principles over the two sides of the Atlantic are also in conflict [17]. The methods may be different, but we must not forget that most of these differences concern the "rounding" of the allocation and are, therefore, small.

One of the methods follows the recommendation of the Venice Commission [13] and is based on the differences from the average constituency size. The Leximin method [15] lexicographically minimizes this difference. This difference may be positive or negative when trying to avoid a service quality that is too low or too high.

We would like to use a similar method, but in our case, it is worse if a worker is overloaded by 10% than if another is underloaded by 25% when compared to the average. Another concept from cooperative game theory has this spirit. The nucleolus [18] focuses on those that are underendowed and lexicographically maximizes their payoffs. Unfortunately, that model has a rather different formulation, and we can only borrow the idea to the apportionment problem.

3.2. The Leximinimax method

The method we introduce focuses on improving service quality below the average level. It is based on apportionment but has a formal connection to the resource allocation mixed-integer linear program (MILP) outlined in (MILP1).

Before we present the formal definitions, we present a simple example. For expositional convenience, we keep track of service quality by calculating its inverse (tasks per worker). We must, therefore, remember that a higher workload per worker implies a lower service quality.

Table 1
Average service quality for different allocations (showing inverse, $1/q$ values).

Allocation (a)	A (48)	B (26)	C (19)
3-1-1	16	26	19
2-2-1	24	13	19
2-1-2	24	26	9.5

Table 2
Average service quality for different allocations in decreasing order (showing inverse, $1/q$ values).

Allocation (a)	Lowest	Medium	Highest
3-1-1	26	19	16
2-2-1	24	19	13
2-1-2	26	24	9.5

Table 3
Letter codes of (inverse) quality values.

Figure	26	24	19	16	13	9.5
Letter code	A	G	I	L	N	O

Table 4
Coded (inverse) quality values.

Allocation (a)	Lowest	Medium	Highest	Code
3-1-1	26	19	16	AIL
2-2-1	24	19	13	GIN
2-1-2	26	24	9.5	AGO

Table 5
Allocations sorted alphabetically by codewords.

Allocation (a)	Code	Lowest	Medium	Highest
2-1-2	AGO	26	24	9.5
3-1-1	AIL	26	19	16
2-2-1	GIN	24	19	13

Example 3. Consider a problem with 3 units (A, B, C) having tasks $p = (48, 26, 19)$ and a total resource of 5. The total number of tasks is 93, so the average worker completes 18.6.

This is a simple problem as it is clear that each unit should get at least one worker, so we only need to allocate the remaining two. Should A get no more, its single worker should complete 48 tasks, way over the average, resulting in a low service quality. So A gets at least 2. Who gets the last worker? Table 1 summarizes our – not necessarily reasonable – options.

We sort the units according to their service qualities in Table 2.

The next step is to sort the inverse quality vectors in a lexicographic order. This ordering is like an alphabetic order. For the sake of the exposition, we map these figures to letters — this step can be skipped as it is really for illustration only. Let us use the code in Table 3

Using these codes, we can translate the vectors of (inverse) service qualities into words. (Table 4) As the last step, we sort the allocations based on the codewords (Table 5). The preferred allocation is at the end of the alphabetic sorting. Therefore, the optimal allocation is $a = (2, 2, 1)$.

The method that we call *Leximinimax* (lexicographically maximizing the minimal service quality) can be applied to an arbitrary number of units and resources.

1. List possible allocations $\{a \in \mathbb{N}^n | a_1 + \cdots a_n = H\}$.[1]
2. We calculate the inverse of service quality $\left(s \in \mathbb{R}^n \mid s_i = \frac{p_i}{a_i}\right)$ for each unit.

[1] In practice, non-monotonic allocations can, for instance, be ignored.

3. We sort the vector of inverse service qualities. That is, we take an n-long permutation $\pi \in S_n$, such that $b = \pi(s)$ and $b_1 \geq b_2 \geq \cdots \geq b_n$
4. The sorted vectors are ordered lexicographically: first by the first values, then, in case of equality, among the second values, and so on (this is equivalent to translating into words and sorting alphabetically). That is, if $b^1, b^2, \ldots, b^p$ are the possible vectors of ordered inverse service quality, then we reindex them in such a way that $b^1 \succ b^2 \succ \cdots \succ b^p$.
5. Take the allocation corresponding to (b^p).

