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Stationary probabilities and the monotone likelihood ratio in bonus-malus systems

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Abstract
The bonus-malus system (BMS) is a widely recognized and commonly employed risk management tool. A well-designed BMS can match expected insurance payments with estimated claims even in a diverse group of risks. Although there has been abundant research on improving bonus-malus (BM) systems, one important aspect has been overlooked: the stationary probability of a BMS satisfies the monotone likelihood ratio property. The monotone likelihood ratio for stationary probabilities allows us to better understand how riskier policyholders are more likely to remain in higher premium categories, while less risky policyholders are more likely to move toward lower premiums. This study establishes this property for BMSs that are described by an ergodic Markov chain with one possible claim and a transition rule $+1/\text{-}d$. We derive this result from the linear recurrences that characterize the stationary distribution; this represents a novel analytical approach in this domain. We also illustrate the practical implications of our findings: in the BM design problem, the premium scale is automatically monotonic.

1. Introduction

In insurance markets, bonus-malus (BM) systems (BMS) play a pivotal role in adjusting premiums according to the claim history of policyholders by rewarding no-claim periods with premium reductions (bonus) and penalizing claim events with increases (malus). The efficiency and fairness of BMS hinge on its parameters, namely the transition rules and the premium scales.

Transition rules determine the long-run distribution of policyholders across different BM classes (premium levels).

The monotone likelihood ratio property (MLRP) is a critical concept in economics, decision theory, and actuarial science, which plays a central role in analyzing risk and pricing models. This property establishes that the likelihood ratio between two outcomes is monotonic with respect to an underlying variable, such as risk level or claim frequency. In the context of bonus-malus systems (BMSs), proving the MLRP for stationary probabilities allows us to better understand how riskier policyholders are more likely to remain in higher premium categories, while less risky policyholders are more likely to move toward lower premiums.

The MLRP is vital for several reasons: it ensures monotonicity in the relationship between claim frequency and premium levels, and it serves as a foundation for stochastic dominance, making it an essential tool in optimizing premium adjustments. This paper aims to demonstrate that the stationary probabilities in a BMS satisfy the MLRP, a result that contributes both to the theoretical understanding of BMS and its practical implementation in insurance pricing.

In this paper, we model BMS as an ergodic Markov chain. We limit our investigation to the case where only one claim is possible in each period and the transition rule is $+1/\text{-}d$: in case of no claim, the policyholder moves one class upward; and moves d classes downward in case of a claim. We prove that the stationary probabilities of this ergodic Markov chain satisfy the MLRP. This property has has many theoretical and practical consequences. One such consequence, discussed in the last part of the paper, is the monotonicity of the premium scale in a proper BM design problem.

1.1 Literature review and theoretical foundation

A BMS is characterized by several parameters, such as the number of BM classes, the transition rules, and possibly the initial BM class. In traditional BM design, the transition rules, which are determined by the number of claims, are fixed, and the objective is to identify the optimal set of premiums (referred to as the premium scale or relativities) assuming some underlying statistical model and an optimality criterion (or decision model). In early works, the transition rule was based solely on the number of claims; see seminal works by Lemaire, 1995 later Tremblay, 1992; Walhin and Paris, 1992; Coene and Doray, 1996; Kaas *et al*., 2001, and a more recent study by Heras *et al*., 2004). Later attention shifted to situations in which transition rules are determined by both the number of claims and their severity (see Sammartini, 1990; Pinquet, 1997; Frangos and Vrontos, 2001; Tzougas *et al*., 2014). Recent research indicates that, when transition rules are also involved in the design problem and are determined inside the mathematical model, they are not outside parameters (Tan *et al*., 2015; Ágoston and Gyetvai, 2020; Boucher, 2022). An interesting experience based system – but not BMS – is presented in Martinek and Arató (2019).

In an actuarial application, we must set premiums for an entire risk community where there are observable and unobservable risk factors. Observable risk factors can be considered directly, but unobservable risk factors can be taken into account via experience rating, as with BMS. Several studies have investigated cases where premiums (for observable factors) and relativities (for unobservable factors) are determined jointly (see Taylor, 1997; Pitrebois *et al*., 2003; Tan, 2015).

Besides adverse selection, moral hazard is another motivation for BMS. Since the premium is higher in lower (worse) BM classes, the policyholder is incentivized to increase their level of effort. It is somewhat surprising that, although moral hazard is frequently mentioned in BMS-related work, theoretical models in BM literature that concentrate on the issue of moral hazard are not too frequent. One refreshing exception is Vandebroek (1993). While economic (and actuarial) theory tends to neglect joint studies on BMS and moral hazard, there is extensive research on the empirical testing of moral hazard in BMS (see, for example, Abbring *et al*., 2003). A special topic within moral hazard is bonus hunger, which has attracted more attention than the general case itself (see von Lanzenauer, 1974; Dellaert et al., 1990; Holtan, 2001).