Perhaps it is not obvious, but the optimal allocation is generically unique. Unfortunately, checking all possible allocations (step 1) is time-consuming, as there are exponentially many ways in which H can be expressed as a sum of n positive integers. Fortunately, a complementary paper [19] show that the Leximinimax method is equivalent to the well-known *Adams apportionment method*. Here, we prove another connection: the Leximinimax method is equivalent to the optimal solution of the linear program described in (MILP1). We demonstrate this by showing that the MILP always produces the apportionment determined by the Adams method. For this, we need to formally define the apportionment problem.

3.3. Apportionment methods

Traditionally, apportionment methods are used to allocate constituencies among geographical or administrative areas, such as states or counties. Here we consider a country with states $N = \{1, 2, \ldots, n\}$ and a house of representatives with $H \in \mathbb{N}_+$ seats to be allocated.

An *apportionment problem* $(\boldsymbol{p}, H)$ is a pair consisting of a vector

$$\boldsymbol{p} = (p_1, p_2, \ldots, p_n)$$

of state populations $p_i \in \mathbb{N}_+$. An *apportionment method* determines the non-negative integers $a_1, a_2, \ldots, a_n$ with $\sum_{i=1}^n a_i = H$, specifying how many constituencies each of the states $1, 2, \ldots, n$ gets. Formally, it is a function M that assigns an *allotment* $\mathbf{a} = (a_1, a_2, \ldots, a_n)$ for each apportionment problem $(\boldsymbol{p}, H)$.

Let $P = \sum_{i=1}^n p_i$ be the population of the country, and let $A = \frac{P}{H}$ denote the average size of a constituency. The fraction $\frac{p_i}{P} H = \frac{p_i}{A}$ is called the *exact quota* of state i. The exact quota is typically not an integer; rounding it up or down is a natural way to obtain an apportionment. Largest remainder methods are based on this idea.

Divisor methods constitute another family of apportionment techniques. An apportionment method is a *divisor method* if there exists a monotone increasing function $f : \mathbb{N} \to \mathbb{R}$, the *divisor criterion*, such that the seats are allocated iteratively to the state with the highest $\frac{p_i}{f(s_i)}$ value in each round, where s_i denotes the number of seats currently allocated to state i. We assume that all of the $\frac{p_i}{f(s_i)}$ values are distinct. Ties are unlikely, for real data no tie-breaking rules are specified.

Common divisor methods include the following (EP stands for Equal Proportions method – aliases are due to reinventions):

Adams method	$f(s) = s$
Danish method	$f(s) = s + 1/3$
Harmonic mean/Dean method	$f(s) = \dfrac{2s(s+1)}{2s+1}$
Huntington-Hill/EP method	$f(s) = \sqrt{s(s+1)}$
Sainte-Laguë/Webster method	$f(s) = s + 1/2$
Jefferson/D'Hondt method	$f(s) = s + 1$

The divisor criteria are listed in pointwise increasing order from Adams to Jefferson/D'Hondt; the methods favour large states over small states in the same order. That is, the Adams method favours small states the most, while the Jefferson/D'Hondt is the most beneficial for large states (see [11,20,21]). The principal advantage of divisor methods is their immunity to paradoxes related to monotonicity, such as the Alabama paradox.

3.4. Apportionment and resource allocation

In this section, we highlight the connection of the apportionment problem to resource allocation. In particular, we show how the Jefferson and Adams methods relate to the mixed-integer linear program defined in (MILP1).

The term 'divisor' in divisor methods originates from the approach of calculating apportionments by dividing each state's population by the size of an average constituency (A), in other words calculating their exact quota. Since these calculations rarely result in whole numbers, Thomas Jefferson suggested taking the lower integer parts of the resulting numbers, then decreasing the divisor until the sum equalled the House size.[2] The Adams method follows a similar process; however, after division, we take the upper integer parts and increase the divisor until the sum matches the House size.

The divisor can be thought of as a market-clearing price. Suppose each state has a budget equal to its population size. Then the divisor calculated by the Jefferson method sets the price of a seat. The amount of seats a state can buy using its budget is the same amount that is allotted to them by the Jefferson method [19,22]. The process of finding the correct price resembles solving a linear program, and this intuition is indeed correct. In fact, the following holds true.