MLRP is a statistical concept that illustrates how the likelihood of observing specific data increases monotonically in relation to a parameter of interest, thus enabling comparisons among various distributions (see, for example, Karlin and Rubin, 1956; Rukhin *et al*., 2009).

In addition to statistics, the MLRP is essential in decision theory, facilitating the ranking of hypotheses according to their likelihoods and ensuring that models correspond with intuitive concepts of risk assessment and behavior. In numerous economic models, the MLRP is a necessary condition to obtain results (see Milgrom, 1981; Jewitt, 1991; Mensch, 2021).

From an actuarial perspective Winter (1992) is of particular interest. In this work, he derives a Pareto-optimal insurance policy in the presence of moral hazard. This is only achievable if the MLRP is valid for the loss distribution. Moral hazard is a common issue in BMS, and the MLRP can be a useful tool for modeling and analyzing it.

The standard mathematical model for BMS analysis is a discrete-time Markov chain. A unique stationary distribution exists if the Markov chain is irreducible and aperiodic (Kemeny and Snell, 1976), which are fair assumptions in the case of a BMS. For a discrete-time Markov chain, the MLRP is merely a sensitivity analysis. This latter field has been the subject of numerous studies (see Schweitzer, 1968; Golub and Meyer, 1986; Funderlic and Meyer, 1986 and Caswell, 2013). Instead, in this paper, we show that the MLRP can be derived from the linear recursions characterizing the stationary distribution.

The remainder of the paper is as follows. In Section 2, we render the necessity foundation for BMS. In Section 3, we introduce linear recursions for stationary probabilities of a BMS. In Lemma 1, we show a key property of this recursion: the order of the linear recursion can be reduced by one, furthermore, in this reduced form all the coefficients are positive. Building on this, we establish our main result in Theorem 1, in which we prove the MLRP for the stationary probabilities of BMS. Section 4 presents an important application of the MLRP (both from a practical and theoretical perspective) in certain BMS design problems, where it ensures that premiums will be monotonic even if this is not explicitly prescribed. This is an obviously desirable property that has not been established to hold in the literature. In this section, we also provide numerical examples to further illustrate the consequences of the MLRP. In Section 5, we summarize our findings.

2. Preliminaries

A BMS with K classes, indexed from 1 to K, is the subject of our investigation. In the BM system, in each year, the participants move u class upward (or get into the top class if they cannot move u classes upward) if they do not have claims and move d classes downward (or get into the bottom class) if they have claims. The probability that a participant has at least a claim in a time period is denoted by q, and we make the standard assumption that the probability of more than one claim during a period is negligible. (Equivalently, further claims do not affect the claimant's class.)

A common tool for modeling BMS is the ergodic Markov chain. In our case, the $+1/\text{-}d$ transition rule ensures the irreducibility and aperiodicity of the Markov chain, so this is not an additional limitation.

The transition matrix $\mathbf{T}$ of this Markov chain is made up of entries t_{ij} that indicate the probability that a customer in class i during a period will move to class j in the next period. The transition matrix will be represented by $\mathbf{T}(q)$, highlighting its dependence on the probability of claims.

The probability that a policyholder belongs to a BM class i during period t is shown by the value $c_{ti}(q)$; the values $c_{ti}(q)$ are arranged into a vector $\mathbf{c}_t(q)$. As t tends to infinity, $\mathbf{c}_t(q)$ tends to a unique stationary distribution $\mathbf{c}(q)$ due to the ergodicity of the Markov chain, which is a solution to the system of equations

$$\mathbf{c}(q)^\top \mathbf{T}(q) = \mathbf{c}(q)^\top \,. \tag{2.1}$$

In other words, $\mathbf{c}(q)$ is the left eigenvector of eigenvalue 1 of matrix $\mathbf{T}(q)$; see Kaas *et al*. (2001), Kemeny and Snell (1976), Lemaire (1995). The target of our investigation is how $c_i(q)$ depends on the probability of claims q.

The MLRP is a well-known concept in statistics, frequently appearing in many decision problems as a necessary condition for establishing other key, theoretical properties. Let X be a random variable whose distribution depends on an external parameter q. Assume that X has a probability density function $f(x|q)$. The MLRP means that the inequality

$$\frac{f(x_0|q_1)}{f(x_0|q_0)} \geq \frac{f(x_1|q_1)}{f(x_1|q_0)}$$

holds for every $x_0 \leq x_1$ and $q_0 \leq q_1$.

If X is a discrete random variable, the MLRP is defined via probabilities (see Rogerson, 1985 for instance): the inequality

$$\frac{P(x_0|q_1)}{P(x_0|q_0)} \geq \frac{P(x_1|q_1)}{P(x_1|q_0)} \tag{2.2}$$

holds for every $x_0 \leq x_1$ and $q_0 \leq q_1$.

We can rearrange (2.2) into the following form:

$$\frac{P(x_0|q_1)}{P(x_1|q_1)} \geq \frac{P(x_0|q_0)}{P(x_1|q_0)}\,, \tag{2.3}$$

which can also be read as the monotonicity of the quantity $\frac{P(x_0|q)}{P(x_1|q)}$ as a function of q.

Expression (2.3) is the key for defining and checking MLRP for BMSs: we have to check that the fraction

$$\frac{c_{K-i}(q)}{c_{K-(i+1)}(q)} \tag{2.4}$$

is monotone in q (decreasing in our case).

3. Monotone likelihood ratio property in BM systems

In our study, we limit our analysis to the case, where $u = 1$, that is, the policyholder moves one class upward if she/he has no claims. This is the most frequent solution in existing BM systems ("In the classical BMS, the −1/+h transition rule is common," Oh *et al*., 2020, p. 143.) From a mathematical point of view, this assumption (along with the obvious $0 < q < 1$) ensures that the Markov chain is ergodic.

The first equation in the system (2.1) is

$$c_K = (1-q)c_K + (1-q)c_{K-1},$$

suppressing the argument q of each function c_i for better readability. The next $d-1$ equations in the system (2.1) are

$$c_{K-(i-1)} = (1-q)c_{K-i} \quad (i = 2, \ldots, d).$$

Compiling these equations, we get

$$c_{K-i} = \frac{q}{(1-q)^i} c_K \quad (i = 1 \ldots d). \tag{2.4}$$

If $i > d$, then the appropriate equation in the system of Equations (2.1) is

$$c_{K-i} = qc_{K-(i-d)} + (1-q)c_{K-(i+1)}\,.$$

Rearranging this, we get

$$c_{K-(i+1)} = \frac{c_{K-i}}{1-q} - \frac{qc_{K-(i-d)}}{1-q}\,. \tag{3.2}$$

Let us introduce the notation $P_i(q) := \frac{(1-q)^i c_{K-i}(q)}{c_K(q)}$. We now see that

$$P_0(q) = 1, \quad P_i(q) = q \ (i = 1 \ldots d), \tag{3.3}$$

and that the relation (3.2) translates to the following linear recurrence on the functions P_i:

$$\frac{P_{i+1}(q)}{(1-q)^{i+1}} = \frac{P_i(q)}{(1-q)(1-q)^i} - \frac{qP_{i-d}(q)}{(1-q)(1-q)^{i-d}} \quad (i > d),$$

that is,

$$P_{i+1}(q) = P_i(q) - q(1-q)^d P_{i-d}(q) \quad (i > d). \tag{3.4}$$

In particular, we can see that each function P_i is a polynomial of degree no greater than i.

The linear recurrence in (3.4) has order d. In Lemma 1, we show that linear recurrence can be transformed to linear recurrence with degree $d-1$.

Lemma 1. *The sequence of polynomials generated by the initial values (3.3) and recurrence (3.4) also satisfies the following linear recurrence relation:*

$$P_{i+1}(q) = qP_i(q) + q(1-q)P_{i-1}(q) + \cdots + q(1-q)^{d-1}P_{i-(d-1)}(q) \quad (i \geq d). \tag{3.5}$$

Proof. We shall use induction. The statement holds for $i = d - 1$, since the P_d defined by (3.5) is the same as the P_d defined in (3.3):

$$\begin{aligned} P_d(q) &= qP_{d-1}(q) + q(1-q)P_{d-2}(q) + \cdots + q(1-q)^{d-2}P_1(q) + q(1-q)^{d-1}P_0(q) \\ &= q\left(q + q(1-q) + \ldots q(1-q)^{d-2}\right) + q(1-q)^{d-1} \\ &= q. \end{aligned}$$

Now suppose that (3.5) holds for $i = n$, that is,

$$P_{n+1}(q) = qP_n(q) + q(1-q)P_{n-1}(q) + \cdots + q(1-q)^{d-1}P_{n-(d-1)}(q).$$