Theorem 4. *Let $H \in \mathbb{N}$ and $\mathbf{p} = (p_1, \ldots, p_n)$ be such that $\frac{p_i}{k_1} \neq \frac{p_j}{k_2}$ whenever $i \neq j$ for any $k_1, k_2 \in \mathbb{N}$. The MILP described in (MILP1) computes Adams apportionment for any such input $(\mathbf{p}, H)$.*

This result was already known by Balinski and Young [11] (see Prop 3.10 on p 105.). Since they did not include proof we provide one in Appendix. The final piece of the puzzle is provided by Kóczy and Sziklai [19] by showing that the Adams and Leximinimax coincide, thereby linking (MILP1) and the Leximinimax solution. Note that it is commonly assumed in apportionment that $\frac{p_i}{k_1} \neq \frac{p_j}{k_2}$ holds true as such cases are highly improbable in real applications and typically require a tie-breaking mechanism to resolve [14].

Since the Adams method mirrors the Jefferson method, one may wonder how the latter is related to resource allocation. The Adams method is beneficial for smaller states, which translates to resource allocation as service quality: ensuring that small administrative units are well-staffed is crucial, as an additional unit of resource makes a larger relative impact there. In contrast, the Jefferson method is advantageous for larger states or administrative units, implying a focus on efficiency. Each unit of resource should be as effectively utilized as possible, prioritizing overall productivity and minimizing waste.

The following MILP yields the same apportionment as the Jefferson method.

$$\min_{q \in \mathbb{R}} q \quad \text{such that} \tag{MILP2}$$

$$p_i q - a_i \geq 0 \quad \text{for all } i$$

$$\sum_i a_i = H$$

$$a_i, q \geq 0 \quad \text{for all } i.$$

By minimizing q, we try to avoid overspending, that is, to allocate too much of the resource to one administrative unit. It is easy to prove the analogue of Theorem 4.

Theorem 5. *Let $H \in \mathbb{N}$ and $\mathbf{p} = (p_1, \ldots, p_n)$ be such that $\frac{p_i}{k_1} \neq \frac{p_j}{k_2}$ whenever $i \neq j$ for any $k_1, k_2 \in \mathbb{N}$. The MILP described in (MILP2) computes Jefferson apportionment for any such input $(\mathbf{p}, H)$.*

[2] This was the original procedure for calculating the Jefferson method, the iterative procedure described in Section 3.3 was invented by D'Hondt.

Table 6
Task types, their weights for salary administrators at HST and their usage in our DEA models.

	Task	Weight	DEA1	DEA2
Core	General payroll accounting	1.0	Output 1	Output 1
	Public employment	1.5		
	Commissioned and scholarships	0.5		
Complicating factors	Mid-month payments	0.5	–	Output 2
	New contracts	1.0		
	Terminated contracts	1.5		
	Benefits	1.3		
	Compensations	1.2		
	Health insurance fund-related cases	1.5		
	Workplace accidents	2.0		
	Deductions	0.3		
	Blocks	1.5		
	Pension enquiries	2.0		
	External audits	2.0		

The proof follows almost verbatim of that of Theorem 4. Again, Kóczy and Sziklai [19] provides a proof that Jefferson and Leximaximin are the same.

Finally, it is worth drawing a parallel between the task in hand and other resource allocation LPs. (MILP1) and (MILP2) show remarkable semblance to the least-core computation of cooperative transferable utility games [23]. What is more, the nucleolus—the lexicographic centre of the game—can be computed by $|N|$ number of consecutive least-core LPs, where N stands for the set of players. Thus, just like in our case, there is a connection between the LP and a lexicographic solution. The MILPs' structure also resembles to n-fold integer programs, which admit fast polynomial time algorithms, see for instance [24,25] or [26].

4. Data

Our task is to optimize the salary accounting organizations of the Hungarian State Treasury (HST) county directorates. In order to determine the elements of the model used for the study, we looked for the key factors that 1) determine the efficiency of these departments and 2) for which reliable data is available. During the discussions with the HST staff, we also obtained a complex picture of HST's organization, activities and tasks, focusing primarily on payroll accounting.

4.1. Identifying inputs and outputs

The aim of our analysis was to examine and compare the relative efficiencies of the county directorates of the HST salary accounting department. Before beginning the study, we have identified the key factors essential for selecting and applying the appropriate model. First, we had to determine the decision-making units (DMU). Next, the inputs used in the activities and the outputs resulting from the activities had to be identified. The elements of the model were determined together with the HST staff during several consultations.

When identifying the DMUs, two possible options were considered: the 19 county boards or the 45 payroll locations should be the DMUs. Considering that it is essential for the definition of DMUs that an organizational unit should be able to make independent decisions about the amount of inputs used, we selected the 19 county directorates as the basic organizational unit to be analysed in the model. The applied technology in all these DMUs is identical, consequently the homogeneity assumption of the DMUs is perfectly met [27]. In our calculations, the names of the organizational units are masked, and the data are transformed for confidential reasons. The transformed data, however, properly characterizes the performance properties of the organizational units.

The core output of a county directorate is the number of items accounted for. Upon closer inspection, it turns out that two subcases required 50% higher and 50% lower effort, respectively. The weighted sum of these core tasks constitutes Output 1 for the DEA model. In addition, there are several common complicating factors and auxiliary activities. The weighted sum of these complicating tasks constitutes Output 2 for the DEA model. Our analysis ignores ad-hoc activities whose load is hard to measure, but these reportedly constituted a far smaller share of the clerks' workload. In total, 14 categories were identified, and corresponding weights were determined (Table 6).

When identifying the inputs, the number of employees and the cost or time of overtime were considered. The workforce comprises several elements: accountants, reviewers, data providers, institution coordinators and managers. Given that the number of accountants determines the number of other employee types, the main input is the *actual number of accountants*. At the outset of the model, we thought that the amount of overtime would be a determining feature of the processes. Still, due to the differences between directorates and the different practices for compensating overtime, we agreed with HST staff that this should not be taken into account. In addition, the studies have shown that the effect of overtime on the results is small.

After identifying the elements of the model, the HST staff collected the defined monthly input and output data for a 12-month period. The data are presented in Tables 8–9 of the Annex for core, complicating and total tasks, respectively. With an appropriately weighted aggregation, we get the total monthly task volumes. This, effectively the sum of the two DEA outputs, is the basis of the optimal allocation.

We want to allocate positions with the aim of minimizing overload. Assuming that positions are fixed over the year, we focus on the months with the highest loads and make the allocations based on these. In the following, we refer to these as the *peak monthly* data.

5. Results

Using the data in the Appendix, we conducted an efficiency analysis of the salary accounting departments of 19 county directorates. The results can be categorized into the following four areas:

1. Relative efficiency calculated for a *single* input (headcount) and a *single* output (peak monthly numbers of core tasks).
2. Relative efficiency calculated based on one input (headcount) and *two* outputs (weighted numbers of core tasks and complicating factors)
3. Optimal allocation of the workforce of 854 among the directorates using the Leximaximin (or Adams) algorithm on the basis of total weighted tasks.
4. Relative efficiency calculated based on the — hypothetical — optimally allocated workforce.

5.1. Relative efficiency with a single output

The relative efficiency of the 19 county directorates was compared on the basis of peak monthly data, with only input and weighted core

Table 7
A summary of the allocation of workers and the relative efficiencies initially and after optimization both for the current headcount and some alternative budget constraints for the combined maximal monthly workload.

	Max	Starting			Optimized				
	total	Head-	Relative eff.		Head-	rel.	Alternative		
	load	count	DEA1	DEA2	count	eff.	headcounts		
Total	11 535	854	(%)		854	(%)	900	850	800
DMU01	2 422	189	86.8	86.8	178	100.0	187	177	166
DMU02	676	49	91.1	94.1	50	92.3	53	50	47
DMU03	427	32	87.5	87.9	32	95.7	33	32	30
DMU04	572	41	86.0	89.5	42	91.9	45	42	40
DMU05	1 011	63	96.0	100.0	74	92.4	79	74	70
DMU06	438	31	99.0	99.8	33	95.2	34	32	30
DMU07	333	23	95.1	96.6	25	95.6	26	25	23
DMU08	351	26	90.6	91.9	26	93.4	28	26	25
DMU09	681	47	91.8	94.0	50	93.9	53	50	47
DMU10	375	32	80.9	82.2	28	94.0	29	28	26
DMU11	220	18	75.8	78.6	17	92.8	18	17	16
DMU12	305	26	79.8	81.2	23	95.1	24	23	21
DMU13	559	49	78.7	78.7	41	97.2	44	41	39
DMU14	853	55	90.2	91.7	63	95.2	66	62	59
DMU15	731	50	100.0	100.0	54	98.3	57	54	50
DMU16	356	26	88.9	90.5	27	93.2	28	26	25
DMU17	208	17	83.6	83.6	16	96.0	17	16	15
DMU18	297	19	91.5	92.2	22	94.6	23	22	21
DMU19	720	60	70.1	71.1	53	95.9	56	53	50