We see that is valid for $i = n + 1$ as well:

$$\begin{aligned} P_{n+2}(q) &= P_{n+1}(q) - q(1-q)^d P_{n+1-d}(q) \\ &= qP_{n+1}(q) + (1-q)P_{n+1}(q) - q(1-q)^d P_{n-(d-1)}(q) \\ &= qP_{n+1}(q) + (1-q)qP_n(q) + (1-q)q(1-q)P_{n-1}(q) + \ldots \\ &\quad + (1-q)q(1-q)^{d-2}P_{n-(d-2)}(q) + (1-q)q(1-q)^{d-1}P_{n-(d-1)}(q) \\ &\quad - q(1-q)^d P_{n-(d-1)}(q) \\ &= qP_{n+1}(q) + q(1-q)P_n(q) + \cdots + q(1-q)^{d-1}P_{n-(d-2)}(q). \end{aligned}$$

□

Remark 2. The characteristic equation of the linear recurrence (3.4) is

$$t^{d+1} - t^d - q(1-q)^d = 0,$$

which can be rearranged as

$$q(1-q)^d = t^d(t-1).$$

From this form it is obvious that $t = 1 - q$ is a root of the characteristic polynomial. Dividing the characteristic polynomial by $t - (1 - q)$, we get

$$t^d - qt^{d-1} - q(1-q)t^{d-2} - \cdots - q(1-q)^{d-1}, \tag{3.6}$$

which is the characteristic polynomial for recursive relation (3.5).

To prove the MLRP of stationary probabilities, we have to prove that $\frac{c_{K-i}(q)}{c_{K-(i+1)}(q)}$ is monotone decreasing in q. Expressing this in terms of the polynomials P_i, we have to prove that

$$\frac{c_{K-i}(q)}{c_{K-(i+1)}(q)} = \frac{\frac{P_i(q)}{(1-q)^i}}{\frac{P_{i+1}(q)}{(1-q)^{i+1}}} = \frac{(1-q)P_i(q)}{P_{i+1}(q)} \tag{3.7}$$

is monotone decreasing in q.

Ágoston and Gyetvai (2022) proved that this is true for $d = 1$. In Theorem 1, we prove that it also holds for $d \geq 2$.

It is well known (and can verified by direct calculation) that if $\frac{a}{b} < \frac{c}{d}$ for some positive numbers a, b, c, and d, then we also have

$$\frac{a}{b} < \frac{a+c}{b+d} < \frac{c}{d}. \tag{3.8}$$

To prove Theorem 1, we will need the following generalization of this inequality.

Lemma 3. *Suppose that $n \geq 2$ and that the positive numbers $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ satisfy the chain of inequalities*

$$\frac{a_1}{b_1} < \frac{a_2}{b_2} < \cdots < \frac{a_n}{b_n}.$$

Then they also satisfy the following:

$$\frac{a_1}{b_1} < \frac{a_1 + a_2 + \cdots + a_n}{b_1 + b_2 + \cdots + b_n} < \frac{a_n}{b_n}.$$

Proof. Noting that

$$\frac{a_1+\cdots+a_n}{b_1+\cdots+b_n}=\frac{b_1}{B}\frac{a_1}{b_1}+\cdots+\frac{b_n}{B}\frac{a_n}{b_n}, \quad \text{where} \quad B=\sum_{i=1}^{n} b_i,$$

we see that $\frac{a_1+\cdots+a_n}{b_1+\cdots+b_n}$ is a convex combination of the fractions $\frac{a_i}{b_i}$ $(i=1,\ldots,n)$; therefore, it must be between the smallest and the largest of these fractions, as claimed. □

We are now ready to prove our main result.

Theorem 4. *The MLRP holds for stationary probabilities of a BMS for every $d \geq 2$.*

Proof. Based on our earlier observations and notation, we need to show that for every $i \geq 0$ and $0<q_1<q_2<1$, we have

$$\frac{(1-q_1)P_i(q_1)}{(1-q_2)P_i(q_2)} > \frac{P_{i+1}(q_1)}{P_{i+1}(q_2)}. \tag{3.9}$$

We will, once again, show this by induction. The inequality (3.9) clearly holds for $i=0,\ldots,d$, since substituting the initial conditions into the inequality (3.3) yields the easily verifiable inequalities

$$\frac{1-q_1}{1-q_2} > \frac{q_1}{q_2} \quad \text{and} \quad \frac{(1-q_1)q_1}{(1-q_2)q_2} > \frac{q_1}{q_2}.$$