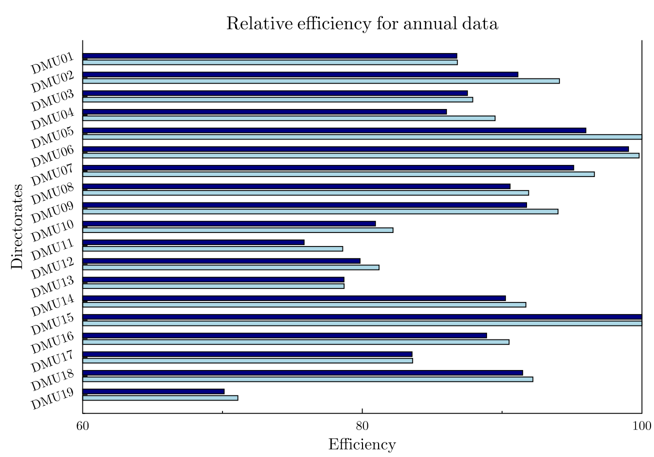

Fig. 1. Relative efficiencies for the initial allocation (1 output (navy) and 2 outputs (light blue)).

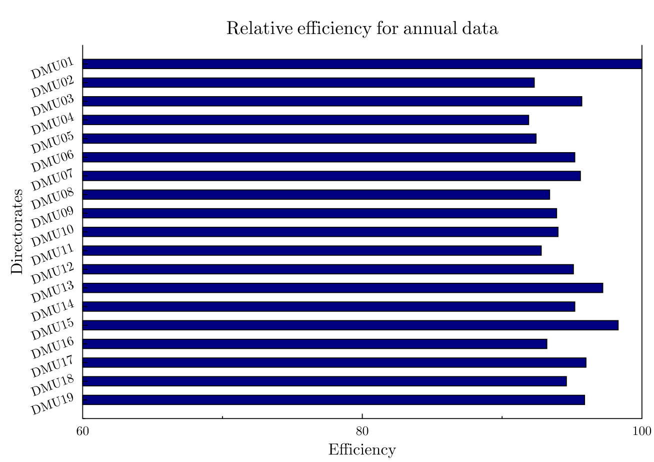

Fig. 2. Relative efficiencies calculated for the optimal allocation.

item numbers taken into account. The results are summarized in Fig. 1 (navy).

It is clear from Table 7, column 4 and Fig. 1 (navy) that only the DMU15 County Directorate proved to be an efficient organizational unit (100% efficiency). This means that based on peak monthly data, DMU15 County used its inputs (the accountants) efficiently, while the other county directorates operated at a lower efficiency, thus using more manpower than necessary.

The DMU06 County Directorate ranks second with 99.0% efficiency, while the DMU05 County Directorate ranks third with 96.0% efficiency. Given that we have used an input-oriented model, these values indicate that instead of 63 people working in the DMU05 County Board, 60 people would be required to perform this task effectively ($63 \times 0.9599 = 60.48$).

The three least efficient counties are DMU19 (70.1%), DMU11 (75.8%) and DMU13 (78.7%).

5.2. Relative efficiency with multiple outputs

Taking the complicating factors into account changes the picture slightly. Directorates that handle more difficult cases are somewhat more efficient than the values in the previous model. Efficiency values that take into account complicating factors are summarized in Table 7, column 5 and Fig. 1 (light blue).

In this case, the enclosed efficiency table does not show a significant change from the previous results. Considering the complicating factors, DMU15 also performed best as the only effective directorate, but now DMU05 joined in this rank. This is not so surprising as DMU05 has a high number of complicating cases so it is unsurprising that it ranks higher now that we take these into account.

DMU06 county ranked third with a 99.8% efficiency rate, ahead of DMU07. It is for the same reason that the order of the middle field also changed, and the efficiency of the least efficient county also improved slightly in the previous case: DMU19 (71.1%).

5.3. The optimal allocation of workforce

Apportionment is based on a single claim vector; seeing the importance of complicating factors, we have taken a weighted sum of all, that is, both core and complicating tasks.

How do we take into account the different monthly data? The purpose of our evaluation is to minimize the overload of accountants. Assuming the headcount is constant throughout the year, overload is the most critical of the months with the most tasks, so this peak load should be considered in the analysis. Accordingly, for each county, we select *highest* monthly task volume as the basis for the calculations.