Now, for the inductive step, suppose that the inequality (3.9) holds for $i=k-d, k-d+1,\ldots,k$; we need to show that it is true for $i=k+1$ as well, that is,

$$\frac{(1-q_1)P_{k+1}(q_1)}{(1-q_2)P_{k+1}(q_2)} > \frac{P_{k+2}(q_1)}{P_{k+2}(q_2)}. \tag{3.10}$$

By our inductive assumption that (3.9) holds for $i<k$, we know that

$$\frac{q_1P_k(q_1)}{q_2P_k(q_2)} < \frac{q_1(1-q_1)P_{k-1}(q_1)}{q_2(1-q_2)P_{k-1}(q_2)} < \cdots < \frac{q_1(1-q_1)^{d-1}P_{k-d+1}(q_1)}{q_2(1-q_2)^{d-1}P_{k-d+1}(q_2)}.$$

Now the recurrence relation (3.5) and repeated application of Lemma 2 yield

$$\begin{aligned}\frac{P_{k+1}(q_1)}{P_{k+1}(q_2)} &= \frac{q_1P_k(q_1)+\cdots+q_1(1-q_1)^{d-2}P_{k+2-d}(q_1)+q_1(1-q_1)^{d-1}P_{k+1-d}(q_1)}{q_2P_k(q_2)+\cdots+q_2(1-q_2)^{d-2}P_{k+2-d}(q_2)+q_2(1-q_2)^{d-1}P_{k+1-d}(q_2)}\\ &> \frac{q_1P_k(q_1)+\cdots+q_1(1-q_1)^{d-2}P_{k+2-d}(q_1)}{q_2P_k(q_2)+\cdots+q_2(1-q_2)^{d-2}P_{k+2-d}(q_2)}.\end{aligned}$$

Rearranging this inequality, we can write

$$\begin{aligned}&\frac{1}{P_{k+1}(q_2)}\left(q_2P_k(q_2)+\cdots+q_2(1-q_2)^{d-2}P_{k+2-d}(q_2)\right) >\\ &\qquad > \frac{1}{P_{k+1}(q_1)}\left(q_1P_k(q_1)+\cdots+q_1(1-q_1)^{d-2}P_{k+2-d}(q_1)\right).\end{aligned} \tag{3.11}$$

Finally, we add (3.11) and the already established inequality $\frac{q_2}{1-q_2} > \frac{q_1}{1-q_1}$ to obtain

$$\begin{aligned}&\frac{q_2}{1-q_2}+\frac{1}{P_{k+1}(q_2)}\left(q_2P_k(q_2)+\cdots+q_2(1-q_2)^{d-2}P_{k+2-d}(q_2)\right) >\\ &\qquad > \frac{q_1}{1-q_1}+\frac{1}{P_{k+1}(q_1)}\left(q_1P_k(q_1)+\cdots+q_1(1-q_1)^{d-2}P_{k+2-d}(q_1)\right),\end{aligned} \tag{3.12}$$

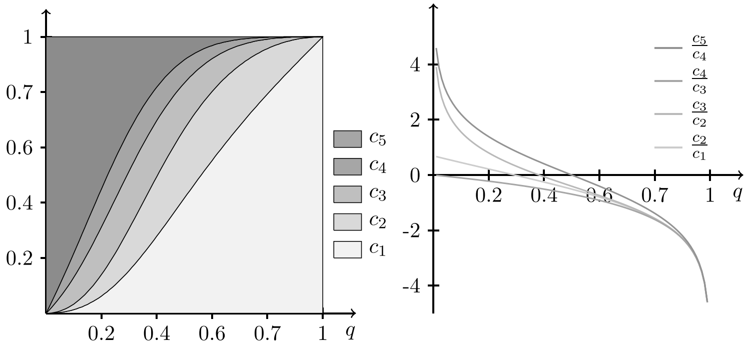

Figure 1. *Stationary probabilities and log likelihood ratios for a five-class BMS with a transition rule +1/-2.*

which in turn can be rearranged as

$$\frac{(1-q_1)P_{k+1}(q_1)}{(1-q_2)P_{k+1}(q_2)} > \frac{q_1P_{k+1}(q_1)+q_1(1-q_1)P_k(q_1)+\cdots+q_1(1-q_1)^{d-1}P_{k+2-d}(q_1)}{q_2P_{k+1}(q_2)+q_2(1-q_2)P_k(q_2)+\cdots+q_2(1-q_2)^{d-1}P_{k+2-d}(q_2)}.$$

Recalling again the recurrence (3.5), we see that this is exactly the desired inequality (3.10), completing our proof. □

Remark 5. The proof also works for the transition rule +1/-1. According to Lemma 1, the recursive relationship is $P_{i+1}(q)=qP_i(q)$, so the inequality (3.9) takes the following form:

$$\frac{(1-q_1)P_i(q_1)}{(1-q_2)P_i(q_2)} > \frac{q_1P_i(q_1)}{q_2P_i(q_2)},$$

which is obviously true if $0<q_1<q_2<1$, so we do not need to use induction.