Our primary result was the effective allocation of the 854 positions available based on the most recent headcount. The results obtained using the Leximinimax (or Adams) algorithm are summarized in Table 7.

At the same time, the allocation method also provides an opportunity to look at the impact of possible headcount changes. Columns 8–10 of Table 7 show the optimal allocation for different aggregate headcount data.

5.4. Relative efficiency for the optimal allocation

Fig. 2 and Table 7 illustrate the optimal, Leximinimax apportionment's impact on efficiency. A total of 854 people were distributed to the county directorates through an optimization algorithm.

We immediately noticed the significant change from previous results. Due to the optimal, Leximinimax (or Adams) apportionment of staff, the values of efficiency indicators equalize, and each county directorate achieves efficiency above 90%. This time, the directorate of the DMU01 counties took the lead from the DMU15 County Board, which was ranked first in the previous models.

We would like to emphasize that these results refer to a hypothetical redistribution of headcount, and the purpose of the study was solely to test the apportionment. The current work contracts and several employee preferences may impede the implementation of the suggested redistribution of employees. A careful management evaluation of the costs and benefits is needed before decisions on workforce redistribution are made. However, it is important to note that the recommended assignment is based on the overload of the workers and not on other management considerations.

Previously, DMU15 was the most efficient county, completing about 14 620 tasks per worker per month. With the new allocation, the most overloaded workers are in DMU01, where they complete about 13 610 tasks in the same period. This corresponds to a 7.4% improvement!

6. Summary

In the previous sections, we presented the efficiency analysis of the salary accounting activities of the Hungarian State Treasury using data envelopment analysis. The results were obtained by solving input-oriented radial models assuming a constant return to scale operation.

Input orientation means that the accounting items as the tasks to be performed are given, and our focus is finding a suitable level of inputs, that is, a suitable level of employment.

Based on the constant return to scale property, we assume that the contribution of one employee to the output is constant and independent of the total size of the workforce. This means that the hiring or dismissal of another employee has the same marginal impact on all departments.

Finally, the radial efficiency indicator expresses the percentage reduction recommended for each county directorate.

Calculations are based on actual data for the period September 2017 - August 2018. Organizations with an efficiency ratio of 100% are considered effective. Less efficiency is an efficiency problem, even if there is only a slight deviation from 100

It is important to emphasize that there are many reasons for inefficient operation, not necessarily due to a lack of work or work organization. Effectiveness can be affected by statutory, labour constraints, staffing specialities, and various local circumstances that affect work performance and are not reflected in the data used for analysis. Exploring these local characteristics is a must when exploring ways to improve efficiency.

Finally, the nine complicating factors were considered by simply adding their values. If this means over-simplification, then it is necessary to quantify (standardize) the impact of the different task types. In the absence of this, no more differentiated examination is possible. However, the results show that although the effect of complicating factors is quantifiable, it does not significantly affect the nature of the result obtained.

Efficiency results can shed light on many reasons for inefficient operation, help to think about aspects of reorganizing tasks and/or human resource management and provide information on decisions to improve payroll organization and technology.

7. Discussion

Several alternative approaches have been considered in developing the study methodology. We briefly discuss these as possible trails for future research.

7.1. Headcounts at the county level or at institutional accounting places (IAP)

The accounting work is organized in the county offices, but most of the offices carry out several IAP's tasks - and IAP nr. 21 and 29 have employees in all the county directorates. This work organization is justified by the special legal regulations of each IAP, as certain accounting tasks require special expertise and are therefore grouped together for practical considerations. It would be obvious to allocate the statuses to the level of the individual IAP instead of the county offices.

We also had the April 2018 data at the IAP level, so we conducted a comparative analysis of county and IAP-based allocations. It is important to note that the more sections, and especially the smaller the task volumes, are assigned to, the less proportional the assignment will be, but the results obtained by the two methods differ marginally, the difference is much smaller than in comparison with the current figures.

7.2. Other positions

Our study aimed to determine the number of accountants in the county, while other jobs, such as data reporting, auditing and management, were ignored, as these can be determined from the number of accountants.

However, one must consider that while the method strives for proportionality, the number of accountants is rounded. With a further conversion - with another inaccuracy - we get the headcount for further jobs. Inaccuracies add up, so there may be significant differences in the workload of these additional jobs, so directly assigning such statuses using the same method may result in a more efficient operation.