For a visual example, let us consider a BMS with 5 classes and a transition rule +1/-2. Figure 1 shows the stationary probabilities and the log-likelihood ratios.

In Example 6, we demonstrate that the MLRP can hold beyond the case examined in Theorem 4. In Example 7, we present a case, where the stationary distribution of the BM classes does not fulfill the MLRP.

Example 6. *Let us consider a BMS in which up to two claims can occur in each period, and the number of claims follows a binomial distribution with parameter (2,q). There are 4 BM classes. If there are no claims, the policyholder moves one class upward; in the event of a claim, the policyholder moves down one class for each claim (i.e., two classes for two claims). The transition matrix is as follows*

$$\begin{pmatrix} q^2+2q(1-q) & (1-q)^2 & 0 & 0 \\ q^2+2q(1-q) & 0 & (1-q)^2 & 0 \\ q^2 & 2q(1-q) & 0 & (1-q)^2 \\ 0 & q^2 & 2q(1-q) & (1-q)^2 \end{pmatrix}$$

The stationary probabilities are

$$\frac{1}{A}\begin{pmatrix} q\left(-3q(1-q)^4-2(1+q)(1-q)^3+(1-q)+1\right) \\ q\left(-2(1-q)^5+(1-q)^3+(1-q)^2\right) \\ q\left((1-q)^5+(1-q)^4\right) \\ (1-q)^6 \end{pmatrix},$$

Table 1. *Stationary probability and likelihood ratios for the BMS described in Example 6.*

q	c_1	c_2	c_3	c_4	$\frac{c_2}{c_1}$	$\frac{c_3}{c_2}$	$\frac{c_4}{c_3}$
0.1	0.0142	0.0510	0.1776	0.7572	3.6029	3.4819	4.2632
0.2	0.1134	0.1730	0.2569	0.4567	1.5261	1.4845	1.7778
0.3	0.3134	0.2630	0.2160	0.2076	0.8391	0.8215	0.9608
0.4	0.5300	0.2654	0.1309	0.0736	0.5008	0.4932	0.5625
0.5	0.7021	0.2128	0.0638	0.0213	0.3030	0.3000	0.3333
0.6	0.8236	0.1459	0.0257	0.0049	0.1771	0.1761	0.1905
0.7	0.9059	0.0853	0.0080	0.0008	0.0941	0.0939	0.0989
0.8	0.9594	0.0389	0.0016	0.0001	0.0406	0.0405	0.0417
0.9	0.9900	0.0099	0.0001	0.0000	0.0100	0.0100	0.0101

where

$$A = q\left(-3q(1-q)^4 - 2(1+q)(1-q)^3 + (1-q) + 1\right) + \\ q\left(-2(1-q)^5 + (1-q)^3 + (1-q)^2\right) + q((1-q)^5 + (1-q)^4) + (1-q)^6$$

is the normalizing factor ensuring that the probabilities sum to one.

To verify the MLRP, we must check that all these fractions are decreasing in q:

$$\frac{q\left(-2(1-q)^5 + (1-q)^3 + (1-q)^2\right)}{q\left(-3q(1-q)^4 - 2(1+q)(1-q)^3 + (1-q) + 1\right)}$$

$$\frac{q\left((1-q)^5 + (1-q)^4\right)}{q\left(-2(1-q)^5 + (1-q)^3 + (1-q)^2\right)}$$

$$\frac{(1-q)^6}{q\left((1-q)^5 + (1-q)^4\right)}$$

Using algebra, we can directly verify that all of the above expressions are decreasing in q. As a numerical illustration, Table 1 shows the stationary probabilities and the likelihood ratios for a few values of q.

Example 7. *Consider a BMS as described in Example 6 except that the claim distribution is special: the probability of no claim is* $0.9(1-q)$*, the probability of one claim is q and the probability of two claims is* $0.1(1-q)$*, where parameter q can take values between 0 and 1. The expected claim amount is* $0.2 + 0.8q$*, which is increases with q. However, it is clear that this underlying distribution does not fulfill the MLRP. The transition matrix is*

$$\begin{pmatrix} 0.1(1-q)+q & 0.9(1-q) & 0 & 0 \\ 0.1(1-q)+q & 0 & 0.9(1-q) & 0 \\ 0.1(1-q) & q & 0 & 0.9(1-q) \\ 0 & 0.1(1-q) & q & 0.9(1-q) \end{pmatrix}$$

As can be seen in Table 2, the stationary probabilities do not satisfy the MLRP.

4. Application: optimal premiums across risk groups

To demonstrate the usefulness of the MLRP, let us consider the following problem: In a risk community, there are N different groups (type) of insured, each is homogeneous with respect to the probability of a claim (the claim amount is 1 unit). The relative proportions of the groups are denoted by ϕ^n (where the superscript n is the group index), that is, $\phi^n \geq 0$, $\sum_n \phi^n = 1$, and q^n stands for the corresponding

Table 2. *Stationary probability and likelihood ratios for the BMS described in Example 2.*

q	c_1	c_2	c_3	c_4	$\frac{c_2}{c_1}$	$\frac{c_3}{c_2}$	$\frac{c_4}{c_3}$
0.1	0.0444	0.1133	0.1600	0.6823	2.5546	**1.4119**	4.2632
0.2	0.0808	0.1460	0.2165	0.5566	1.8064	**1.4824**	2.5714
0.3	0.1402	0.1919	0.2471	0.4208	1.3691	**1.2878**	1.7027
0.4	0.231	0.2393	0.2437	0.2860	1.0363	**1.0180**	1.1739
0.5	0.354	0.2709	0.2063	0.1688	0.7652	**0.7615**	0.8182
0.6	0.4985	0.2712	0.1474	0.0829	0.5440	**0.5434**	0.5625
0.7	0.6467	0.2357	0.0859	0.0318	0.3644	**0.3643**	0.3699
0.8	0.7834	0.1711	0.0373	0.0082	0.2183	**0.2183**	0.2195
0.9	0.9013	0.0890	0.0088	0.0009	0.0988	**0.0988**	0.0989

probability of a claim in group n. Without loss of generality, we assume that $q^1 < q^2 < \ldots < q^N$, and we would like to set premiums π_k for each class $k = 1, \ldots, K$ in such a way that each group pays according to its risk, or at least to get as close to this situation as possible. Formally, this amounts to solving the following nonlinear optimization problem:

$$\min_{\pi} \ \mathbb{E}[L(q^n, \pi)] \tag{4.1a}$$

$$\text{s.t. } \pi_k \le \pi_{k-1} \qquad k = 2, \ldots, K \tag{4.1b}$$

$$\pi_k \ge 0 \qquad k = 1, \ldots, K \tag{4.1c}$$

with

$$\mathbb{E}[L(q^n, \pi)] = \sum_{k=1}^{K} \sum_{n=1}^{N} \phi^n c_k^n L(q^n, \pi_k), \tag{4.2}$$

where the decision variables π_k represent premiums, c_k^n is the stationary probability that an insured in group n is in the BM class k, and $L(x,y)$ is the loss function that measures the deviation of the premium from the claim probability; usually the quadratic deviation $L(x, y) = (x - y)^2$ or the absolute deviation $L(x, y) = |x - y|$. The constraints represent obviously desirable characteristics of the premiums set in our BM system – the question is whether they are necessary to add to the optimization problem. It is clear that the nonnegativity constraints will automatically hold in every optimal solution, for every reasonable loss function, regardless of whether the MLRP is satisfied or not. (For example, it is sufficient that L is monotone in $|x - y|$.) In our next theorem, we argue that for the most commonly used loss functions, if the MLRP holds, then the first set of constraints is also automatically satisfied.

Theorem 8. *Let the loss function L be any function of the form $L(q^n, \pi_k) = \ell(q^n - \pi^k)$, where ℓ is an $\mathbb{R} \to \mathbb{R}$ function with the following properties: ℓ is convex and it has a unique minimizer at 0. Then the constraint (4.1b) is not binding at the optimal solution of (4.1).*