7.3. Management of overload

Our study aimed to minimize the maximum workload to achieve a quality assurance optimum. For this purpose, the month with each county office's highest number of tasks was considered. Is it advisable to use more accountants in a single outbound month, or is it more cost-effective to work this overtime? If the headcount is determined by a peak month, the resulting high number of accountants will work at a low efficiency for the rest of the year.

Alternatively, we can work with the average monthly workload (or the total annual workload, which gives the same result). Such a solution assumes a more flexible working time management and determines the optimal number of employees based on this.

Due to the nature of the task of the accountants, the core monthly workload hardly changes: in the period studied, the workload varied less than 17.5% at the monthly level. On the other hand, the load due to complicating factors is more volatile, and overall, we see a difference of about 40% for different months for the same DMU.

Table 8
Core task numbers for the months of April 2018 to March 2019 in thousands.

Directorate	Apr	May	June	July	Aug	Sept	Oct	Nov	Dec	Jan	Feb	Mar	max
DMU01	1125	1117	1103	1052	1035	1056	1064	1060	1040	1043	1042	1034	1125
DMU02	293	288	284	274	271	278	283	282	277	275	274	271	293
DMU03	196	193	191	179	175	184	187	185	179	180	179	175	196
DMU04	237	235	232	224	220	223	225	223	220	220	220	214	237
DMU05	405	391	379	371	366	365	369	367	362	362	358	355	405
DMU06	196	201	194	181	178	187	190	190	186	185	187	185	201
DMU07	145	149	145	134	133	139	142	140	138	137	136	131	149
DMU08	155	156	153	139	138	147	149	148	145	144	143	142	156
DMU09	285	287	277	268	261	268	270	269	259	260	258	248	287
DMU10	169	168	164	156	154	159	160	159	155	153	152	151	169
DMU11	95	94	91	92	88	92	93	93	91	91	92	90	95
DMU12	132	131	130	127	126	127	128	127	126	125	125	124	132
DMU13	258	258	255	241	239	241	243	241	238	236	235	231	258
DMU14	346	343	339	327	323	320	319	315	308	307	304	301	346
DMU15	339	339	331	308	301	314	318	315	310	304	299	295	339
DMU16	158	157	153	144	141	147	148	147	143	142	141	135	158
DMU17	94	93	91	86	85	89	90	89	88	89	89	88	94
DMU18	135	134	132	125	123	126	128	126	124	123	122	121	135
DMU19	296	290	286	266	265	270	272	272	269	268	267	257	296

7.4. Absolute or relative efficiency

The efficiency of the — up to the constraints of apportionment — optimal allocations we found can be verified like the actual data with a DEA analysis. Because the method seeks task-based distribution, the DEA analysis yields nearly the same efficiency, usually over 95% for each county. This represents an increase in efficiency in most counties compared to the current DEA results. It is important to note that DEA is a *relative* efficiency analysis method, so what happens is that overloaded, seemingly highly efficient counties receive additional calculators, their efficiency decreases, and so the efficiency of every other county increases.

Neither method directly estimates the optimal number of accountants to perform the task. Data from periods when a small amount of overtime was needed can help determine this, as it may be assumed that the branch was operating at peak performance, but this requires further analysis. It should be noted that the efficiency of a branch office may depend not only on the number of employees but also on other circumstances, which are not considered here.

CRediT authorship contribution statement

László Á. Kóczy: Writing – original draft, Project administration, Investigation, Formal analysis, Writing – review & editing, Visualization, Methodology, Funding acquisition, Data curation, Conceptualization. **Tamás Koltai:** Software, Conceptualization, Writing – review & editing, Methodology. **Balázs R. Sziklai:** Conceptualization, Formal analysis, Methodology, Data curation, Writing – review & editing.

Declaration of competing interest

The authors declare the following financial interests/personal relationships which may be considered as potential competing interests: Laszlo A. Koczy reports financial support was provided by National Research Development and Innovation Office. Balazs R. Sziklai reports financial support was provided by National Research Development and Innovation Office. If there are other authors, they declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Appendix

Theorem 4. Let $H \in \mathbb{N}$ and $\mathbf{p} = (p_1, \ldots, p_n)$ be such that $\frac{p_i}{k_1} \neq \frac{p_j}{k_2}$ whenever $i \neq j$ for any $k_1, k_2 \in \mathbb{N}$. The MILP described in (MILP1) computes the Adams apportionment for any such input $(\mathbf{p}, H)$.

Proof. The Adams method is determined by finding a divisor t such that

$$\sum_i \left\lceil \frac{p_i}{t} \right\rceil = H.$$

In this way, the Adams method prescribes $a_i = \left\lceil \frac{p_i}{t} \right\rceil$ number of seats for state i. Suppose $(q^*, \mathbf{a})$ is the optimal solution for the MILP. We prove that $1/q^*$ is a good choice for t.

On one hand $q^* p_i \leq a_i$ implies $\lceil q^* p_i \rceil \leq a_i$ as $a_i \in \mathbb{N}$. Thus,

$$\sum_i \lceil q^* p_i \rceil \leq \sum_i a_i = H.$$

We need to prove that the sum is bounded by H from below as well. We proceed by contradiction. Suppose that $\sum_i \lceil q^* p_i \rceil < H$. It follows that there exist ℓ for which $\lceil q^* p_\ell \rceil \leq a_\ell - 1$. We will show that it is possible to improve on q^* and still satisfy the constraints of the linear program.

Since $\frac{p_i}{k_1} \neq \frac{p_j}{k_2}$ for any $k_1, k_2 \in \mathbb{N}$, it follows that at most one of the $q^* p_i \leq a_i$ inequalities holds with equality. Let m be the index for which $q^* p_m = a_m$ and let

$$\varepsilon = \min \left\{ \frac{a_\ell - 1}{p_\ell} - q^*, \frac{a_m + 1}{p_m} - q^*, \min_{i \neq \ell, m} \left\{ \frac{a_i}{p_i} - q^* \right\} \right\}.$$

By the choice of ε, the constraint $(q^* + \varepsilon) p_i \leq a_i$ still holds for all i except m. Let $\hat{q} = q^* + \varepsilon$, then

$$\begin{aligned} \lceil \hat{q} p_i \rceil &\leq a_i && \text{if } i \notin \{\ell, m\}, \\ \lceil \hat{q} p_m \rceil &\leq a_m + 1 \\ \lceil \hat{q} p_\ell \rceil &\leq a_\ell - 1 \end{aligned}$$

Let $\mathbf{b}$ be such that $b_m = a_m + 1$, $b_\ell = a_\ell - 1$, and $b_i = a_i$ for $i \neq \ell, m$. Then $(\hat{q}, \mathbf{b})$ is a feasible solution for the MILP and $\hat{q} > q^*$ contradicts the optimality of q^*. □

Table 9
Complicating task numbers for the months of April 2018 to March 2019 in thousands.

Directorate	Apr	May	June	July	Aug	Sept	Oct	Nov	Dec	Jan	Feb	Mar	max
DMU01	1298	1222	1242	1036	935	1089	1128	1131	1078	1117	1218	1114	1298
DMU02	358	335	376	321	283	306	318	327	323	306	403	350	383
DMU03	227	208	236	186	165	185	191	193	180	191	244	206	232
DMU04	287	296	307	262	234	250	260	263	244	251	352	286	336
DMU05	497	463	534	452	378	382	402	398	389	393	653	526	606
DMU06	229	237	237	190	171	199	205	202	191	195	240	207	237
DMU07	173	177	187	142	129	145	158	152	151	153	177	150	184
DMU08	194	188	198	144	134	158	160	161	158	161	171	163	195
DMU09	322	334	352	304	258	296	292	295	268	273	423	327	395
DMU10	199	207	208	166	143	163	171	171	163	168	205	190	205
DMU11	126	120	119	103	87	103	109	111	107	107	128	111	126
DMU12	148	141	175	140	123	127	135	133	169	131	171	148	173
DMU13	291	280	304	244	225	241	257	251	246	240	311	264	300
DMU14	376	347	452	374	309	310	322	318	299	298	549	434	507
DMU15	367	366	400	315	282	315	331	329	322	318	382	342	392
DMU16	190	179	203	160	134	154	160	163	151	155	198	163	198
DMU17	114	103	106	85	76	88	92	89	88	88	108	97	114
DMU18	162	152	165	131	115	129	136	133	134	134	155	135	163
DMU19	317	321	434	323	243	259	270	270	262	261	411	324	424

Appendix A. Source data

See Table 8.

Appendix B. Source data

See Table 9.

Data availability

Data will be made available on request.

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