Proof. Consider the (unconstrained) optimization problem in which every constraint of (4.1) is relaxed. In this case, the optimal premiums in each BM class k can be determined independently of the other classes. Let π_k^* denote an optimal premium for class k (keeping in mind that the minimizer might not be unique, but the minimizers form an interval; the argument below holds for any choice of minimizer π_k^*). As a function of π_k, the objective function (4.2) is a univariate convex function that is subdifferentiable at π_k^*, and from the optimality of π_k^*, its left derivative is nonpositive at this point:

$$\sum_{n=1}^{N} \phi^n c_k^n \ell'_-(\pi_k^* - q^n) \le 0. \tag{4.3}$$

Let us search for an index j such that $q^j \leq \pi_k^* < q^{j+1}$. Now, by our convexity assumption, each left derivative in the first j terms of (4.3) is nonpositive, while each of the remaining terms is nonnegative:

$$\sum_{n=1}^{N} \phi^n c_k^n \ell'_-(\pi_k^* - q^n) = \sum_{n=1}^{j} \phi^n c_k^n \underbrace{\ell'_-(\pi_k^* - q^n)}_{\geq 0} + \sum_{n=j+1}^{N} \phi^n c_k^n \underbrace{\ell'_-(\pi_k^* - q^n)}_{\leq 0} \leq 0, \tag{4.4}$$

with at least one of the terms being nonzero (otherwise ℓ would be constant on an interval containing 0, contradicting our assumption).

Introducing the notation $r^n = \frac{c_k^n}{c_{k-1}^n}$, recall that the MLRP implies $r^1 > r^2 > \ldots > r^N > 0$. Now the left-hand side of (4.4) can be bounded from below, term-by-term, as

$$0 \geq \sum_{n=1}^{N} \phi^n c_k^n \ell'_-(\pi_k^* - q^n) = \sum_{n=1}^{N} \phi^n c_{k-1}^n r^n \ell'_-(\pi_k^* - q^n) > \sum_{n=1}^{N} \phi^n c_{k-1}^n r^j \ell'_-(\pi_k^* - q^n). \tag{4.5}$$

Note that the last sum is simply the left derivative of our objective function with respect to π_{k-1} at π_k^*. Because it is strictly negative, the optimal value π_{k-1}^* cannot be smaller than π_k^*. Therefore, at the optimal solutions, $\pi_k^* \leq \pi_{k-1}^*$, as claimed. □

Consider the BMS described in Example 7 and the data in Table 2. For simplicity, let there be two types in equal proportion, with parameter values of 0.1 and 0.2. In this case, the unconstrained model gives the following premium scale for the quadratic loss function and a claim amount 1000: 331.7, 325, 326, and 315.9. We can see that premium scale is not monotonic.

Finally, we would like to draw the attention to the choice of the objective function. Instead of (4.2), the following objective function is often used:

$$\min_{\pi} \sum_{n=1}^{N} \phi^n L\left(\sum_{k=1}^{K} c_k^n q^n, \sum_{k=1}^{K} c_k^n \pi_k\right) \tag{4.6}$$

In (4.6), we aim to minimize the difference between the expected claim amount and the expected premium payment. This sounds reasonable at first but in Example 9 we show that the premium scale is not necessarily monotonic, even if the MLRP is fulfilled for stationary probabilities.

Example 9. *Let us consider the BMS described in Example 1. There are two types with equal proportion and with parameter value 0.1 and 0.2; the claim amount is 1000. We minimize (4.6) without the constraints (4.1b). It is not hard to see that the objective function will be zero, and even for a zero objective value we obtain an underdetermined system of equations with infinitely many solutions. For example, the following premium scale is an optimal solution:* 136.1, 1863.4, *0, and* 136.1. *From an actuarial viewpoint, however, this solution is senseless. With additional constraints, we can certainly ensure a monotonic premium scale, but this solution may still be senseless from an actuarial viewpoint, since the lowest premium could be zero. Further constraints would therefore be needed to achieve a reasonable premium scale. This example shows that, although monotonicity of the premium scale can be enforced using additional constraints, this could potentially obscure inherent issues with the model itself.*

5. Conclusions

We have explored the MLRP within the context of BM systems, establishing its significance in the pricing and risk assessment of insurance policies. Our findings demonstrate that the stationary probabilities of policyholders in a BMS adhere to the MLRP.

We derived the recursive relationships governing the stationary probabilities and using this linear recurrence we proved that stationary probabilities maintain the MLRP under various transition rules, including when multiple downward movements are allowed. Our results extend previous findings, such as those by Ágoston and Gyetvai (2022), to cover more general cases, demonstrating that the MLRP holds even when the number of downward class movements exceeds one.

A practical implication of this result is that the MLRP ensures the monotonicity of premium scale even when this monotonicity is not explicitly prescribed by the system design.

Further research could extend our results in several ways. The transition rules need not be uniform (i.e., claims in different BM classes may result in different downward moves) and certainly more than one claim can occur in each period. It is also interesting what other constraints that are frequently involved in the BM design problem can be shown to be not binding, but hold automatically, under the MLRP assumption.

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